

The Strong Bernstein-Gelfand-Gelfand Resolution for Generalized Kac-Moody Algebras, I — The Existence of the Resolution —

By

Satoshi NAITO***

§1. Introduction

The purpose of this series of works is to show the existence of the strong Bernstein-Gelfand-Gelfand resolution (cf. [1]) of the irreducible highest weight module $L(\Lambda)$ with dominant integral highest weight Λ over a symmetrizable generalized Kac-Moody algebra (=GKM algebra). Here, GKM algebras are a class of contragredient Lie algebras $\mathfrak{g}(A)$ over \mathbf{C} (see [4], [8], or [7, Chapter 1]) associated to a real square matrix $A=(a_{ij})_{i,j \in I}$ indexed by a finite set I which satisfies the conditions:

- (C1) either $a_{ii}=2$ or $a_{ii} \leq 0$;
- (C2) $a_{ij} \leq 0$ if $i \neq j$, and $a_{ij} \in \mathbf{Z}$ if $a_{ii}=2$;
- (C3) $a_{ij}=0$ implies $a_{ji}=0$.

(We call such a matrix a GGCM in this paper.)

These Lie algebras were introduced as a natural generalization of Kac-Moody algebras, and have been studied by Borcherds ([2] and [3]).

In the present paper, we prove the existence of the weak Bernstein-Gelfand-Gelfand (=BGG) resolution by homological methods, following the general lines of [5] and [17, Sections 7–9]. To be more precise, we prove the existence of the following $\mathfrak{g}(A)$ -module resolution of $L(\Lambda)$:

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* Department of Mathematics, Faculty of Science, Shizuoka University, 836 Ohya, Shizuoka 422, Japan.

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$$(\#) \quad 0 \leftarrow L(\Lambda) \leftarrow C_0(\Lambda) \leftarrow C_1(\Lambda) \leftarrow \cdots \leftarrow C_p(\Lambda) \leftarrow C_{p+1}(\Lambda) \leftarrow \cdots,$$

where $C_p(\Lambda) = \sum_{\substack{w \in W(J), \beta \in \mathcal{S}(\Lambda) \\ \ell(w) + \text{ht}(\beta) = p}}^{\oplus} V_{\mathfrak{m}}(w(\Lambda + \rho - \beta) - \rho)$ is the direct sum of generalized Verma modules $V_{\mathfrak{m}}(w(\Lambda + \rho - \beta) - \rho)$ with highest weight $w(\Lambda + \rho - \beta) - \rho$ canonically induced by the irreducible highest weight module $L_{\mathfrak{m}}(w(\Lambda + \rho - \beta) - \rho)$ over the maximal reductive part \mathfrak{m} of a finite type parabolic subalgebra \mathfrak{p} of $\mathfrak{g}(A)$ (see Sections 1.1, 2.1, and 2.2).

Then, using this BGG resolution, we obtain a vanishing theorem for the Lie algebra homology $H_*(\mathfrak{g}(A), L(\Lambda))$ of $\mathfrak{g}(A)$ with coefficients in $L(\Lambda)$, generalizing a result of Kumar [9, Theorem (1.7)] to symmetrizable GKM algebras.

As another consequence of the BGG resolution, we prove the generalization of Bott's "strange equality", which is due to Lepowsky ([11, Corollary 6.7]) in the case where $\mathfrak{g}(A)$ is a symmetrizable Kac-Moody algebra. This determines the dimension of the s -th relative Lie algebra homology $H_s(\mathfrak{g}(A), \mathfrak{m}, L(0))$ of $\mathfrak{g}(A)$ with respect to \mathfrak{m} with coefficients in the trivial one-dimensional module $L(0)$ ($s \geq 0$).

In the succeeding paper [15], we shall describe an explicit construction of the strong BGG resolution, which is equivalent to the above weak BGG resolution, developing the theory of Verma module embeddings.

This paper is organized as follows. In Section 2, after [14], we fix notation and review the notion of the category \mathcal{O}_J corresponding to a certain subset J of I . In Section 3, we briefly sketch the construction of the "weaker" BGG resolution for symmetrizable GKM algebras. That is, a resolution $(\#)$ of $L(\Lambda)$ in which the modules $C_p(\Lambda)$ ($p \geq 0$) have a $\mathfrak{g}(A)$ -module filtration:

$$(\#\#) \quad C_p(\Lambda) = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = 0,$$

such that $M_i/M_{i+1} \cong V_{\mathfrak{m}}(w(\Lambda + \rho - \beta) - \rho)$ ($w \in W(J)$, $\beta \in \mathcal{S}(\Lambda)$, $\ell(w) + \text{ht}(\beta) = p$), $i = 0, 1, \dots, k$.

This construction is carried out basically in the same way as in [5], except for the determination of the highest weights appearing in the modules $C_p(\Lambda)$ ($p \geq 0$). A necessary condition of the highest weights is obtained by imitating the proof of the Weyl-Kac-Borcherds character formula for $L(\Lambda)$ (cf. [2] or [7, Chapter 11]). On the other hand, we can prove as in [5] that the weights in this necessary condition actually appear as highest weights of successive

quotients M_i/M_{i+1} ($0 \leq i \leq k$). Combining these results, we complete the construction of the “weaker” BGG resolution.

In Section 4, we show a vanishing result about the relative Ext bifunctor, which asserts that $C_p(\Lambda)$ ($p \geq 0$) is actually the direct sum of generalized Verma modules $V_m(w(\Lambda + \rho - \beta) - \rho)$ ($w \in W(J)$, $\beta \in \mathcal{S}(\Lambda)$, $\ell(w) + \text{ht}(\beta) = p$). Thus we finally obtain the weak BGG resolution.

In Section 5, as applications of the BGG resolution (and its proof), we get some results on Lie algebra homology of GKM algebras.

§2. Preliminaries and Notation

We generally follow the notation of [7]. For detailed accounts of GKM algebras, see [2] and [7, Chapter 11].

2.1. GKM algebras and parabolic subalgebras. Let $\mathfrak{g}(A)$ be a GKM algebra, over the complex number field \mathbb{C} , associated to a GGCM $A = (a_{ij})_{i,j \in I}$, with Cartan subalgebra \mathfrak{h} , simple roots $\Pi = \{\alpha_i\}_{i \in I}$, and simple coroots $\Pi^\vee = \{\alpha_i^\vee\}_{i \in I}$. Let us fix an element ρ of $\mathfrak{h}^* = \text{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ such that $\langle \rho, \alpha_i^\vee \rangle = (1/2) \cdot a_{ii}$ ($i \in I$).

As is well-known, we have a triangular decomposition: $\mathfrak{g}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ with $\mathfrak{n}^\pm = \sum_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$, where \mathfrak{g}_α is the root space attached to a root $\alpha \in \Delta = \Delta^+ \cup \Delta^-$. Let I^{re} (resp. I^{im}) be the subset $\{i \in I \mid a_{ii} = 2$ (resp. $a_{ii} \leq 0\}$ of the indexing set I . For a subset J of I^{re} , we put $\Delta_J^+ := \Delta^+ \cap (\sum_{i \in J} \mathbb{Z}_{\geq 0} \alpha_i)$, and then $\Delta^+(J) := \Delta^+ \setminus \Delta_J^+$. Now, we define the following Lie subalgebras of $\mathfrak{g}(A)$:

$$\mathfrak{u}^\pm := \sum_{\alpha \in \Delta^\pm(J)} \mathfrak{g}_{\pm\alpha}, \quad \mathfrak{m} := \mathfrak{h} \oplus \sum_{\alpha \in \Delta_J^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}), \quad \mathfrak{p} := \mathfrak{m} \oplus \mathfrak{u}^+.$$

We call $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}^+$ a parabolic subalgebra of $\mathfrak{g}(A)$, and then \mathfrak{m} the maximal reductive part of \mathfrak{p} .

Recall that the Weyl group W of a GKM algebra $\mathfrak{g}(A)$ is by definition the subgroup of $GL(\mathfrak{h}^*)$ generated by fundamental reflections r_i ($i \in I^{re}$). For $w \in W$, we put $\Phi_w := \Delta^+ \cap w(\Delta^-)$. Note that the number of elements of the set Φ_w is equal to the length $\ell(w)$ of w ([13, Proposition 1.2]). Corresponding to a subset J of I^{re} , we define W_J to be the subgroup of W generated by the r_i 's ($i \in J$), and put $W(J) := \{w \in W \mid \Phi_w \subset \Delta^+(J)\}$.

2.2. Category \mathcal{O}_J . From now throughout this paper, we will assume that the GGCM A is symmetrizable. Let J be a subset of I^{re} . In [14,

Section 2], following [12], we defined \mathcal{O}_J to be the category of all \mathfrak{m} -modules M satisfying:

(J1) Viewed as an \mathfrak{h} -module, M is \mathfrak{h} -diagonalizable with finite-dimensional weight spaces;

(J2) there exist a finite number of elements $\mu_i \in \mathfrak{h}^*$ ($1 \leq i \leq s$) such that $\mathcal{P}(M) \subset \bigcup_{i=1}^s D(\mu_i)$, where $\mathcal{P}(M)$ is the set of all weights of M and $D(\mu_i) = \{\mu_i - \beta \mid \beta \in Q_+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i\}$ ($1 \leq i \leq s$);

(J3) Viewed as an \mathfrak{m} -module, M is a direct sum of irreducible highest weight \mathfrak{m} -modules $L_{\mathfrak{m}}(\lambda)$ with highest weight $\lambda \in P_J^+ := \{\mu \in \mathfrak{h}^* \mid \langle \mu, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \ (i \in J)\}$.

Recall that the category \mathcal{O}_J is closed under the operations of taking submodules, quotients, finite direct sums, and tensor products.

Remark. The category \mathcal{O} defined in [7, Chapter 9] is nothing but the category of all $\mathfrak{g}(A)$ -modules satisfying (J1) and (J2).

The following proposition plays a crucial role in the construction of the “weaker” BGG resolution in Section 3.

Proposition 2.1 ([14, Proposition 2.1]). *Let $\Lambda \in P^+ := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \ (i \in I)$, and $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ if $a_{ii} = 2\}$. Fix a subset J of I^e . Then, $L(\Lambda)$ and $(\bigwedge^j \mathfrak{u}^-) \otimes_{\mathbb{C}} L(\Lambda)$ ($j \geq 0$) are in the category \mathcal{O}_J , where $\bigwedge^j \mathfrak{u}^-$ is the exterior algebra of degree j over \mathfrak{u}^- , and is an \mathfrak{m} -module under the adjoint action since $[\mathfrak{m}, \mathfrak{u}^-] \subset \mathfrak{u}^-$.*

Remark. In [5], the above proposition was proved in the case where A is a symmetrizable GCM and the subset J is of finite type (i.e., the submatrix $A_J := (a_{ij})_{i,j \in J}$ of A is a classical Cartan matrix of finite type). In [12], Liu pointed out that, in the case where A is a symmetrizable GCM, the condition that J is of finite type can be removed by using Kac’s complete reducibility theorem ([7, Theorem 10.7]). Also to the case where A is a symmetrizable GGCM, Liu’s proof is applicable since the Chevalley generators e_i, f_i ($i \in I^e$) act locally nilpotently on $\mathfrak{g}(A)$ and $L(\Lambda)$.

Now, for later use, we recall the notion of the algebra \mathcal{E} introduced in [7, Chapter 9], which contains “formal \mathfrak{h} -characters” of \mathfrak{h} -modules in the category \mathcal{O}_J for the case $J = \emptyset$. (Note that in [7, Chapter 9] formal characters

are defined only for $\mathfrak{g}(A)$ -modules in the category \mathcal{O} .) The elements of \mathcal{E} are series of the form $\sum_{\lambda \in \mathfrak{h}^*} c_\lambda e(\lambda)$, where $c_\lambda \in \mathbb{C}$ and $c_\lambda = 0$ for λ outside a finite union of sets of the form $D(\mu)$ ($\mu \in \mathfrak{h}^*$). Here, the elements $e(\lambda)$ are called formal exponentials. They are linearly independent and are in one-to-one correspondence with the elements $\lambda \in \mathfrak{h}^*$. The multiplication in \mathcal{E} is defined by: $e(\lambda_1) \cdot e(\lambda_2) = e(\lambda_1 + \lambda_2)$ ($\lambda_1, \lambda_2 \in \mathfrak{h}^*$).

For an \mathfrak{h} -module M in the category $\mathcal{O}_{\mathcal{O}}$, we define the formal \mathfrak{h} -character $\text{ch}_{\mathfrak{h}} M$ to be $\sum_{\lambda \in \mathfrak{h}^*} (\dim_{\mathbb{C}} M_\lambda) e(\lambda) \in \mathcal{E}$, where M_λ is the weight space of M corresponding to λ .

§3. Construction of the “Weaker” BGG Resolution

We fix a subset J of $I^{re} = \{i \in I \mid a_{ii} = 2\}$.

3.1. Construction of a generalized Verma composition series.

For $\lambda \in P_J^+$, we define the generalized Verma module $V_m(\lambda)$ with highest weight λ as follows: $V_m(\lambda) = U(\mathfrak{g}(A)) \otimes_{U(\mathfrak{p})} L_m(\lambda)$, where $\mathfrak{u}^+(\subset \mathfrak{p})$ acts trivially on the irreducible highest weight \mathfrak{m} -module $L_m(\lambda)$ with highest weight λ . Here, for a Lie algebra \mathfrak{a} , $U(\mathfrak{a})$ denotes the universal enveloping algebra of \mathfrak{a} . Note that when J is an empty set \emptyset , the module $V_m(\lambda)$ is just the Verma module $V(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$, so that $V_m(\lambda)$ is a quotient of $V(\lambda)$.

Definition. Let $\Psi = (\lambda_1, \lambda_2, \dots)$ be a sequence, possibly finite, of elements of P_J^+ . A $\mathfrak{g}(A)$ -module V in the category \mathcal{O} (see the remark before Proposition 2.1) is said to have a *generalized Verma composition series* (=GVCS) of type Ψ if V has a strictly increasing (possibly finite) $\mathfrak{g}(A)$ -module filtration:

$$0 = V_0 \subset V_1 \subset V_2 \subset \dots$$

such that $V = \bigcup_{i \geq 0} V_i$ and the sequence $V_1/V_0, V_2/V_1, \dots$ coincides up to rearrangement with the sequence of generalized Verma modules $V_m(\lambda_1), V_m(\lambda_2), \dots$.

Now, let $(\cdot | \cdot)$ be a standard invariant form on $\mathfrak{g}(A)$ corresponding to the decomposition $A = DB$, where $D = (\varepsilon_i \delta_{ij})_{i, j \in I}$ is a diagonal matrix with $\varepsilon_i > 0$ ($i \in I$), and B is a real symmetric matrix (see [7, Chapter 2]). This induces (through a linear isomorphism $v: \mathfrak{h} \rightarrow \mathfrak{h}^*$) a nondegenerate, W -invariant,

symmetric bilinear form, which is again denoted by $(\cdot|\cdot)$. We fix one such $(\cdot|\cdot)$. Note that for $\alpha \in \Delta^+$, we have $2(\rho|\alpha) \geq (\alpha|\alpha)$ with equality if and only if $\alpha \in \Pi$ (see [7, Chapter 11]).

For a sequence $\Psi = (\lambda_1, \lambda_2, \dots)$ of elements of P_J^+ , and an element $\Lambda \in P^+$, let Ψ_Λ be the subsequence of Ψ consisting of those λ_i in Ψ such that $(\lambda_i + \rho|\lambda_i + \rho) = (\Lambda + \rho|\Lambda + \rho)$.

We can prove the next theorem, arguing basically in the same way as in [5], by making use of Proposition 2.1 instead of [5, Proposition 6.3]. However, in its proof, we need the same additional arguments as in [12, Section 6] concerning the filtration of $C_p(\Lambda)$ ($p \geq 0$), since the subset J of I^e may be arbitrary. For these additional arguments, we need only the fact that Ψ_Λ^p is a finite set for each p ($p \geq 0$), which will be proved in the following subsection (see Proposition 3.2).

Theorem 3.1. *Let $\Lambda \in P^+$. For each $p \in \mathbb{Z}_{\geq 0}$, let $\Psi^p = (\lambda_1, \lambda_2, \dots)$ be the (possibly finite) sequence of elements of P_J^+ such that $(\bigwedge^p u^-) \otimes_c L(\Lambda) \cong \sum_i^{\oplus} L_m(\lambda_i)$ as \mathfrak{m} -modules (such a decomposition exists uniquely by Proposition 2.1). Then, there exists an exact sequence of $\mathfrak{g}(A)$ -homomorphisms:*

$$0 \leftarrow L(\Lambda) \leftarrow C_0(\Lambda) \leftarrow C_1(\Lambda) \leftarrow \dots \leftarrow C_p(\Lambda) \leftarrow C_{p+1}(\Lambda) \leftarrow \dots,$$

where the $\mathfrak{g}(A)$ -module $C_p(\Lambda)$ is in the category \mathcal{O} , and has a GVCS of type Ψ_Λ^p ($p \geq 0$).

3.2. Determination of Ψ_Λ^p ($p \geq 0$). Let us recall some notions concerning roots of GKM algebras. A simple root $\alpha_i \in \Pi$ ($i \in I$) is called real if $a_{ii} = 2$, and imaginary if $a_{ii} \leq 0$. We denote by Π^{re} (resp. Π^{im}) the set of all real (resp. imaginary) simple roots. Let \mathcal{S} be the set of all sums of distinct, pairwise perpendicular (with respect to $(\cdot|\cdot)$), imaginary simple roots. And for each $\beta = \sum_{i \in I} k_i \alpha_i \in Q_+$, we put $\text{ht}(\beta) := \sum_{i \in I} k_i$.

As a necessary condition of Ψ_Λ^p , we get the following, imitating the proof of the character formula for $L(\Lambda)$ (cf. [2] or [7, Chapter 11]).

Proposition 3.2. *Let $\Lambda \in P^+$ and $p \in \mathbb{Z}_{\geq 0}$. If $\lambda \in P_J^+$ is in the sequence Ψ_Λ^p in Theorem 3.1, then*

- (a) $\lambda = w(\Lambda + \rho - \beta) - \rho$, for some (necessarily unique) $w \in W(J)$ and some (necessarily unique) $\beta \in \mathcal{S}(\Lambda) := \{\beta \in \mathcal{S} \mid (\beta|\Lambda) = 0\}$ such that $\ell(w) + \text{ht}(\beta) = p$;

(b) λ occurs with multiplicity one in the sequence Ψ_λ^p .

Proof. First, we remark that since $\langle \Lambda + \rho - \beta, \alpha_i^\vee \rangle \in \mathbf{Z}_{\geq 0}$ ($i \in I^{re}$), $w_1(\Lambda + \rho - \beta_1) - \rho = w_2(\Lambda + \rho - \beta_2) - \rho$ implies $w_1 = w_2$ and $\beta_1 = \beta_2$ ([13, Proposition 1.1]).

We now work in the algebra \mathcal{E} in Section 2.2. Put $\bigwedge^* \mathfrak{n}^- := \sum_{j \geq 0} \bigwedge^j \mathfrak{n}^-$. Then, by Proposition 1.1, we see that the \mathfrak{h} -module $(\bigwedge^* \mathfrak{n}^-) \otimes_{\mathbf{C}} L(\Lambda)$ is in the category $\mathcal{O}_{\mathcal{O}}$, so that $\text{ch}_{\mathfrak{h}}((\bigwedge^* \mathfrak{n}^-) \otimes_{\mathbf{C}} L(\Lambda))$ is in the algebra \mathcal{E} . Clearly, we have

$$\begin{aligned} \text{ch}_{\mathfrak{h}}((\bigwedge^* \mathfrak{n}^-) \otimes_{\mathbf{C}} L(\Lambda)) &= \text{ch}_{\mathfrak{h}}(\bigwedge^* \mathfrak{n}^-) \cdot \text{ch}_{\mathfrak{h}} L(\Lambda) \\ &= \prod_{\alpha \in \Delta^+} (1 + e(-\alpha))^{\text{mult}(\alpha)} \cdot \text{ch}_{\mathfrak{h}} L(\Lambda), \end{aligned}$$

where $\text{mult}(\alpha) := \dim_{\mathbf{C}} \mathfrak{g}_\alpha = \dim_{\mathbf{C}} \mathfrak{g}_{-\alpha}$ ($\alpha \in \Delta^+$). Note that the Weyl group W stabilizes the set $\mathcal{P}(L(\Lambda))$ of all weights of $L(\Lambda)$ and the root system Δ of $\mathfrak{g}(A)$ with their multiplicities, since the Chevalley generators e_i, f_i ($i \in I^{re}$) act locally nilpotently on both of $L(\Lambda)$ and $\mathfrak{g}(A)$. Further, it is obvious that $r_i(\Delta^+ \setminus \{\alpha_i\}) = \Delta^+ \setminus \{\alpha_i\}$ for all $i \in I^{re}$. Therefore, if we define $w(e(\tau)) := e(w(\tau))$ ($\tau \in \mathfrak{h}^*$), we have for $i \in I^{re}$

$$\begin{aligned} & r_i(e(\rho)) \cdot \prod_{\alpha \in \Delta^+} (1 + e(-\alpha))^{\text{mult}(\alpha)} \cdot \text{ch}_{\mathfrak{h}} L(\Lambda) \\ &= r_i(e(\rho)) \cdot r_i(1 + e(-\alpha_i)) \cdot r_i(\prod_{\alpha \in \Delta^+ \setminus \{\alpha_i\}} (1 + e(-\alpha))^{\text{mult}(\alpha)}) \cdot r_i(\text{ch}_{\mathfrak{h}} L(\Lambda)) \\ &= e(\rho) \cdot e(-\alpha_i) \cdot (1 + e(\alpha_i)) \cdot \prod_{\alpha \in \Delta^+ \setminus \{\alpha_i\}} (1 + e(-\alpha))^{\text{mult}(\alpha)} \cdot \text{ch}_{\mathfrak{h}} L(\Lambda) \\ &= e(\rho) \cdot \prod_{\alpha \in \Delta^+} (1 + e(-\alpha))^{\text{mult}(\alpha)} \cdot \text{ch}_{\mathfrak{h}} L(\Lambda). \end{aligned}$$

Hence, in the expression $e(\rho) \cdot \text{ch}_{\mathfrak{h}}((\bigwedge^* \mathfrak{n}^-) \otimes_{\mathbf{C}} L(\Lambda)) = \sum_{\tau \in \mathfrak{h}^*} c_\tau e(\tau + \rho)$, we have

$$(\mathfrak{N}) \quad c_\tau = c_{\tau'}, \text{ if } w(\tau + \rho) = \tau' + \rho \text{ for some } w \in W.$$

In particular, if we put $\Theta := \mathcal{P}((\bigwedge^* \mathfrak{n}^-) \otimes_{\mathbf{C}} L(\Lambda))$, then the set $\rho + \Theta$ ($\subset \rho + \Lambda - Q_+ \subset \mathfrak{h}^*$) is W -invariant. So, for $\lambda \in \Psi_\lambda^p \subset \Theta$, there exists some $w_0 \in W$ such that $\langle w_0^{-1}(\lambda + \rho), \alpha_i^\vee \rangle \in \mathbf{Z}_{\geq 0}$ for all $i \in I^{re}$. We now write $w_0^{-1}(\lambda + \rho) = \mu + \rho$ ($\mu \in \Theta$). Since $\lambda \in \Psi_\lambda^p$, we have $(\mu + \rho | \mu + \rho) = (w_0^{-1}(\lambda + \rho) | w_0^{-1}(\lambda + \rho)) = (\lambda + \rho | \lambda + \rho) = (\Lambda + \rho | \Lambda + \rho)$.

To sum up, $\mu = \Lambda - \sum_{i \in I} k_i \alpha_i$ ($k_i \in \mathbf{Z}_{\geq 0}$) is a weight of $(\bigwedge^* \mathfrak{n}^-) \otimes_{\mathbf{C}} L(\Lambda)$ such

that $(\mu + \rho | \mu + \rho) = (\Lambda + \rho | \Lambda + \rho)$ and $\langle \mu + \rho, \alpha_i^\vee \rangle \in \mathbf{Z}_{\geq 0}$ ($i \in I^{re}$). From this, we conclude that $\mu = \Lambda - \beta_0$ for some $\beta_0 \in \mathcal{S}(\Lambda)$ and that the multiplicity of μ in $(\bigwedge^* \mathfrak{n}^-) \otimes_{\mathbf{C}} L(\Lambda)$ is equal to one, by exactly the same argument that was used in the proof of the character formula for GKM algebras ([2, Section 7] or [7, Theorem 11.13.3]). So, using (\mathfrak{K}) , we deduce that $\lambda = w_0(\Lambda + \rho - \beta_0) - \rho$ and that the multiplicity of λ in $(\bigwedge^* \mathfrak{n}^-) \otimes_{\mathbf{C}} L(\Lambda)$ is equal to one.

Now, it remains to show that $w_0 \in W(J)$ and $\ell(w_0) + \text{ht}(\beta_0) = p$. But this can be shown as in the proof of [14, Lemma 4.2]. \square

For the converse of Proposition 3.2, we have the following.

Proposition 3.3. *Let $\Lambda \in P^+$, $p \in \mathbf{Z}_{\geq 0}$. Put $\lambda = w(\Lambda + \rho - \beta) - \rho$, where $w \in W(J)$ and $\beta \in \mathcal{S}(\Lambda)$ such that $\ell(w) + \text{ht}(\beta) = p$. Then, $L_m(\lambda)$ occurs as \mathfrak{m} -irreducible components of $(\bigwedge^p \mathfrak{u}^-) \otimes_{\mathbf{C}} L(\Lambda)$, and λ is in the sequence Ψ_Λ^p in Theorem 3.1.*

Proof. First, note that we have $(\lambda + \rho | \lambda + \rho) = (w(\Lambda + \rho - \beta) | w(\Lambda + \rho - \beta)) = (\Lambda + \rho - \beta | \Lambda + \rho - \beta) = (\Lambda + \rho | \Lambda + \rho) + (\beta | \beta) - 2(\rho | \beta) - 2(\Lambda | \beta) = (\Lambda + \rho | \Lambda + \rho)$, since $\beta \in \mathcal{S}(\Lambda)$ and $2(\rho | \alpha_i) = (\alpha_i | \alpha_i)$ ($i \in I$).

If we express $\beta = \sum_{k=1}^m \alpha_{i_k}$, where $m = \text{ht}(\beta)$, $\alpha_{i_k} \in \Pi^{im}$ ($1 \leq k \leq m$), and $i_r \neq i_t$ ($1 \leq r \neq t \leq m$), then we have $\lambda = w(\Lambda + \rho - \beta) - \rho = w(-\beta) + w(\rho) - \rho + w(\Lambda) = \sum_{k=1}^m w(-\alpha_{i_k}) + \sum_{\alpha \in \Phi_w} (-\alpha) + w(\Lambda)$, since $w(\rho) - \rho = -\sum_{\alpha \in \Phi_w} \alpha$. Now, we take nonzero root vectors $E_k \in \mathfrak{g}_{-w(\alpha_{i_k})}$ ($1 \leq k \leq m$), $E_\alpha \in \mathfrak{g}_{-\alpha}$ ($\alpha \in \Phi_w$), and a nonzero weight vector $v \in L(\Lambda)_{w(\Lambda)}$ with weight $w(\Lambda)$. Then, it is clear that $v_0 := (E_1 \wedge \cdots \wedge E_m) \wedge (\bigwedge_{\alpha \in \Phi_w} E_\alpha) \otimes v \in (\bigwedge^p \mathfrak{u}^-) \otimes_{\mathbf{C}} L(\Lambda)$ is a nonzero weight vector with weight λ .

Further, we have the following claim:

Claim. We have $e_i(v_0) = 0$ for all $i \in J$, when regarding $(\bigwedge^p \mathfrak{u}^-) \otimes_{\mathbf{C}} L(\Lambda)$ as the tensor product of modules over \mathfrak{m} .

Proof of the claim. It is sufficient to show that $\lambda + \alpha_i$ ($i \in J$) is not a weight of $(\bigwedge^p \mathfrak{u}^-) \otimes_{\mathbf{C}} L(\Lambda)$. Suppose that $\lambda + \alpha_i$ is a weight of $(\bigwedge^p \mathfrak{u}^-) \otimes_{\mathbf{C}} L(\Lambda)$. Then, by exactly the same argument as in the proof of Proposition 3.2, we see that there exists some $w_0 \in W$ such that $(w_0^{-1}(\lambda + \alpha_i + \rho) | \alpha_l) \geq 0$ for all $l \in I^{re}$. We write $\mu = w_0^{-1}(\lambda + \alpha_i + \rho) - \rho$. So, we have $\mu = \Lambda - \sum_k n_k \alpha_k$ ($n_k \in \mathbf{Z}_{\geq 1}$) since $\rho + \mathcal{P}((\bigwedge^* \mathfrak{n}^-) \otimes_{\mathbf{C}} L(\Lambda))$ is W -invariant.

Now, since $\mu = \Lambda - \sum_k n_k \alpha_k$, we have

$$\begin{aligned}
 & (\Lambda + \rho|\Lambda + \rho) - (\lambda + \alpha_i + \rho|\lambda + \alpha_i + \rho) \\
 &= (\Lambda + \rho|\Lambda + \rho) - (w_0^{-1}(\lambda + \alpha_i + \rho)|w_0^{-1}(\lambda + \alpha_i + \rho)) \\
 &= (\Lambda + \rho|\Lambda + \rho) - (\mu + \rho|\mu + \rho) = (\Lambda + \rho|\sum_k n_k \alpha_k) + (\mu + \rho|\sum_k n_k \alpha_k) \\
 &= (\Lambda|\sum_k n_k \alpha_k) + (\mu + 2\rho|\sum_k n_k \alpha_k).
 \end{aligned}$$

Here, $(\Lambda|\alpha_l) \geq 0$ for all $l \in I$, since $\Lambda \in P^+$.

For $j \in I^{re}$, we have

$$\begin{aligned}
 (\mu + 2\rho|\alpha_j) &= (\mu + \rho|\alpha_j) + (\rho|\alpha_j) = (\mu + \rho|\alpha_j) + (1/2) \cdot (\alpha_j|\alpha_j) \\
 &= (w_0^{-1}(\lambda + \alpha_i + \rho)|\alpha_j) + (1/2) \cdot (\alpha_j|\alpha_j) \geq 0.
 \end{aligned}$$

For $j \in I^{im}$ with $n_j \in \mathbf{Z}_{\geq 1}$, we have

$$\begin{aligned}
 (\mu + 2\rho|\alpha_j) &= (\Lambda - \sum_k n_k \alpha_k + 2\rho|\alpha_j) \\
 &= (\Lambda|\alpha_j) - \sum_{k \neq j} n_k (\alpha_k|\alpha_j) + (1 - n_j)(\alpha_j|\alpha_j) \geq 0.
 \end{aligned}$$

Therefore, we get $(\Lambda + \rho|\Lambda + \rho) - (\lambda + \alpha_i + \rho|\lambda + \alpha_i + \rho) \geq 0$.

On the other hand, we have

$$\begin{aligned}
 & (\lambda + \alpha_i + \rho|\lambda + \alpha_i + \rho) - (\Lambda + \rho|\Lambda + \rho) \\
 &= (w(\Lambda + \rho - \beta) + \alpha_i|w(\Lambda + \rho - \beta) + \alpha_i) - (\Lambda + \rho|\Lambda + \rho) \\
 &= (\Lambda + \rho - \beta|\Lambda + \rho - \beta) + 2(\alpha_i|w(\Lambda + \rho - \beta)) + (\alpha_i|\alpha_i) - (\Lambda + \rho|\Lambda + \rho) \\
 &= -2(\Lambda + \rho|\beta) + (\beta|\beta) + 2(\Lambda + \rho - \beta|w^{-1}(\alpha_i)) + (\alpha_i|\alpha_i) \\
 &= 2(\Lambda + \rho - \beta|w^{-1}(\alpha_i)) + (\alpha_i|\alpha_i) \quad (\text{since } \beta \in \mathcal{S}(\Lambda)).
 \end{aligned}$$

Here, note that $w^{-1}(\alpha_i) \in \Delta^+ \cap \sum_{j \in I^{re}} \mathbf{Z} \alpha_j$, since $i \in J \subset I^{re}$ and $w \in W(J) = \{u \in W \mid u^{-1}(\Delta_J^+) \subset \Delta^+\}$. Therefore, we get $(\lambda + \alpha_i + \rho|\lambda + \alpha_i + \rho) - (\Lambda + \rho|\Lambda + \rho) > 0$. This is a contradiction. Thus, we have proved the claim.

Let us recall that the \mathfrak{m} -module $(\bigwedge^p u^-) \otimes_{\mathcal{C}} L(\Lambda)$ is in the category \mathcal{O}_J ($p \geq 0$) from Proposition 2.1. Hence, we can easily deduce from the above claim that the \mathfrak{m} -submodule of $(\bigwedge^p u^-) \otimes_{\mathcal{C}} L(\Lambda)$ generated by the vector v_0 is \mathfrak{m} -module isomorphic to $L_{\mathfrak{m}}(\lambda)$ ($p \geq 0$). \square

By Theorem 3.1, Propositions 3.2 and 3.3, we have the following theorem, extending a result of Liu [12].

Theorem 3.4 (“weaker” BGG resolution). *Let $\mathfrak{g}(A)$ be a GKM algebra associated to a symmetrizable GGCM $A = (a_{ij})_{i,j \in I}$, and let J be a subset of*

$I^e = \{i \in I \mid a_{ii} = 2\}$. Then, for the irreducible highest weight $\mathfrak{g}(A)$ -module $L(\Lambda)$ with highest weight $\Lambda \in P^+$, there exists an exact sequence of $\mathfrak{g}(A)$ -modules and $\mathfrak{g}(A)$ -homomorphisms:

$$0 \leftarrow L(\Lambda) \leftarrow C_0(\Lambda) \leftarrow C_1(\Lambda) \leftarrow \cdots \leftarrow C_p(\Lambda) \leftarrow C_{p+1}(\Lambda) \leftarrow \cdots,$$

where $C_p(\Lambda)$ ($p \geq 0$) has a GVCS of type $(w(\Lambda + \rho - \beta) - \rho)_{\substack{w \in W(J), \beta \in \mathcal{S}(\Lambda) \\ \ell(w) + \text{ht}(\beta) = p}}$. Here, each weight $w(\Lambda + \rho - \beta) - \rho$ occurs with multiplicity one in this sequence.

§ 4. Existence of the Weak BGG Resolution

In this section, we assume that the subset J of I^e is of finite type (i.e., \mathfrak{m} is a finite-dimensional reductive Lie algebra). Define $\mathcal{C}(\mathfrak{g}(A), \mathfrak{m})$ to be the category of all $\mathfrak{g}(A)$ -modules V satisfying (J3) in Section 2.2. That is, $\mathcal{C}(\mathfrak{g}(A), \mathfrak{m})$ is the category of all $\mathfrak{g}(A)$ -modules which are finitely semi-simple as an \mathfrak{m} -module. Note that the category $\mathcal{C}(\mathfrak{g}(A), \mathfrak{m})$ is closed under the operations of taking submodules, direct sums, and tensor products, and that $\mathcal{C}(\mathfrak{g}(A), \mathfrak{m})$ has enough projectives. Therefore, in the category $\mathcal{C}(\mathfrak{g}(A), \mathfrak{m})$, the usual relative Ext bifunctor $\text{Ext}_{(\mathfrak{g}(A), \mathfrak{m})}^j$ is defined ($j \geq 0$) (see [11, Part I]).

Here, we prove that the $\mathfrak{g}(A)$ -module $C_p(\Lambda)$ (in Theorem 3.4) is actually the direct sum of generalized Verma modules $V_{\mathfrak{m}}(w(\Lambda + \rho - \beta) - \rho)$ ($w \in W(J)$, $\beta \in \mathcal{S}(\Lambda)$, $\ell(w) + \text{ht}(\beta) = p$) for $p \geq 0$, by showing $\text{Ext}_{(\mathfrak{g}(A), \mathfrak{m})}^1(V_{\mathfrak{m}}(w_1(\Lambda + \rho - \beta_1) - \rho), V_{\mathfrak{m}}(w_2(\Lambda + \rho - \beta_2) - \rho)) = 0$ if $w_1 \neq w_2$ or $\beta_1 \neq \beta_2$, where $w_i \in W(J)$, $\beta_i \in \mathcal{S}(\Lambda)$, $\ell(w_i) + \text{ht}(\beta_i) = p$ ($i = 1, 2$).

In [17], Rocha-Caridi and Wallach dealt with a certain wide class of graded Lie algebras, that is more general than symmetrizable GKM algebras. So, we can use their results freely. Our main tool is the following.

Lemma 4.1 ([17, Lemma 7.1]). *Let $M = M_0 \supset M_1 \supset \cdots \supset M_s \supset M_{s+1} = 0$ be a filtration of $M \in \mathcal{C}(\mathfrak{g}(A), \mathfrak{m})$ by submodules, such that $M_i/M_{i+1} \cong V_i$ ($0 \leq i \leq s$) as $\mathfrak{g}(A)$ -modules. If $\text{Ext}_{(\mathfrak{g}(A), \mathfrak{m})}^1(V_i, V_j) = 0$ for all $0 \leq i \neq j \leq s$, then $M \cong \sum_{0 \leq i \leq s}^{\oplus} V_i$ as $\mathfrak{g}(A)$ -modules.*

So, we have to prove the following Ext vanishing result.

Proposition 4.2. *Let $\Lambda \in P^+$, $w_i \in W(J)$, and $\beta_i \in \mathcal{S}(\Lambda)$ ($i = 1, 2$). If*

$\text{Ext}_{(\mathfrak{g}(A), \mathfrak{m})}^1(V_{\mathfrak{m}}(w_1(\Lambda + \rho - \beta_1) - \rho), V_{\mathfrak{m}}(w_2(\Lambda + \rho - \beta_2) - \rho)) \neq 0$, then $\ell(w_1) + \text{ht}(\beta_1) < \ell(w_2) + \text{ht}(\beta_2)$.

Before proving Proposition 4.2, we prepare some basic results for real roots and imaginary roots of GKM algebras. Let $\Delta^{re} := W \cdot \Pi^{re}$, $\Delta^{im} := \Delta \setminus \Delta^{re}$, $K := \{\alpha = \sum_{i \in I} k_i \alpha_i \in \mathcal{Q}_+ \setminus \{0\} \mid \langle \alpha, \alpha_j^\vee \rangle \leq 0 \ (j \in I^{re}), \text{ and } \text{supp}(\alpha) \text{ is connected}\} \setminus \bigcup_{j \geq 2} j \cdot \Pi^{im}$, where $\text{supp}(\alpha)$ is the subdiagram of the Dynkin diagram of $A = (a_{ij})_{i, j \in I}$ corresponding to the subset $\{i \in I \mid k_i > 0\}$ of I . Then, we know that $\Delta^{im} \cap \Delta^+ = \bigcup_{w \in W} w(K)$ ([7, Chapter 11]). Moreover, we have

Lemma 4.3. *For $\alpha \in \Delta$, the following are equivalent:*

- (1) $\alpha \in \Delta^{re}$;
- (2) $\langle \alpha | \alpha \rangle > 0$.

Proof. (1) obviously implies (2), since $\langle \alpha_i | \alpha_i \rangle = 2/\varepsilon_i > 0 \ (i \in I^{re})$ and $(\cdot | \cdot)$ is W -invariant. We will show $\alpha \in \Delta^{im}$ implies $\langle \alpha | \alpha \rangle \leq 0$. Since $\Delta^{im} \cap \Delta^+ = \bigcup_{w \in W} w(K)$, we may assume that $\alpha = \sum_{j \in I} k_j \alpha_j \in K$. Then, we have $\langle \alpha | \alpha \rangle = \sum_{j \in I} k_j \langle \alpha | \alpha_j \rangle$. Note that for $i \in I$, $\langle \alpha | \alpha_i \rangle = \varepsilon_i^{-1} \cdot \langle \alpha, \alpha_i^\vee \rangle$, where $\varepsilon_i > 0$. If $i \in I^{re}$, $\langle \alpha, \alpha_i^\vee \rangle \leq 0$ since $\alpha \in K$. And if $i \in I^{im}$, $\langle \alpha, \alpha_i^\vee \rangle = \sum_{j \in I} k_j \langle \alpha_j, \alpha_i^\vee \rangle = \sum_{j \in I} k_j a_{ij} \leq 0$ since $a_{ii} \leq 0$. So, we have $\langle \alpha | \alpha \rangle \leq 0$. \square

For $\alpha \in \Delta^{re}$, we define a reflection r_α with respect to α by: $r_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \ (\lambda \in \mathfrak{h}^*)$, where $\alpha^\vee = 2v^{-1}(\alpha)/\langle \alpha | \alpha \rangle$ is the dual real root of α . Note that if $\alpha = w(\alpha_i) \ (w \in W, \alpha_i \in \Pi^{re})$, then $r_\alpha = wr_i w^{-1} \in W$. Then, generalizing [17, Lemma 8.2], we have

Lemma 4.4. *Let $\mu \in P^+$, $w \in W$, and $\beta \in \Delta^+$. Then, the following are equivalent:*

- (1) $2\langle w(\mu + \rho), v^{-1}(\beta) \rangle = m\langle \beta | \beta \rangle$ for some $m \in \mathbf{Z}_{\geq 1}$,
- (2) we have either of the following two cases:
 - (a) $\beta \in \Delta^{re}$ and $\ell(r_\beta w) > \ell(w)$;
 - (b) $w^{\mathfrak{g}-1}(\beta) \in \Pi^{im}$ and $\langle w^{-1}(\beta) | \mu \rangle = 0$.

Proof. We can show that (2) implies (1) in exactly the same way as in the proof of [17, Lemma 8.2]. So, we only show that (1) implies (2). Now, suppose that $\beta \notin \Delta^{re}$, so that $\beta \in \Delta^{im} \cap \Delta^+$. Then, by Lemma 4.3, we have $\langle \beta | \beta \rangle \leq 0$. On the other hand, from the assumption we have $m\langle \beta | \beta \rangle = 2\langle w(\mu + \rho) | \beta \rangle = 2\langle \mu + \rho | w^{-1}(\beta) \rangle$. Since $\Delta^{im} \cap \Delta^+$ is W -invariant, we get

$w^{-1}(\beta) \in \Delta^{im} \cap \Delta^+$. Put $\alpha := w^{-1}(\beta)$. Then, we have

$$\begin{aligned} m(\alpha|\alpha) &= m(\beta|\beta) = 2(\mu + \rho|\alpha) = 2(\mu|\alpha) + 2(\rho|\alpha) \\ &\geq 2(\mu|\alpha) + (\alpha|\alpha), \end{aligned}$$

so that $(m-1)(\alpha|\alpha) \geq 2(\mu|\alpha)$ with equality only if $\alpha \in \Pi$. Here, $(\mu|\alpha) \geq 0$ since $\mu \in P^+$ and $\alpha \in \Delta^+$. Therefore, we deduce $w^{-1}(\beta) = \alpha \in \Pi^{im}$ and $(\mu|w^{-1}(\beta)) = 0$, since $(\alpha|\alpha) = (\beta|\beta) \leq 0$.

Hence, we may assume that $\beta \in \Delta^{re}$. But, in this case, we can show that $\ell(r_\beta w) > \ell(w)$ in the same way as in the proof of [17, Lemma 8.2]. \square

Proof of Proposition 4.2. Put $\mu_i := w_i(\Lambda + \rho - \beta_i) - \rho$ ($i = 1, 2$). We have $\text{Ext}_{\mathfrak{g}(A), \mathfrak{m}}^1(V_{\mathfrak{m}}(\mu_1), V_{\mathfrak{m}}(\mu_2)) \neq 0$ by assumption. Then, by [17, Theorem 7.5], there exists an element $\mu \in P_J^+$ ($\mu \neq \mu_1$) such that $[V_{\mathfrak{m}}(\mu_2) : L(\mu)] \cdot [V_{\mathfrak{m}}(\mu) : L(\mu_1)] \neq 0$. Here, for $\mathfrak{g}(A)$ -modules $V \in \mathcal{O}$ and $L(\tau)$ ($\tau \in \mathfrak{h}^*$), $[V : L(\tau)]$ is the multiplicity of $L(\tau)$ in V (see [4, Definition 3.5]). So, we have $[V(\mu_2) : L(\mu)] \cdot [V(\mu) : L(\mu_1)] \neq 0$. Therefore, we have only to show the following claim:

Claim. Let β be an element of the set $\sum_{i \in I^{im}} \mathbf{Z}_{\geq 0} \alpha_i$, and let $w \in W$. If, for some $\mu \in \mathfrak{h}^*$ ($\mu \neq w(\Lambda + \rho - \beta) - \rho$), $[V(w(\Lambda + \rho - \beta) - \rho) : L(\mu)] \neq 0$, then $\mu = w_0(\Lambda + \rho - \beta_0) - \rho$ for some $w_0 \in W$ and some $\beta_0 \in \sum_{i \in I^{im}} \mathbf{Z}_{\geq 0} \alpha_i$ such that $\ell(w_0) + \text{ht}(\beta_0) > \ell(w) + \text{ht}(\beta)$.

Proof of the claim. Since $[V(w(\Lambda + \rho - \beta) - \rho) : L(\mu)] \neq 0$, by [8, Theorem 2], there are a sequence $\gamma_1, \dots, \gamma_k$ of positive roots and a sequence m_1, \dots, m_k of positive integers such that

$$\begin{aligned} 2\langle w(\Lambda + \rho - \beta) - m_1\gamma_1 - \dots - m_{i-1}\gamma_{i-1}, v^{-1}(\gamma_i) \rangle &= m_i(\gamma_i|\gamma_i) \quad (1 \leq i \leq k), \text{ and} \\ w(\Lambda + \rho - \beta) - \rho - \mu &= \sum_{1 \leq i \leq k} m_i \gamma_i. \end{aligned}$$

Clearly, we may assume that $k = 1$. That is, we may assume that there exist a positive root γ and a positive number m such that

$$2\langle w(\Lambda + \rho - \beta), v^{-1}(\gamma) \rangle = m(\gamma|\gamma) \text{ and } w(\Lambda + \rho - \beta) - \rho - \mu = m\gamma.$$

Then, by Lemma 4.4, we have either of the following two cases:

- (a) $\gamma \in \Delta^{re}$ and $\ell(r_\gamma w) > \ell(w)$;

(b) $w^{-1}(\gamma) \in \Pi^{im}$ and $(w^{-1}(\gamma)|\Lambda - \beta) = 0$.

In Case (a), we have $(r_\gamma w)(\Lambda + \rho - \beta) = r_\gamma(w(\Lambda + \rho - \beta)) = w(\Lambda + \rho - \beta) - 2/(\gamma|\gamma) \cdot \langle w(\Lambda + \rho - \beta), v^{-1}(\gamma) \rangle \gamma = w(\Lambda + \rho - \beta) - m\gamma = \mu + \rho$, with $\ell(r_\gamma w) > \ell(w)$.

In Case (b), by putting $\alpha := w^{-1}(\gamma) \in \Pi^{im}$, we have $\mu = w(\Lambda + \rho - \beta) - m\gamma - \rho = w(\Lambda + \rho - \beta) - w(m\alpha) - \rho = w(\Lambda + \rho - (\beta + m\alpha)) - \rho$, with $\text{ht}(\beta + m\alpha) > \text{ht}(\beta)$.

Thus, the claim has been proved. □

By Theorem 3.4, Lemma 4.1, and Proposition 4.2, we have the following main theorem.

Theorem 4.5 (weak BGG resolution). *Let $\Lambda \in P^+$ and let $J (\subset I^e)$ be a subset of finite type. Then, for the irreducible highest weight module $L(\Lambda)$ with highest weight Λ over a symmetrizable GKM algebra $\mathfrak{g}(A)$, there exists an exact sequence of $\mathfrak{g}(A)$ -modules and $\mathfrak{g}(A)$ -homomorphisms:*

$$0 \leftarrow L(\Lambda) \leftarrow C_0(\Lambda) \leftarrow C_1(\Lambda) \leftarrow \cdots \leftarrow C_p(\Lambda) \leftarrow C_{p+1}(\Lambda) \leftarrow \cdots ,$$

where $C_p(\Lambda) = \sum^{\oplus}_{\substack{w \in W(J), \beta \in \mathcal{S}^e(\Lambda) \\ \ell(w) + \text{ht}(\beta) = p}} V_m(w(\Lambda + \rho - \beta) - \rho)$ is the direct sum of generalized Verma modules $V_m(w(\Lambda + \rho - \beta) - \rho)$ with highest weight $w(\Lambda + \rho - \beta) - \rho$.

§5. Applications

5.1. Extension of Kostant’s homology theorem. In [14], we derived Kostant’s homology theorem from the Weyl-Kac-Borcherds character formula for $L(\Lambda)$ over a symmetrizable GKM algebra $\mathfrak{g}(A)$ under the condition $(\widehat{C}1)$ on the GGCM $A = (a_{ij})_{i,j \in I}$ that $a_{ii} = 2$ or $a_{ii} = 0$ ($i \in I$). Here, we extend Kostant’s homology theorem to arbitrary symmetrizable GKM algebras. This completely determines the Lie algebra homology $H_p(\mathfrak{u}^-, L(\Lambda))$ ($p \geq 0$) of \mathfrak{u}^- with coefficients in $L(\Lambda)$ as an \mathfrak{m} -module. Note that in this subsection, we assume only that J is a subset of I^{re} (not necessarily of finite type).

First, we recall the following.

Proposition 5.1 ([14, Proposition 3.2]). *Let $\Lambda \in P^+$ and $p \in \mathbf{Z}_{\geq 0}$. Then, the \mathfrak{m} -module $H_p(\mathfrak{u}^-, L(\Lambda))$ is in the category \mathcal{O}_J , and every \mathfrak{m} -irreducible*

component of $H_p(u^-, L(\Lambda))$ is of the form $L_m(\mu)$ ($\mu \in P_J^+$) with $(\mu + \rho | \mu + \rho) = (\Lambda + \rho | \Lambda + \rho)$.

Remark. The above proposition was proved by Liu [12] in the case where A is a symmetrizable GCM. The proof needs no modifications also in the case where A is a symmetrizable GGCM.

Now, we can easily deduce the following proposition from Propositions 3.2, 3.3, and 5.1 by the well-known Euler-Poincaré principle (cf. the proof of [14, Proposition 4.2]).

Proposition 5.2. *Let $\Lambda \in P^+$, $p \in \mathbb{Z}_{\geq 0}$. We put $\lambda = w(\Lambda + \rho - \beta) - \rho$, where $w \in W(J)$ and $\beta \in \mathcal{S}(\Lambda)$ such that $\ell(w) + \text{ht}(\beta) = p$. Then, $L_m(\lambda)$ occurs as m -irreducible components of $H_p(u^-, L(\Lambda))$.*

By Proposition 5.1 together with Proposition 3.2, and by Proposition 5.2, we obtain the following theorem, which generalizes Kostant’s famous theorem on Lie algebra homology to arbitrary symmetrizable GKM algebras.

Theorem 5.3. *Let $\Lambda \in P^+$ and let J be a subset of I^{re} . Then, as m -modules ($p \geq 0$),*

$$H_p(u^-, L(\Lambda)) \cong \sum_{\substack{w \in W(J), \beta \in \mathcal{S}(\Lambda) \\ \ell(w) + \text{ht}(\beta) = p}}^{\oplus} L_m(w(\Lambda + \rho - \beta) - \rho).$$

Here, the above sum is a direct sum of inequivalent irreducible highest weight m -modules.

Remark. In [14], we proved the above theorem under the condition $(\widehat{C}1)$ on the GGCM $A = (a_{ij})_{i,j \in I}$ that for each $i \in I$, either $a_{ii} = 2$ or $a_{ii} = 0$ (see [14, Theorem 4.1]). And then, using it, we got [14, Theorem 5.1]. However, in the proof of [14, Theorem 5.1], we required the condition $(\widehat{C}1)$ only to ensure the validity of [14, Theorem 4.1]. So, since we have proved Theorem 5.3 (without any condition on the symmetrizable GGCM A), [14, Theorem 5.1] now holds without the condition $(\widehat{C}1)$ on the GGCM A .

5.2. Homology vanishing theorem. From now till the end of this paper, we will assume that the subset J of I^{re} is of finite type. As in Section

4, let $\mathcal{C}(\mathfrak{g}(A), \mathfrak{m})$ be the category of all $\mathfrak{g}(A)$ -modules which are finitely semi-simple as an \mathfrak{m} -module. Note that under the adjoint action, the GKM algebra $\mathfrak{g}(A)$ itself is in $\mathcal{C}(\mathfrak{g}(A), \mathfrak{m})$ since \mathfrak{m} is a finite-dimensional reductive Lie algebra. In this subsection, we use the notation of [11, Part I]. Let \mathfrak{b} be a Lie algebra, and \mathfrak{a} a Lie subalgebra such that \mathfrak{b} is a finitely semi-simple \mathfrak{a} -module under the adjoint action. Then, for (left) \mathfrak{b} -modules V_1 and V_2 which are both finitely semi-simple as an \mathfrak{a} -module, $\text{Tor}_j^{\mathfrak{b}, \mathfrak{a}}(V_1^t, V_2)$ is defined as in [11, Part I] ($j \geq 0$), where V_1^t denotes the right \mathfrak{b} -module associated to V_1 in a natural way by using the unique anti-automorphism of $U(\mathfrak{b})$ which is -1 on \mathfrak{b} .

Here, we obtain a vanishing theorem for the Lie algebra homology $H_*(\mathfrak{g}(A), L(\Lambda))$ of $\mathfrak{g}(A)$ with coefficients in $L(\Lambda)$. Our dependence on the ideas in the papers [9,10] of Kumar will be clear to any informed reader.

Recall that for an \mathfrak{h} -diagonalizable module $M = \sum_{\tau \in \mathfrak{h}^*} M_\tau$ with finite-dimensional weight spaces M_τ , we put $M_c^* := \sum_{\tau \in \mathfrak{h}^*} M_\tau^*$, where $M_\tau^* = \text{Hom}_{\mathbb{C}}(M_\tau, \mathbb{C})$ (see [14, Section 5]).

Theorem 5.4. *Let $\Lambda \in P^+$, $\mu \in P_J^+$, and let $L^*(\Lambda) = \{L(\Lambda)\}_c^*$ be the irreducible lowest weight $\mathfrak{g}(A)$ -module with lowest weight $-\Lambda \in \mathfrak{h}^*$. We assume that J is a subset of I^e of finite type. Then, as \mathbb{C} -vector spaces:*

(a) *If $\mu \neq w(\Lambda + \rho - \beta) - \rho$ for any $w \in W(J)$ and $\beta \in \mathcal{S}(\Lambda)$, we have for all $n \geq 0$*

$$\text{Tor}_n^{\mathfrak{g}(A)}(L^*(\Lambda)^t, V_m(\mu)) = 0.$$

(b) *If $\mu = w_0(\Lambda + \rho - \beta_0) - \rho$ for some (necessarily unique) $w_0 \in W(J)$ and $\beta_0 \in \mathcal{S}(\Lambda)$, we have for all $n \geq 0$*

$$\text{Tor}_n^{\mathfrak{g}(A)}(L^*(\Lambda)^t, V_m(\mu)) \cong L_m(0) \otimes_{\mathbb{C}} H_{n - (\ell(w_0) + \text{ht}(\beta_0))}(\mathfrak{m}, L_m(0)).$$

Proof. First, we can easily show that

$$\text{Tor}_p^{\mathfrak{g}(A), \mathfrak{m}}(L^*(\Lambda)^t, V_m(\mu)) \cong \text{Tor}_p^{(\mathfrak{p}, \mathfrak{m})} (L^*(\Lambda)^t, L_m(\mu)) \quad (p \geq 0)$$

as \mathbb{C} -vector spaces by a standard argument. Now, for an \mathfrak{m} -module V , we denote by $V^{\mathfrak{m}}$ the space of \mathfrak{m} -invariants in V . Then, by [11, Proposition 4.11] applied to the case $\mathfrak{b} = \mathfrak{p}$, $\mathfrak{a} = \mathfrak{m}$, and $\mathfrak{c} = \mathfrak{u}^+$, we see that $\text{Tor}_p^{(\mathfrak{p}, \mathfrak{m})}(L^*(\Lambda)^t, L_m(\mu))$ is the space of \mathfrak{m} -invariants in the p -th homology of the following

complex:

$$\{L_m(\mu) \otimes_{U(\mathfrak{u}^+)} ((U(\mathfrak{p}) \otimes_{U(\mathfrak{m})} \wedge^j(\mathfrak{p}/\mathfrak{m})) \otimes_{\mathcal{C}} L^*(\Lambda))\}_{j \geq 0},$$

which is easily seen to be $(L_m(\mu) \otimes_{\mathcal{C}} H_p(\mathfrak{u}^+, L^*(\Lambda)))^m$. Here, by using the Chevalley involution (i.e., the involutive automorphism ω of $\mathfrak{g}(A)$ such that $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$ ($i \in I$), $\omega(h) = -h$ ($h \in \mathfrak{h}$), we can show that

$$H_p(\mathfrak{u}^+, L^*(\Lambda)) \cong \{H_p(\mathfrak{u}^-, L(\Lambda))\}_{\mathcal{C}}^* \text{ as } \mathfrak{m}\text{-modules } (p \geq 0)$$

(cf. the proof of [10, Lemma (2.8)]).

On the other hand, by Theorem 5.3, we have

$$H_p(\mathfrak{u}^-, L(\Lambda)) \cong \sum_{\substack{w \in W(J), \beta \in \mathcal{S}'(\Lambda) \\ \ell(w) + \text{ht}(\beta) = p}}^{\oplus} L_m(w(\Lambda + \rho - \beta) - \rho)$$

as \mathfrak{m} -modules ($p \geq 0$). So, since $L_m(\tau)$ is finite-dimensional for $\tau \in P_J^+$ as is well-known, we see that $H_p(\mathfrak{u}^-, L(\Lambda))$ is finite-dimensional, so that $\{H_p(\mathfrak{u}^-, L(\Lambda))\}_{\mathcal{C}}^* = \{H_p(\mathfrak{u}^-, L(\Lambda))\}^*$ ($p \geq 0$). Therefore, by using the fact that $V_1^* \otimes_{\mathcal{C}} V_2 \cong \text{Hom}_{\mathcal{C}}(V_1, V_2)$ as \mathfrak{m} -modules for finite-dimensional \mathfrak{m} -modules V_1 and V_2 , we have

$$\begin{aligned} \text{Tor}_p^{(\mathfrak{p}, \mathfrak{m})}(L^*(\Lambda)^t, L_m(\mu)) &\cong (\{H_p(\mathfrak{u}^-, L(\Lambda))\}^* \otimes_{\mathcal{C}} L_m(\mu))^m \\ &\cong \text{Hom}_{U(\mathfrak{m})}(L_m(\mu), H_p(\mathfrak{u}^-, L(\Lambda))) \quad (p \geq 0) \end{aligned}$$

as \mathcal{C} -vector spaces. Hence, again by Theorem 5.3 above, we get

in Case (a), $\text{Tor}_p^{(\mathfrak{g}(A), \mathfrak{m})}(L^*(\Lambda)^t, V_m(\mu)) = 0 \quad (p \geq 0)$,

in Case (b),

$$\text{Tor}_p^{(\mathfrak{g}(A), \mathfrak{m})}(L^*(\Lambda)^t, V_m(\mu)) \cong \begin{cases} 0 & (p \neq \ell(w_0) + \text{ht}(\beta_0)), \\ L_m(0) & (p = \ell(w_0) + \text{ht}(\beta_0)). \end{cases}$$

Now, by [6, Section 6] (adapted to our situation), there exists a spectral sequence $\{E_{p,q}^r\}$ such that

$$E_{p,q}^2 = H_p(\mathfrak{g}(A), \mathfrak{m}, L^*(\Lambda) \otimes_{\mathcal{C}} V_m(\mu)) \otimes_{\mathcal{C}} H_q(\mathfrak{m}, L_m(0))$$

$$\Rightarrow E_{p+q} = H_{p+q}(\mathfrak{g}(A), L^*(\Lambda) \otimes_{\mathbb{C}} V_{\mathfrak{m}}(\mu)).$$

Note that $H_p(\mathfrak{g}(A), \mathfrak{m}, L^*(\Lambda) \otimes_{\mathbb{C}} V_{\mathfrak{m}}(\mu)) = \text{Tor}_p^{\mathfrak{g}(A), \mathfrak{m}}(L^*(\Lambda)^t, V_{\mathfrak{m}}(\mu))$ and that $H_{p+q}(\mathfrak{g}(A), L^*(\Lambda) \otimes_{\mathbb{C}} V_{\mathfrak{m}}(\mu)) = \text{Tor}_{p+q}^{\mathfrak{g}(A)}(L^*(\Lambda)^t, V_{\mathfrak{m}}(\mu))$ by [11, Propositions 4.2 and 4.3]. Therefore,

in Case (a), $\text{Tor}_n^{\mathfrak{g}(A)}(L^*(\Lambda)^t, V_{\mathfrak{m}}(\mu)) = E_n = 0,$

in Case (b), $\text{Tor}_n^{\mathfrak{g}(A)}(L^*(\Lambda)^t, V_{\mathfrak{m}}(\mu)) = E_n = E_{\ell(w_0) + \text{ht}(\beta_0), n - \ell(w_0) - \text{ht}(\beta_0)}^2 = L_{\mathfrak{m}}(0) \otimes_{\mathbb{C}} H_{n - \ell(w_0) - \text{ht}(\beta_0)}(\mathfrak{m}, L_{\mathfrak{m}}(0)).$

Thus, we have proved the theorem. □

Remark. In the above proof, we have also shown that if $P_j^+ \ni \mu \neq w(\Lambda + \rho - \beta) - \rho$ for any $w \in W(J)$ and $\beta \in \mathcal{S}(\Lambda)$, we have

$$\text{Tor}_n^{\mathfrak{g}(A), \mathfrak{m}}(L^*(\Lambda)^t, V_{\mathfrak{m}}(\mu)) = 0 \text{ for all } n \geq 0.$$

As an application of Theorem 4.5, we obtain the following theorem, which generalizes a result of Kumar [9, Theorem (1.7)] to arbitrary symmetrizable GKM algebras.

Theorem 5.5. *Let $\Lambda_1, \Lambda_2 \in P^+$. Assume that*

$$\Lambda_1 - \Lambda_2 \notin \mathcal{S}(\Lambda_1) - \mathcal{S}(\Lambda_2) := \{\beta_1 - \beta_2 \mid \beta_1 \in \mathcal{S}(\Lambda_1), \beta_2 \in \mathcal{S}(\Lambda_2)\}.$$

Then, we have

$$\text{Tor}_n^{\mathfrak{g}(A)}(L^*(\Lambda_1)^t, L(\Lambda_2)) = 0 \text{ for all } n \geq 0.$$

Proof. By Theorem 4.5, there exists an exact sequence of $\mathfrak{g}(A)$ -homomorphisms:

$$0 \leftarrow L(\Lambda_2) \leftarrow C_0(\Lambda_2) \leftarrow C_1(\Lambda_2) \leftarrow \cdots \leftarrow C_p(\Lambda_2) \leftarrow C_{p+1}(\Lambda_2) \leftarrow \cdots,$$

where $C_p(\Lambda_2) = \sum_{\substack{w \in W(J), \beta \in \mathcal{S}(\Lambda_2) \\ \ell(w) + \text{ht}(\beta) = p}}^{\oplus} V_{\mathfrak{m}}(w(\Lambda_2 + \rho - \beta) - \rho)$ is a direct sum of generalized Verma modules ($p \geq 0$).

On the other hand, by Theorem 5.4, we get $\text{Tor}_n^{\mathfrak{g}(A)}(L^*(\Lambda_1)^t, V_{\mathfrak{m}}(w(\Lambda_2 + \rho - \beta) - \rho)) = 0$ for any $w \in W(J)$ and $\beta \in \mathcal{S}(\Lambda_2)$, since $\Lambda_1 - \Lambda_2 \notin \mathcal{S}(\Lambda_1) - \mathcal{S}(\Lambda_2)$

(see [13, Proposition 1.1]). So, we have $\text{Tor}_n^{\mathfrak{g}(A)}(L^*(\Lambda_1)^t, C_p(\Lambda_2)) = 0$ for all $p \geq 0$. Therefore, we see by a standard spectral sequence argument that $\text{Tor}_n^{\mathfrak{g}(A)}(L^*(\Lambda_1)^t, L(\Lambda_2)) = 0$. \square

Corollary 5.6. *Let $\Lambda_1, \Lambda_2 \in P^+$ be such that $\Lambda_1 - \Lambda_2 \notin \mathcal{S}(\Lambda_1) - \mathcal{S}(\Lambda_2)$. Assume that J is a subset of I^{re} of finite type. Then,*

$$\text{Tor}_n^{\mathfrak{g}(A), \mathfrak{m}}(L^*(\Lambda_1)^t, L(\Lambda_2)) = 0 \text{ for all } n \geq 0.$$

Proof. By the remark after Theorem 5.4 and the assumption on Λ_1, Λ_2 , we have $\text{Tor}_n^{\mathfrak{g}(A), \mathfrak{m}}(L^*(\Lambda_1)^t, V_{\mathfrak{m}}(w(\Lambda_2 + \rho - \beta) - \rho)) = 0$ for any $w \in W(J)$ and $\beta \in \mathcal{S}(\Lambda_2)$. So, the corollary now follows in the same way as the above theorem from Theorem 4.5. \square

By putting $\Lambda_1 = 0$ in Theorem 5.5, we get a homology vanishing theorem.

Theorem 5.7. *Let $\Lambda \in P^+$. Assume that $\Lambda \notin \mathcal{S}(\Lambda) - \mathcal{S} = \{\beta_1 - \beta_2 \mid \beta_1 \in \mathcal{S}(\Lambda), \beta_2 \in \mathcal{S}\}$. Then, we have*

$$H_n(\mathfrak{g}(A), L(\Lambda)) = 0 \text{ for all } n \geq 0.$$

Remark. When $\mathfrak{g}(A)$ is a Kac-Moody algebra (i.e., A is a GCM), \mathcal{S} consists of only one element $0 \in \mathfrak{h}^*$. So, the above theorem generalizes a result of Kumar [9].

5.3. Bott’s “strange equality”. By putting $\Lambda_1 = 0$ in Corollary 5.6, we have $H_n(\mathfrak{g}(A), \mathfrak{m}, L(\Lambda)) = 0$ ($n \geq 0$) for $\Lambda \in P^+$ such that $\Lambda \notin \mathcal{S}(\Lambda) - \mathcal{S}$. However, for $\Lambda \in \mathcal{S}(\Lambda) - \mathcal{S}$, the relative Lie algebra homology $H_*(\mathfrak{g}(A), \mathfrak{m}, L(\Lambda))$ does not necessarily vanish. Actually, we have the following, as another application of Theorem 4.5.

Theorem 5.8. *Assume that the subset J of I^{re} is of finite type. Then, the dimension of each relative Lie algebra homology space $H_s(\mathfrak{g}(A), \mathfrak{m}, L(0))$ of $\mathfrak{g}(A)$ with respect to \mathfrak{m} with coefficients in the trivial one-dimensional module $L(0)$ is given as follows:*

$$\begin{aligned} \dim_{\mathbb{C}} H_s(\mathfrak{g}(A), \mathfrak{m}, L(0)) &= 0 && \text{if } s (\geq 0) \text{ is odd,} \\ \dim_{\mathbb{C}} H_{2p}(\mathfrak{g}(A), \mathfrak{m}, L(0)) &= \text{the number of elements} \end{aligned}$$

of the set $\{(w, \beta) \in W(J) \times \mathcal{S} \mid \ell(w) + \text{ht}(\beta) = p\}$

= the number of m -irreducible components in the p -th Lie algebra homology $H_p(\mathfrak{u}^-, L(0))$ of \mathfrak{u}^- with coefficients in $L(0)$ ($p \geq 0$).

Proof. This follows from exactly the same argument as for [11, Corollary 6.7], in which we use Theorem 4.5 instead of [11, Theorem 5.1]. \square

LIST OF NOTATION

\mathbf{Z} = the set of integers

$\mathbf{Z}_{\geq 0}$ = $\{k \in \mathbf{Z} \mid k \geq 0\}$

$\mathbf{Z}_{\geq 1}$ = $\{k \in \mathbf{Z} \mid k \geq 1\}$

\mathbf{C} = the set of complex numbers

GGCM = a real square matrix $A = (a_{ij})_{i,j \in I}$ satisfying (C1)–(C3) (see §1)

$U(\mathfrak{a})$ = the universal enveloping algebra of a Lie algebra \mathfrak{a}

$\mathfrak{g}(A)$ = the generalized Kac-Moody algebra associated to a GGCM $A = (a_{ij})_{i,j \in I}$

\mathfrak{h} = the Cartan subalgebra of $\mathfrak{g}(A)$

$V^* = \text{Hom}_{\mathbf{C}}(V, \mathbf{C})$ = the dual of a finite-dimensional vector space V over \mathbf{C}

$\langle \cdot, \cdot \rangle$ = a pairing between a finite-dimensional vector space V and its dual V^*

Π = the set of simple roots of $\mathfrak{g}(A)$

Π^\vee = the set of simple coroots of $\mathfrak{g}(A)$

ρ = an element of \mathfrak{h}^* such that $\langle \rho, \alpha_i^\vee \rangle = (1/2) \cdot a_{ii}$ ($i \in I$)

$Q_+ = \sum_{i \in I} \mathbf{Z}_{\geq 0} \alpha_i$

$\text{supp}(\alpha)$ ($\alpha = \sum_{i \in I} k_i \alpha_i \in Q_+$) = the subdiagram of the Dynkin diagram of $A = (a_{ij})_{i,j \in I}$ corresponding to the subset $\{i \in I \mid k_i > 0\}$ of I

$\text{ht}(\alpha)$ ($\alpha = \sum_{i \in I} k_i \alpha_i \in Q_+$) = $\sum_{i \in I} k_i$

Δ = the set of roots of $\mathfrak{g}(A)$

Δ^+ = the set of positive roots of $\mathfrak{g}(A)$

Δ^- = the set of negative roots of $\mathfrak{g}(A)$

\mathfrak{g}_α ($\alpha \in \Delta$) = the root space attached to a root α

$\text{mult}(\alpha)$ ($\alpha \in \Delta$) = $\dim_{\mathbf{C}} \mathfrak{g}_\alpha$

e_i, f_i ($i \in I$) = the Chevalley generators of $\mathfrak{g}(A)$

ω = the involutive automorphism of $\mathfrak{g}(A)$ such that $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$ ($i \in I$), $\omega(\mathfrak{h}) = -\mathfrak{h}$ ($\mathfrak{h} \in \mathfrak{h}$)

$I^{re} = \{i \in I \mid a_{ii} = 2\}$

$I^m = \{i \in I \mid a_{ii} \leq 0\}$

Δ_J^+ ($J \subset I^{re}$) = $\Delta^+ \cap (\sum_{i \in J} \mathbf{Z} \alpha_i)$

$\Delta^+(J)$ ($J \subset I^{re}$) = $\Delta^+ \setminus \Delta_J^+$

$$\mathfrak{n}^\pm = \sum_{\alpha \in \Delta^\pm} \mathfrak{g}_{\pm\alpha}$$

$$\mathfrak{u}^\pm = \sum_{\alpha \in \Delta^+(J)} \mathfrak{g}_{\pm\alpha}$$

$\bigwedge^j \mathfrak{n}^\pm$ ($j \geq 0$) = the exterior algebra of degree j over \mathfrak{n}^\pm

$$\bigwedge^* \mathfrak{n}^- = \sum_{j \geq 0} \bigwedge^j \mathfrak{n}^-$$

$\bigwedge^j \mathfrak{u}^\pm$ ($j \geq 0$) = the exterior algebra of degree j over \mathfrak{u}^\pm

$$\mathfrak{m} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$$

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}^+$$

$$P^+ = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \ (i \in I), \text{ and } \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \ (i \in I^e)\}$$

$$P_J^+ \ (J \subset I^e) = \{\mu \in \mathfrak{h}^* \mid \langle \mu, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \ (i \in J)\}$$

$L(\lambda)$ ($\lambda \in \mathfrak{h}^*$) = the irreducible highest weight $\mathfrak{g}(A)$ -module with highest weight λ

$L_m(\lambda)$ ($\lambda \in P_J^+$) = the irreducible highest weight \mathfrak{m} -module with highest weight λ

$V_m(\lambda)$ ($\lambda \in P_J^+$) = the generalized Verma module with highest weight λ (see § 3.1)

$V(\lambda)$ ($\lambda \in \mathfrak{h}^*$) = the Verma module with highest weight λ (see § 3.1)

$\mathcal{P}(M)$ = the set of weights of an \mathfrak{h} -module M

$$D(\mu) \ (\mu \in \mathfrak{h}^*) = \mu - Q_+$$

\mathcal{O} = the category of all $\mathfrak{g}(A)$ -modules V satisfying (J1) and (J2) (see § 2.2)

GVCS: see the Definition in § 3.1.

$[V : L(\tau)]$ ($\tau \in \mathfrak{h}^*$) = the multiplicity of $L(\tau)$ in a $\mathfrak{g}(A)$ -module V in the category \mathcal{O} (see [4, Definition 3.5])

\mathcal{O}_J ($J \subset I^e$) = the category of all \mathfrak{m} -modules M satisfying (J1)–(J3) (see § 2.2)

\mathcal{E} : see § 2.2.

$\text{ch}_{\mathfrak{h}} M$: see § 2.2.

r_i ($i \in I^e$) = the fundamental reflection with respect to α_i determined by $r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ ($\lambda \in \mathfrak{h}^*$)

W ($\subset GL(\mathfrak{h}^*)$) = the Weyl group of $\mathfrak{g}(A)$ generated by the r_i 's ($i \in I^e$)

$l(w)$ ($w \in W$) = the length of w

$$\Phi_w \ (w \in W) = \Delta^+ \cap w(\Delta^-)$$

$$W(J) \ (J \subset I^e) = \{w \in W \mid \Phi_w \subset \Delta^+(J)\}$$

$\Pi^{re} = \{\alpha_i \in \Pi \mid i \in I^{re}\}$ = the set of real simple roots of $\mathfrak{g}(A)$

$\Pi^{im} = \{\alpha_i \in \Pi \mid i \in I^{im}\}$ = the set of imaginary simple roots of $\mathfrak{g}(A)$

$\Delta^{re} = W \cdot \Pi^{re}$ = the set of real roots of $\mathfrak{g}(A)$

$\Delta^{im} = \Delta \setminus \Delta^{re}$ = the set of imaginary roots of $\mathfrak{g}(A)$

$$K = \{\alpha = \sum_{i \in I} k_i \alpha_i \in Q_+ \setminus \{0\} \mid \langle \alpha, \alpha_j^\vee \rangle \leq 0 \ (j \in I^e), \text{ and } \text{supp}(\alpha) \text{ is connected}\} \setminus \bigcup_{j \geq 2} j \cdot \Pi^{im}$$

$\alpha^\vee (\alpha \in \Delta^{re}) =$ the dual real root of α (see §4)

$r_\alpha (\alpha \in \Delta^{re}) =$ the reflection with respect to α determined by $r_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha (\lambda \in \mathfrak{h}^*)$ (see §4)

$(\cdot | \cdot) =$ a standard invariant form on \mathfrak{h}^* (see §3.1)

$v =$ the linear isomorphism from \mathfrak{h} onto \mathfrak{h}^* determined by $\langle v(h_1), h_2 \rangle = (h_1 | h_2) (h_1, h_2 \in \mathfrak{h})$

$\mathcal{S} =$ the set of all sums of distinct, pairwise perpendicular (with respect to $(\cdot | \cdot)$), imaginary simple roots

$\mathcal{S}(\Lambda) (\Lambda \in P^+) = \{\beta \in \mathcal{S} \mid (\beta | \Lambda) = 0\}$

$\Psi^p (p \geq 0) =$ the sequence of elements λ_i of P_J^+ such that $(\bigwedge^p u^-) \otimes_{\mathbb{C}} L(\Lambda) \cong \sum_i^{\oplus} L_m(\lambda_i)$ as \mathfrak{m} -modules

$\Psi_\Lambda^p (p \geq 0, \Lambda \in P^+) =$ the subsequence of Ψ^p consisting of those λ_i in Ψ^p such that $(\lambda_i + \rho | \lambda_i + \rho) = (\Lambda + \rho | \Lambda + \rho)$

$\mathcal{G}(\mathfrak{g}(A), \mathfrak{m}) =$ the category of all $\mathfrak{g}(A)$ -modules which are direct sums of $L_m(\lambda)$'s ($\lambda \in P_J^+$) as an \mathfrak{m} -module (see §4)

$\text{Ext}_{(\mathfrak{g}(A), \mathfrak{m})}^j (j \geq 0) =$ the relative Ext bifunctor defined in the category $\mathcal{G}(\mathfrak{g}(A), \mathfrak{m})$ (see [11, Part I])

$V^t =$ the right \mathfrak{b} -module associated to a left \mathfrak{b} -module V in a natural way (see §5.2)

$\text{Tor}_j^{\mathfrak{b}, \mathfrak{a}} (j \geq 0) =$ the relative Tor bifunctor defined in the category $\mathcal{G}(\mathfrak{b}, \mathfrak{a})$ (see [11, Part I])

$M_c^* = \sum_{\tau \in \mathfrak{h}^*}^{\oplus} \text{Hom}_{\mathbb{C}}(M_\tau, \mathbb{C})$, where $M = \sum_{\tau \in \mathfrak{h}^*}^{\oplus} M_\tau$ is an \mathfrak{h} -diagonalizable module with finite-dimensional weight spaces $M_\tau (\tau \in \mathfrak{h}^*)$

$L^*(\Lambda) (\Lambda \in P^+) = \{L(\Lambda)\}_c^* =$ the irreducible lowest weight $\mathfrak{g}(A)$ -module with lowest weight $-\Lambda$

$V^{\mathfrak{m}} = \{v \in V \mid \mathfrak{m}(v) = 0 \text{ for all } \mathfrak{m} \in \mathfrak{m}\} =$ the space of \mathfrak{m} -invariants in an \mathfrak{m} -module V

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