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# High-Energy Behavior of Total Scattering Cross Sections for 3-body Quantum Systems

By

Hiroshi T. ITO*

## § 1. Introduction

In this paper, we investigate the high-energy behavior of total scattering cross sections with 2-body initial states for a 3-body system.

A 3-body Schrödinger operator is given by

$$\tilde{H}=-\sum_{1\leq j\leq 3}(2m_j)^{-1}\Delta_{r_j}+\sum_{1\leq i<j\leq 3}V_{ij}(r_i-r_j)\qquad \text{in } L^2(\boldsymbol{R}^{3N}). \tag{1.1}$$

Here $m_j>0$ and $r_j\in\boldsymbol{R}^N$ ($N\geqq 3$) are the mass and the position of the $j$-th particle, respectively, and $V_{ij}$ is the interaction between the $i$-th particle and the $j$-th particle. All $V_{ij}$ are real-valued functions and satisfy the following condition for some $\ell\in\boldsymbol{N}\cup\{0\}$ or $\ell=3/2$ ($\boldsymbol{N}=\{1, 2, \cdots\}$):

$(\mathrm{V})_\ell$ $\qquad V_{ij}(x)\in C^{2\ell+2}(\boldsymbol{R}^N)$ and there exists a $\delta>\ell+((N+1)/2)$

such that

$$|\partial_x^\gamma V_{ij}(x)|\leqq C\langle x\rangle^{-\delta} \tag{1.2}$$

for all multi-indices $\gamma$ with $|\gamma|\leqq 2\ell+2$, where $\langle x\rangle:=(1+|x|^2)^{1/2}$.

Let $H$ be the Schrödinger operator obtained by separating the kinetic energy of the center of mass from $\tilde{H}$. $H$ acts in $\mathcal{H}:=L^2(\boldsymbol{R}^{2N})$, and its explicit form depends on the coordinates of $\boldsymbol{R}^{2N}$. We adopt the Jacobi coordinates. A partition of the set $\{1, 2, 3\}$ into nonempty disjoint subsets is called a cluster decomposition. We call $\{(1), (2), (3)\}$ (resp. $\{(i, j), k\}$, $i<j$) a 3-cluster decomposition (resp. a 2-cluster decomposition). We denote by $\boldsymbol{A}_2$ the set of all 2-cluster decompositions. For $a\in\boldsymbol{A}_2$ with $a=\{(i, j), k\}$, we define the Jacobi coordinates $(x_a, y_a)$ by

$$x_a=r_i-r_j,\qquad y_a=r_k-\frac{m_ir_i+m_jr_j}{m_i+m_j}. \tag{1.3}$$

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* Department of Mathematics, Kyoto University, Kyoto 606, Japan.

Let $b\in \boldsymbol{A}_2$. Then $x_b$, $y_b$ are linear combinations of $x_a$, $y_a$. By the coordinates, $H$ is expressed as

$$H=H_0+V=-(2m_a)^{-1}\Delta_{x_a}-(2n_a)^{-1}\Delta_{y_a}+V\ , \tag{1.4}$$

where $V=\sum_{1\leq p<q\leq 3} V_{pq}$ and $V_{pq}=V_{pq}(r_p-r_q)$. Note that $r_p-r_q$ is expressed as a linear combination of $x_a$, $y_a$. $m_a$ and $n_a$ are the reduced masses defined by $m_a{}^{-1}:=m_i{}^{-1}+m_j{}^{-1}$, $n_a{}^{-1}:=m_k{}^{-1}+(m_i+m_j)^{-1}$, respectively.

Under assumption $(\mathrm{V})_0$, $H$ is self-adjoint with domain $D(H)=H^2(\boldsymbol{R}^{2N})$, the Sobolev space of order 2.

For $a\in \boldsymbol{A}_2$ with $a=\{(i,\ j),\ k\}$, the cluster Hamiltonian $H_a$ is defined by

$$H_a:=H_0+V_{ij},\qquad D(H_a)=H^2(\boldsymbol{R}^{2N})\ . \tag{1.5}$$

$H_a$ is expressed as $H_a=h_a\otimes \mathrm{Id}+\mathrm{Id}\otimes T_a$ according to the decomposition $\mathcal{H}=L^2(\boldsymbol{R}^N_{x_a})\otimes L^2(\boldsymbol{R}^N_{y_a})$, where $h_a:=-(2m_a)^{-1}\Delta_{x_a}+V_{ij}$ and $T_a:=-(2n_a)^{-1}\Delta_{y_a}$ are self-adjoint in $L^2(\boldsymbol{R}^N_{x_a})$ with $D(h_a)=H^2(\boldsymbol{R}^N_{x_a})$ and in $L^2(\boldsymbol{R}^N_{y_a})$ with $D(T_a)=H^2(\boldsymbol{R}^N_{y_a})$, respectively.

Let $d_a$ be the number of strictly negative bound state energies (counting multiplicity) of $h_a$. It is known that under assumption $(\mathrm{V})_0$, $d_a$ is finite (cf. [RS] IV, XIII. 3). We set the set of the 2-body channels with negative bound states energies:

$$\Gamma_2:=\{\alpha=(a,\ k)\ ;\ a\in \boldsymbol{A}_2,\ 1\leqq k\leqq d_a,\ k\in \boldsymbol{N}\}, \tag{1.6}$$

and write $D(\alpha)=a$ for $\alpha=(a,\ k)\in\Gamma_2$. For each 2-body channel $\alpha=(a,\ k)\in\Gamma_2$ we denote by $\lambda_\alpha(<0)$ the $k$-th negative eigenvalue of $h_a$ and by $\psi_\alpha$ the eigenfunction of $h_a$ with eigenvalue $\lambda_\alpha$ such that $\{\psi_\alpha\}$ $(\alpha\in\Gamma_2,\ D(\alpha)=a)$ is an orthonormal system for each $a\in \boldsymbol{A}_2$. For each $\alpha=(a,\ k)\in\Gamma_2$ the channel Hamiltonian $H_\alpha$ in $L^2(\boldsymbol{R}^N_{y_a})$ and the channel identification operator $J_\alpha\in \boldsymbol{B}\ (L^2(\boldsymbol{R}^N_{y_a}),\ \mathcal{H})$ are defined by

$$H_\alpha=\lambda_\alpha+T_a,\qquad J_\alpha u=\psi_\alpha\otimes u\ , \tag{1.7}$$

respectively. Here we denote by $\boldsymbol{B}(X,\ Y)$ the space of all bounded linear operators from $X$ to $Y$. Under assumption $(\mathrm{V})_0$ the channel wave operators:

$$W_\alpha^\pm:=\operatorname*{s-lim}_{t\to\pm\infty} e^{itH}J_\alpha e^{-itH_\alpha}\in \boldsymbol{B}(L^2(\boldsymbol{R}^N_{y_a}),\ \mathcal{H}) \tag{1.8}$$

exist (see, e.g. [RS] III, Theorem XI. 35).

We set

$$\Gamma:=\Gamma_2\cup\{0\}\ , \tag{1.9}$$

where $0\in\Gamma$ stands for the 3-body channel.

For the 3-body channel the channel Hamiltonian is $H_0$ and the channel wave operators are defined by

(1.10) $$W_0^{\pm} := s\text{-}\lim_{t\to\pm\infty} e^{itH} e^{-itH_0} \in \boldsymbol{B}(\mathcal{H}) \qquad (\boldsymbol{B}(\mathcal{H}) := \boldsymbol{B}(\mathcal{H}, \mathcal{H})).$$

The existence of the channel wave operators is also known (see, e.g. [RS] III, Theorem XI. 35).

Throughout this paper we only consider the case where the initial channel belongs to $\Gamma_2$ and the final channel belongs to $\Gamma$. The scattering operator $S_{\alpha\to\beta}$ for scattering $\alpha\to\beta$ ($\alpha\in\Gamma_2$, $\beta\in\Gamma$) is defined by

(1.11) $$S_{\alpha\to\beta} = W_\beta^{+*} W_\alpha^{-},$$

where $A^*$ denotes the adjoint of an operator $A$. Since the intertwining property, $\exp(itH_\beta)S_{\alpha\to\beta} = S_{\alpha\to\beta}\exp(itH_\alpha)$, holds, $S_{\alpha\to\beta}$ is decomposable by a family of operators $\{S_{\alpha\to\beta}(\lambda)\}$, $\lambda>\lambda_{\beta\alpha} := \max(\lambda_\alpha, \lambda_\beta)$ (cf. [AJS], 15-3). The representation formula of $T_{\alpha\to\beta}(\lambda) := S_{\alpha\to\beta}(\lambda) - \delta_{\beta\alpha}$ will be given in the next section and the Appendix, where $\delta_{\beta\alpha}$ is Kronecker's delta and we set $\lambda_\beta = 0$ if $\beta=0$. $T_{\alpha\to\beta}(\lambda)$ is defined for a.e. $\lambda>\lambda_{\beta\alpha}$ as an operator in $\boldsymbol{B}(L^2(S^{N-1}), L^2(S^\beta))$, where $S^\beta := S^{N-1}$ (the unit sphere in $\boldsymbol{R}^N$) for $\beta\neq 0$ and $S^\beta = S^{2N-1}$ for $\beta=0$.

If $\beta\neq 0$, $T_{\alpha\to\beta}(\lambda)$ is well-defined as a norm continuous function of $\lambda>0$ and is of Hilbert-Schmidt class with kernel $T_{\alpha\to\beta}(\lambda, \theta, \omega)$, and $T_{\alpha\to\beta}(\lambda, \cdot, \omega)$ is $L^2(S^{N-1})$-valued strongly continuous function of $\lambda>0$ and $\omega\in S^{N-1}$ (Proposition 2.3).

To treat the case $\beta=0$, we need the following condition in addition of $(\mathrm{V})_0$:

(Z) For each $a=\{(i, j), k\}\in \boldsymbol{A}_2$, $-1$ is not an eigenvalue of the following bounded operator on $L^2(\boldsymbol{R}^N_{x_a})$:

(1.12) $$V_{ij}^{1/2}(-(2m_a)^{-1}\Delta_{x_a} - 0 - i0)^{-1}|V_{ij}|^{1/2},$$
$$(:= \lim_{\varepsilon\downarrow 0} V_{ij}^{1/2}(-(2m_a)^{-1}\Delta_{x_a} - 0 - i\varepsilon)^{-1}|V_{ij}|^{1/2}),$$

where $V_{ij}^{1/2} := |V_{ij}(x_a)|^{1/2}\operatorname{sgn} V_{ij}(x_a)$ and the existence of the norm limit is known (cf. [GM], Proposition 3.1)).

Assumption (Z) implies the absence of zero eigenvalue of $h_a$ (cf. [GM], Proposition (3.4)), and assumption $(\mathrm{V})_0$ implies the absence of positive eigenvalues (cf. [RS], XIII. 13). Therefore, it follows that the set of all eigenvalues of $h_a$ coincides with the set $\{\lambda_\alpha ; \alpha\in\Gamma_2, D(\alpha)=a\}$ under assumptions (Z), $(\mathrm{V})_0$.

Under assumptions $(\mathrm{V})_0$ and (Z), $T_{\alpha\to 0}(\lambda)$, is of Hilbert-Schmidt class with kernel $T_{\alpha\to 0}(\lambda, \theta, \omega)$ for all large $\lambda>0$ and the integral

(1.13) $$\int_{S^{2N-1}} |T_{\alpha\to 0}(\lambda, \theta, \omega)|^2 d\theta$$

is continuous in $\lambda\gg 1$ and $\omega\in S^{N-1}$ (Proposition 2.4).

Now we give the following definition (see [AJS], p. 627):

**Definition.** The total scattering cross section $\sigma_{\alpha\to\beta}(\lambda, \omega)$ for scattering $\alpha\to\beta$ ($\alpha\in\Gamma_2$, $\beta\in\Gamma$) at energy $\lambda\gg 1$ and incident direction $\omega\in S^{N-1}$ is defined by

$$(1.14)\qquad \sigma_{\alpha\to\beta}(\lambda, \omega):=(2\pi)^{N-1}[2n_a(\lambda-\lambda_\alpha)]^{(1-N)/2}\int_{S^\beta}|T_{\alpha\to\beta}(\lambda, \theta, \omega)|^2 d\theta\,,$$

under assumption $(\mathrm{V})_0$ for $\beta\neq 0$ and under assumptions $(\mathrm{V})_0$ and (Z) for $\beta=0$. The total scattering csoss section for an initial channel $\alpha$ at energy $\lambda\gg 1$ and incident direction $\omega\in S^{N-1}$ is defined by

$$(1.15)\qquad \sigma_\alpha(\lambda, \omega):=\sum_{\beta\in\Gamma}\sigma_{\alpha\to\beta}(\lambda, \omega)$$

under assumptions $(\mathrm{V})_0$ and (Z).

For $a=\{(i, j), k\}\in \boldsymbol{A}_2$ we define the intercluster potential $I_a$ by

$$(1.16)\qquad I_a(x_a, y_a)=V-V_{ij}(x_a)$$

and set

$$(1.17)\qquad W_a(x_a\,;\omega, \eta):=\int_{\boldsymbol{R}} I_a(x_a, t\omega+\eta)dt$$

for $\omega\in S^{N-1}$ and $\eta\in\Pi_\omega:=\{\xi\in\boldsymbol{R}^N\,;\xi\cdot\omega=0\}$. $(\cdot, \cdot)_a$ and $\|\cdot\|_a$ denote the $L^2$-scalar product and the $L^2$-norm in $L^2(\boldsymbol{R}^N_{x_a})$, respectively.

Now we state our main results.

**Theorem 1.1.** *Let* $\alpha\in\Gamma_2$ $(a=D(\alpha))$ *and* $\beta\in\Gamma$, *and let the notations be as above.*

(i) *Assume* $(\mathrm{V})_\ell$ *with* $\ell\in\boldsymbol{N}\cup\{0\}$ *and* $\beta\in\Gamma_2$ *with* $D(\beta)\neq a$. *Then*

$$(1.18)\qquad \sigma_{\alpha\to\beta}((1/2)n_a v^2+\lambda_\alpha, \omega)=O(v^{-2\ell-1})$$

*uniformly in* $\omega\in S^{N-1}$ *as* $v\to+\infty$.

(ii) *Assume* $(\mathrm{V})_0$ *and* $\beta\in\Gamma_2$ *with* $D(\beta)=a$. *Then*

$$(1.19)\qquad \sigma_{\alpha\to\beta}((1/2)n_a v^2+\lambda_\alpha, \omega)=v^{-2}\int_{\Pi_\omega}|(W_a(\cdot\,;\omega, \eta)\phi_\alpha, \phi_\beta)_a|^2 d\eta+o(v^{-2})$$

*uniformly in* $\omega\in S^{N-1}$ *as* $v\to+\infty$.

(iii) *Assume* $(\mathrm{V})_0$, (Z) *and* $\beta=0$. *Then*

$$(1.20)\qquad \sigma_{\alpha\to\beta}((1/2)n_a v^2+\lambda_\alpha, \omega)=v^{-2}\int_{\Pi_\omega}\|P^c(h_a)W_a(\cdot\,;\omega, \eta)\phi_\alpha\|_a{}^2 d\eta+o(v^{-2}),$$

$$(1.21)\qquad \sigma_\alpha((1/2)n_a v^2+\lambda_\alpha, \omega)=v^{-2}\int_{\Pi_\omega}\|W_a(\cdot\,;\omega, \eta)\phi_\alpha\|_a{}^2 d\eta+o(v^{-2}),$$

*uniformly in* $\omega\in S^{N-1}$ *as* $v\to+\infty$, *where* $P^c(h_a)$ *is the orthogonal projection onto the continuity subspace of* $L^2(\boldsymbol{R}^N_{x_a})$ *with respect to* $h_a$.

**Theorem 1.2.** *If we replace assumption* $(\mathrm{V})_0$ *by assumption* $(V)_{3/2}$ *in* (ii), (iii) *of the above theorem, all the remainder terms* $o(v^{-2})$ *in* (1.19), (1.20), (1.21) *can be replaced by* $O(v^{-3})$.

There are several literature on 3-body total cross sections ([APS], [ES], [AJS], [AS], [I], [IT]). In particular, bounds on the total cross sections at high energies for many-body systems are discussed in [APS], but the asymptotic behavior are not discussed in it. In [APS] and [ES] the approach to study the total cross sections is the time-dependent one, while our approach is the time-independent one and is based on the representation formula of the scattering matrix and some resolvent estimates, which is proved by using multiple commutator methods ([JMP]). A similar approach is carried out in [I]. In [Ha] the convergence of Born series for (2-cluster)→(2-cluster) $T$-matrix for $n(\leqq 4)$-body systems at high energies is proved.

The organization of this paper is as follows. In Section 2 and in the Appendix, we shall review some properties of $T_{\alpha\to\beta}(\lambda, \theta, \omega)$ and prove the optical theorem (Theorem 2.5). The proof of Theorems 1.1 and 1.2 will be given in Section 3. A proof of Proposition 3.1, which is crucial for a proof of Theorem 1.1, will be given in Section 5 by using the abstract commutator methods (Theorem 4.2) in Section 4.

## §2. Representation Formula of $T_{\alpha\to\beta}(\lambda, \theta, \omega)$

In this section we will give the representation formula of $T_{\alpha\to\beta}(\lambda, \theta, \omega)$ for $\alpha\in\Gamma_2$ $(a=D(\alpha))$ and $\beta\in\Gamma$, and will prove the optical theorem.

We first consider the case $\beta\in\Gamma_2$ with $b=D(\beta)$. The next lemmas are crucial for our representation formula of $T_{\alpha\to\beta}(\lambda, \theta, \omega)$. We write $R(z)=(H-z)^{-1}$ for $\mathrm{Im}\, z\neq 0$.

**Lemma 2.1** ([M], [PSS]). *Assume* $(V)_0$ *and* $s>1/2$. *Then the norm limits*

$$R(\lambda\pm i0):=\lim_{\varepsilon\downarrow 0} R(\lambda\pm i\varepsilon) \tag{2.1}$$

*exist in* $\boldsymbol{B}(L_s^2(\boldsymbol{R}^{2N}), L_{-s}^2(\boldsymbol{R}^{2N}))$ *for* $\lambda>0$, *and the convergence is uniform on each compact subset in* $(0, \infty)$, *where* $L_t^2(\boldsymbol{R}^{2N})$ $(t\in\boldsymbol{R})$ *is the weighted* $L^2$*-space*:

$$L_t^2(\boldsymbol{R}^{2N}):=L^2(\boldsymbol{R}^{2N}; \langle x_a; y_a\rangle^{2t} dx_a dy_a),$$

*where* $\langle x_a; y_a\rangle:=(1+|x_a|^2+|y_a|^2)^{1/2}$. *(Since* $m_a|x_a|^2+n_a|y_a|^2=m_b|x_b|^2+n_b|y_b|^2$ *for any* $b\in\boldsymbol{A}_2$, *the definition of the space* $L_t^2(\boldsymbol{R}^{2N})$ *is independent of the choice of* $a\in\boldsymbol{A}_2$.)

**Lemma 2.2.** *Assume* $(\mathrm{V})_\ell$. *Then for any* $s>0$ *and any multi-index* $\gamma$ *with* $|\gamma|\leqq 2\ell+4$, $\phi_\beta$ *satisfies* $\partial_{r_b}^{\gamma}\phi_\beta\in L_s^2(\boldsymbol{R}_{x_b}^N)$.

For the proof of Lemma 2.1, see [PSS]. Lemma 2.1 (called the limiting absorption principle) holds for $n(\geqq 2)$-body systems under milder conditions on the potentials ([M], [PSS], [ABG], [T]). Lemma 2.2 is known as the exponential decay of eigenfunctions. For the proof, see [Ag] (see also [RS] IV, XIII. 11). Now we give the spectral representation of $H_\beta$ (cf. [AJS], 16-2).

For $\lambda>\lambda_\beta$ we define a map $Z_\beta(\lambda)\in \boldsymbol{B}(L_s^2(\boldsymbol{R}_{y_b}^N), L^2(S^{N-1}))$, $s>1/2$, by

$$(Z_\beta(\lambda)f)(\omega)=C_\beta(\lambda)\int e^{-i(2n_b(\lambda-\lambda_\beta))^{1/2}\omega\cdot y_b}f(y_b)dy_b\,, \tag{2.2}$$

where $\omega\in S^{N-1}$ and

$$C_\beta(\lambda)=(2\pi)^{-N/2}n_b^{1/2}(2n_b(\lambda-\lambda_\beta))^{(N-2)/4}\,. \tag{2.3}$$

Then the map $Z_\beta$, defined by

$$(Z_\beta f)(\lambda, \omega)=(Z_\beta(\lambda)f)(\omega)\,, \tag{2.3$'$}$$

can be extended to a unitary operator from $L^2(\boldsymbol{R}_{y_b}^N)$ to $L^2((\lambda_\beta, \infty); L^2(S^{N-1}))$ and

$$(Z_\beta H_\beta f)(\lambda, \cdot)=\lambda(Z_\beta f)(\lambda, \cdot)$$

in $L^2(S^{N-1})$ for a. e. $\lambda>\lambda_\beta$ if $f\in D(H_\beta)$. $Z_\alpha$ is defined in the same way.

We define $G_{\beta\alpha}(\lambda, \omega)=G_{\beta\alpha}(\lambda, \omega; y_b)$ by

$$G_{\beta\alpha}(\lambda, \omega)=\int\overline{\phi_\beta(x_b)}(K(\lambda)e_\alpha(\lambda, \omega))(x_b, y_b)dx_b\,, \tag{2.4}$$

where $K(\lambda)=-I_\alpha+I_bR(\lambda+i0)I_\alpha$, and

$$e_\alpha(\lambda, \omega)=\phi_\alpha(x_a)e^{i(2n_a(\lambda-\lambda_\alpha))^{1/2}\omega\cdot y_a}\,.$$

From $(\mathrm{V})_0$ and Lemmas 2.1, 2.2, it follows that $G_{\beta\alpha}(\lambda, \omega)$ is $L_s^2(\boldsymbol{R}_{y_b}^N)$-valued strongly continuous function of $(\lambda, \omega)\in(0, \infty)\times S^{N-1}$ if $1/2<s<\delta-(N/2)$.

**Proposition 2.3.** *Let $\alpha, \beta\in\Gamma_2$ and assume $(\mathrm{V})_0$. Then $Z_\beta S_{\alpha\to\beta}Z_\alpha^*$ is decomposable by a family of operators $\{S_{\alpha\to\beta}(\lambda)\}$, $\lambda>0$:*

$$(Z_\beta S_{\alpha\to\beta}Z_\alpha^* h)(\lambda)=S_{\alpha\to\beta}(\lambda)h(\lambda) \quad in\ L^2(S^{N-1}) \tag{2.5}$$

*for a. e. $\lambda>0$, where $h\in L^2((0, \infty); L^2(S^{N-1}))$, which is considered to be embedded in $L^2((\lambda_\alpha, \infty); L^2(S^{N-1}))$ by regarding $h(\lambda)=0$ for $\lambda\in(\lambda_\alpha, 0]$. Furthermore, $T_{\alpha\to\beta}(\lambda):=S_{\alpha\to\beta}(\lambda)-\delta_{\alpha\beta}\in\boldsymbol{B}(L^2(S^{N-1}))$ is continuous in $\lambda>0$ with respect to the Hilbert-Schmidt norm and its kernel $T_{\alpha\to\beta}(\lambda, \theta, \omega)$, $\theta, \omega\in S^{N-1}$, is given by*

$$T_{\alpha\to\beta}(\lambda, \theta, \omega)=2\pi iC_\alpha(\lambda)(Z_\beta(\lambda)G_{\beta\alpha}(\lambda, \omega))(\theta)\,. \tag{2.6}$$

*In particular, $T_{\alpha\to\beta}(\lambda, \cdot, \omega)$ is $L^2(S^{N-1})$-valued continuous function of $(\lambda, \omega)$ and the kernel $(\mathrm{Re}\, T_{\alpha\to\alpha}(\lambda))(\theta, \omega)$ of $(1/2)(T_{\alpha\to\alpha}(\lambda)+T_{\alpha\to\alpha}(\lambda)^*)$ is continuous in $(\lambda, \theta, \omega)$.*

*Proof.* The first half of this proposition and (2.6) can be proved in almost the same way as in the 2-body case. For the proof of the first half, see, for example, Proposition 2.4 of [I]. (2.6) yields

(2.6)′ $$(\mathrm{Re}\, T_{\alpha\to\alpha}(\lambda))(\theta, \omega)=\pi i C_\alpha(\lambda)^2((R(\lambda+i0)-R(\lambda-i0))I_a e_\alpha(\lambda, \omega), I_a e_\alpha(\lambda, \theta)).$$

We fix $s$ with $1/2<s<\delta-(N/2)$. Then, by $(\mathrm{V})_0$ and Lemma 2.2, $I_a e_\alpha(\lambda, \omega)$ is $L_s^2(\boldsymbol{R}^{2N})$-valued strongly continuous function of $(\lambda, \omega)$. Thus, the last half of the proposition follows from (2.6)′ and Lemma 2.1. ∎

We next consider the case $\beta=0$. To give the spectral representation of $H_0$, we define a unitary operator $U$ on $\mathcal{H}$ by

(2.7) $$(Uf)(x_a, y_a)=(2m_a)^{-N/4}(2n_a)^{-N/4}f((2m_a)^{-1/2}x_a, (2n_a)^{-1/2}y_a)$$

and define an operator $Z_0(\lambda)\in\boldsymbol{B}(L_s^2(\boldsymbol{R}^{2N}), L^2(S^{2N-1}))$, $\lambda>0$, $s>1/2$, by

(2.8) $$(Z_0(\lambda)f)(\theta)=C_0(\lambda)\int e^{-i\lambda^{1/2}\theta\cdot X}(Uf)(X)dX,$$

where $\theta\in S^{2N-1}$, $X=(x_a, y_a)$, $dX=dx_a dy_a$ and

$$C_0(\lambda)=2^{-N/2}(2\pi)^{-N}(2\lambda)^{(N-1)/2}.$$

Then the map $Z_0$, defined by

$$(Z_0 f)(\lambda, \theta)=(Z_0(\lambda)f)(\theta),$$

can be extended to a unitary operator from $\mathcal{H}$ to $L^2((0, \infty); L^2(S^{2N-1}))$ and give the spectral representation of $H_0$: For each $f\in D(H_0)$,

(2.9) $$(Z_0 H_0 f)(\lambda, \cdot)=\lambda(Z_0 f)(\lambda, \cdot) \quad \text{in } L^2(S^{2N-1})$$

for a. e. $\lambda>0$.

**Proposition 2.4.** *Assume* $(\mathrm{V})_0$, (Z) *and* $\alpha\in\Gamma_2$, $\beta=0$. *Then,* $Z_0 S_{\alpha\to0} Z_\alpha^*$ *is decomposable by a family of operators* $\{T_{\alpha\to0}(\lambda)\}$, $\lambda>0$. $T_{\alpha\to0}(\lambda)\in\boldsymbol{B}(L^2(S^{N-1}), L^2(S^{2N-1}))$ *is continuous in* $\lambda\gg1$ *with respect to the Hilbert-Schmidt norm. Let* $T_{\alpha\to0}(\lambda, \theta, \omega)$, $\theta\in S^{2N-1}$, $\omega\in S^{N-1}$, *be its integral kernel. Then,* $T_{\alpha\to0}(\lambda, \cdot, \omega)$ *is* $L^2(S^{2N-1})$-*valued continuous function of* $\lambda\gg1$ *and* $\omega\in S^{N-1}$.

Most of these results are obtained in [AS]. In [AS], it is shown that $T_{\alpha\to\beta}(\lambda)$ can be defined for all $\lambda>0$ but a closed null set, while the boundedness of this set is not proved in it. Thus, for completeness, we give a proof of Proposition 2.4 in the Appendix.

Owing to the following theorem, we need not study directly the asymptotic behavior of $\sigma_{\alpha\to0}(\lambda, \omega)$ as $\lambda\to\infty$.

**Theorem 2.5.** *Assume* $(\mathrm{V})_0$, (Z) *and* $\alpha\in\Gamma_2$. *Then, for each* $\lambda\gg1$ *and* $\omega\in S^{N-1}$, *the following relation holds:*

(2.11) $$\sigma_\alpha(\lambda, \omega)=-2(2\pi)^{N-1}(2n_a(\lambda-\lambda_\alpha))^{(1-N)/2}(\mathrm{Re}\,T_{\alpha\to\alpha}(\lambda))(\omega, \omega)$$

(*see* (1.15)).

*Proof.* Under assumption $(V)_0$, it is known that for $\alpha\neq\beta$ ($\alpha, \beta\in\Gamma$), Ran $W_\alpha^\pm$ is orthogonal to Ran $W_\beta^\pm$ (Ran=Range) and the asymptotic completeness holds (see (1.8), (1.10)). Moreover, if we assume (Z) as well, we have

$$\sum_{\alpha\in\Gamma}\oplus\,\mathrm{Ran}\,W_\alpha^\pm=\mathcal{H}_{ac}(H),$$

where $\mathcal{H}_{ac}(H)$ denotes the absolute continuity subspace of $\mathcal{H}$ with respect to $H$ (cf. [E]). This yields

(2.12) $$\sum_{\beta\in\Gamma}S_{\alpha\to\beta}{}^*S_{\alpha\to\beta}=\mathrm{Id}$$

and so

(2.13) $$\sum_{\beta\in\Gamma}T_{\alpha\to\beta}(\lambda)^*T_{\alpha\to\beta}(\lambda)=-T_{\alpha\to\alpha}(\lambda)-T_{\alpha\to\alpha}(\lambda)^*.$$

This equality holds for each $\lambda\gg1$ since all terms are continuous by Propositions 2.3 and 2.4. From (2.13) we have

(2.14) $$\sum_{\beta\in\Gamma}\int_{S^\beta}\overline{T_{\alpha\to\beta}(\lambda, \theta, \omega')}T_{\alpha\to\beta}(\lambda, \theta, \omega)d\theta=-2(\mathrm{Re}\,T_{\alpha\to\alpha}(\lambda))(\omega', \omega).$$

This equality holds for each $\omega, \omega'\in S^{N-1}$ and $\lambda\gg1$ since both sides are continuous by Propositions 2.3 and 2.4. Hence, putting $\omega'=\omega$, we get the desired result. ∎

*Remark.* This theorem is called the optical theorem ([AJS], p. 628). We can show that $\sigma_\alpha(\lambda, \omega)$ is well-defined for a.e. $(\lambda, \omega)\in(0, \infty)\times S^{N-1}$ and (2.11) holds for a.e. $(\lambda, \omega)\in(0, \infty)\times S^{N-1}$ under assumptions $(V)_0$ without (Z) (see Proposition 2.3 in [IT]).

## §3. Proof of Theorem 1.1

**3.1.** In this subsection we assume $(V)_\ell$ with $\ell\in\boldsymbol{N}\cup\{0\}$. Let $\alpha, \beta\in\Gamma_2$ with $D(\alpha)=a$, $D(\beta)=b$, $a\neq b$. We write $x=x_a$, $y=y_a$ for simplicity. For $v>0$ we define

(3.1) $$\lambda(v):=(1/2)n_av^2+\lambda_\alpha.$$

If $v$ is large, we can take $v'=v'(v)>0$ such that

(3.2) $$(1/2)n_bv'^2+\lambda_\beta=\lambda(v).$$

Throughout this section, we assume $v\gg1$. By (2.4) and (2.6) we have

(3.3) $$T_{\alpha\to\beta}(\lambda(v), \theta, \omega)=C(v)([-I_a+I_bR(\lambda+i0)I_a](\phi_\alpha e^{in_av\omega\cdot y}), \phi_\beta e^{in_bv'\theta\cdot y_b}),$$

where $\psi_a=\psi_a(x)$, $\psi_\beta=\psi_\beta(x_b)$ and

$$C(v):=i(2\pi)^{-N+1}(n_a n_b)^{(N-1)/2}(vv')^{(N-2)/2}. \tag{3.4}$$

Here notice that the R. H. S. of (3.3) is well-defined for all $(\lambda, \omega, \theta)\in(0, \infty)\times S^{N-1}\times S^{N-1}$ by $(V)_\ell$ and $a\neq b$.

For each $v\gg 1$ and $\omega\in S^{N-1}$, we define a self-adjoint operator $L(v, \omega)$ in $\mathcal{H}$ by

$$L(v, \omega):=v^{-1}(H-\lambda_a)-i\omega\cdot\nabla_y=e^{-in_a v\omega\cdot y}v^{-1}(H-\lambda(v))e^{in_a v\omega\cdot y}. \tag{3.5}$$

By (3.5) and Lemma 2.1, the norm limit

$$(L(v, \omega)-i0)^{-1}:=\lim_{\varepsilon\downarrow 0}(L(v, \omega)-i\varepsilon)^{-1}\in \boldsymbol{B}(L_s^2(\boldsymbol{R}^{2N}), L_{-s}^2(\boldsymbol{R}^{2N})) \tag{3.6}$$

exists for $s>1/2$. Then $T_{\alpha\to\beta}(\lambda(v), \theta, \omega)$ is written as

$$\begin{aligned} T_{\alpha\to\beta}&(\lambda(v), \theta, \omega)\\ &=C(v)([-I_a+v^{-1}I_b(L(v, \omega)-i0)^{-1}I_a]\psi_a, \psi_\beta e^{in_b v'\theta\cdot y_b-in_a v\omega\cdot y}). \end{aligned} \tag{3.7}$$

For $k$, $s\in\boldsymbol{R}$ we define

$$H_s^k(\boldsymbol{R}^{2N})=\{f\in\mathcal{S}'(\boldsymbol{R}^{2N})\,;\,\|f\|_{k,s}:=\|\langle x\,;\,y\rangle^s(-\Delta+1)^{k/2}f\|_{L^2(\boldsymbol{R}^{2N})}<\infty\},$$

where $\mathcal{S}'$ is the tempered distributions and $\Delta$ is the $2N$-dimensional Laplacian.

Instead of $(\mathrm{V})_\ell$, we assume the following condition $(\mathrm{U})_\ell$, $\ell\in\boldsymbol{N}\cup\{0\}$, to prove the next proposition:

$(\mathrm{U})_\ell$ Each $V_{ij}(x)$ $(1\leqq i<j\leqq 3)$ is a real-valued $C^{2\ell+2}$-function on $\boldsymbol{R}^N$ and satisfies

$$|D_x^\gamma V_{ij}(x)|\leqq C\langle x\rangle^{-\min(|\gamma|, \ell+2)}, \qquad |\gamma|\leqq 2\ell+2.$$

Since $N\geqq 3$, it is obvious that $(\mathrm{V})_\ell$ implies $(\mathrm{U})_\ell$ for $\ell\in\boldsymbol{N}\cup\{0\}$. The following proposition will be proved in Section 5.

**Proposition 3.1.** (i) *Assume* $(\mathrm{U})_\ell$ *and let* $k$ *be an integer with* $0\leqq k\leqq\ell$ *and* $s$ *a real with* $k+(1/2)<s$. *Then there exists a* $v_0>0$ *such that*

$$\sup_{\substack{0<\varepsilon<1, v\geqq v_0\\ \omega\in S^{N-1}}}\|(L(v, \omega)-i\varepsilon)^{-1}\|_{\boldsymbol{B}_{k,s}}<\infty \tag{3.8}$$

*and the norm limit*

$$(L(v, \omega)-i0)^{-1}:=\lim_{\varepsilon\downarrow 0}(L(v, \omega)-i\varepsilon)^{-1} \tag{3.9}$$

*exists in* $\boldsymbol{B}_{k,s}:=\boldsymbol{B}(H_s^k(\boldsymbol{R}^{2N}), H_{-s}^k(\boldsymbol{R}^{2N}))$ *uniformly in* $v\geqq v_0$ *and* $\omega\in S^{N-1}$. *In particular, the operator norm* $\|(L(v, \omega)-i0)^{-1}\|_{\boldsymbol{B}_{k,s}}$ *is uniformly bounded in* $v\geqq v_0$ *and* $\omega\in S^{N-1}$.

(ii) *Assume* $(U)_1$. *Then*

(3.10) $$\|(L(v,\omega)-i0)^{-1}-(-i\omega\cdot\nabla_y-i0)^{-1}\|_{B(H_2^4,L_{-2}^2)}=O(v^{-1})$$

*uniformly in* $\omega\in S^{N-1}$ *as* $v\to+\infty$ *where* $H_2^4=H_2^4(\boldsymbol{R}^{2N})$, $L_{-2}^2=L_{-2}^2(\boldsymbol{R}^{2N})$

*Remark.* The norm limit

(3.10)′ $$(-i\omega\cdot\nabla_y-i0)^{-1}:=\lim_{\varepsilon\downarrow 0}(-i\omega\cdot\nabla_y-i\varepsilon)^{-1}\in\boldsymbol{B}_{0,s},\qquad s>1/2,$$

exists uniformly in $\omega\in S^{N-1}$ and

(3.10)″ $$((-i\omega\cdot\nabla_y-i0)^{-1}u)(x,\eta+t\omega)=i\int_{-\infty}^{t}u(x,\eta+s\omega)ds,$$

where $\eta\in\Pi_\omega=\{\eta\in\boldsymbol{R}^N;\eta\cdot\omega=0\}$. Indeed, in one dimensional case, $\langle q\rangle^{-s}(-i(d/dq)-i\varepsilon)^{-1}\langle q\rangle^{-s}$, $\varepsilon>0$, $s>1/2$, is an integral operator with Hilbert-Schmidt kernel $K(\varepsilon;q,t)=i\langle q\rangle^{-s}\exp(\varepsilon(t-q))\chi(t,q)\langle t\rangle^{-s}$, where $\chi(t,q)=1$ (resp. $=0$) for $t\leqq q$ (resp. $q\leqq t$), and converges to a Hilbert-Schmidt operator with kernel $K(+0;q,t)=i\langle q\rangle^{-s}\chi(t,q)\langle t\rangle^{-s}$ w.r.t. the Hilbert-Schmidt norm as $\varepsilon\downarrow 0$.

*Proof of Theorem* 1.1 (i). Since $a\neq b$, $y$ is written as $y=mx_b+ny_b$ for some constants $m\neq 0$, $n\neq 0$. Thus, by (3.7), we have

(3.11) $$\begin{aligned}&T_{\alpha\to\beta}(\lambda(v),\theta,\omega)\\&=C(v)\int e^{(-in_bv'\theta+in_avn\omega)\cdot y_b}dy_b\int e^{in_avm\omega\cdot x_b}(f_1+v^{-1}f_2)dx_b,\end{aligned}$$

where

(3.12) $$\begin{aligned}&f_1=f_1(x_b,y_b)=-\bar{\phi}_\beta(x_b)I_a\phi_\alpha(x),\\&f_2=f_2(x_b,y_b)=\bar{\phi}_\beta(x_b)([I_b(L(v,\omega)-i0)^{-1}I_a]\phi_\alpha)(x_b,y_b).\end{aligned}$$

By $(\mathrm{V})_\ell$, Lemma 2.2 and Proposition 3.1, the following estimates can be verified:

(3.13) $$\begin{aligned}&\partial_{x_b}^{\gamma_1}\partial_{y_b}^{\gamma_2}f_1\in L^1(\boldsymbol{R}^{2N}),\qquad |\gamma_1|+|\gamma_2|\leqq\ell+1,\\&\sup_{v\gg1,\,\omega\in S^{N-1}}\|\partial_{x_b}^{\gamma_1}\partial_{y_b}^{\gamma_2}f_2\|_{L^1(R^{2N})}<\infty,\qquad |\gamma_1|+|\gamma_2|\leqq\ell,\\&(\sup_{v\gg1}\cdots:=\sup_{v\geq v_1}\cdots\qquad\text{for some large } v_1).\end{aligned}$$

Hence, by integration by part in $x_b$, we can write

(3.14) $$T_{\alpha\to\beta}(\lambda(v),\theta,\omega)=C(v)v^{-\ell-1}(F_{y_b}g)(n_bv'\theta-n_avn\omega),$$

where $(F_{y_b}g)(\xi)$ is the Fourier transform of

(3.15) $$g(y_b)=\int e^{in_avm\omega\cdot x_b}h(v,\omega;x_b,y_b)dx_b,$$

where

$$h=(2\pi)^{N/2}\{(i/n_am)^{\ell+1}(\omega\cdot\nabla_{x_b})^{\ell+1}f_1+(i/n_am)^\ell(\omega\cdot\nabla_{x_b})^\ell f_2\}.$$

In the same way as (3.13), we have

(3.16) $$\sup_{v\gg 1,\, \omega\in S^{N-1}} \|\langle x_b\rangle^N\langle y_b\rangle h(v,\, \boldsymbol{\omega}\,;\, \cdot,\, \cdot)\|_{L^2(\boldsymbol{R}^{2N})}<\infty\, .$$

Thus, by (1.14), (3.14), we have

(3.17) $$\sigma_{\alpha\to\beta}(\lambda(v),\, \omega)=(2\pi)^{N-1}(n_a n_b v v')^{1-N}C(v)^2 v^{-2\ell-2} \times\int_{|\xi|=n_b v'}|(F_{y_b}g)(\xi-n_a v n\boldsymbol{\omega})|^2 dS_\xi\, ,$$

where $dS_\xi$ is the Lebesgue measure on the sphere $\{\xi\in\boldsymbol{R}^N\,;\, |\xi|=n_b v'\}$. On the other hand, by the trace theorem ([GM], Proposition 2.1) we get

(3.18) $$\begin{aligned}\int_{|\xi|=n_b v'}|(F_{y_b}g)(\xi-n_a v n\boldsymbol{\omega})|^2 dS_\xi &\leqq C\|(F_{y_b}g)(\cdot-n_a v n\boldsymbol{\omega})\|^2_{H^1(\boldsymbol{R}^N)}\\ &\leqq C\|g\|^2_{L^2_1(\boldsymbol{R}^N_{y_b})}\\ &\leqq C\|\langle x_b\rangle^N\langle y_b\rangle h(v,\, \boldsymbol{\omega}\,;\, \cdot,\, \cdot)\|^2_{L^2(\boldsymbol{R}^{2N})},\end{aligned}$$

where we have used the Schwarz inequality in the last step. Therefore, by (3.4), (3.16) and (3.17), we obtain (1.18). ■

**3.2.** In this subsection we assume $(\mathrm{V})_0$. We begin with the following lemma.

**Lemma 3.2.** *Let $\alpha,\, \beta\in\Gamma_2$ with $D(\alpha)=a$, $D(\beta)=b$. Then, for each $\lambda>0$ and each $\omega\in S^{N-1}$, $\sigma_{\alpha\to\beta}(\lambda,\, \omega)$ can be represented as*

(3.19) $$\sigma_{\alpha\to\beta}(\lambda,\, \omega)=2(2\pi)^N C_\alpha(\lambda)^2(2n_a(\lambda-\lambda_\alpha))^{(1-N)/2}I(\lambda,\, \omega)\, ,$$

*where*

$$I(\lambda,\, \omega):=\mathrm{Im}\,(E_\beta R(\lambda+i0)I_a e_\alpha,\, I_a e_\alpha)+\mathrm{Im}\,(E_\beta I_b R(\lambda+i0)I_a e_\alpha,\, R(\lambda+i0)I_a e_\alpha)\, ,$$

*$e_\alpha:=\phi_\alpha(x)e^{i(2n_a(\lambda-\lambda_\alpha))^{1/2}\omega\cdot y}$, and $E_\beta=J_\beta J_\beta^*$ (see (1.7)).*

*Remark.* $I(\lambda,\, \omega)$ is well-defined by Lemma 2.1 and the following:

(3.19)′ $$\begin{gathered}\langle x\,;\, y\rangle^s I_a e_\alpha\in L^2(\boldsymbol{R}^{2N})\, ,\\ \langle x\,;\, y\rangle^{-s}E_\beta\langle x\,;\, y\rangle^s\in\boldsymbol{B}(L^2(\boldsymbol{R}^{2N}))\, ,\\ \langle x\,;\, y\rangle^s E_\beta I_b\langle x\,;\, y\rangle^s\in\boldsymbol{B}(L^2(\boldsymbol{R}^{2N}))\, ,\end{gathered}$$

for some $s>1/2$, which follow from $(\mathrm{V})_0$ and Lemma 2.2.

*Proof.* We first note that

(3.20) $$T_{\alpha\to\beta}(\lambda,\, \theta,\, \omega)=2\pi i C_\alpha(\lambda)(Z_\beta(\lambda)J_\beta{}^*K(\lambda)e_\alpha)(\theta)\, ,$$

where $K(\lambda):=-I_a+I_b R(\lambda+i0)I_a$ (see (2.4), (2.6)). Thus we have

$$\sigma_{\alpha\to\beta}(\lambda,\ \omega)=(2\pi)^{N+1}C_\alpha(\lambda)^2(2n_\alpha(\lambda-\lambda_\alpha))^{(1-N)/2}$$
$$\times(Z_\beta(\lambda)^*Z_\beta(\lambda)J_\beta^*K(\lambda)e_\alpha,\ J_\beta^*K(\lambda)e_\alpha).$$

Here we regard $Z_\beta(\lambda)^*\in\boldsymbol{B}(L^2(S^{N-1}),\ L_{-s}^2(\boldsymbol{R}_{y_b}^N))$, $s>1/2$, by regarding $L_{-s}^2$ as $(L_s^2)^*$. Therefore we get

$$\sigma_{\alpha\to\beta}(\lambda,\ \omega)=-i(2\pi)^N C_\alpha(\lambda)^2(2n_\alpha(\lambda-\lambda_\alpha))^{(1-N)/2} \tag{3.21}$$
$$\times([R_b(\lambda+i0)-R_b(\lambda-i0)]E_\beta K(\lambda)e_\alpha,\ K(\lambda)e_\alpha),$$

where $R_b(\lambda\pm i0)=(H_b-(\lambda\pm i0))^{-1}:=\lim_{\varepsilon\downarrow 0}(H_b-(\lambda\pm i\varepsilon))^{-1}$ (see (1.5), Lemma 2.1) is a bounded operator from $L_s^2(\boldsymbol{R}^{2N})$ to $L_{-s}^2(\boldsymbol{R}^{2N})$, $s>1/2$, and we have used the following two relations in the last step:

$$Z_\beta(\lambda)^*Z_\beta(\lambda)=(2\pi i)^{-1}\{(H_\beta-(\lambda+i0))^{-1}-(H_\beta-(\lambda-i0))^{-1}\}, \tag{3.22}$$
$$J_\beta(H_\beta-(\lambda\pm i0))^{-1}=R_b(\lambda\pm i0)J_\beta.$$

Furthermore, the resolvent equation $R_b(z)I_bR(z)=R_b(z)-R(z)$ yields the following:

$$R_b(\lambda+i0)E_\beta K(\lambda)=-E_\beta R(\lambda+i0)I_a.$$

Thus, by this together with $R_b(\lambda+i0)E_\beta=E_\beta R_b(\lambda+i0)$, we can get the following relation:

$$\begin{aligned}K(\lambda)^*[R_b(\lambda+i0)-R_b(\lambda-i0)]E_\beta K(\lambda)&\\ =I_aE_\beta R(\lambda+i0)I_a-I_aR(\lambda-i0)E_\beta I_a&\\ -I_aR(\lambda-i0)I_bE_\beta R(\lambda+i0)I_a&\\ +I_aR(\lambda-i0)E_\beta I_bR(\lambda+i0)I_a,&\end{aligned} \tag{3.23}$$

where $K(\lambda)^*:=-I_a+I_aR(\lambda-i0)I_b$. This relation together with (3.21) implies the desired result. ■

*Proof of Theorem* 1.1 (ii). Let $a=b$. Then, by Lemma 3.2, we have

$$\sigma_{\alpha\to\beta}(\lambda(v),\ \omega)=(2/v^2)\,\mathrm{Im}\,(E_\beta(L(v,\ \omega)-i0)^{-1}I_a\phi_\alpha,\ I_a\phi_\alpha) \tag{3.24}$$
$$+(2/v^3)\,\mathrm{Im}\,(E_\beta I_b(L(v,\ \omega)-i0)^{-1}I_a\phi_\alpha,\ (L(v,\ \omega)-i0)^{-1}I_a\phi_\alpha),$$

where we have used $E_\beta\exp(in_av\omega\cdot y)=\exp(in_av\omega\cdot y)E_\beta$, which follows from $a=b$ and the definition of $E_\beta$. By Proposition 3.1 (i) with $k=0$ and (3.19)′ the second term is $O(v^{-3})$ uniformly in $\omega\in S^{N-1}$ as $v\to+\infty$. Next we will show that

$$\|((L(v,\ \omega)-i0)^{-1}-(-i\omega\cdot\nabla_y-i0)^{-1})I_a\phi_\alpha\|_{0,-s}=o(1) \tag{3.25}$$

uniformly for $\omega\in S^{N-1}$ as $v\to+\infty$ for $s>1/2$. By the resolvent equation we have

$$(L(v,\omega)-i\varepsilon)^{-1}I_a\psi_a-(-i\omega\cdot\nabla_y-i\varepsilon)^{-1}I_a\psi_a$$
$$=-v^{-1}(L(v,\omega)-i\varepsilon)^{-1}\cdot(H-\lambda_a)(H_0-i)^{-1}\cdot(-i\omega\cdot\nabla_y-i\varepsilon)^{-1}(H_0-i)I_a\psi_\alpha$$

for each $\varepsilon>0$, and this yields

$$\|((L(v,\omega)-i\varepsilon)^{-1}-(-i\omega\cdot\nabla_y-i\varepsilon)^{-1})I_a\psi_a\|\leqq C(\varepsilon)v^{-1} \tag{3.26}$$

for each $\varepsilon>0$. We write

$$\|((L(v,\omega)-i0)^{-1}-(-i\omega\cdot\nabla_y-i0)^{-1})I_a\psi_a\|_{0,-s}$$
$$\leqq\|((L(v,\omega)-i0)^{-1}-(L(v,\omega)-i\varepsilon)^{-1})I_a\psi_a\|_{0,-s}$$
$$+\|((-i\omega\cdot\nabla_y-i0)^{-1}-(-i\omega\cdot\nabla_y-i\varepsilon)^{-1})I_a\psi_a\|_{0,-s}$$
$$+\|((L(v,\omega)-i\varepsilon)^{-1}-(-i\omega\cdot\nabla_y-i\varepsilon)^{-1})I_a\psi_a\|\,.$$

Then (3.25) follows from (3.26), (3.10)′ and Proposition 3.1 (i). Thus by (3.19)′ and (3.25) the first term of the R. H. S. of (3.24) is

$$(2/v^2)\,\mathrm{Im}\,(E_\beta(-i\omega\cdot\nabla_y-i0)^{-1}I_a\psi_a,\ I_a\psi_\alpha)+o(v^{-2}) \tag{3.27}$$

uniformly in $\omega\in S^{N-1}$ as $v\to+\infty$. Hence,

$$\sigma_{a\to\beta}(\lambda(v),\omega)=(2/v^2)\,\mathrm{Im}\,(E_\beta(-i\omega\cdot\nabla_y-i0)^{-1}I_a\psi_a,\ I_a\psi_\alpha)+o(v^{-2}), \tag{3.28}$$

uniformly in $\omega\in S^{N-1}$ as $v\to+\infty$. Since

$$((-i\omega\cdot\nabla_y-i0)^{-1}I_a\psi_\alpha)(x,\ \eta+t\omega)=i\psi_a(x)F(x,\ \eta+t\omega), \tag{3.29}$$

where $\eta\in\Pi_\omega$, $t\in\boldsymbol{R}$ and

$$F(x,\ \eta+t\omega)=\int_{-\infty}^{t}I_a(x,\ \eta+s\omega)ds\,,$$

we have

$$2i\,\mathrm{Im}\,(E_\beta(-i\omega\cdot\nabla_y-i0)^{-1}I_a\psi_a,\ I_a\psi_\alpha)$$
$$=i\int_{\Pi_\omega}d\eta\int\psi_\alpha(x)\overline{\psi_\beta(x)}dx\int\overline{\psi_a(x')}\psi_\beta(x')dx'$$
$$\times\int_{-\infty}^{\infty}\frac{d}{dt}(F(x,\ \eta+t\omega)F(x',\ \eta+t\omega))dt$$
$$=i\int_{\Pi_\omega}|(W_a(\cdot\,;\omega,\ \eta)\psi_a,\ \psi_\beta)_a|^2d\eta\,.$$

This completes the proof. ■

### 3.3.

*Proof of Theorem* 1.1 (iii). We assume $(\mathrm{V})_0$ and (Z). Theorem 2.5 yields

$$(3.30)\qquad \sigma_\alpha(\lambda(v),\omega)=-2(2\pi)^{N-1}(n_a v)^{1-N}(\mathrm{Re}\,T_{\alpha\to\alpha}(\lambda(v)))(\omega,\omega)$$
$$=2v^{-2}\,\mathrm{Im}\,((L(v,\omega)-i0)^{-1}I_a\phi_a,\ I_a\phi_a).$$

Thus, by (3.25), we have

$$(3.31)\qquad \sigma_\alpha(\lambda(v),\omega)=2v^{-2}\,\mathrm{Im}\,((-i\omega\cdot\nabla_y-i0)^{-1}I_a\phi_\alpha,\ I_a\phi_\alpha)+o(v^{-2})$$
$$=v^{-2}\int_{\Pi_\omega}\|W_a(\cdot\,;\omega,\eta)\phi_\alpha\|_a^2\,d\eta+o(v^{-2})\qquad\text{(see (3.29))}$$

uniformly in $\omega\in S^{N-1}$ as $v\to+\infty$. This proves (1.21). Under assumptions $(V)_0$ and (Z), the set $\{\lambda_\alpha;\ \alpha\in\Gamma_2,\ D(\alpha)=a\}$ coincides with the set of all eigenvalues of $h_a$. Thus, by (1.15), (3.31), (1.18), (1.19), we obtain

$$\sigma_{\alpha\to0}(\lambda(v),\omega)=v^{-2}\int_{\Pi_\omega}\|P^c(h_a)W_a(\cdot\,;\omega,\eta)\phi_\alpha\|_a^2\,d\eta+o(v^{-2})$$

uniformly in $\omega\in S^{N-1}$ as $v\to\infty$. This completes the proof. ■

**3.4.**

*Proof of Theorem* 1.2. We assume $(V)_{3/2}$. Then, $I_a\phi_\alpha\in H^2_{\frac{3}{2}}(\mathbf{R}^{2N})$ and $(U)_1$ is satisfied. Thus, by replacing (3.25) by (3.10) in the above proofs, we get the desired results. ■

## § 4. Abstract Theory for Resolvent Estimates

In this section we give an abstract theorem for the proof of Proposition 3.1. This theorem is a slight extension of Theorem 2.2 of [JMP] (see also [J], [I]). Throughout this section we work on an abstract Hilbert space $\boldsymbol{H}$ and denote by $\|\ \|$ the operator norm of bounded operators on $\boldsymbol{H}$.

**Definition 4.1.** (I) Let $A$ be a self-adjoint operator in $\boldsymbol{H}$ and $d\in\boldsymbol{N}$. We denote by $S_d(A)$ the set of all self-adjoint operators $K$ in $\boldsymbol{H}$ satisfying the following properties (A-i)~(A-iv).

(A-i) $D(K)\cap D(A)$ is a core for $K$.

(A-ii) $\exp(itA)$ leaves $D(K)$ invariant, and for each $f\in D(K)$

$$\sup_{|t|\le1}\|K\exp(itA)f\|<\infty\,.$$

(A-iii) Let $K^{(0)}=K$. There are self-adjoint operators $iK^{(1)},\cdots,i^dK^{(d)}$ satisfying the following:

$$D(i^jK^{(j)})\supset D(K)\qquad(j=1,\cdots,d)\,,$$

the form $i[i^{j-1}K^{(j-1)},A]$ defined on $D(K)\cap D(A)$ is bounded from below and closable, and the self-adjoint operator associated with its closure is $i^jK^{(j)}$ ($j=1,\cdots,d$). Here, $[\ ,\ ]$ means the commutator: $([B,C]f,g)=(Cf,B^*g)-$

$(Bf, C^*g)$.

(A-iv) The form $[K^{(d)}, A]$, defined on $D(K)\cap D(A)$, extends to a bounded operator from $\boldsymbol{H}_{+2}$ to $\boldsymbol{H}_{-2}$, which is denoted by $[K^{(d)}, A]_0$, where $\boldsymbol{H}_{+2}$ is the domain $D(K)$ with the graph norm $\|f\|_{+2}:=\|(K+i)f\|$ and $\boldsymbol{H}_{-2}$ is the dual of $\boldsymbol{H}_{+2}$ obtained via the inner product in $\boldsymbol{H}$.

(II) Let $A$ be a self-adjoint operator in $\boldsymbol{H}$ and $d\in N$. We denote by $B_d(A)$ the set of all bounded operators $W$ on $\boldsymbol{H}$ satisfying the following property (A-v).

(A-v) Let $W^{(0)}=W$. There are bounded operators $W^{(1)}, \cdots, W^{(d)}$ on $\boldsymbol{H}$ satisfying the following properties:

The form $[W^{(j-1)}, A]$, defined on $D(A)$, extends to the bounded operator $W^{(j)}$ $(j=1, \cdots, d)$.

For $K\in S_d(A)$ and $W\in B_d(A)$, we set

$$|||K|||_{S_d(A)}:=\sum_{j=1}^{d}\|K^{(j)}(K+i)^{-1}\|+\|(K+i)^{-1}[K, A]_0(K+i)^{-1}\|\,, \tag{4.1}$$

$$|||W|||_{B_d(A)}:=\sum_{j=1}^{d}\|W^{(j)}\|\,. \tag{4.2}$$

To state our main results in this section, we prepare some notations. Let $I$ be a compact interval in $\boldsymbol{R}$ and $I_\pm:=\{z\in\boldsymbol{C}\,;\,\mathrm{Re}\,z\in I,\ 0<\pm\mathrm{Im}\,z<1\}$. We fix a smooth function $\chi(t)$ on $\boldsymbol{R}$ such that $0\leqq\chi\leqq 1$, $\chi(t)=1$ on $I$ and $\mathrm{supp}\,\chi$ (supp= support) is contained in a small neighborhood of $I$.

**Theorem 4.2.** *Let $I$, $\chi$ be as above, and $A$ a self-adjoint operator in $\boldsymbol{H}$, $d\in\boldsymbol{N}$, and $K_1, \cdots, K_d\in S_d(A)$. Furthermore, if $d\geqq 2$, let $W_1, \cdots, W_{d-1}\in B_d(A)$. Assume $K_j$ satisfies:*

$$\chi(K_j)iK_j^{(1)}\chi(K_j)\geqq C_0\chi(K_j)^2\,, \qquad j=1, \cdots, d\,, \tag{4.3}$$

*for some $C_0>0$. We define $D(z)$, $z\in\boldsymbol{C}\setminus\boldsymbol{R}$, by*

$$D(z):=\langle A\rangle^{-s}(K_1-z)^{-1}\langle A\rangle^{-s}$$

*for $d=1$,*

$$D(z):=\langle A\rangle^{-s}(K_1-z)^{-1}W_1(K_2-z)^{-1}\cdots W_{d-1}(K_d-z)^{-1}\langle A\rangle^{-s}$$

*for $d\geqq 2$, where $s$ is a real with $s>d-(1/2)$ and $\langle A\rangle:=(1+|A|^2)^{1/2}$. Then, the following* (i), (ii) *and* (iii) *hold.*

(i)

$$\sup_{z\in I_\pm}\|D(z)\|\leqq C<\infty \tag{4.4}$$

(ii)

$$\|D(z)-D(z')\|\leqq C\,|z-z'|^{\delta_0}\,, \tag{4.5}$$

*for* $z, z'\in I_{\pm}$, *where* $\delta_0=(1+(sd/(s-d+1/2))^{-1}$.

(iii) *The norm limits* $D(\lambda\pm i0):=\lim_{\varepsilon\downarrow 0} D(\lambda\pm i\varepsilon)$ *exist in* $\boldsymbol{B}(\boldsymbol{H})$ *uniformly in* $\lambda\in I$.

*Moreover, if* $A, K_1, \cdots, K_d$ (*and* $W_1, \cdots, W_{d-1}$ *if* $d\geqq 2$) *depend on a parameter* $\nu$ *such that* $I$, $\chi$ *and* $C_0$ *can be taken independently of* $\nu$ *and that* $|||K_j|||_{S_d(A)}$ $(j=1, \cdots, d)$ (*and* $|||W_j|||_{B_d(A)}$ $(j=1, \cdots, d-1)$ *if* $d\geqq 2$) *remain bounded in* $\nu$, *then* $C$ *can be taken independently of* $\nu$.

For $0<|\varepsilon|\ll 1$ the operator

$$Q_j(\varepsilon):=\sum_{m=1}^{d}\frac{\varepsilon^m}{m!}K_j^{(m)}, \qquad j=1, \cdots, d, \tag{4.6}$$

is $K_j$-bounded with $K_j$-bound$<1$ by (A-iii). Thus $K_j+Q_j(\varepsilon)$ is a closed operator with $D(K_j+Q_j(\varepsilon))=D(K_j)$.

**Lemma 4.3** ([JMP], Lemma 3.1). *There exists a* $\varepsilon_1>0$ *such that the following properties hold for* $0<\pm\varepsilon<\varepsilon_1$, $z\in I_{\pm}$, $j=1, \cdots, d$:

(i) $K_j+Q_j(\varepsilon)-z$ *has a bounded inverse* $G_{j,z}(\varepsilon)\in\boldsymbol{B}(\boldsymbol{H})$

(ii) $G_{j,z}(\varepsilon)$ *satisfies the following estimates*:

$$\|G_{j,z}(\varepsilon)\|\leqq C|\varepsilon|^{-1}, \tag{4.7}$$

$$\|(K_j+i)G_{j,z}(\varepsilon)\|+\|G_{j,z}(\varepsilon)(K_j+i)\|\leqq C|\varepsilon|^{-1}, \tag{4.8}$$

$$\|(K_j+i)G_{j,z}(\varepsilon)\langle A\rangle^{-1}\|+\|\langle A\rangle^{-1}G_{j,z}(\varepsilon)(K_j+i)\|\leqq C|\varepsilon|^{-1/2}, \tag{4.9}$$

*where* $C$ *is independent of* $\pm\varepsilon\in(0, \varepsilon_1)$, $z\in I_{\pm}$, $j=1, \cdots, d$.

(iii) *The form* $[A, G_{j,z}(\varepsilon)]$, *defined on* $D(A)$, *extends to a bounded operator* $[A, G_{j,z}(\varepsilon)]_0$ *on* $\boldsymbol{H}$. $G_{j,z}(\varepsilon)$ *maps* $D(A)$ *into* $D(A)\cap D(K_j)$.

(iv) *For each* $z\in I_+$ (*resp.* $I_-$), $G_{j,z}(\varepsilon)\in C^1((0, \varepsilon_1); \boldsymbol{B}(\boldsymbol{H}))$ (*resp.* $C^1((-\varepsilon_1, 0); \boldsymbol{B}(\boldsymbol{H}))$ *and*

$$\frac{d}{d\varepsilon}G_{j,z}(\varepsilon)=[G_{j,z}(\varepsilon), A]_0+\frac{\varepsilon^d}{d!}G_{j,z}(\varepsilon)[K_j^{(d)}, A]_0 G_{j,z}(\varepsilon). \tag{4.10}$$

*Moreover, if* $A, K_1, \cdots, K_d$ *depends on a parameter* $\nu$ *such that* $I$, $\chi$ *and* $C_0$ *can be taken independently of* $\nu$ *and that* $|||K_j|||_{S_d(A)}$ $(j=1, \cdots, d)$ *remain bounded in* $\nu$, *then* $\varepsilon_1$, $C$ *can be taken independently of* $\nu$.

For properties (i)$\sim$(iv) of the lemma it suffices to prove them for each $j$. For the proof, see [JMP]. The last part can be shown by carefully checking the estimates carried out in [JMP] (see also [M], [PSS]).

**Lemma 4.4** ([I], Lemma 3.5). *Let* $f_k(\varepsilon)=|\log\varepsilon|$ *for* $k=0$ *and* $f_k(\varepsilon)=\varepsilon^{-k}$ *for* $k\in\boldsymbol{N}$. *Assume that a* $\boldsymbol{B}(\boldsymbol{H})$*-valued* $C^1$*-function* $X(\varepsilon)$, $0<\varepsilon<\varepsilon_1$ $(\varepsilon_1>0)$, *satisfies*:

(4.11) $$\|(d/d\varepsilon)X(\varepsilon)\| \leqq C_1(\|X(\varepsilon)\|^p\cdot\varepsilon^{-q}+f_k(\varepsilon)+1),$$

(4.12) $$\|X(\varepsilon)\| \leqq C_2\varepsilon^{-r},$$

*where* $p$, $q$, $r$, $C_1$, $C_2$ *are constants satisfying* $0\leqq p<1$, $0\leqq q<1$, $r\geqq 0$, $C_1$, $C_2>0$. *Then* $X(\varepsilon)$ *satisfies the following estimates*:

(4.13) $$\|X(\varepsilon)\|=C\cdot\varepsilon^{-k+1} \qquad when\ k\geqq 2,$$

(4.14) $$\|X(\varepsilon)\|\leqq C|\log\varepsilon| \qquad when\ k=1,$$

(4.15) $$\|X(\varepsilon)\|\leqq C \qquad when\ k=0,$$

*where* $C=C(C_1, C_2, \varepsilon_1, p, q, r)>0$. *Furthermore, when* $k=0$, *the norm limit* $X(0):=\lim_{\varepsilon\downarrow 0}X(\varepsilon)$ *exists in* $\boldsymbol{B}(\boldsymbol{H})$.

For the proof, see [I].

For the proof of Theorem 4.2 for the case $d=1$, see [M], [PSS]. We can also prove this theorem for $d\geqq 2$ in the same way as in Theorem 3.3 in [I]. The proof is a slight modification of the proof of Theorem 2.2 in [JMP]. But, for the sake of completeness, we give the proof of Theorem 4.2 for $d\geqq 2$.

*Proof of Theorem* 4.2. We give only the proof for the case $z\in I_+$.

(i) For multi-indices of nonnegative integers $\alpha=(\alpha_1, \cdots, \alpha_{d-1})$, $\beta=(\beta_1, \cdots, \beta_{d-1})$ we write $|\alpha|=\alpha_1+\cdots+\alpha_{d-1}$, and $\alpha\leqq\beta$ if and only if $\alpha_j\leqq\beta_j$ for all $j$. Let $M_\alpha$ be a family of all multi-indices $\beta$ with $\alpha\leqq\beta$, $|\beta|=|\alpha|+1$. Namely $\beta\in M_\alpha$ implies that $\alpha_j=\beta_j-1$ for some $j$ and $\beta_i=\alpha_i$ for $i\neq j$. We set

$$F_z^\alpha(\varepsilon):=\langle A\rangle^{-s}G_{1,z}(\varepsilon)W_1^{(\alpha_1)}G_{2,z}(\varepsilon)\cdots W_{d-1}^{(\alpha_{d-1})}G_{d,z}(\varepsilon)\langle A\rangle^{-s}$$

for $z\in I_+$, $\varepsilon>0$, $\alpha=(\alpha_1, \cdots, \alpha_{d-1})$ with $|\alpha|\leqq d$.

By Lemma 4.3 (iv), we have for $|\alpha|\leqq d-1$.

(4.16)
$$\begin{aligned}\frac{d}{d\varepsilon}F_z^\alpha(\varepsilon)&=\langle A\rangle^{-s}\Big(\frac{d}{d\varepsilon}G_{1,z}(\varepsilon)\Big)W_1^{(\alpha_1)}G_{2,z}(\varepsilon)\cdots W_{d-1}^{(\alpha_{d-1})}G_{d,z}(\varepsilon)\langle A\rangle^{-s}\\&\quad+\cdots+\langle A\rangle^{-s}G_{1,z}(\varepsilon)W_1^{(\alpha_1)}G_{2,z}(\varepsilon)\cdots W_{d-1}^{(\alpha_{d-1})}\Big(\frac{d}{d\varepsilon}G_{d,z}(\varepsilon)\Big)\langle A\rangle^{-s}\\&=\langle A\rangle^{-s}\{[G_{1,z}(\varepsilon), A]_0W_1^{(\alpha_1)}G_{2,z}(\varepsilon)\cdots W_{d-1}^{(\alpha_{d-1})}G_{d,z}(\varepsilon)\\&\quad+\cdots+G_{1,z}(\varepsilon)W_1^{(\alpha_1)}G_{2,z}(\varepsilon)\cdots W_{d-1}^{(\alpha_{d-1})}[G_{d,z}(\varepsilon), A]_0\}\langle A\rangle^{-s}\\&\quad+\frac{\varepsilon^d}{d!}\langle A\rangle^{-s}\{G_{1,z}(\varepsilon)[K_1^{(d)}, A]_0G_{1,z}(\varepsilon)\cdots W_{d-1}^{(\alpha_{d-1})}G_{d,z}(\varepsilon)\\&\quad+\cdots+G_{1,z}(\varepsilon)W_1^{(\alpha_1)}\cdots W_{d-1}^{(\alpha_{d-1})}G_{d,z}(\varepsilon)[K_d^{(d)}, A]_0G_{d,z}(\varepsilon)\}\langle A\rangle^{-s}\\&=I_1(\varepsilon)+I_2(\varepsilon).\end{aligned}$$

First we estimate $I_2(\varepsilon)$. Since $s>1$ and $W_j^{(\alpha_j)}$ $(j=1, \cdots, d-1)$, $(K_n+i)^{-1}[K_n^{(d)}, A]_0(K_n+i)^{-1}$ $(n=1, \cdots, d)$ are bounded by (A-v), (A-iv), we have

$$\|I_2(\varepsilon)\| \leqq C \cdot \varepsilon^d \cdot \varepsilon^{-1/2} \cdot \varepsilon^{-d+1} \cdot \varepsilon^{-1/2} \leqq C \tag{4.17}$$

by Lemma 4.3 (ii).

Next we estimate $I_1(\varepsilon)$. Noting that $G_{n,z}(\varepsilon)$ maps $D(A)$ into $D(A)$ and $W_j^{(\alpha_j)}$ maps $D(A)$ into $D(A)$, as follows from (A-v) and Lemma 4.3 (iii), we have, by elementary computation,

$$I_1(\varepsilon)=[F_z^{\alpha}(\varepsilon), A]-\sum_{\beta\in M_\alpha} F_z^{\beta}(\varepsilon)\,. \tag{4.18}$$

Since $\|\langle A\rangle^s F_z^{\alpha}(\varepsilon)\|$, $\|F_z^{\alpha}(\varepsilon)\langle A\rangle^s\| \leqq C \cdot \varepsilon^{-d+(1/2)}$ by Lemma 4.3 (ii) and

$$\|F_z^{\alpha}(\varepsilon)\langle A\rangle\| \leqq \|F_z^{\alpha}(\varepsilon)\|^{1-(1/s)} \|F_z^{\alpha}(\varepsilon)\langle A\rangle^s\|^{1/s},$$

$$\|\langle A\rangle F_z^{\alpha}(\varepsilon)\| \leqq \|F_z^{\alpha}(\varepsilon)\|^{1-(1/s)} \|\langle A\rangle^s F_z^{\alpha}(\varepsilon)\|^{1/s}$$

by interpolation, we have

$$\begin{aligned}\|[F_z^{\alpha}(\varepsilon), A]\| &\leqq \|F_z^{\alpha}(\varepsilon)\langle A\rangle\| + \|\langle A\rangle F_z^{\alpha}(\varepsilon)\| \\ &\leqq C \cdot \|F_z^{\alpha}(\varepsilon)\|^{1-(1/s)} \varepsilon^{(-d+(1/2))/s}\,.\end{aligned}$$

Thus we get

$$\|I_1(\varepsilon)\| \leqq C(\|F_z^{\alpha}(\varepsilon)\|^{1-(1/s)} \varepsilon^{(-d+(1/2))/s} + \sum_{\beta\in M_\alpha} \|F_z^{\beta}(\varepsilon)\|)\,. \tag{4.19}$$

Therefore $F_z^{\alpha}(\varepsilon)$ satisfies by (4.16), (4.17) and (4.19)

$$\left\|\frac{d}{d\varepsilon} F_z^{\alpha}(\varepsilon)\right\| \leqq C(\|F_z^{\alpha}(\varepsilon)\|^{1-(1/s)} \varepsilon^{-m} + \sum_{\beta\in M_\alpha} \|F_z^{\beta}(\varepsilon)\| + 1) \tag{4.20}$$

for all multi-indices $\alpha$ with $|\alpha| \leqq d-1$ where $m=(d-(1/2))/s$.

Furthermore, it follows from Lemma 4.3 (ii) that

$$\|F_z^{\gamma}(\varepsilon)\| \leqq C \varepsilon^{-d+1} \tag{4.21}$$

for all multi-indices $\gamma$ with $|\gamma| \leqq d$.

Let $|\alpha|=d-1$. Then we have by (4.20) and (4.21)

$$\left\|\frac{d}{d\varepsilon} F_z^{\alpha}(\varepsilon)\right\| \leqq C(\|F_z^{\alpha}(\varepsilon)\|^{1-(1/s)} \varepsilon^{-m} + \varepsilon^{-d+1} + 1)\,.$$

Applying Lemma 4.4 ($p=1-(1/s)$, $q=m$; $p$, $q\in[0, 1)$ by $s>d-(1/2)$), we have

$$\|F_z^{\alpha}(\varepsilon)\| \leqq C \varepsilon^{-d+2}\,. \tag{4.22}$$

Next let $|\alpha|=d-2$. Then $|\beta|=d-1$ for $\beta\in M_\alpha$. Thus we obtain by (4.20)

$$\left\|\frac{d}{d\varepsilon} F_z^{\alpha}(\varepsilon)\right\| \leqq C(\|F_z^{\alpha}(\varepsilon)\|^{1-(1/s)} \varepsilon^{-m} + \varepsilon^{-d+2} + 1)\,.$$

Applying Lemma 4.4, we have

$$\|F_z^{\alpha}(\varepsilon)\| \leqq C\varepsilon^{-d+3}.$$

Continuing, we have for $|\alpha|=0$

$$\left\|\frac{d}{d\varepsilon}F_z^{\alpha}(\varepsilon)\right\| \leqq C(\|F_z^{\alpha}(\varepsilon)\|^{1-(1/8)}\varepsilon^{-m}+|\log\varepsilon|+1). \tag{4.23}$$

Thus we have the following estimate, by Lemma 4.4,

$$\sup_{z\in I_{+},\, 0<\varepsilon<1}\|F_z(\varepsilon)\| \leqq C<\infty, \tag{4.24}$$

where $F_z(\varepsilon):=F_z^{\alpha}(\varepsilon)$ for $|\alpha|=0$.

We set $R_n(z)=(K_n-z)^{-1}$. Since $\lim_{\varepsilon\downarrow 0}\|Q_n(\varepsilon)R_n(z)\|=0$ for each $z\in \boldsymbol{C}\setminus\boldsymbol{R}$ by (A-iii), $1+Q_n(\varepsilon)R_n(z)$ has a bounded inverse, and so

$$G_{n,z}(\varepsilon)=R_n(z)(1+Q_n(\varepsilon)R_n(z))^{-1}$$

holds for each $z\in\boldsymbol{C}\setminus\boldsymbol{R}$ when $\varepsilon>0$ is small. Therefore we get

$$\lim_{\varepsilon\downarrow 0} G_{n,z}(\varepsilon)=R_n(z) \qquad (n=1,\cdots,d)$$

in the norm of $\boldsymbol{B}(\boldsymbol{H})$ for each $z\in\boldsymbol{C}\setminus\boldsymbol{R}$, and so we have by (4.24)

$$\sup_{z\in I_+}\|D(z)\| \leqq C.$$

(ii) By (4.23), (4.24) we obtain

$$\left\|\frac{d}{d\varepsilon}F_z(\varepsilon)\right\| \leqq C(\varepsilon^{-m}+1).$$

Integrating this, we have, by noting $0<m<1$,

$$\|F_z(\varepsilon)-F_z(0)\| \leqq C\cdot\varepsilon^{1-m}. \tag{4.25}$$

On the other hand $G_{n,z}(\varepsilon)$ is differentiable in $z\in I_+$ for each $\varepsilon>0$ by Lemma 4.3. We have the following estimate by Lemma 4.3 (ii):

$$\begin{aligned}\left\|\frac{d}{dz}F_z(\varepsilon)\right\| &\leqq \|\langle A\rangle^{-s}G_{1,z}(\varepsilon)^2W_1\cdots G_{d,z}(\varepsilon)\langle A\rangle^{-s}\|\\ &\quad+\cdots+\|\langle A\rangle^{-s}G_{1,z}(\varepsilon)W_1\cdots W_{d-1}G_{d,z}(\varepsilon)^2\langle A\rangle^{-s}\|\\ &\leqq C\cdot\varepsilon^{-d},\end{aligned}$$

which implies

$$\|F_z(\varepsilon)-F_{z'}(\varepsilon)\| \leqq C\cdot\varepsilon^{-d}|z-z'| \tag{4.26}$$

for $z, z'\in I_+$, $\varepsilon>0$. Let $\varepsilon=|z-z'|^{\delta_1}$, $\delta_1=(1-m)^{-1}\delta_0$ (see (4.5) for $\delta_0$). Then by (4.25), (4.26) we have

$$\|F_z(0)-F_{z'}(0)\| \leqq \|F_z(0)-F_z(\varepsilon)\| + \|F_z(\varepsilon)-F_{z'}(\varepsilon)\| + \|F_{z'}(\varepsilon)-F_{z'}(0)\|$$
$$\leqq C \cdot |z-z'|^{\delta_0}.$$

Thus we have proved (ii). (iii) follows from (ii).

The uniformity of the choice of $C$ can be obtained if one takes into consideration the last part of Lemma 4.3 and the proof carried out above. ∎

## § 5. Proof of Proposition 3.1

In this section we will prove Proposition 3.1 by applying Theorem 4.2. Throughout this section we assume $(\mathrm{U})_\ell$ and fix an integer $k$ with $0 \leqq k \leqq \ell$. Furthermore we define a set $\Omega := \{(v, \omega); v > v_0, \omega \in S^{N-1}\}$ for $v_0 > 0$. Let $A_0$ be the generator of dilations on $\boldsymbol{R}^{2N}$:

(5.1) $$A_0 = (1/2i)(x \cdot \nabla_x + \nabla_x \cdot x + y \cdot \nabla_y + \nabla_y \cdot y),$$

which is self-adjoint in $\mathcal{H}$ with a core $\mathcal{S} = \mathcal{S}(\boldsymbol{R}^{2N})$, the Schwartz space of rapidly decreasing functions. Thus, the operator $A(\tau)$, $\tau = (v, \omega) \in \Omega$, defined by

(5.2) $$A(\tau) := (n_a v)^{-1} A_0 + \omega \cdot y$$
$$= (n_a v)^{-1} \exp(-i n_a v \omega \cdot y) A_0 \exp(i n_a v \omega \cdot y),$$

is self-adjoint in $\mathcal{H}$ with a core $\mathcal{S}$. For notational brevity we write $L(\tau) = L(v, \omega)$ for $\tau = (v, \omega) \in \Omega$. Then, a simple calculation yields

(5.3) $$i[L(\tau), A(\tau)] = 2(n_a v)^{-1} L(\tau) + (n_a v^2)^{-1} (i[V, A_0] - 2V + 2\lambda_a) + 1$$

on $\mathcal{S}$, where $i[V, A_0]$ is an operator of multiplication:

(5.4) $$i[V, A_0] = -\sum_{1 \leqq i < j \leqq 3} V_{ij}^{(1)}(r_i - r_j),$$

where

(5.5) $$V_{ij}^{(n)}(x) = (x \cdot \nabla_x)^n V_{ij}(x), \qquad n \in \boldsymbol{N}.$$

**Lemma 5.1.** (i) *$L(\tau) \in S_{\ell+1}(A(\tau))$ for each $\tau \in \Omega$ (see Definition 4.1) and*

(5.6) $$\sup_{\tau \in \Omega} |||L(\tau)|||_{S_{\ell+1}(A(\tau))} < \infty.$$

(ii) *Fix a smooth function $\chi(t)$ on $\boldsymbol{R}$ with $\chi = 1$ on $[-1/2, 1/2]$ and $\operatorname{supp} \chi \subset [-1, 1]$. Then,*

(5.7) $$\chi(L(\tau)) i[L(\tau), A(\tau)] \chi(L(\tau)) \geqq (1/2) \chi(L(\tau))^2$$

*for all $\tau \in \Omega$ if $v_0 \gg 1$.*

*Proof.* (i) Since $\mathcal{S}$ is a common core for $L(\tau)$ and $A(\tau)$, (A-i) follows. (A-ii) can be easily verified by (5.2). By (5.3), (5.4), (5.5) and $(\mathrm{U})_\ell$ we see that the $n$-th $(0\leqq n\leqq \ell+2)$ commutator

$$i^{n}[[\cdots[L(\tau), A(\tau)], \cdots], A(\tau)] \tag{5.8}$$

on $\mathcal{S}$ can be uniquely extended to a self-adjoint operator $i^{n}L^{(n)}(\tau)$ with domain $H^{2}(\boldsymbol{R}^{2N})$. Thus, taking account of the fact that $\mathcal{S}$ is a common core for $L(\tau)$ and $A(\tau)$, we can verify (A-iii), (A-iv). Therefore, we see that $L(\tau)\in S_{\ell+1}(A(\tau))$ for each $\tau\in\Omega$. (5.6) can be verified by using (5.3).

(ii) For $v_0\gg 1$, we get, by (5.3),

$$\begin{aligned}&\chi(L(\tau))i[L(\tau), A(\tau)]\chi(L(\tau))\\ &\quad\geqq(-(2/n_a v)-(1/n_a v^2)\|i[V, A_0]-2V+2\lambda_a\|+1)\chi(L(\tau))^2\\ &\quad\geqq(1/2)\chi(L(\tau))^2 .\end{aligned}$$

This completes the proof. ■

For $\tau\in\Omega$ and $z\in\boldsymbol{C}\setminus\boldsymbol{R}$, we write $R(\tau, z)=(L(\tau)-z)^{-1}$. It is not difficult to check, by using $\nabla R(\tau, i)=R(\tau, i)\nabla-R(\tau, i)v^{-1}(\nabla V)R(\tau, i)$ and $(\mathrm{U})_\ell$, the following estimate:

$$\sup_{\tau\in\Omega}\|R(\tau, i)\|_{B(H^m, H^m)}<\infty \tag{5.9}$$

for $0\leqq m\leqq 2\ell+2$, where $H^m=H_0^m(\boldsymbol{R}^{2N})$.

**Lemma 5.2.** *Let $m$ be an integer with $0\leqq m\leqq 2\ell+2$. Then*

$$\sup_{\tau\in\Omega} v^{-1}\|R(\tau, i)\|_{B(H^m, H^{m+1})}<\infty . \tag{5.10}$$

*Proof.* Let $L_0(\tau):=v^{-1}(H_0-\lambda_a)-i\omega\cdot\nabla_\eta$ for $\tau=(v, \omega)\in\Omega$. Then, by (5.9), $(U)_\ell$ and the resolvent equation

$$R(\tau, i)=(L_0(\tau)-i)^{-1}-v^{-1}(L_0(\tau)-i)^{-1}VR(\tau, i),$$

it suffices to show that

$$\sup_{\tau=(v,\omega)\in\Omega} v^{-1}\|(L_0(\tau)-i)^{-1}\|_{B(H^m, H^{m+1})}<\infty , \tag{5.11}$$

which is, by the Fourier transform, reduced to the following estimate:

$$\begin{aligned}\sup_{\substack{\tau=(v,\omega)\in\Omega\\ \xi,\eta\in\boldsymbol{R}^N}} &|v^{-1}(|\xi|+|\eta|+1)\\ &\times(|v^{-1}((1/2m_a)\xi^2+(1/2n_a)\eta^2-\lambda_a)+\omega\cdot\eta|+1)^{-1}|<\infty .\end{aligned} \tag{5.12}$$

Taking account of $\lambda_a<0$ and $2ab\leqq a^2+b^2$ for $a, b\in\boldsymbol{R}$, we have

$$|\xi|+|\eta|+1\leqq v^{-1}((1/2m_a)\xi^2+(1/2n_a)\eta^2-\lambda_a)+\omega\cdot\eta+((m_a/2)+2n_a)v+1 ,$$

which yields (5.12). This completes the proof. ■

*Proof of Proposition* 3.1 (i).

*Case* $k=0$. We first consider the case $k=0$. For the proof, it sufficies to show that

$$\sup_{\tau\in\Omega,\ 0<\varepsilon<1} \|X_s R(\tau, i\varepsilon)X_s\| < \infty\,, \tag{5.13}$$

$$\lim_{\varepsilon,\varepsilon'\downarrow 0}\sup_{\tau\in\Omega} \|X_s[R(\tau, i\varepsilon)-R(\tau, i\varepsilon')]X_s\| = 0\,, \tag{5.14}$$

where $X_s := (1+|x|^2+|y|^2)^{-s/2}$ and $1/2<s\leqq 1$. By the resolvent equation we have

$$R(\tau, i\varepsilon)=R(\tau, i)+(i\varepsilon-i)R(\tau, i)^2+(i\varepsilon-i)^2R(\tau, i)R(\tau, i\varepsilon)R(\tau, i)\,. \tag{5.15}$$

Thus, we have only to prove the following estimates:

$$\sup_{\tau\in\Omega,\ 0<\varepsilon<1} \|\langle A(\tau)\rangle^{-s}R(\tau, i\varepsilon)\langle A(\tau)\rangle^{-s}\| < \infty\,, \tag{5.16}$$

$$\lim_{\varepsilon,\varepsilon'\downarrow 0}\sup_{\tau\in\Omega} \|\langle A(\tau)\rangle^{-s}[R(\tau, i\varepsilon)-R(\tau, i\varepsilon')]\langle A(\tau)\rangle^{-s}\| = 0\,, \tag{5.17}$$

$$\sup_{\tau\in\Omega} \|X_s R(\tau, \pm i)\langle A(\tau)\rangle^{s}\| < \infty\,. \tag{5.18}$$

(5.16) and (5.17) follow from Theorem 4.2 and Lemma 5.1. By interpolation, it suffices to prove (5.18) for $s=1$. We have

$$X_1R(\tau, \pm i)A(\tau)=X_1A(\tau)R(\tau, \pm i)-X_1R(\tau, \pm i)[L(\tau), A(\tau)]R(\tau, \pm i) \tag{5.19}$$

on $\mathcal{S}$. By (5.3) the operator norm of the second term in the R. H. S. is uniformly bounded in $\tau\in\Omega$, and the norm of the first term is also uniformly bounded in $\tau\in\Omega$ by Lemma 5.2 with $m=0$ because

$$A(\tau)=(n_a v i)^{-1}(x\cdot\nabla_x+y\cdot\nabla_y+N)+\omega\cdot y, \qquad \tau=(v, \omega)\,. \tag{5.20}$$

Thus we have proved Proposition 3.1 (i) for $k=0$.

*Case* $k\geqq 1$. We next consider the case $1\leqq k\leqq \ell$. We may assume $k+(1/2)<s\leqq k+1$. For the proof, it suffices to prove

$$\sup_{\tau\in\Omega,\ 0<\varepsilon<1} \|X_s D^\gamma R(\tau, i\varepsilon)\langle D\rangle^{-k}X_s\| < \infty\,, \tag{5.21}$$

and

$$\lim_{\varepsilon,\varepsilon'\downarrow 0}\sup_{\tau\in\Omega} \|X_s D^\gamma[R(\tau, i\varepsilon)-R(\tau, i\varepsilon')]\langle D\rangle^{-k}X_s\| = 0\,, \tag{5.22}$$

where $|\gamma|\leqq k$ and $D:=(\partial_x, \partial_y)$, $\langle D\rangle := (-\Delta+1)^{1/2}$. Taking account of (5.13), (5.14) and

$$X_{-s}D^\gamma\langle D\rangle^{-k}X_s\in \boldsymbol{B}(\mathcal{H}) \qquad \text{for } |\gamma|\leqq k\,, \tag{5.23}$$

and using

(5.23)′ $$DR(\tau, i\varepsilon)=R(\tau, i\varepsilon)D-R(\tau, i\varepsilon)v^{-1}(DV)R(\tau, i\varepsilon)$$

repeatedly, we see that the proof of (5.21) and (5.22) are reduced to the proof of the following:

(5.24) $$\sup_{\tau\in\Omega,\ 0<\varepsilon<1}\|X_sR(\tau, i\varepsilon)(D^{\gamma_1}V)R(\tau, i\varepsilon)\cdots(D^{\gamma_m}V)R(\tau, i\varepsilon)X_s\|<\infty\,,$$

(5.25) $$\lim_{\varepsilon,\varepsilon'\downarrow 0}\sup_{\tau\in\Omega}\|X_s\{R(\tau, i\varepsilon)(D^{\gamma_1}V)R(\tau, i\varepsilon)\cdots(D^{\gamma_m}V)R(\tau, i\varepsilon) -R(\tau, i\varepsilon')(D^{\gamma_1}V)R(\tau, i\varepsilon')\cdots(D^{\gamma_m}V)R(\tau, i\varepsilon')\}X_s\|=0\,,$$

where $1\leqq m\leqq k$ and $\sum_{j=1}^{m}|\gamma_j|\leqq k$. Using (5.15) repeatedly, we have

(5.26) $$R(\tau, i\varepsilon)=\sum_{0\leq m\leq 2k+1}(i\varepsilon-i)^m R(\tau, i)^{m+1} +(i\varepsilon-i)^{2k+2}R(\tau, i)^{k+1}R(\tau, i\varepsilon)R(\tau, i)^{k+1}\,.$$

Thus, replacing $R(\tau, i\varepsilon)$ in (5.24), (5.25) with the R. H. S. of (5.26) and taking account of $\|R(\tau, i)\|\leqq 1$, we can reduce the proof of (5.21), (5.22) to that of the following:

(5.27) $$\sup_{\tau\in\Omega,\ 0<\varepsilon<1}\|X_sU_1R(\tau, i\varepsilon)U_2\cdots R(\tau, i\varepsilon)U_{m+1}X_s\|<\infty\,,$$

(5.28) $$\lim_{\varepsilon,\varepsilon'\downarrow 0}\sup_{\tau\in\Omega}\|X_sU_1\{R(\tau, i\varepsilon)U_2\cdots R(\tau, i\varepsilon) -R(\tau, i\varepsilon')U_2\cdots R(\tau, i\varepsilon')\}U_{m+1}X_s\|=0\,.$$

where $1\leqq m\leqq k+1$, and each $U_j=U_j(\tau)$ has the form:

(5.29) $$U_j=R(\tau, i)^{k+1}Y_1\cdots Y_q \quad\text{or}\quad Y_1\cdots Y_qR(\tau, i)^{k+1}\,,$$

where $Y_j=R(\tau, i)$ or $D^{\gamma}V$ for $|\gamma|\leqq k$.

Consequently, taking account of (5.16), (5.17) in the case $m=1$ it suffices to prove the following:

(5.30) $$\sup_{\tau\in\Omega}\{\|X_sU_1\langle A(\tau)\rangle^s\|+\|\langle A(\tau)\rangle^sU_{m+1}X_s\|<\infty\,,$$

(5.31) $$\sup_{\tau\in\Omega,\ 0<\varepsilon<1}\|\langle A(\tau)\rangle^{-s}R(\tau, i\varepsilon)U_2R(\tau, i\varepsilon)\cdots U_mR(\tau, i\varepsilon)\langle A(\tau)\rangle^{-s}\|<\infty \quad (m\geqq 2)\,,$$

(5.32) $$\lim_{\varepsilon,\varepsilon'\downarrow 0}\sup_{\tau\in\Omega}\|\langle A(\tau)\rangle^{-s}\{R(\tau, i\varepsilon)U_2R(\tau, i\varepsilon)\cdots U_mR(\tau, i\varepsilon) -R(\tau, i\varepsilon')U_2R(\tau, i\varepsilon')\cdots U_mR(\tau, i\varepsilon')\}\langle A(\tau)\rangle^{-s}\|=0 \quad (m\geqq 2).$$

By (U)$_t$ and (5.3), we can easily see that $U_j(\tau)\in B_{t+1}(A(\tau))$ and

(5.33) $$\sup_{\tau\in\Omega}|||U_j(\tau)|||_{B_{t+1}(A(\tau))}<\infty$$

for each $j$. Hence, (5.31), (5.32) follow from Lemma 5.1 and Theorem 4.2 with $d=m$, $K_j(\tau)=L(\tau)$ $(1\leqq j\leqq m)$ and $W_j(\tau)=U_{j+1}(\tau)$ $(1\leqq j\leqq m-1)$.

Finally we prove that $\|X_s U_1 \langle A(\tau)\rangle^s\|$, $\tau\in\Omega$, is uniformly bounded. The uniform boundedness of $\|\langle A(\tau)\rangle^s U_{m+1} X_s\|$ can be proved similarly. We write $U_1 A(\tau)^{k+1}=A(\tau)^{k+1}U_1+[U_1, A(\tau)^{k+1}]$ on $\mathcal{S}$. $A(\tau)^{k+1}$ is written as

(5.34) $$\sum_{\substack{|\gamma_1+\gamma_2|\leqq k+1\\ |\gamma_3|\leqq k+1}} C(\tau;\gamma_1,\gamma_2,\gamma_3)x^{\gamma_1}y^{\gamma_2}(v^{-1}D)^{\gamma_3}$$

for $\tau=(v, \omega)$, where the constants $C(\tau;\gamma_1,\gamma_2,\gamma_3)$ are uniformly bounded in $\tau\in\Omega$. Since $U_1$ is written as (5.29), it follows from Lemma 5.2, (5.9) and (U), that for $|\gamma|\leqq k+1$,

(5.35) $$\sup_{\tau\in\Omega}\|(v^{-1}D)^\gamma U_1\|<\infty .$$

This together with (5.34) implies

(5.36) $$\sup_{\tau\in\Omega}\|X_{k+1}A(\tau)^{k+1}U_1\|<\infty .$$

Now, by induction we have

(5.37) $$[U_1, A(\tau)^{k+1}]=\sum_{j=0}^{k} C_j A(\tau)^j[\cdots[U_1, A(\tau)], A(\tau)]\cdots], A(\tau)],$$

where the multiple commutators are $(k+1-j)$ fold commutators, and each $C_j$ is independent of $\tau\in\Omega$. By (5.3) we see that

(5.37)′ $$\sup_{\tau\in\Omega}\|(v^{-1}D)^\gamma[\cdots[U_1, A(\tau)], A(\tau)]\cdots], A(\tau)]\|<\infty ,$$

in the same way as (5.35), where $|\gamma|\leqq k+1$ and the multiple commutator is $(k+1-j)$ fold commutator $(0\leqq j\leqq k)$. Thus, by (5.34) we obtain,

(5.38) $$\sup_{\tau\in\Omega}\|X_{k+1}[U_1, A(\tau)^{k+1}]\|<\infty .$$

Therefore, we get (5.30) by (5.36), (5.38) and interpolation. This completes the proof of Proposition 3.1 (i). ■

*Proof of Proposition* 3.1 (ii). We denote by $T(\tau)$ the self-adjoint operator $-i\omega\cdot\nabla_y$ in $\mathcal{H}$ for each $\tau=(v, \omega)\in\Omega$. Then, for each $\tau=(v, \omega)\in\Omega$, we have

(5.39) $$i[T(\tau), A(\tau)]=(n_a v)^{-1}T(\tau)+1 \qquad \text{on } \mathcal{S} .$$

Hence, it follows that $T(\tau)\in S_m(A(\tau))$ and

(5.40) $$\sup_{\tau\in\Omega}\|T(\tau)\|_{S_m(A(\tau))}<\infty ,$$

for any integer $m\geqq 0$, and that

(5.41) $$\chi(T(\tau))i[T(\tau), A(\tau)]\chi(T(\tau))\geqq(1/2)\chi(T(\tau))^2$$

for all $\tau\in\Omega$, where $\chi$ is the same as in Lemma 5.1. Now we set $r(\tau, z):=(T(\tau)-z)^{-1}$ for $\operatorname{Im} z\neq 0$ and write

$$[R(\tau, i\varepsilon)-r(\tau, i\varepsilon)]=-v^{-1}R(\tau, i\varepsilon)r(\tau, i\varepsilon)(H_0-\lambda_\alpha)$$
$$-v^{-1}R(\tau, i\varepsilon)Vr(\tau, i\varepsilon).$$

Since $\|\langle A(\tau)\rangle^2(H_0-\lambda_\alpha)\|_{B(H_2^4, L^2)}$ and $\|\langle A(\tau)\rangle^2\|_{B(H_2^4, L^2)}$ are uniformly bounded in $\tau\in\Omega$, for the proof of Proposition 3.1 (ii) we have only to prove the following:

(5.42) $$\sup_{\tau\in\Omega,\ 0<\varepsilon<1}\|X_2R(\tau, i\varepsilon)r(\tau, i\varepsilon)\langle A(\tau)\rangle^{-2}\|<\infty,$$

(5.43) $$\sup_{\tau\in\Omega,\ 0<\varepsilon<1}\|X_2R(\tau, i\varepsilon)Vr(\tau, i\varepsilon)\langle A(\tau)\rangle^{-2}\|<\infty.$$

We only prove (5.43). (5.42) is proved similary. By the relation

$$R(\tau, i\varepsilon)=R(\tau, i)+(i\varepsilon-i)R(\tau, i)^2+(i\varepsilon-i)^2R(\tau, i)^2R(\tau, i\varepsilon),$$

the proof of (5.43) is reduced to proving the uniform boundedness of the following norms:

(5.44) $$\|X_2R(\tau, i)^m\langle A(\tau)\rangle\|\|\langle A(\tau)\rangle^{-1}V\langle A(\tau)\rangle\|\|\langle A(\tau)\rangle^{-1}r(\tau, i\varepsilon)\langle A(\tau)\rangle^{-1}\|$$

for $m=1$, 2 and

(5.45) $$\|X_2R(\tau, i)^2\langle A(\tau)\rangle^2\|\|\langle A(\tau)\rangle^{-2}R(\tau, i\varepsilon)Vr(\tau, i\varepsilon)\langle A(\tau)\rangle^{-2}\|.$$

Both of the first factors in (5.44) and (5.45) are uniformly bounded in $\tau$ in the same way as (5.30). The second factor in (5.44) is uniformly bounded by $(\mathrm{U})_\iota$. By (5.40), (5.41) and Theorem 4.2 with $d=1$, $K_1=T(\tau)$, $A=A(\tau)$ we can prove the uniform boundedness of the last factor in (5.44), and by (5.40), (5.41), Lemma 5.1 and Theorem 4.2 with $d=2$, $K_1=L(\tau)$, $K_2=T(\tau)$, $W_1=V$ and $A=A(\tau)$ we can prove the uniform boundedness of the second factor in (5.45). Hence, we have shown that (5.44) and (5.45) are uniformly bounded. This completes the proof of Proposition 3.1 (ii). ■

## Appendix

In this appendix we will prove Proposition 2.4 by supplementing the proof of Proposition 2 in [AS]. Throughout the discussion in this appendix we always assume $(\mathrm{V})_0$ and (Z). Then $\kappa:=-\max\{\mu;\ \mu\in\bigcup_{c\in A_2}\sigma_P(h_c)\}$ is strictly positive, where $\sigma_P(h_c)$ denotes the point spectrum of $h_c$. For $c=\{(i, j), k\}\in \boldsymbol{A}_2$, we denote by $V_c$ the potential $V_{ij}(x_c)$. Fix arbitrary $r_0\gg 1$ and let $\chi_1(\lambda)$ and $\chi_0(\lambda)$ be the characteristic functions for $[0, r_0+(\kappa/4)]$ and $I_0:=[r_0-(\kappa/4), r_0+(\kappa/4)]$, respectively. Recall that $a\in A_2$ is the 2-cluster decomposition associated with the initial channel $\alpha\in\Gamma_2: D(\alpha)=a$. We put off the proof of the following lemma.

**Lemma A-1.** *Let* $c, d\in \boldsymbol{A}_2$ *with* $d\neq a$. *Then the operator*

$$F_{cd}(z):=\chi_1(T_c)\langle x_c\rangle V_c R(z) V_d\langle x_a\rangle^{-2\delta}\langle y_a\rangle^{\delta-(1/2)}$$

*has the norm limits* $F_{cd}(\lambda\pm i0):=\lim_{\varepsilon\downarrow 0} F_{cd}(\lambda\pm i\varepsilon)$ *in* $\boldsymbol{B}(\mathcal{H})$ *uniformly in* $\lambda\in I_0$ *(see* (1.2) *for* $\delta$).

For the proof of the next lemma, see, for example, [GM] (Proposition 2.2).

**Lemma A-2.** *The operator* $Z_0(\lambda)\langle x_c\rangle^{-1}$, $\lambda>0$, *defined on* $\mathcal{S}$, *extends to a bounded operator from* $\mathcal{H}$ *to* $L^2(S^{2N-1})$ *for each* $c\in \boldsymbol{A}_2$, *and* $Z_0(\lambda)\langle x_c\rangle^{-1}f$, $f\in\mathcal{H}$, *is strongly continuous in* $\lambda>0$.

*Proof of Proposition* 2.4. By $\chi_0(H_0)\chi_1(T_c)=\chi_0(H_0)$ for any $c\in \boldsymbol{A}_2$, $Z_0(\lambda)\chi_0(H_0)=\chi_0(\lambda)Z_0(\lambda)$ and Lemmas A-1, A-2, we get the norm limit

$$\begin{aligned}(\mathrm{A1})\qquad & Z_0(\lambda)[-I_a+VR(\lambda+i0)I_a]J_\alpha Z_\alpha(\lambda)^*\\ &=\lim_{\varepsilon\downarrow 0} Z_0(\lambda)[-I_a+VR(\lambda+i\varepsilon)I_a]J_\alpha Z_\alpha(\lambda)^*\\ &=-Z_0(\lambda)\langle x_a\rangle^{-1}\langle x_a\rangle I_a J_\alpha Z_\alpha(\lambda)^*\\ &\quad+\sum_{\substack{c\\ d\neq a}} Z_0(\lambda)\langle x_c\rangle^{-1}F_{cd}(\lambda+i0)\langle x_a\rangle^{2\delta}\langle y_a\rangle^{-\delta+(1/2)}J_\alpha Z_\alpha(\lambda)^*\end{aligned}$$

for $\lambda\in I_0$. Here we note that $I_a\langle x_a\rangle J_\alpha Z_\alpha(\lambda)^*$, $\langle x_a\rangle^{2\delta}\langle y_a\rangle^{-\delta+(1/2)}J_\alpha Z_\alpha(\lambda)^*$ is continuous in $\lambda\in I_0$ w.r.t. the Hilbert-Schmidt norm. Thus, by noting that $Z_0(\lambda)\langle x_c\rangle^{-1}$ is strongly continuous in $\lambda$ and $F_{cd}(\lambda+i0)$ is norm continuous in $\lambda\in I_0$, we see that

$$(\mathrm{A2})\qquad 2\pi i Z_0(\lambda)[-I_a+VR(\lambda+i0)I_a]J_\alpha Z_\alpha(\lambda)^*$$

is continuous in $\lambda\in I_0$ with respect to the Hilbert-Schmidt norm as an operator from $L^2(S^{N-1})$ to $L^2(S^{2N-1})$. Furthermore, in almost the same way as in the 2-body case, we can see that the above operator is equal to $T_{\alpha\to 0}(\lambda)$. Thus the integral kernel of $T_{\alpha\to 0}(\lambda)$ is given by

$$(\mathrm{A3})\qquad T_{\alpha\to 0}(\lambda,\theta,\omega)=2\pi i C_\alpha(\lambda)(Z_0(\lambda)f(\lambda,\omega,\cdot))(\theta),$$

where

$$f(\lambda,\omega,X)=([-I_a+VR(\lambda+i0)I_a]\phi_\alpha e^{i(2n_a(\lambda-\lambda_\alpha))^{1/2}\omega\cdot y_a})(X),$$

$X=(x_a, y_a)$. Moreover, we can also see that $T_{\alpha\to 0}(\lambda,\cdot,\omega)$ is $L^2(S^{2N-1})$-valued continuous function in $\lambda\in I_0$ and $\omega\in S^{N-1}$. Thus we have proved Proposition 2.4, accepting Lemma A-1. ∎

Finally we prove Lemma A-1. The proof is given for the $+$ case and divided into several steps. From now on, $c$ and $d$ denote 2-cluster decompositions.

*Step* 1. Let us begin with introducing notations. For $c\in \boldsymbol{A}_2$ we denote by

$A_c$, $B_c$ and $C_c$ the multiplication operators $|V_c(x_c)|^{1/2}$, $|V_c(x_c)|^{1/2}\operatorname{sgn} V_c(x_c)$ and $\langle y_c\rangle^{-\delta/4}$, respectively. New Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are defined by

$$\mathcal{H}_1 := \mathcal{H}\oplus\mathcal{H}\oplus\mathcal{H}\,, \qquad \mathcal{H}_2 := \mathcal{H}_1\oplus\mathcal{H}_1\,.$$

We associate each $\mathcal{H}$ in $\mathcal{H}_1$ with each 2-cluster decomposition. Each operator $Q$ on $\mathcal{H}_1$ is an operator valued $3\times 3$ matrix with the $c-d$ component $Q^{cd}$ ($c, d \in \boldsymbol{A}_2$), which is an operator on $\mathcal{H}$. We label two copies of $\mathcal{H}_1$ in $\mathcal{H}_2$ 0 and 1, respectively, and denote by $R_{ij}$ ($0\leqq i, j\leqq 1$) the $i-j$ component of an operator $R$ on $\mathcal{H}_2$. Of course, each $R_{ij}$ is an operator on $\mathcal{H}_1$.

We introduce several operators.

$$Y(z)\in\boldsymbol{B}(\mathcal{H}_1):\ \ Y(z)^{cd}=B_cR(z)C_d$$

$$G(z)\in\boldsymbol{B}(\mathcal{H}_1):\ \ G(z)^{cd}=B_cE_cR_c(z)C_d\delta_{cd},$$

$$J(z)=(Id,\, G(z))\in\boldsymbol{B}(\mathcal{H}_2,\, \mathcal{H}_1)$$

$$K(z)={}^t(K_0(z),\, K_1(z))\in\boldsymbol{B}(\mathcal{H}_1,\, \mathcal{H}_2):$$

$$K_0(z)^{cd}=B_c(\mathrm{Id}-E_c)R_c(z)C_d, \qquad K_1(z)^{cd}=C_c^{-1}E_cC_d.$$

$$D_{00}(z)\in\boldsymbol{B}(\mathcal{H}_1):\ \ D_{00}(z)^{cd}=B_c(\mathrm{Id}-E_c)R_c(z)A_d(\mathrm{Id}-\delta_{cd}),$$

$$D_{01}(z)=D_{00}(z)G(z)\in\boldsymbol{B}(\mathcal{H}_1),$$

$$D_{10}(z)\in\boldsymbol{B}(\mathcal{H}_1):\ \ D_{10}(z)^{cd}:=C_c^{-1}E_cA_d(\mathrm{Id}-\delta_{cd}),$$

$$D_{11}(z)=D_{10}(z)G(z)\in\boldsymbol{B}(\mathcal{H}_1),$$

$$D(z)\in\boldsymbol{B}(\mathcal{H}_2):\ D(z)_{ij}=D_{ij}(z).$$

$$N\in\boldsymbol{B}(\mathcal{H}_2):\ N_{10}=D_{10}(z), \qquad N_{ij}=0\ \ ((i,\, j)\neq(1,\, 0)),$$

$$W(z)=(Id-N)(D(z)-N)\in\boldsymbol{B}(\mathcal{H}_2).$$

Here $R_c(z):=(H_c-z)^{-1}$, $E_c:=P^P(h_c)\otimes\mathrm{Id}$, $P^p(h_c)$ being the orthogonal projection onto the subspace spanned by eigenvectors of $h_c$, and $\delta_{cd}$ is Kronecker's delta. For each $z\in\boldsymbol{C}\setminus\boldsymbol{R}$, $Id+W(z)$ has a bounded inverse and $Y(z)$ can be written as ([AS], p. 1572)

$$Y(z)=J(z)(\mathrm{Id}+W(z))^{-1}(\mathrm{Id}-N)K(z). \tag{A4}$$

*Step* 2. $W(z)$ has the following properties: (i) $W(z)$, $z\in\boldsymbol{C}\setminus\boldsymbol{R}$, is a compact operator and has the norm limit $W(\lambda+i0):=\lim_{\varepsilon\downarrow 0}W(\lambda+i\varepsilon)$ in $\boldsymbol{B}(\mathcal{H}_2)$ uniformly on any compact set in $(0, \infty)$, (ii) There exists a closed null set $e_0$ in $(0, \infty)$ such that $Id+W(\lambda+i0)$ is invertible in $\lambda\in(0, \infty)\setminus e_0$.

We will show that $e_0$ is a bounded set by proving

$$\|W(\lambda+i0)\| \longrightarrow 0 \qquad \text{as } \lambda\to\infty\,. \tag{A5}$$

The following lemma is important for the proof of (A5).

**Lemma A-3.** (i) ([GM], Proposition (2.4), [Ha], Corollary 5.5) *Let* $\delta>1$ *and* $R_0(z):=(H_0-z)^{-1}$. *Then the norm limit*

$$\langle x_c\rangle^{-\delta}R_0(\lambda+i0)\langle x_d\rangle^{-\delta}=\lim_{\varepsilon\downarrow 0}\langle x_c\rangle^{-\delta}R_0(\lambda+i\varepsilon)\langle x_d\rangle^{-\delta}$$

*exists in* $\boldsymbol{B}(\mathcal{H})$ *uniformly for* $\lambda$ *in any compact set in* $\boldsymbol{R}$ *for any* $c$, $d$ *and, moreover, we have*

$$\lim_{\lambda\to\infty}\|\langle x_c\rangle^{-\delta}R_0(\lambda+i0)\langle x_d\rangle^{-\delta}\|=0 \qquad \textit{for } c\neq d\,.$$

(ii) (cf. [AJS], Lemma 16.15) *The norm limit*

$$B_c(\mathrm{Id}-E_c)R_c(\lambda+i0)A_c=\lim_{\varepsilon\downarrow 0}B_c(\mathrm{Id}-E_c)R_c(\lambda+i\varepsilon)A_c$$

*exists in* $\boldsymbol{B}(\mathcal{H})$ *uniformly for* $\lambda$ *in* $\boldsymbol{R}$, *and*

$$\sup_{\lambda\in\boldsymbol{R}}\|B_c(\mathrm{Id}-E_c)R_c(\lambda+i0)A_c\|<\infty$$

*for any* $c$.

For the proofs, see [GM], [Ha] and [AJS].

In order to prove (A5), we will show that each $D_{ij}(\lambda+i\varepsilon)$, $0\leqq i,\ j\leqq 1$, has the norm limit $D_{ij}(\lambda+i0)$ in $\boldsymbol{B}(\mathcal{H}_1)$ as $\varepsilon\downarrow 0$ and all norms $\|D_{00}(\lambda+i0)\|$, $\|D_{01}(\lambda+i0)\|$ and $\|D_{11}(\lambda+i0)\|$ go to zero as $\lambda\to\infty$. Then the relation

$$W(\lambda+i0)=(\mathrm{Id}-N)(D(\lambda+i0)-N)$$

yields (A5).

By the resolvent equation, we have

$$\begin{aligned}(\mathrm{A6})\qquad & B_c(\mathrm{Id}-E_c)R_c(\lambda+i\varepsilon)A_d\\ &=-B_cE_cR_0(\lambda+i\varepsilon)A_d+[\mathrm{Id}-B_c(\mathrm{Id}-E_c)R_c(\lambda+i\varepsilon)A_c]B_cR_0(\lambda+i\varepsilon)A_d.\end{aligned}$$

For $c\neq d$, by $(\mathrm{V})_l$, Lemmas 2.2 and A-3, the R. H. S. has the norm limit in $\boldsymbol{B}(\mathcal{H})$ as $\varepsilon\downarrow 0$ and the norm limit goes to zero as $\lambda\to\infty$. This proves the existence of the norm limit $D_{00}(\lambda+i0)$ and

$$(\mathrm{A7})\qquad \lim_{\lambda\to\infty}\|D_{00}(\lambda+i0)\|=0\,.$$

To prove the existence of $D_{01}(\lambda+i0)$ and

$$(\mathrm{A8})\qquad \lim_{\lambda\to\infty}\|D_{01}(\lambda+i0)\|=0\,.$$

we must estimate the following operators (see (A6))

$$\begin{aligned}(\mathrm{A9})\qquad & -B_cE_cR_0(\lambda+i\varepsilon)V_dR_d(\lambda+i\varepsilon)E_dC_d\\ &+[\mathrm{Id}-B_c(\mathrm{Id}-E_c)R_c(\lambda+i\varepsilon)A_c]B_cR_0(\lambda+i\varepsilon)V_dR_d(\lambda+i\varepsilon)E_dC_d\end{aligned}$$

for $c \neq d$. The norm of the operator in $[\cdots]$ is uniformly bounded in $\varepsilon \in (0, 1]$ and $\lambda \gg 1$ by Lemma A-3 (ii). Furthermore, by the resolvent equation, we have

$$B_c R_0(\lambda+i\varepsilon) V_d R_d(\lambda+i\varepsilon) E_d C_d$$
$$= B_c R_0(\lambda+i\varepsilon) E_d C_d - B_c E_d R_d(\lambda+i\varepsilon) E_d C_d .$$

Then, the first term and the second have norm limits in $\boldsymbol{B}(\mathcal{H})$ and the norms of these limits go to zero as $\lambda \to \infty$ by Lemma A-3 (i) and the well known fact (cf. [GM], Proposition (2.3)):

$$\text{(A10)} \qquad \lim_{\lambda \to \infty} \| \langle y_d \rangle^{-1} (-\Delta_{y_d} - \lambda - i0)^{-1} \langle y_d \rangle^{-1} \| = 0 .$$

The first term in (A9) has the norm limit and the norm of this limit goes to zero as $\lambda \to \infty$ in the same way as above. Thus we have proved (A8). By (A10) and $C_c^{-1} E_c A_d B_d C_d^{-1} E_d \in \boldsymbol{B}(\mathcal{H})$ for $c \neq d$, $C_c^{-1} E_c A_d B_d E_d R_d(\lambda+i\varepsilon) C_d$ has the norm limit in $\boldsymbol{B}(\mathcal{H})$ as $\varepsilon \downarrow 0$ and the limit goes to zero as $\lambda \to \infty$ for $c \neq d$. This implies the existence of $D_{11}(\lambda+i0)$ and

$$\lim_{\lambda \to \infty} \| D_{11}(\lambda+i0) \| = 0 .$$

Thus we have proved (A5).

*Step* 3. Finally we prove Lemma A-1. We assume $r_0 - (\kappa/4) > \sup e_0$. Since

$$F_{cd}(z) := \chi_1(T_c) \langle x_c \rangle A_c$$
$$\times (J(z)(Id + W(z))^{-1} (\mathrm{Id} - N) K(z))^{cd} V_d C_d^{-1} \langle x_a \rangle^{-2\delta} \langle y_a \rangle^{\delta - (1/2)},$$

it suffices to show that

$$\chi_1(T_c) \langle x_c \rangle A_c G(\lambda+i\varepsilon)^{cf} = \langle x_c \rangle V_c E_c \chi_1(T_c) R_c(\lambda+i\varepsilon) C_c \delta_{cf}$$

and

$$K_0(\lambda+i\varepsilon)^{gd} V_d C_d^{-1} \langle x_a \rangle^{-2\delta} \langle y_a \rangle^{\delta - (1/2)}$$
$$= B_g (\mathrm{Id} - E_g) R_g(\lambda+i\varepsilon) \langle x_a \rangle^{-\delta} V_d \langle x_a \rangle^{-\delta} \langle y_a \rangle^{\delta - (1/2)} ,$$

have norm limits in $\boldsymbol{B}(\mathcal{H})$ as $\varepsilon \downarrow 0$ uniformly in $\lambda \in I_0$ for any $c, f, g, d \in \mathcal{A}_2$ with $d \neq a$. By the choice of $\kappa$ and $I_0$, the first has norm limit. Since $B_g(\mathrm{Id} - E_g) R_g(\lambda+i\varepsilon) \langle x_a \rangle^{-\delta} = B_g(\mathrm{Id} - E_g) R_0(\lambda+i\varepsilon) \langle x_a \rangle^{-\delta} - B_g(\mathrm{Id} - E_g) R_g(\lambda+i\varepsilon) \times A_g B_g R_0(\lambda+i\varepsilon) \langle x_a \rangle^{-\delta}$ and $V_d \langle x_a \rangle^{-\delta} \langle y_a \rangle^{\delta - (1/2)}$ is bounded by $a \neq d$, we see that the second has the norm limit by Lemma A-3. This completes the proof of Lemma A-1. ■

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