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# Pairings in Categories

By

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### Abstract

We study a categorical generalization of some of the basic results of the homotopy theory which have been obtained in the category of topological spaces with base point. They are some of the results of Gottlieb, Hilton, Hoo, Lim, Varadarajan, Woo and Kim and others. To carry out categorical approach, we define pseudo-product, pseudo-coproduct and homotopy relation in general categories. These concepts enable us to define pairings and copairings in categories. Some of the classical results mentioned above are generalized to the categories with product, coproduct and zero object.

## Introduction

The purpose of this paper is to obtain a categorical generalization of the definitions and the basic properties of the following objects; square lemma for the homotopy set $[A, X]$, cyclic map and cocyclic map, Gottlieb group, Varadarajan set, Woo-Kim group, pairings and copairings, homotopy set of the axes of pairings and its dual.

To generalize these concepts to other categories, we define *homotopy relations* in categories. But our homotopy relations are more general than those in *algebraic homotopy theory*, so we do not assume homotopy theory in a strict sense (e.g. Baues [1]); we simply consider an equivalence relation among morphisms, called *homotopy*, which are preserved by induced morphisms. We also define *pseudo-product* and *pseudo-coproduct*, which are not assumed to have universal property unlike usual product and coproduct in a category.

In §1 we define homotopy relation, homotopy preserving functor, homotopy natural transformation, pseudo-product, pseudo-coproduct etc. in categories. These concepts are necessary for the various categorical definitions

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in the subsequent sections, especially for the categorical definition of pairings and copairings which are basic concepts of our categorical study.

In §2 we define pairing and related concepts in general categories. We study fundamental properties of perpendicular relations. We define the homotopy set of the axes of pairings, which is a categorical generalization of Gottlieb group [3], Varadarajan set [13] and Woo and Kim group [14] (cf. [10,12]). Dula [2] defined cyclic maps and cocyclic maps in general categories. Our results generalize some of the results of Dula [2].

In §3 we consider the duals of §2.

In §4 we work in categories $\mathscr{C}$ with product, coproduct and zero object. We show that some of the basic results of Hilton, Varadarajan and others can be generalized to other categories by proving these results in general categories. Our first results is a categorical generalization of square lemma (cf. Theorem 1.5 and Proposition 14.13 of Hilton [4] and Theorem 2.7 of [11]).

**Theorem 4.1.** *Let $\mu : X \square Y \to Z$ be a pairing and $\theta : A \to H \square R$ a copairing in $\mathscr{C}$. Let $\alpha : H \to X$, $\beta : R \to X$, $\gamma : H \to Y$ and $\delta : R \to Y$ be morphisms in $\mathscr{C}$. Then the following relation holds in $[A, Z]$:*

$$(\alpha \dot{+} \beta) \underset{\cdot}{+} (\gamma \dot{+} \delta) = (\alpha \underset{\cdot}{+} \gamma) \dot{+} (\beta \underset{\cdot}{+} \delta),$$

*where $\underset{\cdot}{+}$ and $\dot{+}$ are the pairings induced by $\mu$ and $\theta$ respectively.*

The above theorem implies categorical generalization of some of the earlier works of Gottlieb [3], Hoo [5], Lim [6,7], Varadarajan [13], Woo and Kim [14] and others (cf. [10,11,12]).

In Theorem 4.4, we prove that: *If $g : Y \to X$ is a cyclic morphism, then the image of $g_* : [A, Y] \to [A,X]$ is contained in the center of $[A,X]$ for any co-Hopf object $A$ in $\mathscr{C}$.* The dual result holds for cocyclic morphisms.

In Theorems 4.7 and 4.8 we prove the following result and its dual: *If $A$ is a co-grouplike object in $\mathscr{C}$, then $(1_X)^{\perp}(A,X)$, the homotopy set of cyclic morphisms from $A$ to $X$, is an abelian subgroup which is contained in the center of a group $[A,X]$.*

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## §1. Category with Homotopy Relation

Let $\mathscr{C}$ be a category [8,9]. The *composite* of two morphisms $f : X \to Y$ and $g : Y \to Z$ is denoted by $g \circ f : X \to Z$. An *identity morphism* in $\mathscr{C}$ is denoted by $1_X : X \to X$ for any object $X$ of $\mathscr{C}$.

A *homotopy relation* in $\mathscr{C}$ is an equivalence relation $f \simeq g$ among morphisms $f, g : X \to Y$ in $\mathscr{C}$, which satisfies the following two conditions.

(1.1) If $f \simeq g$, then $h \circ f \simeq h \circ g$ for any morphism $h : Y \to Z$.
(1.2) If $f \simeq g$, then $f \circ w \simeq g \circ w$ for any morphism $w : W \to X$.

We denote by $[f] : X \to Y$ the homotopy class containing the morphism $f : X \to Y$. We denote by $[X, Y]$ the set of all the homotopy classes of morphisms from $X$ to $Y$, namely $[X, Y] = \{[f] : X \to Y\}$.

Two objects $X$ and $Y$ are said to have the *same homotopy type* when there exist morphisms $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$.

Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between categories with homotopy relation. We call $F : \mathscr{C} \to \mathscr{D}$ a *homotopy preserving functor* when $f \simeq g : X \to Y$ implies $F(f) \simeq F(g) : F(X) \to F(Y)$.

Let $F, G : \mathscr{C} \to \mathscr{D}$ be homotopy preserving functors between categories with homotopy relation. A *homotopy natural transformation* $\tau : F \dot{\to} G$ is a family of morphisms

$$\{\tau(X) : F(X) \to G(X) \mid X \in \mathscr{C}\}$$

which satisfies $G(f) \circ \tau(X) \simeq \tau(Y) \circ F(f)$ for each objects $X$, $Y$ and each morphism $f : X \to Y$ in $\mathscr{C}$. In this situation we call the morphism $\tau = \tau(X) : F(X) \to G(X)$ a *homotopy natural morphism*.

For contravariant functors, we define homotopy preserving functors and homotopy natural transformations similarly.

If $\mathscr{A}$ and $\mathscr{B}$ are categories with homotopy relation, then the product category $\mathscr{A} \times \mathscr{B}$ [9] has a homotopy relation induced by those of $\mathscr{A}$ and $\mathscr{B}$. There are homotopy preserving functors

$$P_1 : \mathscr{A} \times \mathscr{B} \to \mathscr{A} \text{ and } P_2 : \mathscr{A} \times \mathscr{B} \to \mathscr{B}$$

which are defined by $P_1(A,B) = A$ and $P_2(A,B) = B$.

A *pseudo-product* is a homotopy preserving functor $\sqcap : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ (we write $\sqcap(X,Y) = X \sqcap Y$) with homotopy natural transformations $i_1 : P_1 \dot{\to} \sqcap$ and $i_2 : P_2 \dot{\to} \sqcap$. Therefore we have a homotopy commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i_1} & X \sqcap Y & \xleftarrow{i_2} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \sqcap g} & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow{i_1} & Z \sqcap W & \xleftarrow{i_2} & W \end{array}$$

for any objects $X, Y, Z, W$ and any morphisms $f : X \to Z$, $g : Y \to W$ in $\mathscr{C}$.

A *pseudo-coproduct* is a homotopy preserving functor $\sqcup : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ (we write $\sqcup(A,B) = A \sqcup B$) with homotopy natural transformations $q_1 : \sqcup \dot{\to} P_1$ and $q_2 : \sqcup \dot{\to} P_2$.

*Remarks* 1.3. (1) Since $1_X \sqcap 1_Y = 1_{X \sqcap Y}$ (resp. $1_X \sqcup 1_Y = 1_{X \sqcup Y}$) holds for any objects $X$ and $Y$, the homotopy type of $X \sqcap Y$ (resp. $X \sqcup Y$) depends only on the homotopy types of $X$ and $Y$.
(2) (Y. Hirashima) The conditions (1.1) and (1.2) are equivalent to the condition that the quotient category $\mathscr{C}/\simeq$ is well-defined (cf. §8 of Chapter II of Mac Lane [9]). A homotopy relation in a category $\mathscr{C}$ can be defined by a functor $F$ from $\mathscr{C}$ to any category $\mathscr{D}$ by defining $f \simeq g : X \to Y$ if and only if $F(f) = F(g) : F(X) \to F(Y)$. Moreover, any homotopy relation is obtained in this manner using a quotient functor $F : \mathscr{C} \to \mathscr{D}(=\mathscr{C}/\simeq)$.

Then a homotopy preserving functor $F : \mathscr{C} \to \mathscr{D}$ is characterized by a functor $\bar{F} : \mathscr{C}/\simeq \to \mathscr{D}/\simeq$ between quotient categories. A homotopy natural transformation $\tau : F \dot{\to} G$ is characterized by a natural transformation $\bar{\tau} : \bar{F} \dot{\to} \bar{G}$ between the induced functors $\bar{F}, \bar{G} : \mathscr{C}/\simeq \to \mathscr{D}/\simeq$.

## §2. Pseudo-product and Pairing

We call a morphism $\mu : X \sqcap Y \to Z$ a *pairing* with *axes* $f : X \to Z$ and $g : Y \to Z$ if the relations $\mu \circ i_1 \simeq f$ and $\mu \circ i_2 \simeq g$ hold in $\mathscr{C}$, where $i_1 : X \to X \sqcap Y$ and $i_2 : Y \to X \sqcap Y$ are the homotopy natural morphisms for pseudo-product.

Let us fix a pseudo-product $\sqcap$ throughout the rest of this section. We write $f \perp g$ (and say that $f$ is *perpendicular* or *orthogonal* to $g$) when there exists a pairing $\mu: X \sqcap Y \to Z$ with axes $f: X \to Z$ and $g: Y \to Z$ in $\mathscr{C}$. If $f_0 \simeq f_1: X \to Z$ and $g_0 \simeq g_1: Y \to Z$, then we see that $f_0 \perp g_0$ if and only if $f_1 \perp g_1$.

We call a morphism $g: Y \to Z$ a *cyclic morphism* in $\mathscr{C}$ when $1_Z \perp g$.

We call an object $X$ a *Hopf object* in $\mathscr{C}$ when $1_X \perp 1_X$.

**Theorem 2.1.** *If $f \perp g$ for morphisms $f: X \to Z$ and $g: Y \to Z$, then the following results hold.*
(1) *$(f \circ f') \perp (g \circ g')$ for any morphisms $f': X' \to X$ and $g': Y' \to Y$.*
(2) *$(w \circ f) \perp (w \circ g)$ for any morphism $w: Z \to W$.*

*Proof.* Since $f \perp g$, there is a pairing $\mu: X \sqcap Y \to Z$ with axes $f$ and $g$. Then $\mu \circ (f' \sqcap g')$ and $w \circ \mu$ are pairings for $(f \circ f') \perp (g \circ g')$ and $(w \circ f) \perp (w \circ g)$ respectively.

**Proposition 2.2.** (1) *If $g: Y \to Z$ is a cyclic morphism, then $g \circ g': Y' \to Z$ is a cyclic morphism for any $g': Y' \to Y$.*
(2) *Let $t: Z \to W$ be a morphism with a right homotopy inverse, that is, there exists a morphism $i: W \to Z$ such that $t \circ i \simeq 1_W$. If $g: Y \to Z$ is a cyclic morphism, them $t \circ g: Y \to W$ is a cyclic morphism.*
(3) *If $g: Y \to Z$ is a cyclic morphism, then $f \perp g$ for any morphism $f: X \to Z$.*

*Proof.* (1) If $1_Z \perp g$, then we have $1_Z \perp (g \circ g')$ by Theorem 2.1(1).
(2) If $1_Z \perp g$, then $(t \circ 1_Z \circ i) \perp (t \circ g)$ by Theorem 2.1(1)(2) or $1_W \perp (t \circ g)$.
(3) Since $1_Z \perp g$, we have $(1_Z \circ f) \perp g$ by Theorem 2.1(1) or $f \perp g$. q.e.d.

Let $v: V \to Z$ be a fixed morphism in $\mathscr{C}$. We call a morphism $g: Y \to Z$ a *$v$-cyclic morphism* when $v \perp g$. We define the homotopy set of $v$-cyclic morphisms by

$$v^{\perp}(Y,Z) = \{[g]: Y \to Z \mid v \perp g\} \subset [Y,Z].$$

The set $v^{\perp}(Y,Z)$ is *homotopy invariant*, that is, it depends only on homotopy types of $V$, $Y$, $Z$ and the homotopy class of $v$.

**Proposition 2.3.** (1) *If $v \perp g$ for morphisms $v: V \to Z$ and $g: Y \to Z$, then* $\mathrm{Im}(g_*: [A,Y] \to [A,Z]) \subset v^{\perp}(A,Z)$ *for any $A$ in $\mathscr{C}$.*

(2) $v^{\perp}(Y,Z) \subset (v \circ d)^{\perp}(Y,Z)$ *for any morphisms* $v: V \to Z$ *and* $d: D \to V$.

*Proof.* (1) The relation $v \perp g$ implies $v \perp (g \circ \alpha)$ for any morphism $\alpha: A \to Y$ by Theorem 2.1(1). It follows that $g_*([\alpha]) = [g \circ \alpha] \in v^{\perp}(A,Z)$.
(2) Let $\alpha \in v^{\perp}(Y,Z)$. Then we have $v \perp \alpha$ and hence $(v \circ d) \perp \alpha$ by Theorem 2.1(1). Then we have $\alpha \in (v \circ d)^{\perp}(Y,Z)$.

**Theorem 2.4.** *An object* $Z$ *is a Hopf object in* $\mathscr{C}$ *if and only if* $v^{\perp}(Y,Z) = [Y,Z]$ *for any morphism* $v: V \to Z$ *and any object* $Y$.

*Proof.* If $Z$ is a Hopf object, then the relation $1_Z \perp 1_Z$ holds. Let $\alpha: Y \to Z$ be any morphism. Then we have $(1_Z \circ v) \perp (1_Z \circ \alpha)$ by Theorem 2.1(1) and hence $v \perp \alpha$. It follows that $[\alpha] \in v^{\perp}(Y,Z)$.

Conversely, if $v^{\perp}(Y,Z) = [Y,Z]$ for any morphism $v: V \to Z$ and any object $Y$, then we have $(1_Z)^{\perp}(Z,Z) = [Z,Z]$. It follows that $1_Z \perp 1_Z$.

**Example 2.5.** (Category of graded $K$-algebras) We note that Proposition 4.1 (p.181) of Mac Lane [8] gives a condition for morphisms to be perpendicular in this category (We consider the equality "=" as a homotopy relation and the tensor product $\Lambda \otimes \Sigma$ as a pseudo-product): *If* $f: \Lambda \to \Omega$ *and* $g: \Sigma \to \Omega$ *are homomorphisms of graded* $K$*-algebras such that always*

$$(f\lambda)(g\sigma) = (-1)^{\deg\lambda \deg\sigma}(g\sigma)(f\lambda),$$

*there is a unique homomorphism* $h: \Lambda \otimes \Sigma \to \Omega$ *of graded algebras with* $h(\lambda \otimes 1) = f(\lambda)$, $h(1 \otimes \sigma) = g(\sigma)$.

*Remarks* 2.6. (1) Proposition 2.2 (1) and (2) are categorical generalization of Lemmas 1.3 and 1.4 of Varadarajan [13]. Theorem 2.4 is a categorical generalization of Proposition 3.3 of Lim [6]. cf. Proposition 1.9 of Varadarajan [13]. For other results, see [10,11,12].
(2) Special cases of the homotopy set $v^{\perp}(Y,Z)$ were defined and studied in the category of topological spaces with base point by the following authors; $(1_X)^{\perp}(S^n,X) = G_n(X)$ by Gottlieb [3]; $(1_X)^{\perp}(Y,X) = G(Y,X)$ by Varadarajan [13]; $f^{\perp}(S^n,X) = G_n^f(X,A,*)$ for any map $f: A \to X$ by Woo and Kim [14]. Cyclic maps in the category of topological spaces were studied by Hoo [5] and Lim [6] and the other mathematicians.

## §3. Pseudo-coproduct and Copairing

We now consider the duals of the results in §2. The proofs are omittedhere, since they are given by dualizing the corresponding proofs in §2.

We call a morphism $\theta: A \to H \sqcup R$ a *copairing* with *coaxes* $h: A \to H$ and $r: A \to R$ if the relations $q_1 \circ \theta \simeq h$ and $q_2 \circ \theta \simeq r$ hold in $\mathscr{C}$, where $q_1: H \sqcup R \to H$ and $q_2: H \sqcup R \to R$ are homotopy natural morphisms for pseudo-coproduct.

Let us fix a pseudo-coproduct $\sqcup$ throughout the rest of this section. We write $h \top r$ (and say that $h$ is *coperpendicular* or *co-orthogonal* to $r$) when there exists a copairing $\theta: A \to H \sqcup R$ with coaxes $h: A \to H$ and $r: A \to R$ in $\mathscr{C}$. If $h_0 \simeq h_1: A \to H$ and $r_0 \simeq r_1: A \to R$, then we see that $h_0 \top r_0$ if and only if $h_1 \top r_1$.

We call a morphism $r: A \to R$ a *cocyclic morphism* in $\mathscr{C}$ when $1_A \top r$.

We call an object $A$ a *co-Hopf object* in $\mathscr{C}$ when $1_A \top 1_A$.

**Theorem 3.1.** *If $h \top r$ for morphisms $h: A \to H$ and $r: A \to R$, then the following results hold.*

(1) *$(h' \circ h) \top (r' \circ r)$ for any morphisms $h': H \to H'$and $r': R \to R'$.*

(2) *$(h \circ d) \top (r \circ d)$ for any morphism $d: D \to A$.*

**Proposition 3.2.** (1) *If $r: A \to R$ is a cocyclic morphism, then $r' \circ r: A \to R'$ is a cocyclic morphism for any $r': R \to R'$.*

(2) *Let $i: X \to A$ be a morphism with left homotopy inverse, that is, there exist a morphism $t: A \to X$ such that $t \circ i \simeq 1_X$. If $r: A \to R$ is a cocyclic morphism, then $r \circ i: X \to R$ is a cocyclic morphism.*

(3) *If $r: A \to R$ is a cocyclic morphism, then $h \top r$ for any morphism $h: A \to H$.*

Let $u: A \to U$ be a fixed morphism. We call a morphism $r: A \to R$ *a $u$-cocyclic morphism* when $u \top r$. We define the homotopy set of $u$-cocyclic morphisms by

$$u^{\top}(A,R) = \{[r]: A \to R \mid u \top r\} \subset [A,R].$$

This set $u^{\top}(A,R)$ is also a homotopy invariant.

**Proposition 3.3.** (1) *If $u \top r$ for morphisms $u: A \to U$ and $r: A \to R$, then* $\mathrm{Im}(r^*: [R,X] \to [A,X]) \subset u^{\top}(A,X)$ *for any $X$ in $\mathscr{C}$.*

(2) $u^{\top}(A,X) \subset (w \circ u)^{\top}(A,X)$ *for any morphisms* $u: A \to U$ *and* $w: U \to W$.

**Theorem 3.4.** *An object* $A$ *is a co-Hopf object in* $\mathscr{C}$ *if and only if* $u^{\top}(A,R) = [A,R]$ *for any morphism* $u: A \to U$ *and any object* $R$ *in* $\mathscr{C}$.

*Remarks* 3.5 (1) (Baues; Dula [2]) Let $F$ be a homotopy preserving functor. If the pseudo-product is preserved by $F$, namely $F(X \sqcap Y) = F(X) \sqcap F(Y)$, then $f \perp g$ implies $F(f) \perp F(g)$. Similar remark applies to pseudo-coproduct and homotopy preserving contravariant functor.
(2) The relations $f \perp g$ and $h \top r$, and hence the sets $v^{\perp}(Y,Z)$ and $u^{\top}(A,R)$, depend on the choice of pseudo-product $\sqcap$ and pseudo-coproduct $\sqcup$.
(3) Professor P.J. Eccles named the relation $h \top r$ *coperpendicular*.
(4) Proposition 3.2(1) is a categorical generalization of Lemma 7.2 of Varadarajan [13]. Theorem 3.4 is a categorical generalization of Proposition 3.2 of Lim [7]. For other results, see [10,11,12].
(5) In the category of topological spaces with base point, we have $(1_A)^{\top}(A,C) = DG(A,C)$ of Varadarajan [13]. Lim [7] studied cocyclic maps and the properties of the set $DG(A,C)$ in this category.
(6) We have homotopy invariant families of subsets

$$\{v^{\perp}(Y,Z) \mid v: V \to Z \text{ for all } V\} \text{ of } [Y,Z] \text{ and}$$

$$\{u^{\top}(A,X) \mid u: A \to U \text{ for all } U\} \text{ of } [A,X].$$

## §4. Category with Product, Coproduct and Zero Object

In this section we prove various results on pairings and copairings for any category which has product, coproduct and zero object.

Let $\mathscr{C}$ be a category with homotopy relation. In this section we assume that $\mathscr{C}$ has product $X \times Y$ and coproduct $X \vee Y$ and zero object (= null object = both initial and final object, cf. Mac Lane [9]). We denote by $*$ the zero object. We denote also by $*: X \to Y$ the morphism which factors through $*$, namely, $*: X \to * \to Y$.

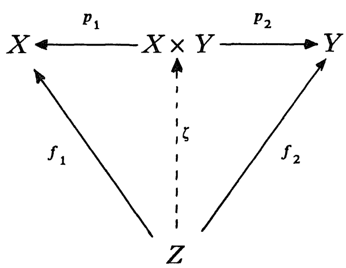
A product diagram

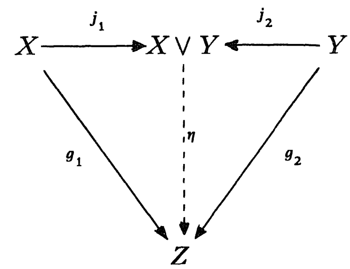
$$X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$$

and a coproduct diagram

$$X \xrightarrow{j_1} X \vee Y \xleftarrow{j_2} Y$$

have universal property. Therefore the following diagrams can be uniquely filled in by dotted arrows so that the diagrams are *strictly* commutative in the category $\mathscr{C}$ for any morphisms $f_1 : Z \to X$, $f_2 : Z \to Y$ and $g_1 : X \to Z$, $g_2 : Y \to Z$:




We write $\zeta = (f_1, f_2)$ and $\eta = <g_1, g_2>$. We remark that

$$(f_1' \times f_2') \circ (f_1, f_2) = (f_1' \circ f_1, f_2' \circ f_2),\ (f_1, f_2) \circ f = (f_1 \circ f, f_2 \circ f),$$

$$<g_1, g_2> \circ (g_1' \vee g_2') = <g_1 \circ g_1', g_2 \circ g_2'>,\ g \circ <g_1, g_2> = <g \circ g_1, g \circ g_2>$$

for any morphisms $f_1' : X \to X'$, $f_2' : Y \to Y'$, $f : A \to Z$ and $g_1' : X' \to X$, $g_2' : Y' \to Y$, $g : Z \to W$.

Let $X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$ be a product diagram. The zero object $*$ enables us to define "inclusion morphisms $i_1 = (1_X, *) : X \to X \times Y$ and $i_2 = (*, 1_Y) : Y \to X \times Y$. We have $p_1 \circ i_1 = 1_X$, $p_2 \circ i_1 = *$, $p_1 \circ i_2 = *$ and $p_2 \circ i_2 = 1_Y$. Dually, we define morphisms $q_1 = <1_X, *> : X \vee Y \to X$ and $q_2 = <*, 1_Y> : X \vee Y \to Y$. We have $q_1 \circ j_1 = 1_X$, $q_1 \circ j_2 = *$, $q_2 \circ j_1 = *$ and $q_2 \circ j_2 = 1_Y$.

If the product and the coproduct are homotopy preserving, then we can define a pseudo-product and a pseudo-coproduct by the product and the coproduct respectively, namely

$X \sqcap Y = X \times Y$ (pseudo-product = product),

$X \sqcup Y = X \vee Y$ (pseudo-coproduct = coproduct).

**In the rest of this section we assume that** *the category $\mathscr{C}$ has product, coproduct and zero object* **and that** *the pseudo-product is given by the product and the pseudo-coproduct is given by the coproduct.* (So the product and coproduct must be homotopy preserving ones.)

A pairing $\mu : X \sqcap Y \to Z$ defines a pairing of homotopy sets

$$\underset{\cdot}{+} : [A,X]\times[A,Y] \to [A,Z]$$

by the formula $\alpha \underset{\cdot}{+} \beta = \mu \circ (\alpha,\beta) : A \to Z$ for any morphisms $\alpha : A \to X$ and $\beta : A \to Y$ in $\mathscr{C}$. We have $\alpha \underset{\cdot}{+} * = f_*(\alpha)$ and $* \underset{\cdot}{+} \beta = g_*(\beta)$ for any morphisms $\alpha : A \to X$ and $\beta : A \to Y$, since $(\alpha, *) = i_1 \circ \alpha$ and $(*, \beta) = i_2 \circ \beta$.

A copairing $\theta : A \to H \sqcup R$ defines a pairing of homotopy sets

$$\dot{+} : [H,Z] \times [R,Z] \to [A,Z]$$

by the formula $\alpha \dot{+} \beta = <\alpha,\beta> \circ \theta : A \to Z$ for any morphisms $\alpha : H \to Z$ and $\beta : R \to Z$ in $\mathscr{C}$. We have $\alpha \dot{+} * = h^*(\alpha)$ and $* \dot{+} \beta = r^*(\beta)$ for any morphisms $\alpha : H \to Z$ and $\beta : R \to Z$, since $<\alpha,*> = \alpha \circ q_1$ and $<*,\beta> = \beta \circ q_2$.

**Theorem 4.1.** *Let $\mu : X \sqcap Y \to Z$ be a pairing and $\theta : A \to H \sqcup R$ a copairing in $\mathscr{C}$. Let $\alpha : H \to X$, $\beta : R \to X$, $\gamma : H \to Y$ and $\delta : R \to Y$ be morphisms in $\mathscr{C}$. Then the following relation holds in $[A,Z]$:*

$$(\alpha \dot{+} \beta) \underset{\cdot}{+} (\gamma \dot{+} \delta) = (\alpha \underset{\cdot}{+} \gamma) \dot{+} (\beta \underset{\cdot}{+} \delta).$$

*Proof.* (We actually show that the equation holds as morphisms in $\mathscr{C}$.) Define a map $\Psi : H \sqcup R \to X \sqcap Y$ by

$$\Psi = (<\alpha,\beta>, <\gamma,\delta>) = <(\alpha,\gamma), (\beta,\delta)>,$$

where $<\alpha,\beta> : H \sqcup R \to X$, $<\gamma,\delta> : H \sqcup R \to Y$, $(\alpha,\gamma) : H \to X \sqcap Y$ and $(\beta,\delta) : R \to X \sqcap Y$. We see that two definitions of the above $\Psi$ coincide by universality of product and coproduct (cf. Lemma 14.12 of Hilton [4] and p.74 of Mac Lane [9]). We have

$$\text{(E)} \qquad \mu \circ (\Psi \circ \theta) = (\mu \circ \Psi) \circ \theta.$$

The left hand side of the equation (E) is

$$\mu \circ (\Psi \circ \theta) = \mu \circ (<\alpha,\beta> \circ \theta, <\gamma,\delta> \circ \theta) = \mu \circ (\alpha \dot{+} \beta, \gamma \dot{+} \delta) = (\alpha \dot{+} \beta) \underset{\cdot}{+} (\gamma \dot{+} \delta).$$

The right hand side of the equation (E) is

$$(\mu \circ \Psi) \circ \theta = <\mu \circ (\alpha,\gamma), \mu \circ (\beta,\delta)> \circ \theta = <\alpha \underset{\cdot}{+} \gamma, \beta \underset{\cdot}{+} \delta> \circ \theta = (\alpha \underset{\cdot}{+} \gamma) \dot{+} (\beta \underset{\cdot}{+} \delta).$$

Therefore the result follows.

**Theorem 4.2.** *Let $\mu : X \sqcap Y \to Z$ be a pairing with axes $f : X \to Z$ and $g : Y \to Z$ in $\mathscr{C}$. Let $\theta : A \to H \sqcup R$ be a copairing with coaxes $h : A \to H$ and $r : A \to R$ in $\mathscr{C}$. Then we have the following relations in $[A,Z]$ for any morphisms $\alpha : H \to X$, $\beta : R \to X$, $\gamma : H \to Y$ and $\delta : R \to Y$ in $\mathscr{C}$.*

(1) $h^*(\alpha) \underset{\cdot}{+} r^*(\delta) = f_*(\alpha) \dot{+} g_*(\delta)$

(2) $r^*(\beta) \underset{\cdot}{+} h^*(\gamma) = g_*(\gamma) \dot{+} f_*(\beta)$

*Proof.* By Theorem 4.1 (set $\beta=\gamma=*$ for (1) and $\alpha=\delta=*$ for (2)).

**Proposition 4.3.** *Let $\mu : X \sqcap Y \to Z$ be a pairing with axes $f : X \to Z$ and $g : Y \to Z$ and $\theta : A \to H \sqcup R$ a copairing with coaxes $h : A \to H$ and $r : A \to R$ in $\mathscr{C}$.*

(1) *If $X=Y$ and $f \simeq g$, then*

$$h^*(\alpha) \underset{\cdot}{+} r^*(\beta) = r^*(\beta) \underset{\cdot}{+} h^*(\alpha)$$

*in $[A,Z]$ for any morphisms $\alpha : H \to X$ and $\beta : R \to X$.*

(2) *If $H=R$ and $h \simeq r$, then*

$$f_*(\beta) \dot{+} g_*(\delta) = g_*(\delta) \dot{+} f_*(\beta)$$

*in $[A,Z]$ for any morphisms $\beta : H \to X$ and $\delta : H \to Y$ in $\mathscr{C}$.*

*Proof.* These are direct consequences of Theorem 4.2. q.e.d

The homotopy set $[A,X]$ has a binary operation $\dot{+}$ when $A$ is a co-Hopf object and it has a binary operation $\underset{\cdot}{+}$ when $X$ is a Hopf object. If $A$ is a co-Hopf object and $X$ is a Hopf object, then the two binary operations $\dot{+}$ and $\underset{\cdot}{+}$ in $[A,X]$ coincide and these binary operations are abelian by Theorem 4.2 (set $h \simeq r \simeq 1_A$ and $f \simeq g \simeq 1_X$). Moreover, it is associative (set $\beta = *$ in Theorem 4.1).

Let $S$ be a set with a binary operation $\cdot$. We define the *center* of $S$ by

$$\{z \in S \mid z \cdot x = x \cdot z \text{ for any } x \in S\} \subset S.$$

**Theorem 4.4.** (1) *Let $X$ be a Hopf object in $\mathscr{C}$. If $r : A \to R$ is a cocyclic morphism, then the image of $r^* : [R,X] \to [A,X]$ is contained in the center of $[A,X]$.*

(2) *Let $A$ be a co-Hopf object in $\mathscr{C}$. If $g : Y \to X$ is a cyclic morphism,*

*then the image of* $g_*: [A,Y] \to [A,X]$ *is contained in the center of* $[A,X]$.

*Proof.* (1) By Proposition 4.3(1) (set $f \simeq g \simeq 1_X$ and $h \simeq 1_A$).
(2) By Proposition 4.3(2) (set $h \simeq r \simeq 1_A$ and $f \simeq 1_X$). q.e.d.

We have a "canonical" morphism $j: X \sqcup Y \to X \sqcap Y$ which is defined by $j = \langle i_1, i_2 \rangle$ for the inclusion morphisms $i_1: X \to X \sqcap Y$ and $i_2: Y \to X \sqcap Y$.

**Theorem 4.5.** *Let* $f: X \to Z$, $v: V \to Z$, $g: Y \to V$ *and* $w: W \to V$ *be morphisms and* $\theta: A \to X \sqcup Y$ *a copairing in* $\mathscr{C}$. *Then* $f \perp v$ *and* $g \perp w$ *implies* $\{f \dotplus (v \circ g)\} \perp (v \circ w)$.

*Proof.* Let $\mu_1: X \sqcap V \to Z$ and $\mu_2: Y \sqcap W \to V$ be pairings for $f \perp v$ and $g \perp w$ respectively. Then the composition of morphisms

$$\mu_1 \circ (1_X \sqcap \mu_2) \circ \Xi \circ (j \sqcap 1_W) \circ (\theta \sqcap 1_W): A \sqcap W \to Z$$

is a pairing for $\{f \dotplus (v \circ g)\} \perp (v \circ w)$, where

$$\Xi: (X \sqcap Y) \sqcap W \to X \sqcap (Y \sqcap W)$$

is the natural isomorphism of products. q.e.d.

Dually we have the following results.

**Theorem 4.6.** *Let* $h: A \to H$, $r: A \to R$, $u: H \to U$ *and* $d: H \to D$ *be morphisms and* $\mu: D \sqcap R \to Z$ *a pairing in* $\mathscr{C}$. *Then* $u \top d$ *and* $h \top r$ *implies* $(u \circ h) \top \{(d \circ h) \dotplus r\}$.

We call a co-Hopf object $A$ with a comultiplication $\theta: A \to A \cup A$ a *homotopy associative* co-Hopf object when

$$(\theta \sqcup 1_A) \circ \theta \simeq (1_A \sqcup \theta) \circ \theta: A \to A \sqcup A \sqcup A.$$

We call an object $A$ a *co-grouplike object* in $\mathscr{C}$ when $A$ is a homotopy associative co-Hopf object in $\mathscr{C}$ with an inverse $\nu: A \to A$, namely, $1_A \dotplus \nu \simeq * \simeq \nu \dotplus 1_A$. If $A$ is a co-grouplike object, then $[A,X]$ is a group for any object $X$ in $\mathscr{C}$ (cf. Chapter 14 of Hilton [4]).

**Theorem 4.7.** *If $A$ is a co-grouplike object in $\mathscr{C}$, then $(1_X)^{\perp}(A,X)$ is an abelian subgroup which is contained in the center of a group $[A,X]$.*

*Proof.* We remark that $f \perp g$ if and only if $g \perp f$ for any morphisms $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ in our category, since we can define "switching morphism" $T: X \times Y \rightarrow Y \times X$ by the universality of product and coproduct. Then we can use Theorem 4.5 for the proof of this theorem.

Now, by Theorem 4.5 (set $v=w=1_X$ and $f,g:A \rightarrow X$), we see that the subset $(1_X)^{\perp}(A,X)$ is closed under the operation $\dot{+}$ induced by the co-Hopf structure of $A$. Moreover, $(1_X)^{\perp}(A,X)$ is contained in the center of $[A,X]$ by Theorem 4.4(2).

Let $[\alpha]$ be an element of $(1_X)^{\perp}(A,X)$. Then we have $\alpha \perp 1_X$ and hence $(\alpha \circ \nu) \perp 1_X$ for the inverse $\nu: A \rightarrow A$ by Theorem 2.1(1). Therefore we have $\dot{-}[\alpha]=[\alpha \circ \nu] \in (1_X)^{\perp}(A,X)$. q.e.d

By a dual argument we have the following results.

We call a Hopf object $Z$ with a multiplication $\mu: Z \sqcap Z \rightarrow Z$ a *homotopy associative* Hopf object when

$$\mu \circ (\mu \sqcap 1_Z) \simeq \mu \circ (1_Z \sqcap \mu): Z \sqcap Z \sqcap Z \rightarrow Z.$$

We call an object $Z$ a *grouplike object* in $\mathscr{C}$ when $Z$ is a homotopy associative Hopf object in $\mathscr{C}$ with an inverse $\nu: Z \rightarrow Z$, namely, $1_Z \dot{+} \nu \simeq * \simeq \nu \dot{+} 1_Z$. If $Z$ is a grouplike object, then $[A,Z]$ is a group for any object $A$ in $\mathscr{C}$.

**Theorem 4.8.** *Let $Z$ be a grouplike object and $A$ any object in $\mathscr{C}$. Then $(1_Z)^{\top}(A,Z)$ is an abelian subgroup which is contained in the center of a group $[A,Z]$.*

Finally we list eight categories as examples to which our categorical results apply; (i) category of equivariant topological spaces with base point, (ii) category of topological spaces with a homotopy relation induced by a functor, (iii) category of metric spaces with uniform homotopy, (iv) category of uniform spaces, (v) category of topological spaces with $\theta$-continuous maps, (vi) category of fibrewise pointed topological spaces over $B$, (vii) category of simplicial sets, (viii) model category with zero object.

All the results in this section hold in the above categories with their own homotopy relations, since these categories have product, coproduct and zero object.

## References

[1] Baues, H.J., *Algebraic homotopy*, Cambridge Stud. Adv. Math., Cambridge Univ. Press, Cambridge, 1989.

[2] Dula, G., Hopf invariants of morphisms in general categories, *Preprint*.

[3] Gottlieb, D.H., Evaluation subgroups of homotopy groups, *Amer. J. Math.*, **91** (1969), 729–756.

[4] Hilton, P., *Homotopy theory and duality*, Notes on Math. Appl., Gordon and Breach, New York London Paris, 1965.

[5] Hoo, C.S., Cyclic maps from suspensions to suspensions, *Canad. J. Math.*, **24** (1972), 789–791.

[6] Lim K.L., On cyclic maps, *J. Austral. Math. Soc. Ser. A*, **32** (1982), 349–357.

[7] ———, Cocyclic maps and coevaluation subgroups, *Canad. Math. Bull.*, **30** (1987), 63–71.

[8] Mac Lane, S., *Homology*, Grundlehren Math. Wiss. in Einzeldarstellungen **114**, Springer-Verlag, Berlin Göttingen New York, 1963.

[9] ———, *Categories for the working mathematician*, Graduate Texts in Math. **5**, Springer-Verlag, New York, 1971.

[10] Oda, N., The homotopy set of the axes of pairings, *Canad. J. Math.*, **42** (1990), 856–868.

[11] ———, Pairings and copairings in the category of topological spaces, *Publ. RIMS, Kyoto Univ.*, **28** (1992), 83–97.

[12] ———, Pairings of homotopy sets over and under *B*, *Canad. Math. Bull.*, **36** (1993), 231–240.

[13] Varadarajan, K., Generalized Gottlieb groups, *J. Indian Math. Soc.*, **33** (1969), 141–164.

[14] Woo, M.H. and Kim, J.-R., Certain subgroups of homotopy groups, *J. Korean Math. Soc.*, **21** (1984), 109–120.