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General Integral Representation of the Holomorphic Functions on the Analytic Subvariety

By

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§1. Introduction

Henkin[1] and Ramirez[2] obtained an integral representation of holomorphic functions for strictly pseudoconvex domains in $\boldsymbol{C}^n$. Range and Siu[3] gave a generalization of Henkin-Ramirez's formula to the domains in $\boldsymbol{C}^n$ with piecewise smooth strictly pseudoconvex boundaries Sommer[4] proved an integral formula of Weil type for analytic polyhedra in $\boldsymbol{C}^n$. Sergeev and Henkin[5] also obtained an integral representation for the strictly pseudoconvex ployhedra. Stout[6] and Hatziafratis[7] have respectively proved integral formulas for strictly pseudoconvex domains in codimension-one and codimension-m complex submanifolds of $\boldsymbol{C}^n$. The formula which was given by Stout is valid not only for nonsingular hypersurfaces, but also for certain subvarieties which may possess sufficiently restricted singular points. Hatziafratis' work is based on the results of Stout.

In this paper we derive integral formulas which include all the above ([1]–[7]) integral formulas for holomorphic functions. The papers of Stout[6], Hatziafratis[7] and the author[8] are most relevant references to this work.

§2. Definitions, Symbols and Terms

Definition 1 (Polyhedral domain[8]) Let Ω be a domain of holomorphy in $\boldsymbol{C}^n$. An open set $D\subset\subset\Omega$ is called a polyhedral domain if there is a neighbourhood $U_{\bar{D}}$ of $\bar{D}$ and holomorphic mappings:

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$$X_\alpha: U_{\bar{D}} \to C^{m_\alpha},\ \alpha=1,2,\cdots,N,\ \sum_{\alpha=1}^{N} m_\alpha \geqslant n,$$

and $D_\alpha \subset\subset C^{m_\alpha}$, $\alpha=1,2,\cdots,N$, such that

$$D=X_1^{-1}(D_1) \bigcap\cdots\bigcap X_N^{-1}(D_N).$$

If $P_1,\cdots,P_N$ are differentiable functions in the neighbourhoods $\theta_1,\cdots,\theta_N$ of $\partial D_1,\cdots,\partial D_N$ respectively, and

$$D_\alpha \bigcap \theta_\alpha=\{z\in\theta_\alpha: P_\alpha(z)<0\},\ \alpha=1,2,\cdots,N,$$

then $\partial D\subseteq X_1^{-1}(\theta_1)\bigcup\cdots\bigcup X_N^{-1}(\theta_N)$ and a point $z\in X_1^{-1}(\theta)\bigcup\cdots\bigcup X_N^{-1}(\theta_N)$ belongs to D if and only if $z\in X_\alpha^{-1}(\theta_\alpha)$ and $P[X_\alpha(z)]<0$ for some $\alpha: 1\leqslant\alpha\leqslant N$. D is called a non-degenerate polyhedral domain, if we can choose the functions X_α and P_α so that

$$d(P_{\alpha_1}\cdot X_{\alpha_1})(z)\wedge\cdots\wedge d(P_{\alpha_l}\cdot Z_{\alpha_l})(z)\neq 0.$$

Whenever $P_{\alpha_1}[X_{\alpha_1}(z)]=\cdots=P_{\alpha_l}[X_{\alpha_l}(z)]=0$, for all $1\leqslant\alpha_1<\cdots<\alpha_l\cdots\leqslant N$.

In this paper we only consider non-degenerate polyhedral domains.

A nondegenerate pohyhedral domain will be called a strictly pseudoconvex polyhedron if $P_\alpha(\alpha=1,2,\cdots,N)$ are strictly plurisubharmonic functions; and called a holomorphic polyhedron (including Weil polyhedron) if the mapping $P_\alpha(\alpha=1,2,\cdots,N)$ are pluriharmonic functions (or usual harmonic functions when $n=1$), i.e. $P_\alpha(\alpha=1,2,\cdots,N)$ are twice continuously differentiable and $\partial^2 P_\alpha/\partial z_j\partial\bar{z}_k=0$, $j,k=1,2,\cdots,n$. There exist continuously differentiable support functions for the nondegenerate polyhedral domains, support functions holomorphic in z for strictly pseudoconvex polyhedrons, and holomorphic support functions for the holomorphic polyhedron.

Definition 2 (Space with slits[9]) A compact metric space R is called a slit space or a space with slit S if S is a nonempty closed subsed of R each point of which is an accumulation point of $R-S$ and $R-S$ is homeomorphic to a topological product $X\times Y$, where X is a connected $\tilde{m}$-dimensional differential manifold of class C^2, called the base space, and Y is a compact set, called the side space. The homeomorphism $\varphi: X\times Y\to R-S$ is called the coordinate function.

Example. The closure R of any bounded domain in a $\tilde{n}$-dimensional

Euclidean space $E^{\bar{n}}$ can be considered a slit space with the boundary as its slit. Y is then a set consisting of a single point.

Definition 3. A sequence of spaces $R_1 \supset R_2 \supset \cdots \supset R_k$ is called a chain of slit spaces, if each R_v is a slit space with R_{v+1} as its slit ($v=1,\cdots,k-1$).

Firstly, we consider the following two types of bounded domains $D \subset \Omega$ in $\boldsymbol{C}^n$:

⟨I⟩ Its boundary ∂D consists of a chain of slit spaces, and this chain can be written as:

$$\partial D = \sigma^{(1)} \supset \sigma^{(2)} \supset \cdots \supset \sigma^{(\beta)} = \sigma_1^{(\beta-1)} \supset \sigma_1^{(\beta-1)} \supset \cdots \supset \sigma_k^{(\beta-1)}$$

where $\sigma_{v+1}^{(\beta-1)}$ is the slit of $\sigma_\gamma^{(\beta-1)}$, $\sigma^{(i+1)}$ is the slit of $\sigma^{(i)}$, $\sigma_v^{(\beta-1)} = \bigcup_{j_1<\cdots<j_v} \sigma_{j_1\cdots j_v}^{(\beta-1)}$, $\sigma_{j_1\cdots j_v}^{(\beta-1)}$ is of real dimension $2n-\beta-v+1$; $\sigma^{(i)} = \bigcup_{k_1<\cdots<k_i} \sigma_{k_1\cdots k_i}^{(0)}$, and $\sigma_{k_1\cdots k_i}^{(0)}$ is of real dimension $2n-i$. $\sigma_k^{(\beta-1)}$ is called the distinguished boundary of D.

Example. The closed bicylinder $R=\{(z_1,z_2): |z_1| \leqslant 1,\ |z_2| \leqslant 1\}$ can be considered a space with the boundary $R_1=\{(z_1,z_2): |z_1|=1,\ |z_2| \leqslant 1$ and $|z_1| \leqslant 1,\ |z_2|=1\}$ as slit. Moreover the boundary R_1 can also be considered a space with slit $R_2=\{(z_1,z_2): |z_1|=1,\ |z_2|=1\}$.

⟨II⟩ Its boundary ∂D consists of a chain of slit spaces, and this chain can be written as:

$$\partial D = \sigma^{(1)} \supset \cdots \supset \sigma^{(s)} \supset \sigma^{(t)} \supset \cdots \supset \sigma^{(\eta)} \supset \sigma^{(\beta)} = \sigma_1^{(\beta-1)} \supset \sigma_2^{(\beta-1)} \supset \cdots \supset \sigma_k^{(\beta-1)},$$

where $\sigma^{(\beta)}$ is slit of $\sigma^{(\eta)}$, and the dimensions of $\sigma^{(\eta)}$ may be at least one dimension greater than the dimensions of $\sigma^{(\beta)}$. $\sigma_k^{(\beta-1)}$ is also called the distinguished boundary of D.

Example. The closure of all invariant subspaces of the classical domain[9] consists of a chain of slit spaces mentioned above.

Secondly, if $F_1,\cdots,F_m$ are holomorphic functions in the neighbourhood $U_{\bar{D}}$ of $\bar{D}$, and set

$$Z(F_1,\cdots,F_m) = \{z \in U_{\bar{D}}:\ F_1(z) = \cdots = F_m(z) = 0\}.$$

We asume that $Z(F_1,\cdots,F_m)$ meet ∂D transversally. We set $\tilde{D} = Z(F_1,\cdots,F_m) \bigcap D$, and consider

$\langle \mathrm{I}' \rangle \quad \partial\tilde{D}=\tilde{\sigma}^{(1)}\supset\tilde{\sigma}^{(2)}\supset\cdots\supset\tilde{\sigma}^{(\beta)}=\tilde{\sigma}_1^{(\beta-1)}\supset\tilde{\sigma}_2^{(\beta-1)}\supset\cdots\supset\tilde{\sigma}_k^{(\beta-1)},$

$\langle \mathrm{II}' \rangle \quad \partial\tilde{D}=\tilde{\sigma}^{(1)}\supset\cdots\supset\tilde{\sigma}^{(s)}\supset\tilde{\sigma}^{(t)}\supset\cdots\supset\tilde{\sigma}^{(\eta)}\supset\tilde{\sigma}^{(\beta)}=\tilde{\sigma}_1^{(\beta-1)}\supset\cdots\supset\tilde{\sigma}_k^{(\beta-1)},$

where $\tilde{\sigma}^{(\theta)}=Z(F_1,\cdots,F_m)\bigcap\sigma^{(\theta)}$ and $\tilde{\sigma}_j^{(\beta-1)}=Z(F_1,\cdots,F_m)\bigcap\sigma_k^{(\beta-1)}$.

When $m=0$, $\langle \mathrm{I}' \rangle$ and $\langle \mathrm{II}' \rangle$ coincide with $\langle \mathrm{I} \rangle$ and $\langle \mathrm{II} \rangle$ respectively.

According to Hefer's theorem, we have

$F_l(\zeta)-F_l(z)=\sum_{j=1}^{n}(\zeta_j-z_j)h_{lj}(\zeta,z), l=1,2,\cdots,m,$ where h_{lj} are holomorphic functions on a neighbourhood of $\bar{D}\times\bar{D}$.

§3. Some Lemmas

In what follows let D be a nondegenerate polyhedral domain.

Lemma 1. Let M_1 be a continuously differentiable support function for $\bar{D}$, then we have

$$\bar{\partial}_\zeta \det_{(n)}\left(\frac{N_1}{M_1}, h_1,\cdots,h_k,h_{k+1},\bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right),\cdots,\bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right)\right)$$

$$=F_{k+1}(\zeta)\det_{(n)}\left(\frac{N_1}{M_1}, h_1,\cdots,h_k,\bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right),\cdots,\bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right)\right) \tag{1}$$

on $\bar{D}-Z(F_{k+1})$.

Proof. Since $M_1=M_1(\zeta,z)=\sum_{j=1}^{n}(\zeta_j-z_j)N_{1j}(\zeta,z)$, i.e. $\sum_{j=1}^{n}(\zeta_j-z_j)\frac{N_{1j}}{M_1}=1$, we obtain $\sum_{j=1}^{n}(\zeta_j-z_j)\bar{\partial}_\zeta(N_{1j}/M_1)=0$. Thus we have the following determinant of $(n+1)\times(n+1)$:

$$\begin{vmatrix} 1 & 0 & \cdots & 0 & F_{k+1} & 0 & \cdots & 0 \\ \dfrac{N_{11}}{M_1} & h_{11} & \cdots & h_{k1} & h_{k+11} & \bar{\partial}_\zeta\left(\dfrac{N_{11}}{M_1}\right) & \cdots & \bar{\partial}_\zeta\left(\dfrac{N_{11}}{M_1}\right) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \dfrac{N_{1n}}{M_1} & h_{1n} & \cdots & h_{kn} & h_{k+11} & \bar{\partial}_\zeta\left(\dfrac{N_{1n}}{M_1}\right) & \cdots & \bar{\partial}_\zeta\left(\dfrac{N_{1n}}{M_1}\right) \end{vmatrix} = 0, \qquad (2)$$

on $Z(F_1,\cdots,F_k)-Z(F_{k+1})$. Taking it into account that $F_1,\cdots,F_m$ are the holomorphic functions and $\bar{\partial}_\zeta\bar{\partial}_\zeta(N_{1j}/M_1)=0$, by (2) we have

$$(-1)^{k+1}F_{k+1}(\zeta)\det{}_{(n)}\left(\frac{N_1}{M_1},h_1,\cdots,h_k,\bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right),\cdots,\bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right)\right)$$

$$=\det{}_{(n)}\left(h_1,\cdots,h_k,h_{k+1},\bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right),\cdots,\bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right)\right)$$

$$=(-1)^{k+1}\det{}_{(n)}\left(\bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right),h_1,\cdots,h_{k+1},\bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right),\cdots,\bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right)\right)$$

$$=(-1)^{k+1}\bar{\partial}_\zeta\det{}_{(n)}\left(\frac{N_1}{M_1},h_1,\cdots,h_{k+1},\bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right),\cdots,\bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right)\right).$$

Thus we obtain (1).

Since

$$\bar{\partial}_\zeta\left(\frac{N_{1j}}{M_1}\right)=\frac{\bar{\partial}_\zeta N_{1j}}{M_1}-\frac{N_{1j}}{M_1}\cdot\frac{\bar{\partial}_\zeta M_1}{M_1},$$

we can apply the properties of the determinant and write (1) as:

$$\bar{\partial}_\zeta\frac{1}{M_1^{n-k-1}}\det{}_{(n)}\left(N_1,h_1,\cdots,h_k,h_{k+1},\bar{\partial}_\zeta N_1,\cdots,\bar{\partial}_\zeta N_1\right)$$

$$=\frac{F_{k+1}(\zeta)}{M_1^{n-k}}\det{}_{(n)}\left(N_1,h_1,\cdots,h_k,\bar{\partial}_\zeta N_1,\cdots,\bar{\partial}_\zeta N_1\right). \qquad (3)$$

Especially when $k=0$, we have

$$\bar{\partial}_\zeta \det_{(n)}\left(\frac{N_1}{M_1}, h_1, \frac{\bar{\partial}_\zeta N_1}{M_1}, \cdots, \frac{\bar{\partial}_\zeta N_1}{M_1}\right) = \frac{F_1(\zeta)}{M_1^n} \det_{(n)}(N_1, \bar{\partial}_\zeta N_1, \cdots, \bar{\partial}_\zeta N_1) \tag{4}$$

on $\bar{D} - Z(F_1)$.

Lemma 2. If

$$B_{k+1}^F(\zeta) = \frac{(n-k-1)!}{|\nabla_{k+1}^F(\zeta)|^2} \sum_{1 \leq j_1 < \cdots < j_{k+1} \leq n}$$

$$(-1)^{j_1 + \cdots + j_{k+1}} \frac{\partial(\overline{F_1, \cdots, F_{k+1}})}{\partial(\zeta_{j_1}, \cdots, \zeta_{j_{k+1}})} \bigwedge_{l \neq j_1, \cdots, j_{k+1}} d\zeta_l,$$

where

$$|\nabla_{k+1}^F(\zeta)|^2 = \sum_{1 \leq j_1 < \cdots < j_{k+1} \leq n} \frac{\partial(F_1, \cdots, F_{k+1})}{\partial(\zeta_{j_1}, \cdots, \zeta_{j_{k+1}})}^2 \neq 0$$

(especially $B_0^F(\zeta) = n!\ \omega(\zeta)$, $\omega(\zeta) = d\zeta_1 \wedge \cdots \wedge d\zeta_n$), then we have

$$B_{k+1}^F(\zeta) \wedge dF_{k+1} = \frac{(-1)^{n+k}}{n-k} B_k^F(\zeta) \tag{5}$$

on D.

Remark. When $k=0$, as $|\nabla_1^F(\zeta)|^2 = \sum_{j=1}^{n} \left|\frac{\partial F_1}{\partial \zeta_j}\right|^2$, so (5) may be written as

$$B_1^F(\zeta) \wedge dF_1 = \frac{(n-1)!}{|\nabla_1^F(\zeta)|^2} \sum_{j=1}^{n} (-1)^j \frac{\overline{\partial F_1}}{\partial \zeta_j} d\zeta_1 \wedge \cdots \wedge [d\zeta_j] \wedge \cdots \wedge d\zeta_n \wedge$$

$$\sum_{j=1}^{n} \frac{\partial F_1}{\partial \zeta_j} d\zeta_j$$

$$= (-1)^n (n-1)! d\zeta_1 \wedge \cdots \wedge d\zeta_n = \frac{(-1)^n}{n} B_0^F(\zeta).$$

The proof of Lemma 2[7]. First of all, Notice that $|\nabla_{k+1}^F(\zeta)| \neq 0$ implies $|\nabla_k^F(\zeta)| \neq 0$. Since we can assume that

$$\frac{\partial(F_1,\cdots,F_k)}{\partial(\zeta_{n-k+1},\cdots,\zeta_n)}(\zeta_0)\neq 0,$$

where $\zeta_0\in Z(F_1,\cdots,F_k)$, according to the implicit function theorem, restricted to $Z(F_1,\cdots,F_k)$ locally at a point ζ_0, we have

$$\zeta_{n-k+j}=\tilde{F}_j(\zeta^{(k)}),\ j=1,\cdots.k,\zeta^{(k)}=(\zeta_1,\cdots,\zeta_{n-k})$$

such that

$$F_j(\zeta^{(k)},\ \tilde{F}_1(\zeta^{(k)}),\cdots,\tilde{F}_k(\zeta^{(k)}))\equiv 0,\ j=1,\cdots,k,$$

therefore, the following equations are true,

$$(*)\qquad \frac{\partial F_j}{\partial\zeta_{n-k+1}}\cdot\frac{\partial\tilde{F}_1}{\partial\zeta_i}+\cdots+\frac{\partial F_j}{\partial\zeta_n}\cdot\frac{\partial\tilde{F}_k}{\partial\zeta_i}=-\frac{\partial F_j}{\partial\zeta_i},\ i=1,\cdots,n-k.$$

For a fixed sequence $1\leqslant j_1<\cdots<j_k\leqslant n$, let us assume that

$$1\leqslant j_1<\cdots<j_l\leqslant n-k<n-k+1\leqslant j_{l+1}<\cdots<j_k\leqslant n.$$

Then it follows from (*) that

$$\frac{\partial(\tilde{F}_1,\cdots,\hat{j}_{l+1},\cdots,\hat{j}_k,\cdots,\tilde{F}_k)}{\partial(\zeta_{j_1},\cdots,\zeta_{j_l})}=(-1)^l\delta_1\frac{\partial(F_1,\cdots,F_k)}{\partial(\zeta_{j_1},\cdots,\zeta_{j_k})}\left(\frac{\partial(F_1,\cdots,F_k)}{\partial(\zeta_{n-k+1},\cdots,\zeta_n)}\right)^{-1},$$

where $\delta_1=(-1)^{n(k-l)}(-1)^{j_{l+1}+\cdots+j_k}(-1)^{(k-l)(k-l-1)/2}$.

So we have

$$\bigwedge_{\substack{i=1\\ i\neq j_1,\cdots,j_k}}^{n} d\zeta_i=d\zeta_1\wedge\cdots\hat{j}_1\cdots\hat{j}_l\cdots\wedge d\zeta_{n-k}\wedge d\zeta_{n-k+1}\wedge\cdots\hat{j}_{l+1}\cdots\hat{j}_k\cdots\wedge d\zeta_n$$

$$=\delta_2\frac{\partial(\tilde{F}_1,\cdots,\hat{j}_{l+1},\cdots,\hat{j}_k,\cdots,\tilde{F}_k)}{\partial(\zeta_{j_1},\cdots,\zeta_{j_l})}\bigwedge_{i=1}^{n-k} d\zeta_i\ ,$$

where $\delta_2=(-1)^{(k-n)l}(-1)^{j_1+\cdots+j_l}(-1)^{l(l-1)/2}$.

From the above we obtain

$$(**)\quad (-1)^{j_1+\cdots+j_k} \bigwedge_{\substack{i=1\\ i\neq j_1,\cdots,j_k}}^{n} d\zeta_i$$

$$=\delta\frac{\partial(F_1,\cdots,F_k)}{\partial(\zeta_{j_1},\cdots,\zeta_{j_k})}\left(\frac{\partial(F_1,\cdots,F_k)}{\partial(\zeta_{n-k+1},\cdots,\zeta_n)}\right)^{-1}\bigwedge_{i=1}^{n-k} d\zeta_i\ ,$$

where $\delta=(-1)^{k(n-k)}(-1)^{k(k+1)/2}$.

It follows from (**) that

$$(***)\quad B_k^F(\zeta)=\frac{(n-k)\delta}{|\nabla_k^F(\zeta)|^2}\sum_{j\leqslant j_1<\cdots j_k\leqslant n}\overline{\frac{\partial(F_1,\cdots,F_k)}{\partial(\zeta_{j_1},\cdots,\zeta_{j_k})}}\cdot\frac{\partial(F_1,\cdots,F_k)}{\partial(\zeta_{j_1},\cdots,\zeta_{j_k})}$$

$$\left(\frac{\partial(F_1,\cdots,F_k)}{\partial(\zeta_{n-k+1},\cdots,\zeta_n)}\right)^{-1}\bigwedge_{i=1}^{n-k} d\zeta_i$$

$$=\delta(n-k)!\left(\frac{\partial(F_1,\cdots,F_k)}{\partial(\zeta_{n-k+1},\cdots,\zeta_n)}\right)^{-1}\bigwedge_{i=1}^{n-k} d\zeta_i\ .$$

Using the above expression, we obtain

$$B_{k+1}^F\wedge dF_{k+1}=\frac{(n-k-1)!(-1)^{n+k}}{|\nabla_{k+1}^F(\zeta)|^2}\sum_{1\leqslant j_1<\cdots<j_k\leqslant n}(-1)^{j_1+\cdots+j_k}$$

$$\Big[\sum_{i=1}^{n}\frac{\partial F_{k+1}}{\partial\zeta_i}\overline{\frac{\partial(F_1,\cdots,F_k,F_{k+1})}{\partial(\zeta_{j_1},\cdots,\zeta_{j_k},\zeta_i)}}\Big]\bigwedge_{\substack{i=1\\ i\neq j_1,\cdots,j_k}}^{n} d\zeta_i$$

$$=\frac{(n-k-1)!(-1)^{n+k}\delta}{|\nabla_{k+1}^F(\zeta)|^2}\sum_{j\leqslant j_1<\cdots j_k\leqslant n}\sum_{i=1}^{n}\frac{\partial F_{k+1}}{\partial\zeta_i}\overline{\frac{\partial(F_1,\cdots,F_k,F_{k+1})}{\partial(\zeta_{i_1},\cdots,\zeta_{i_k},\zeta_i))}}\ \cdot$$

$$\frac{\partial(F_1,\cdots,F_k)}{\partial(\zeta_{j_1},\cdots,\zeta_{j_k})}\left(\frac{\partial(F_1,\cdots,F_k)}{\partial(\zeta_{n-k+1},\cdots,\zeta_n)}\right)^{-1}\bigwedge_{i=1}^{n-k} d\zeta_i$$

$$=\frac{(-1)^{n+k}\delta(n-k)!}{n-k}\left(\frac{\partial(F_1,\cdots,F_k)}{\partial(\zeta_{n-k+1},\cdots,\zeta_n)}\right)^{-1}\bigwedge_{i=1}^{n-k} d\zeta_i=\frac{(-1)^{n+k}}{n-k}B_k^F(\zeta).$$

Lemma 3. Let $D_k=Z(F_1,\cdots,F_k)\cap D$, $D_0=D$, $D_m=\tilde{D}$ and M_1 a continuously differentiable support function. If $f(z)$ is a holomorphic function on $\bar{D}_k$ and $|\nabla_{k+1}^F(\zeta)|\neq 0$, then

$$c_{k+1}\int_{\partial D_{k+1}}\frac{f(\zeta)}{M_1^{n-k+1}}\ \det{}_{(n)}(N_1,h_1,\cdots,h_{k+1},\bar{\partial}_\zeta N_1,\cdots,\bar{\partial}_\zeta N_1)\wedge B_{k+1}^F(\zeta)$$

$$=c_k\int_{\partial D_k}\frac{f(\zeta)}{M_1^{n-k}}\det{}_{(n)}(N_1,h_1,\cdots,h_k,\bar{\partial}_\zeta N_1,\cdots,\bar{\partial}_\zeta N_1)\wedge B_k^F(\zeta), \tag{6}$$

where $\quad c_k=(-1)^{k(n+1)}(-1)^{k(k+1)/2}/(n-k)!\ (2\pi i)^{n-k}$.

Proof. Let $(\partial D_k)_\varepsilon=\{\zeta\in\partial D_k:|F_{k+1}(\zeta)|>\varepsilon\}$. By lemma 1 and lemma 2, taking account that $c_{k+1}=(-1)^{n+k}(n-k)2\pi i c_k$, we obtain

$$c_k\int_{(\partial D_k)_\varepsilon}\frac{f(\zeta)}{M_1^{n-k}}\det{}_{(n)}(N_1,h_1,\cdots,h_k,\bar{\partial}_\zeta N_1,\cdots,\bar{\partial}_\zeta N_1)\wedge B_k^F(\zeta)$$

$$=c_k\int_{(\partial D_k)_\varepsilon} d_\zeta\ [\frac{f(\zeta)}{M_1^{n-k-1}}\det{}_{(n)}(N_1,h_1,\cdots,h_k,h_{k+1},\bar{\partial}_\zeta N_1,\cdots,\bar{\partial}_\zeta N_1)$$

$$\wedge(-1)^{n+k}(n-k)B_{k+1}^F(\zeta)\wedge\frac{dF_{k+1}}{F_{k+1}}]$$

$$=(n-k)(-1)^{n+k}c_k\int_{\partial(\partial D_k)_\varepsilon}\frac{f(\zeta)}{M_1^{n-k-1}}\ \det{}_{(n)}(N_1,h_1,\cdots,h_{k+1},\bar{\partial}_\zeta N_1,\cdots,\bar{\partial}_\zeta N_1)$$

$$\wedge B_{k+1}^F(\zeta)\wedge\frac{dF_{k+1}}{F_{k+1}}$$

$$=\frac{c_{k+1}}{2\pi i}\int_{|\tau|=\varepsilon}[\int_{\{\zeta\in\partial D_k:F_{k+1}(\zeta)=\tau\}}\frac{f(\zeta)}{M_1^{n-k-1}}\ \det{}_{(n)}(N_1,h_1,\cdots,h_{k+1},$$

$$\bar{\partial}_\zeta N_1,\cdots,\bar{\partial}_\zeta N_1)\ \wedge B_{k+1}^F(\zeta)]\frac{d\tau}{\tau}. \tag{7}$$

Let $s_0=\partial D_{k+1}=\{\zeta\in\partial D_k:F_{k+1}(\zeta)=0\}$ and $\nu(\zeta)$ be the normal direction at $\zeta\in s_0$. We consider the smooth mapping $f:(\zeta,\tau)\to\zeta+\tau\nu(\zeta)$, $s_0\times\{|\tau|<\varepsilon\}\to V_0\equiv\{\zeta\in\partial D_k:F_{k+1}(\zeta)=\tau\}|_{|\tau|<\varepsilon}$. Since s_0 is compact and the Jacobian $Jf|_{(\zeta,0)}\neq 0$ for every $(\zeta,0)$ in $\boldsymbol{R}^{2n-2k-1}$, there is the inverse f^{-1}. Here ε is chosen to be sufficiently small. From the above, we conclude that $\{\zeta\in\partial D_k:F_{k+1}(\zeta)=\tau\}$ is (for $\tau\in\boldsymbol{C}$, $|\tau|<\varepsilon$ and ε a small positive number) diffeomorphic to ∂D_{k+1}.

When $\varepsilon\to 0$, the left side and right side of (7) tend to the right side and left side of (6) respectively.

Corollary 1. Let $f(z)$ be a holomorphic function on $\bar{D}_1$, then when $z\in D_1$, and $|\nabla_1^F(\zeta)|\neq 0$ on ∂D_1, we have

$$f(z)=c_1\int_{\partial D_1}\frac{f(\zeta)}{M_1^{n-1}}\det{}_{(n)}(N_1,h_1,\bar{\partial}_\zeta N_1,\cdots,\bar{\partial}_\zeta N_1)\wedge B_1^F(\zeta). \tag{8}$$

Proof. Applying remark for lemma 2 and (6), we have

$$c_1\int_{\partial D_1}\frac{f(\zeta)}{M_1^{n-1}}\det{}_{(n)}(N_1,h_1,\bar{\partial}_\zeta N_1,\cdots,\bar{\partial}_\zeta N_1)\wedge B_1^F(\zeta)$$

$$=c_0\int_{\partial D_0}\frac{f(\zeta)}{M_1^n}\det{}_{(n)}(N_1,\bar{\partial}_\zeta N_1,\cdots,\bar{\partial}_\zeta N_1)\wedge n!\omega(\zeta)$$

$$=\frac{1}{(2\pi i)^n}\int_{\partial D}\frac{f(\zeta)}{M_1^n}\det{}_{(n)}(N_1,\bar{\partial}_\zeta N_1,\cdots,\bar{\partial}_\zeta N_1)\wedge\omega(\zeta). \tag{9}$$

By Canchy-Fantappie formula, the right-hand side of (9) equals to $f(z)$, and (8) is obtained.

Remark. In fact, representation (8) is more evident than that in [6].

Corollary 2. Let $f(z)$ be a holomorphic function on $\bar{\tilde{D}}$, then when $z\in\tilde{D}$ and $|\nabla_m^F(\zeta)|\neq 0$ on $\partial\tilde{D}$, we have

$$f(z)=c_m\int_{\partial\tilde{D}}\frac{f(\zeta)}{M_1^{n-m}}\det{}_{(n)}(N_1,h_1,\cdots,h_m,\bar{\partial}_\zeta N_1,\cdots,\bar{\partial}_\zeta N_1)\wedge B_m^F(\zeta). \tag{10}$$

Proof. By lemma 3 and its corollary 1, we obtain (10).

Lemma 4. Let $T_0^{(i)}=\sum_{j=1}^n(\zeta_j-z_j)S_{0j}^{(i)}(\zeta,z)\neq 0(\zeta\neq z)$ for some continuous functions $S_{0j}^{(i)}(i=1,2)$ on D, and let $T_l=\sum_{j=1}^n(\zeta_j-z_j)s_{lj}(\zeta,z)(l=1,2,\cdots,n-k-1)$ be continuously differentiable support functions for $\bar{D}$. Then we have

$$\det{}_{(n)}\left(\frac{S_0^{(1)}}{T_0^{(1)}},h_1,\cdots,h_k,\bar{\partial}_\zeta\left(\frac{S_1}{T_1}\right),\cdots,\bar{\partial}_\zeta\left(\frac{S_{n-k-1}}{T_{n-k-1}}\right)\right)$$

$$=\det_{(n)}\left(\frac{S_0^{(2)}}{T_0^{(2)}},h_1,\cdots,h_k,\bar{\partial}_\zeta(\frac{S_1}{T_1}),\cdots,\bar{\partial}_\zeta(\frac{S_{n-k-1}}{T_{n-k-1}})\right) \tag{11}$$

on D_k.

Proof. Since $\sum_{j=1}^{n}(\zeta_j-z_j)S_{lj}/T_l=1$, then

$$\sum_{j=1}^{n}(\zeta_j-z_j)\bar{\partial}_\zeta(S_{lj}/T_l)=0, \qquad l=1,\cdots,n-k-1. \tag{12}$$

Since

$$\sum_{j=1}^{n}(\zeta_j-z_j)\left(\frac{S_{0j}^{(1)}}{T_0^{(1)}}-\frac{S_{0j}^{(2)}}{T_0^{(2)}}\right)=0, \tag{13}$$

and on D_k

$$0=F_l(\zeta)-F_l(z)=\sum_{j=1}^{n}(\zeta_j-z_j)h_{lj}(\zeta,z), l=1,\cdots,k, \tag{14}$$

thus by (12)–(14), we obtain

$$\det_{(n)}\left(\frac{S_0^{(1)}}{T_0^{(1)}}-\frac{S_0^{(2)}}{T_0^{(2)}},h_1,\cdots,h_k,\bar{\partial}_\zeta(\frac{S_1}{T_1}),\cdots,\bar{\partial}_\zeta(\frac{S_{n-k-1}}{T_{n-k-1}})\right)=0,$$

on D_k, i.e. we have (11) on D_k.

Lemma 5. Let $T_l=\sum_{j=1}^{n}(\zeta_j-z_j)S_{lj}(\zeta,z), l=0,1,2,\cdots,n-k-1$, be continuous differentiable support functions for $\bar{D}$, then

$$\det_{(n)}\left(\frac{S_0}{T_0},h_1,\cdots,h_k,\bar{\partial}_\zeta(\frac{S_1}{T_1}),\cdots,\bar{\partial}_\zeta(\frac{S_l}{T_l}),\cdots,\bar{\partial}_\zeta(\frac{S_{n-k-1}}{T_{n-k-1}})\right)$$
$$-\det_{(n)}\left(\frac{S_0}{T_0},h_1,\cdots,h_k,\bar{\partial}_\zeta(\frac{S_1}{T_1}),\cdots,\bar{\partial}_\zeta(\frac{\tilde{S}_l}{\tilde{T}_l}),\cdots,\bar{\partial}_\zeta(\frac{S_{n-k-1}}{T_{n-k-1}})\right) \tag{15}$$

is the exact differential form of $\bar{\partial}$ on D_k. Here $\tilde{S}_l$ and $\tilde{T}_l$ are of the same properties as S_l and T_l respectively.

Proof. Since

$$\bar{\partial}_\zeta \det{}_{(n)}\left(\frac{S_l}{T_l},h_1,\cdots,h_k,\bar{\partial}_\zeta(\frac{S_1}{T_1}),\cdots,\frac{\tilde{S}_l}{\tilde{T}_l},\cdots,\bar{\partial}_\zeta(\frac{S_{n-k-1}}{T_{n-k-1}})\right)$$

$$=\det{}_{(n)}\left(\bar{\partial}_\zeta(\frac{S_l}{T_l}),h_1,\cdots,h_k,\bar{\partial}_\zeta(\frac{S_1}{T_1}),\cdots,\frac{\tilde{S}_l}{\tilde{T}_l},\cdots,\bar{\partial}_\zeta(\frac{S_{n-k-1}}{T_{n-k-1}})\right)$$

$$+(-1)^{l-1}\det{}_{(n)}\left(\frac{S_l}{T_l},h_1,\cdots,h_k,\bar{\partial}_\zeta(\frac{S_1}{T_1}),\cdots,\bar{\partial}_\zeta(\frac{\tilde{S}_l}{\tilde{T}_l}),\cdots,\bar{\partial}_\zeta(\frac{S_{n-k-1}}{T_{n-k-1}})\right)$$

$$=(-1)^l\det{}_{(n)}\left(\frac{\tilde{S}_l}{\tilde{T}_l},h_1,\cdots,h_k,\bar{\partial}_\zeta(\frac{S_1}{T_1}),\cdots,\bar{\partial}_\zeta(\frac{S_l}{T_l}),\cdots,\bar{\partial}_\zeta(\frac{S_{n-k-1}}{T_{n-k-1}})\right)$$

$$+(-1)^{l-1}\det{}_{(n)}\left(\frac{S_l}{T_l},h_1,\cdots,h_k,\bar{\partial}_\zeta(\frac{S_1}{T_1}),\cdots,\bar{\partial}_\zeta(\frac{\tilde{S}_l}{\tilde{T}_l}),\cdots,\bar{\partial}_\zeta(\frac{S_{n-k-1}}{T_{n-k-1}})\right), \tag{16}$$

by lemma 4, replacing $\tilde{S}_l/\tilde{T}_l$ and S_l/T_l by S_0/T_0 in two determiants of the right-hand side of (16), we can conclude that (15) is a exact differential form of $\bar{\partial}$.

Corollary. With the identical assumptions of lemma 5,

$$\det{}_{(n)}\left(\frac{S_0}{T_0},h_1,\cdots,h_k,\bar{\partial}_\zeta(\frac{S_1}{T_1}),\cdots,\bar{\partial}_\zeta(\frac{S_{n-k-1}}{T_{n-k-1}})\right)$$

$$-\det{}_{(n)}\left(\frac{S_0}{T_0},h_1,\cdots,h_k,\bar{\partial}_\zeta(\frac{\tilde{S}_l}{\tilde{T}_l}),\cdots,\bar{\partial}_\zeta(\frac{S_{n-k-1}}{T_{n-k-1}})\right) \tag{17}$$

is a exact differential form of $\bar{\partial}$. Here $\tilde{S}_1,\cdots,\tilde{S}_{n-k-1}$ and $\tilde{T}_1,\cdots,\tilde{T}_{n-k-1}$ are of the same properties as $S_1,\cdots,S_{n-k-1}$ and $T_1,\cdots,T_{n-k-1}$ respectively.

§4. Main Theorems

Let $\{U_j\}_{j=1}^N$ be a finite open covering of an open neighbourhood U of

∂D, and let $X_j\colon U_j \to \boldsymbol{R}(1 \leqslant j \leqslant N)$ be C^1 functions, such that

(i) $D \bigcap U_{\partial D} = \{z \in U\colon$ for $1 \leqslant j \leqslant N$ either $z \notin U_j$ or $X_j(z) < 0\}$,

(ii) for $1 \leqslant i_1, < \cdots < i_l \leqslant N$, $dX_{i_1}, \cdots, dX_{i_l}$ are linearly independent over $\boldsymbol{R}$ at every point of $\bigcap_{\nu=1}^{l} U_{i_\nu}$.

For every ordered subset $\{j_1, \cdots, j_\theta\}$ of $\{1, \cdots, N\}$, define

$$\sigma^{(0)}_{j_1 \cdots j_\theta} = \{z \in \partial D \bigcap_{\alpha=1}^{\theta} U_{j_\alpha}\colon X_{j_1}(z) = \cdots = X_{j_\theta}(z) = 0\}$$

and choose the orientation on $\sigma^{(0)}_{j_1 \cdots j_\theta}$ such that the orientation is skew symmetric in $(j_1, \cdots, j_\theta)$ and the following equations hold when D is given the natural orientation:

$$\partial D = \bigcup_{j=1}^{N} \sigma^{(0)}_j, \sigma^{(\theta)} = \bigcup_{j_1 < \cdots < j_\theta} \sigma^{(0)}_{j_1 \cdots j_\theta}, \partial \sigma^{(0)}_{j_1 \cdots j_\theta} = \bigcup_{j=1}^{N} \sigma^{(0)}_{j_1 \cdots j_\theta j}.$$

$\tilde{\sigma}^{(0)}_{j_1 \cdots j_\theta}$, $\partial \tilde{D}$, $\tilde{\sigma}^{(\theta)}$, $\partial \tilde{\sigma}^{(0)}_{j_1 \cdots j_\theta}$ is defined as above, and it is easy to verify the following (cf. [3]):

$$\partial\Big(\sum_{\Theta} (-1)^\theta \tilde{\sigma}^{(\theta)} \times \Delta^{(\theta)}_0\Big) = \sum_{\Theta} \tilde{\sigma}^{(\theta)} \times \Delta^{(\theta-1)} - \tilde{\sigma}^{(1)} \times \Delta^{(0)}_0, \tag{18}$$

where $\Theta = \{j_1, \cdots, j_\theta\}$ is a ordered subset of $\{1, \cdots, N\}$, $j_1 < \cdots < j_\theta$;

$$\Delta = \{\mu = (\mu_0, \mu_1, \cdots, \mu_N) \in \boldsymbol{R}^{N+1}\colon \mu_j \geqslant 0, \sum_{j=0}^{N} \mu_j = 1\}, \tag{19}$$

$$\Delta^{(\theta)}_{0 j_1 \cdots j_\theta} = \{\mu \in \Delta\colon \mu_0 + \mu_{j_1} + \cdots + \mu_{j_0} = 1, \Delta^{(0)}_0 = (1, 0, \cdots, 0), \tag{20}$$

$$\Delta^{(\theta)}_0 = \bigcup_{j_1 < \cdots < j_\theta} \Delta^{(0)}_{0 j_1 \cdots j_\theta}, \Delta^{(\theta-1)} = \bigcup_{j_1 < \cdots < j_\theta} \Delta^{(\theta-1)}_{j_1 \cdots j_\theta}. \tag{21}$$

Theorem 1. Let D be a nondegenesate polyhedral domain in $\boldsymbol{C}^n$ whose boundary can be written as a chain of slit spaces;

$$\sigma^{(1)} \supset \sigma^{(2)} \supset \cdots \supset \sigma^{(\beta)}. \tag{22}$$

Assume that $Z(F_1,\cdots,F_m)$ meets ∂D transversally yielding a chain of slit spaces;

$$\partial\tilde{D}=\tilde{\sigma}^{(1)}\supset\tilde{\sigma}^{(2)}\supset\cdots\supset\tilde{\sigma}^{(\beta)}, \tag{23}$$

and that $|\nabla_m^F(\zeta)|\neq 0$ on $\partial\tilde{D}$. Then

$$f(z)=c_m\sum_{\Theta}\int_{\tilde{\sigma}^{(\theta)}\times\Delta^{(\theta-1)}} f(\zeta)\det_{(n)}(Q,h_1,\cdots,h_m,\partial_{\bar{\zeta}\mu}Q,\cdots,\partial_{\bar{\zeta}\mu}Q)\ \wedge\ B_m^F(\zeta)\ \text{if}\ \beta>1 \tag{24}$$

and

$$f(z)=c_m\int_{\partial\tilde{D}}\frac{f(\zeta)}{M_0^{n-m}}\det_{(n)}(N^0,h_1,\cdots,h_m,\partial_{\bar{\zeta}}N^0,\cdots,\partial_{\bar{\zeta}}N^0)\ \wedge\ B_m^F(\zeta)\ \text{if}\ \beta=1, \tag{25}$$

for any holomorphic function on $\bar{\tilde{D}}$ and $z\in\tilde{D}$. Here

$$Q=(Q_1,\cdots,Q_n)^t,\ Q_p=\mu_0\frac{N_p^0(\zeta,z)}{M_0}+\sum_{j\in\Theta}\mu_j\frac{N_{jp}(\zeta,z)}{M_j}\quad\text{on}\quad\Delta_0^{(\theta)},$$

where $M_0=\sum_{p=1}^{n}(\zeta_p-z_p)N_p^0(\zeta,z)\neq 0$ (when $\zeta\neq z$), $M_j=\sum_{p=1}^{n}(\zeta_p-z_p)N_{jp}(\zeta,z)\neq 0$ (when $\zeta\neq z$), i.e. M_0, M_j are the continuously differentiable support functions; and $Q_p=\sum_{j\in\Theta}\mu_jN_{jp}(\zeta,z)/M_j$ on $\Delta^{(\theta-1)}$.

Proof. Since $\sum_{j=1}^{n}(\zeta_j-z_j)Q_j(\zeta,z,\mu)\equiv 1$ on $\Delta_0^{(\theta)}$, then $\sum_{j=1}^{n}(\zeta_j-z_j)\partial_{\bar{\zeta}\mu}Q(\zeta,z,\mu)=0$. According to Hefer's theorem, we have $0=F_l(\zeta)-F_l(z)=\sum_{j=1}^{n}(\zeta_j-z_j)h_{lj}(\zeta,z)$ on $\partial\tilde{D}\times\tilde{D}$, $l=1,\cdots,m$. By (***) we have $\bar{\partial}_\zeta B_m^F(\zeta)=0$. As a result we obtain

$$\begin{aligned}&d[\det_{(n)}(Q,h_1,\cdots,h_m,\partial_{\bar{\zeta}\mu}Q,\cdots,\partial_{\bar{\zeta}\mu}Q)\ \wedge\ B_m^F(\zeta)]\\&=\det_{(n)}(\partial_{\bar{\zeta}\mu}Q,h_1,\cdots,h_m,\partial_{\bar{\zeta}\mu}Q,\cdots,\partial_{\bar{\zeta}\mu}Q)\wedge B_m^F(\zeta)=0\end{aligned} \tag{26}$$

on $\tilde{\sigma}^{(\theta)}\times\Delta_0^{(\theta)}$. Using Stokes' theorem and taking account of (18) and (26), we obtain

$$c_m \sum_{\Theta} \int_{\tilde{\sigma}^{(\theta)} \times \Delta^{(\theta-1)}} f(\zeta) \det_{(n)}(Q, h_1, \cdots, h_m, \partial_{\bar{\zeta}\mu} Q, \cdots, \partial_{\bar{\zeta}\mu} Q) \wedge B_m^F(\zeta)$$

$$= c_m \int_{\tilde{\sigma}^{(1)} \times \Delta_0^{(0)}} f(\zeta) \det_{(n)}(Q, h_1, \cdots, h_m, \partial_{\bar{\zeta}\mu} Q, \cdots, \partial_{\bar{\zeta}\mu} Q) \wedge B_m^F(\zeta)$$

$$= c_m \int_{\partial \tilde{D}} f(\zeta) \det_{(n)}\left(\frac{N^0}{M_0}, h_1, \cdots, h_m, \partial_{\bar{\zeta}}\left(\frac{N^0}{M_0}\right), \cdots, \partial_{\bar{\zeta}}\left(\frac{N^0}{M_0}\right)\right) \wedge B_m^F(\zeta)$$

$$= c_m \int_{\partial \tilde{D}} \frac{f(\zeta)}{M_0^{n-m}} \det_{(n)}(N^0, h_1, \cdots, h_m, \partial_{\bar{\zeta}} N^0, \cdots, \partial_{\bar{\zeta}} N^0) \wedge B_m^F(\zeta). \tag{27}$$

Applying corollary 2 of lemma 3 to the right-hand side of (27), we obtain (24).

Remark. Obviously, (24) includes the generalizations of Range and Siu's formula[3], and of Sergeev and Henkin's formula[5] on the analytic subvariety.

Theorem 2. Let $f(z)$ be a holomorphic function on $\bar{\tilde{D}}$, then, for $z \in \tilde{D}$, we have

$$f(z) = c_m \int_{\partial \tilde{D}} f(\zeta) \det_{(n)}\left(\frac{S_0}{T_0}, h_1, \cdots, h_m, \bar{\partial}_{\zeta}\left(\frac{S_1}{T_1}\right), \cdots, \bar{\partial}_{\zeta}\left(\frac{S_{n-m-1}}{T_{n-m-1}}\right)\right) \wedge B_m^F(\zeta). \tag{28}$$

Remark 1. When $m=0$, $c_m = 1/n!\,(2\pi i)^n$, $B_0^F(\zeta) = n!\omega(\zeta)$ in (28), and (28) can be rewritten as:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial D} f(\zeta) \det_{(n)}\left(\frac{S_0}{T_0}, \bar{\partial}_{\zeta}\left(\frac{S_1}{T_1}\right), \cdots, \bar{\partial}_{\zeta}\left(\frac{S_{n-1}}{T_{n-1}}\right)\right) \wedge \omega(\zeta). \tag{29}$$

This is the generalized Cauchy-Fantappie formula for the bounded domains in $\boldsymbol{C}^n$. In fact, let $S_1 = \cdots = S_{n-m-1} = S_0$, (29) is the Cauchy-Fantappie formula.

Remark 2. If $S_1 = \cdots = S_{n-m-1} = S_0$, then $T_1 = \cdots = T_{n-m-1} = T_0$. Thus (28) is the generalization of the generalized Cauchy-Fantappie formula on analytic subvarieties.

Remark 3. For fixed $z \in \tilde{D}$, we consider the following surface in $\boldsymbol{C}^{2n-2m}$

$$M_z = \{(\zeta, w): \sum_{j=1}^{n} (\zeta_j - z_j) w_j = 1,\ \zeta \in \partial \tilde{D}\}.$$

We make the following assumptions: $w_j=w_j(\zeta,z)$, $(\zeta,z)\in\partial\tilde{D}\times\tilde{D}$, $j=1,\cdots,n$, which belong to $C^1(\partial\tilde{D})$ in ζ, and the condition $\sum_{j=1}^{n}(\zeta_j-z_j)w_j=1$ is fulfilled. $\tilde{C}_0$ denotes a cycle on M_z and cycle $\tilde{C}_0$ represents a homology class $\mathfrak{h}\in\mathscr{H}_{2n-2m-1}(M_z)$. Then, for any holomorphic function $f(z)$ in $\bar{\tilde{D}}$ and any cycle $\tilde{C}\in\mathfrak{h}$, we have

$$f(z)=c_m\int_{\tilde{C}}f(\zeta)\det{}_{(n)}(\frac{S_0}{T_0},h_1,\cdots,h_m,d(\frac{S_1}{T_1}),\cdots,d(\frac{S_{n-m-1}}{T_{n-m-1}}))\wedge B_m^F(\zeta). \tag{30}$$

Proof of theorem 2. According to the corollary of lemma 5, we have

$$\begin{aligned}&\int_{\partial\tilde{D}}f(\zeta)\det{}_{(n)}(\frac{S_0}{T_0},h_1,\cdots,h_m,\bar{\partial}_\zeta(\frac{S_1}{T_1}),\cdots,\bar{\partial}_\zeta(\frac{S_{n-m-1}}{T_{n-m-1}}))\wedge B_m^F(\zeta)\\&=\int_{\partial\tilde{D}}f(\zeta)\det{}_{(n)}(\frac{S_0}{T_0},h_1,\cdots,h_m,\bar{\partial}_\zeta(\frac{S_0}{T_0}),\cdots,\bar{\partial}_\zeta(\frac{S_0}{T_0}))\wedge B_m^F(\zeta)\\&=\int_{\partial\tilde{D}}f(\zeta)\det{}_{(n)}(\frac{S_0}{T_0},h_1,\cdots,h_m,\frac{\bar{\partial}_\zeta S_0}{T_0},\cdots,\frac{\bar{\partial}_\zeta S_0}{T_0})\wedge B_m^F(\zeta)\\&=\int_{\partial\tilde{D}}\frac{f(\zeta)}{T_0^{n-m}}\det{}_{(n)}(S_0,h_1,\cdots,h_m,\bar{\partial}_\zeta S_0\cdots,\bar{\partial}_\zeta S_0)\wedge B_m^F(\zeta).\end{aligned}$$

By further applying corollary 2 of lemma 3, we obtain (28).

Theorem 3. Let D be a bounded domain with piecewise smooth boundaries in $\mathbb{C}^n$, the boundary ∂D of D consisting of a chain of slit spaces

$$\partial D=\sigma^{(1)}\supset\cdots\supset\sigma^{(s)}\supset\sigma^{(t)}\supset\cdots\supset\sigma^{(n)}\supset\sigma^{(\beta)}.$$

Assume that $\sigma^{(\beta)}$ be a $2n-\beta$ dimensional boundary chain, i.e. there is a $2n-\beta+1$ dimensional chain τ_0, such that $\partial\tau_0=\sigma^{(\beta)}$. Correspondingly

$$\partial\tilde{D}=\tilde{\sigma}^{(1)}\supset\cdots\supset\tilde{\sigma}^{(s)}\supset\tilde{\sigma}^{(t)}\supset\cdots\supset\tilde{\sigma}^{(n)}\supset\tilde{\sigma}^{(\beta)},$$

$\partial\tau_m=\tilde{\sigma}^{(\beta)}$, and when $\zeta\in\partial\tilde{D}$, $|\nabla_m^F(\zeta)|\neq0$ and

$$\operatorname{rank}\frac{\partial(N_\beta^1,\cdots,N_\beta^n)}{\partial(\bar{\zeta}_1,\cdots,\bar{\zeta}_n)}\leqslant n-m-\beta. \tag{31}$$

Then, for a holomorphic function $f(z)$ on $\bar{\tilde{D}}$ we have

$$f(z)=c_m\int_{\tilde{\sigma}^{(\beta)}\times\Delta^{(\beta-1)}} f(\zeta)\det_{(n)}(Q,h_1,\cdots,h_m,\partial_{\bar{\zeta}\mu}Q,\cdots,\partial_{\bar{\zeta}\mu}Q)\ \wedge\ B_m^F(\zeta),\ \textit{for}\ z\in\tilde{D}. \tag{32}$$

Proof. Let

$$c=\partial(\tau_0\times\Delta^{(\beta-1)})=\sigma^{(\beta)}\times\Delta^{(\beta-1)}+\varepsilon_0\tau_0\times\partial\Delta^{(\beta-1)},$$

$$\tilde{c}=\partial(\tau_m\times\Delta^{(\beta-1)})=\tilde{\sigma}^{(\beta)}\times\Delta^{(\beta-1)}+\varepsilon\tau_m\times\partial\Delta^{(\beta-1)},$$

where $\varepsilon_0,\varepsilon=\pm1$. Thus on $\Delta^{(\beta-1)}$, we have

$$\begin{aligned}&\det_{(n)}(Q,h_1,\cdots,h_m,\partial_{\bar{\zeta}\mu}Q,\cdots,\partial_{\bar{\zeta}\mu}Q)\\&\qquad=\chi_0(\zeta,\mu)+\chi_1(\zeta,\mu)+\cdots+\chi_{\beta-1}(\zeta,\mu),\end{aligned} \tag{33}$$

where $\chi_r(\zeta,\mu)$ are differential forms, the degrees of $d\mu_\theta$ and $\overline{d\zeta_j}$ are r and $n-m-r-1$ respectively. By (31), $\chi_r(\zeta,\mu)=0$, if $r<\beta-1$, and by the degree reasons, we have

$$\int_{\tau_m\times\partial\Delta^{(\beta-1)}} f(\zeta)\chi_{\beta-1}(\zeta,\mu)\ \wedge\ B_m^F(\zeta)=0.$$

Thus

$$\int_{\tau_m\times\partial\Delta^{(\beta-1)}} f(\zeta)\det_{(n)}(Q,h_1,\cdots,h_m,\partial_{\bar{\zeta}\mu}Q,\cdots,\partial_{\bar{\zeta}\mu}Q)\ \wedge\ B_m^F(\zeta)=0.$$

Therefore we obtain

$$\begin{aligned}&\int_{\tilde{\sigma}^{(\beta)}\times\Delta^{(\beta-1)}} f(\zeta)\det_{(n)}(Q,h_1,\cdots,h_m,\partial_{\bar{\zeta}\eta}Q,\cdots,\partial_{\bar{\zeta}\eta}Q)\ \wedge\ B_m^F(\zeta)\\&\qquad=\int_{\tilde{C}} f(\zeta)\det_{(n)}(Q,h_1,\cdots,h_m,\partial_{\bar{\zeta}\mu}Q,\cdots,\partial_{\bar{\zeta}\mu}Q)\wedge B_m^F(\zeta).\end{aligned} \tag{34}$$

On the other hand, let $\tilde{C}_1=\partial(\tilde{D}\times\Delta_0^{(0)})=\partial\tilde{D}\times\Delta_0^{(0)}$. Since $\tilde{C}$ and $\tilde{C}_1$ are the cycles of real dimension $2n-2m-1$, and

$$d[\det_{(n)}(Q,h_1,\cdots,h_m,\partial_{\bar\zeta_\mu}Q,\cdots,\partial_{\bar\zeta_\mu}Q)\wedge B_m^F(\zeta)]$$
$$=\det_{(n)}(\partial_{\bar\zeta_\mu}Q,h_1,\cdots,h_m,\partial_{\bar\zeta_\mu}Q,\cdots,\partial_{\bar\zeta_\mu}Q)\wedge B_m^F(\zeta)=0$$

on $\partial\tilde D=\tilde\sigma^{(1)}$, then

$$\int_{\tilde C} f(\zeta)\det_{(n)}(Q,h_1,\cdots,h_m,\partial_{\bar\zeta_\mu}Q,\cdots,\partial_{\bar\zeta_\mu}Q)\wedge B_m^F(\zeta)$$
$$=\int_{\tilde C_r} f(\zeta)\det_{(n)}(Q,h_1,\cdots,h_m,\partial_{\bar\zeta_\mu}Q,\cdots,\partial_{\bar\zeta_\mu}Q)\wedge B_m^F(\zeta)$$
$$=\int_{\partial\tilde D\times\Delta_0^{(0)}} f(\zeta)\det_{(n)}(Q,h_1,\cdots,h_m,\partial_{\bar\zeta_\mu}Q,\cdots,\partial_{\bar\zeta_\mu}Q)\wedge B_m^F(\zeta)$$
$$=\int_{\partial\tilde D}\frac{f(\zeta)}{M_0^{(n-m)}}\det_{(n)}(N^0,h_1,\cdots,h_m,\partial_{\bar\zeta}N^0,\cdots,\partial_{\bar\zeta}N^0)\wedge B_m^F(\zeta). \tag{35}$$

Applying (34), (35) and (25), we obtain (32).

Theorem 4. Let D be a nondegenerate polyhedral domain in C^n, such that its boundary ∂D consists of a chain of slit spaces

$$\partial D=\sigma_1^{(0)}\supset\sigma_2^{(0)}\supset\cdots\supset\sigma_k^{(0)}\supset\sigma_{k+1}^{(0)}\supset\sigma_{k+2}^{(0)}\supset\cdots\supset\sigma_\beta^{(0)},$$

where $\sigma_{k+1}^{(0)},\cdots,\sigma_\beta^{(0)}$ are the boundary surfaces of polyhedral type which is defined by pluriharmonic functions. Let $\Phi_{j_\upsilon}(\zeta,z)=\sum_{p=1}^{n}(\zeta_p-z_p)\varphi_{j_\upsilon p}(\zeta,z)$ be the holomorphic support functions on $\sigma_{k+1}^{(0)}$ and $H_{j_\upsilon p}=\varphi_{j_\upsilon p}/\Phi_{j_\upsilon}$. Let

$$\partial\tilde D=\tilde\sigma_1^{(0)}\supset\tilde\sigma_2^{(0)}\supset\cdots\supset\tilde\sigma_k^{(0)}\supset\tilde\sigma_{k+1}^{(0)}\supset\tilde\sigma_{k+2}^{(0)}\supset\cdots\supset\tilde\sigma_\beta^{(0)}$$

be the corresponding chain of slits of $\partial\tilde D$ and assume that $|\nabla_m^F(\zeta)|\neq 0$ on $\partial\tilde D$. Then for a holomorphic function $f(z)$ in $\bar{\tilde D}$ we have

$$f(z)=c_m^0\sum_{\substack{J_\theta\\(k+1\leqslant\theta\leqslant n-m)}}\int_{\tilde\sigma_{J_\theta}^{(0)}\times\Delta_{J_{\theta-k}}} f(\zeta)\sum_{\rho=1}^{k+1}(-1)^{k+1-\rho}\det_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,$$
$$[H_{j_\rho}],\cdots,H_{j_{k+1}},Q,\partial_{\bar\zeta_\mu}Q,\cdots,\partial_{\bar\zeta_\mu}Q)\wedge B_m^F(\zeta),\ \text{for } z\in\tilde D \tag{36}$$

where $C_m^0=(-1)^{mn}(-1)^{\frac{m(m+1)}{2}}/(n-m)!(2\pi i)^{n-m}$, $J_\theta=\{j_1,\cdots,j_\theta\}\subset\{1,\cdots,N\}$, $j_1<\cdots<j_\theta$; $J_{\theta-k}=\{j_1,\cdots,j_{\theta-k}\}\subset\{1,\cdots,N\}$, $j_1<\cdots<j_{\theta-k}$.

Proof. It is easy to verify the following

$$\partial\Big(\sum_{\substack{J_\theta\\(k+1\leqslant\theta\leqslant N)}}(-1)^\theta\tilde{\sigma}_{J_\theta}^{(0)}\times\Delta_{0J_{\theta-k}}\Big)=\sum_{\substack{J_\theta\\(k+1\leqslant\theta\leqslant N)}}\tilde{\sigma}_{J_\theta}^{(\theta)}\times\Delta_{J_{\theta-k}}-$$

$$\sum_{j_1<\cdots<j_{k+1}}\tilde{\sigma}_{j_1\cdots j_{k+1}}^{(0)}\times\Delta_0. \tag{37}$$

Since

$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0\\ Q_1 & h_{11} & \cdots & h_{m1} & H_{j_11} & \cdots & H_{j_{k+1}1} & \partial_{\bar{\zeta}\mu}Q_1 & \cdots & \partial_{\bar{\zeta}\mu}Q_1\\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\ Q_n & h_{1n} & \cdots & h_{mn} & H_{j_1n} & \cdots & H_{j_{k+1}n} & \partial_{\bar{\zeta}\mu}Q_n & \cdots & \partial_{\bar{\zeta}\mu}Q_n \end{vmatrix}=0$$

on $\tilde{\sigma}_{J_\theta}^{(0)}\times\Delta_{0J_{\theta-k}}(k+1\leqslant\theta\leqslant N)$, then

$$\sum_{\rho=1}^{k+1}(-1)^{k+1-\rho}\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,[H_{j_\rho}],\cdots,H_{j_{k+1}},Q,\partial_{\bar{\zeta}\mu}Q,\cdots,\partial_{\bar{\zeta}\mu}Q)$$

$$=\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,H_{j_{k+1}},\partial_{\bar{\zeta}\mu}Q,\cdots,\partial_{\bar{\zeta}\mu}Q).$$

Then

$$d\Big[\sum_{\rho=1}^{k+1}(-1)^{k+1-\rho}\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,[H_{j_\rho}],\cdots,H_{j_{k+1}},Q,\partial_{\bar{\zeta}\mu}Q,\cdots,\partial_{\bar{\zeta}\mu}Q)$$

$$\wedge\ B_m^F(\zeta)\Big]=\partial_{\bar{\zeta}\mu}\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,H_{j_{k+1}},\partial_{\bar{\zeta}\mu}Q,\cdots,\partial_{\bar{\zeta}\mu}Q)\ \wedge\ B_m^F(\zeta)=0. \tag{38}$$

It is from (37), (38) and Stokes' theorem, that

$$C_m^0\sum_{\substack{J_\theta\\(k+1\leqslant\theta\leqslant n-m)}}\int_{\tilde{\sigma}_{J_\theta}^{(0)}\times\Delta_{J_{\theta-k}}}f(\zeta)\sum_{\rho=1}^{k+1}(-1)^{k+1-\rho}\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,$$

$$[H_{j_\rho}],\cdots,H_{j_{k+1}},Q,\partial_{\bar{\zeta}\mu}Q,\cdots,\partial_{\bar{\zeta}\mu}Q)\ \wedge\ B_m^F(\zeta)$$

$$=C_m^0\sum_{j_1<\cdots<j_{k+1}}\int_{\tilde{\sigma}_{j_1\cdots j_{k+1}}^{(0)}\times\Delta_0}f(\zeta)\sum_{\rho=1}^{k+1}(-1)^{k+1-\rho}\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,$$

$$[\mathrm{H}_{j_\rho}],\cdots,H_{j_{k+1}},Q,\partial_{\bar\zeta\mu}Q,\cdots,\partial_{\bar\zeta\mu}Q)\ \wedge\ B^F_m(\zeta)$$

$$=C^0_m\sum_{j_1<\cdots<j_{k+1}}\int_{\partial^{(0)}_{j_1\cdots j_{k+1}}}\frac{f(\zeta)}{M_0^{n-m-1}}\sum_{\rho=1}^{k+1}(-1)^{k+1-\rho}\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,$$

$$[H_{j_\rho}],\cdots,H_{j_{k+1}},N^0,\partial_{\bar\zeta}N^0,\cdots,\partial_{\bar\zeta}N^0)\ \wedge\ B^F_m(\zeta). \tag{39}$$

Since

$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & G & \cdots & G \\ \dfrac{N^0_1}{M_0} & h_{11} & \cdots & h_{m1} & H_{j_1 1} & \cdots & H_{j_k 1} & \partial_{\bar\zeta}N^0_1 & \cdots & \partial_{\bar\zeta}N^0_1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \dfrac{N^0_n}{M_1} & h_{1n} & \cdots & h_{mn} & H_{j_1 n} & \cdots & H_{j_k n} & \partial_{\bar\zeta}N^0_n & \cdots & \partial_{\bar\zeta}N^0_n \end{vmatrix}=0 \tag{40}$$

where $G=\sum_{j=1}^{n}(\zeta_j-z_j)\partial_{\bar\zeta}N^0_j$, then

$$\frac{1}{\mathrm{M}_0^{n-m-k}}\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,h_{j_k},\partial_{\bar\zeta}N^0,\cdots,\partial_{\bar\zeta}N^0)-\frac{n-m-k}{M^{n-m-k+1}}$$

$$G\wedge\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,H_{j_k},N^0,\partial_{\bar\zeta}N^0,\cdots,\partial_{\bar\zeta}N^0)$$

$$=\sum_{\rho=1}^{k}(-1)^{k-\rho}\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,[H_{j_\rho}],\cdots,H_{j_k},$$

$$\frac{N^0}{M_0^{n-m-k+1}},\partial_{\bar\zeta}N^0,\cdots,\partial_{\bar\zeta}N^0). \tag{41}$$

On the other hand, we have

$$\bar\partial_\zeta\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,[H_{j_\rho}],\cdots,H_{j_{k+1}},\frac{N^0}{M_0^{n-m-k}},\partial_{\bar\zeta}N^0,\cdots,\partial_{\bar\zeta}N^0)$$

$$=\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,[H_{j_\rho}],\cdots,H_{j_{k+1}},\bar\partial_\zeta(\frac{N^0}{M_0^{n-m-k}}),\partial_{\bar\zeta}N^0,\cdots,\partial_{\bar\zeta}N^0)$$

$$=\frac{1}{M_0^{n-m-k}}\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,[H_{j_\rho}],\cdots,H_{j_{k+1}},\partial_{\bar\zeta}N^0,\cdots,\partial_{\bar\zeta}N^0)$$

$$=\frac{n-m-k}{M_0^{n-m-k+1}}\,G\wedge \det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,[H_{j_\rho}],\cdots,H_{j_{k+1}},N^0,\partial_{\bar\zeta}N^0,\cdots,\partial_{\bar\zeta}N^0) \tag{42}$$

on $\partial\tilde{D}$.

Moreover, since $\partial\tilde\sigma^{(0)}_{j_1\cdots[j_\rho]\cdots j_{k+1}}=\bigcup_{j_\rho}(-1)^{k+1-\rho}\tilde\sigma^{(0)}_{j_1\cdots j_{k+1}}$, and

$$\sum_{j_1<\cdots<j_{k+1}}=\frac{1}{(k+1)!}\sum_{j_1,\cdots,j_{k+1}},$$

then by Stokes' formula and (41), (42), we obtain

$$C_m^0\sum_{j_1<\cdots<j_{k+1}}\int_{\tilde\sigma^{(0)}_{j_1\cdots j_{k+1}}} f(\zeta)\sum_{\rho=1}^{k+1}(-1)^{k+1-\rho}\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,$$

$$[H_{j_\rho}],\cdots,H_{j_{k+1}},\frac{N^0}{M_0^{n-m-k}},\partial_{\bar\zeta}N^0,\cdots,\partial_{\bar\zeta}N^0)\ \wedge\ B_n^F(\zeta)$$

$$=C_m^0\frac{1}{(k+1)!}\sum_{\rho=1}^{k+1}\sum_{j_1,\cdots,[j_\rho],\cdots,j_{k+1}}\sum_{j_\rho}\int_{\tilde\sigma^{(0)}_{j_1\cdots j_{k+1}}}(-1)^{k+1-\rho}f(\zeta)\det{}_{(n)}(h_1,$$

$$\cdots,h_m,H_{j_1},\cdots,[H_{j_\rho}],\cdots,H_{j_{k+1}},\frac{N^0}{M_0^{n-m-k}},\partial_{\bar\zeta}N^0,\cdots,\partial_{\bar\zeta}N^0)\ \wedge B_m^F(\zeta)$$

$$=C_m^0\frac{1}{(k+1)!}\sum_{\rho=1}^{k+1}\sum_{j_1,\cdots,[j_\rho],\cdots,j_{k+1}}\int_{\tilde\sigma^{(0)}_{j_1\cdots[j_\rho]\cdots j_{k+1}}} d[f(\zeta)\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,$$

$$[H_{j_\rho}],\cdots,H_{j_{k+1}},\frac{N^0}{M_0^{n-m-k}},\partial_{\bar\zeta}N^0,\cdots,\partial_{\bar\zeta}N^0)\ \wedge B_m^F(\zeta)]$$

$$=C_m^0\frac{1}{k!}\sum_{j_1,\cdots,j_k}\int_{\tilde\sigma^{(0)}_{j_1\cdots j_k}} f(\zeta)[\frac{1}{M_0^{n-m-k}}\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,H_{j_k},$$

$$\partial_{\bar\zeta}N^0,\cdots,\partial_{\bar\zeta}N^0)\ \wedge B_m^F(\zeta)-\frac{n-m-k}{M_0^{n-m-k+1}}\,G\wedge\det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,$$

$$H_{j_k},N^0,\partial_{\bar\zeta}N^0,\cdots,\partial_{\bar\zeta}N^0)\ \wedge B_m^F(\zeta)$$

$$=C_m^0 \sum_{j_1<\cdots<j_k} \int_{\tilde{\sigma}^{(0)}_{j_1\cdots j_k}} f(\zeta) \sum_{\rho=1}^{k} (-1)^{k-\rho} \det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,[H_{j_\rho}],\cdots,H_{j_k},$$

$$\frac{N^0}{M_0^{n-m-k+1}},\partial_{\bar{\zeta}}N^0,\cdots,\partial_{\bar{\zeta}}N^0,\cdots,\partial_{\bar{\zeta}}N^0)\wedge B_m^F(\zeta). \tag{43}$$

Using (43) repeatedly we obtain

$$C_m^0 \sum_{j_1<\cdots<j_{k+1}} \int_{\tilde{\sigma}^{(0)}_{j_1\cdots j_{k+1}}} f(\zeta) \sum_{\rho=1}^{k+1} (-1)^{k+1-\rho} \det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,[H_{j_\rho}],\cdots,$$

$$H_{j_{k+1}},\frac{N^0}{M_0^{n-m-k}},\partial_{\bar{\zeta}}N^0,\cdots,\partial_{\bar{\zeta}}N^0)\wedge B_m^F(\zeta)$$

$$=C_m^0 \sum_{j_1} \int_{\tilde{\sigma}^{(0)}_{j_1}} f(\zeta)\det{}_{(n)}(h_1,\cdots,h_m,\frac{N^0}{M_0^{n-m}},\partial_{\bar{\zeta}}N^0,\cdots,\partial_{\bar{\zeta}}N^0)\wedge B_m^F(\zeta)$$

$$=C_m\int_{\partial\tilde{D}} \frac{f(\zeta)}{M_0^{n-m}}\det{}_{(n)}(N^0,h_1,\cdots,h_m,\partial_{\bar{\zeta}}N^0,\cdots,\partial_{\bar{\zeta}}N^0)\wedge B_m^F(\zeta). \tag{44}$$

Applying (25) to the right-hand side of (44), we have

$$f(z)=C_m^0 \sum_{j_1<\cdots<j_{k+1}} \int_{\tilde{\sigma}^{(0)}_{j_1\cdots j_{k+1}}} \frac{f(\zeta)}{M_0^{n-m-k}} \sum_{\rho=1}^{k+1} (-1)^{k+1-\rho} \det{}_{(n)}(h_1,\cdots,h_m,$$

$$H_{j_1},\cdots,[H_{j_\rho}],\cdots,h_{j_{k+1}},N^0,\partial_{\bar{\zeta}}N^0,\cdots,\partial_{\bar{\zeta}}N^0)\wedge B_m^F(\zeta). \tag{45}$$

(39) and (45) imply (36).

Remark. When $k=n-m-1$, (45) can be rewritten as:

$$f(z)=C_m^0 \sum_{j_1<\cdots<j_{n-m}} \int_{\tilde{\sigma}^{(0)}_{j_1\cdots j_{n-m}}} \frac{f(\zeta)}{M_0} \sum_{\rho=1}^{n-m} (-1)^{n-m-\rho} \det{}_{(n)}(h_1,\cdots,h_m,H_{j_1},\cdots,$$

$$[H_{j_\rho}],\cdots,H_{j_{n-m}},N^0)\wedge B_m^F(\zeta)$$

$$=C_m^0 \sum_{j_1<\cdots<j_{n-m}} \int_{\tilde{\sigma}^{(0)}_{j_1\cdots j_{n-m}}} f(\zeta)\det{}_{(n)}(h_1,\cdots,h_m,$$

$$H_{j_1},\cdots,H_{j_{n-m}}),\wedge B_m^F(\zeta). \tag{46}$$

Let D be a holomorphic polyhedron. Then (45) and (46) are generalizations[8] of the integral representation formulas of holomorphic functions for analytic polyhedrons[4] in analytic subvarieties (the generalization of Weil's integral representation in analytic subvarieties[7] is also included).

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