

Asymptotic Behaviour of Time-Inhomogeneous Evolutions on von Neumann Algebras*

By

Alberto FRIGERIO* and Gabriele GRILLO†

Abstract

We consider a sequence τ_n of dynamical maps of a von Neumann algebra \mathcal{M} into itself, each of which has a faithful normal invariant state ω_n , and we investigate conditions under which the time-evolved $\varphi_n = \varphi_0 \circ \tau_1 \cdots \circ \tau_n$ of an arbitrary normal initial state φ_0 is such that $\lim_{n \rightarrow \infty} \|\varphi_n - \omega_n\| = 0$. This is proved under conditions on the spectral gap of τ_n extended to a contraction on the GNS space of (\mathcal{M}, ω_n) , and on the difference (in a sense to be made precise below) between ω_n and ω_{n-1} ; we do not require detailed balance of τ_n w. r. t. ω_n . We also give conditions on the sequence of relative Hamiltonians h_n between ω_n and ω_{n-1} ensuring that the result holds. Finally, we prove that the techniques of the present paper do not admit a simple generalization to C^* -algebras and non-normal states.

§ 1. Introduction

By “time-inhomogeneous evolution” on a von Neumann algebra \mathcal{M} we mean a sequence $\{\tau_n : n=1, 2, \dots\}$ of completely positive weakly* continuous linear maps of \mathcal{M} into itself, with $\tau_n(1)=1$ (*dynamical maps* in the sequel). To help intuition, τ_n may be regarded as the map describing evolution of the observables of a physical system from times t_{n-1} to time t_n , where $0=t_0 < t_1 < \dots < t_n \rightarrow \infty$. We assume that each τ_n has a unique (faithful normal) invariant state ω_n , and we investigate under which conditions, for any initial normal state φ_0 on \mathcal{M} , the time-evolved state $\varphi_n = \varphi_0 \circ \tau_1 \cdots \circ \tau_n$ becomes indistinguishable from ω_n in the limit as $n \rightarrow \infty$.

Several results exist in the literature for the case when all τ_n are the same map with a faithful normal invariant state, or $\tau_n = \exp[(t_n - t_{n-1})\mathcal{L}]$, \mathcal{L} being the generator of a dynamical semigroup (asymptotic behaviour of dynamical

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* Dipartimento di Scienze dell'Informazione, Università di Milano, via Comelico 39/41, 20135 Milano, Italy.

† Dipartimento di Matematica e Informatica, Università di Udine, via Zanon 6, 33100 Udine, Italy.

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semigroups with a faithful normal invariant state) [1-7].

Our generalization is primarily motivated by such problems as simulated annealing [8-10], where a time-inhomogeneous evolution of a (fictitious classical) physical system is used to minimize a nonnegative function U on a space X (interpreted as the energy function of the system); then the instantaneous invariant states ω_n are Gibbs states with energy function U and inverse temperatures β_n diverging to $+\infty$. More complicated situations where also U depends on "time" n have been considered in connection with adaptive algorithms [11]. Some noncommutative generalizations of the above results have been studied in [12, 13]. As compared with our previous work on the same subject [12, 13], the class of evolutions for which this asymptotic indistinguishability holds is extended to cover situations in which the maps τ_n need not be symmetric with respect to their invariant states ω_n . In order to prove our results, we need two kinds of assumptions:

- i) an estimate on the spectral gap of τ_n extended to a contraction operator on the GNS space of (\mathcal{M}, ω_n) ;
- ii) an estimate on the difference (in a suitable sense to be defined in §3) between ω_n and ω_{n-1} .

In particular, we need $\omega_{n-1} \leq \lambda_n \omega_n$ for suitable constants $\lambda_n > 0$ for all n .

We do not address ourselves to the question *i*), and we just remind the reader that results have been obtained in [10, 14] for finite classical systems, in [12] for finite quantum systems, in [15] for some infinite quantum systems and in [16, 17] for a class of infinite classical systems. Concerning *ii*) we give sufficient conditions on the sequence of relative Hamiltonians h_n between ω_n and ω_{n-1} ensuring that the above mentioned difference is small enough to allow application of our general argument.

The paper is organized as follows. In §2 we collect some preliminary results on von Neumann algebras and on dynamical maps which we require in the following. The general argument is outlined in §3. In §4 we investigate conditions on the relative Hamiltonians under which the arguments of §3 can be applied. In §5 we explore the possibility of generalizing the arguments in §3; we show that there is no simple generalization to the case where the ω_n are disjoint states on a C^* -algebra \mathcal{A} . However, we show that, at the price of some complication in the statement of the conditions, the assumption that $\omega_{n-1} \leq \lambda_n \omega_n$ can be relaxed.

§2. Preliminaries

Let \mathcal{M} be a von Neumann algebra of operators on a separable Hilbert space \mathcal{H} , with a cyclic and separating vector Ω . Denote by ω and by ω' the faithful normal states on \mathcal{M} and on the commutant \mathcal{M}' respectively defined by

$$\omega(a) = \langle \Omega, a\Omega \rangle : a \in \mathcal{M}; \tag{2.1}$$

$$\omega'(a') = \langle \Omega, a'\Omega \rangle : a' \in \mathcal{M}'. \tag{2.2}$$

Let Δ_Ω and J be the modular operator and the modular involution canonically associated with the pair (\mathcal{M}, Ω) by the Tomita-Takesaki theory, and let $V = \overline{\Delta_\Omega^{1/4} \mathcal{M}_+ \Omega}$ be the natural positive cone. For each normal state φ on \mathcal{M} there exists a unique vector Φ in V such that

$$\varphi(a) = \langle \Phi, a\Phi \rangle : a \in \mathcal{M}. \tag{2.3}$$

The relative modular operator $\Delta_{\varphi, \Omega}$ is defined by

$$\Delta_{\varphi, \Omega} = S_{\varphi, \Omega}^* \bar{S}_{\varphi, \Omega}, \tag{2.4}$$

where

$$S_{\varphi, \Omega} a \Omega = a^* \Phi : a \in \mathcal{M}. \tag{2.5}$$

Denote by $S(\mathcal{M})$ the set of all normal states on \mathcal{M} and by $S_\omega(\mathcal{M})$ the set of those normal states on \mathcal{M} which are majorized by a scalar multiple of ω .

Lemma 2.1. *For any $\varphi \in S(\mathcal{M})$ the following conditions are equivalent :*

- i) $\varphi \in S_\omega(\mathcal{M})$;
- ii) *there exist a (unique) element $x = x_\varphi$ of \mathcal{M}'_+ such that*

$$\varphi(a) = \langle x_\varphi \Omega, a\Omega \rangle : a \in \mathcal{M}; \tag{2.6}$$

iii) *the Connes cocycle $\{(D\varphi : D\omega)_t = \Delta_{\varphi, \Omega}^{it} \Delta_\Omega^{-it} : t \in \mathbf{R}\} \subseteq \mathcal{M}$ extends to an analytic function on the strip $z \in \mathbf{C} : -1/2 < \text{Im } z < 0$, continuous on the boundaries, with values in \mathcal{M} .*

Moreover, one has

$$\Phi = (D\varphi : D\omega)_{-1/2} \Omega, \tag{2.7}$$

$$x_\varphi = J[(D\varphi : D\omega)_{-1/2}]^* (D\varphi : D\omega)_{-1/2} J. \tag{2.8}$$

An immediate consequence of the equivalence $i) \Leftrightarrow ii)$ is the following

Corollary 2.2. *$S_\omega(\mathcal{M})$ is norm-dense in $S(\mathcal{M})$.*

Let $h = h^* \in \mathcal{M}$. Then the expression

$$\Omega(h) = \sum_{k=0}^{\infty} (-1)^k \int_0^{1/2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{k-1}} dt_k \Delta_\Omega^{tk} h \Delta_\Omega^{t_{k-1}-t_k} h \cdots \Delta_\Omega^{t_1-t_2} h \Omega \tag{2.9}$$

is well-defined, and h is said to be the relative Hamiltonian between the state ω^h given by

$$\omega^h(a) = \frac{\langle \Omega(h), a\Omega(h) \rangle}{\langle \Omega(h), \Omega(h) \rangle} : a \in \mathcal{M} \tag{2.10}$$

and ω . Note that we have adopted the same conventions concerning sign and normalization as in Donald [18], which are different from those of Araki [19].

Given ω and h , the perturbed state ω^h is the unique faithful state in $S(\mathcal{M})$ maximizing the function

$$\varphi \longmapsto \langle \Phi, \log \Delta_{\omega, \varphi} \Phi \rangle - \varphi(h), \tag{2.11}$$

where $\langle \Phi, \log \Delta_{\omega, \varphi} \Phi \rangle \leq 0$ is known as the relative entropy of φ with respect to ω (the opposite sign convention is also used in the literature). This variational characterization of ω^h makes sense also for self-adjoint operators h affiliated with \mathcal{M} which are bounded from below but unbounded from above and may possibly have $+\infty$ as an eigenvalue (see Donald [18]). With this extended notion of ω^h , for each φ in $S_\omega(\mathcal{M})$ there exists a unique h such that $\varphi = \omega^h$. However, given ω and h , the state ω^h need not be in $S_\omega(\mathcal{M})$; a sufficient condition for $\omega^h \in S_\omega(\mathcal{M})$ is that $\Delta_{\omega^h}^t h \Delta_{\omega^h}^{-t} \in \mathcal{M}$ for all $t \in [0, 1/2]$ (cf. Lemma 4.1 below).

Definition 2.3. A dynamical map τ on \mathcal{M} is a completely positive weakly*-continuous linear map of \mathcal{M} into itself with $\tau(1)=1$.

Lemma 2.4. [1, 20] *Let τ be a dynamical map on \mathcal{M} , leaving ω invariant. Then there exists a dynamical map τ' on \mathcal{M}' , leaving ω' invariant, such that*

$$\langle \tau'(a')\Omega, a\Omega \rangle = \langle a'\Omega, \tau(a)\Omega \rangle: \quad a \in \mathcal{M}, \quad a' \in \mathcal{M}'. \tag{2.12}$$

Proof (Sketch). If φ is in $S_\omega(\mathcal{M})$ and τ leaves ω invariant, then $\varphi \circ \tau$ is in $S_\omega(\mathcal{M})$. Define τ' by linear extension of

$$\tau'(x_\varphi) = x_{\varphi \circ \tau}: \quad \varphi \in S_\omega(\mathcal{M}). \tag{2.13}$$

Then τ' is a positive weakly*-continuous linear map of \mathcal{M}' into itself, satisfying (2.12), and $\tau'(1)=1$ since

$$\langle \tau'(1)\Omega, a\Omega \rangle = \langle \Omega, \tau(a)\Omega \rangle = \langle \Omega, a\Omega \rangle: \quad a \in \mathcal{M},$$

and $\omega' \circ \tau' = \omega'$ since

$$\langle \Omega, \tau'(a')\Omega \rangle = \langle \tau(1)\Omega, a'\Omega \rangle = \langle \Omega, a'\Omega \rangle: \quad a' \in \mathcal{M}'.$$

Complete positivity is shown as follows: let $a_1, \dots, a_n \in \mathcal{M}$, $x_1, \dots, x_n \in \mathcal{M}'$. Since τ is completely positive, one has

$$\begin{aligned} 0 &\leq \sum_{i,j=1}^n \langle x_i\Omega, \tau(a_i^* a_j) x_j\Omega \rangle = \sum_{i,j=1}^n \langle x_j^* x_i \Omega, \tau(a_i^* a_j)\Omega \rangle \\ &= \sum_{i,j=1}^n \langle \tau'(x_j^* x_i)\Omega, a_i^* a_j\Omega \rangle = \sum_{i,j=1}^n \langle a_i\Omega, \tau'(x_j^* x_i) a_j\Omega \rangle. \end{aligned}$$

Since $\mathcal{M}\Omega$ is dense in \mathcal{H} , also τ' is completely positive.

Lemma 2.5. [21, 22] *Let τ be a dynamical map on \mathcal{M} , leaving ω invariant. Then there exists a contraction T on \mathcal{K} such that*

$$T(a\Omega) = \tau(a)\Omega : \quad a \in \mathcal{M}, \tag{2.14}$$

$$T^*(a'\Omega) = \tau'(a')\Omega : \quad a' \in \mathcal{M}'. \tag{2.15}$$

Proof (Sketch). By the Kadison-Schwarz inequality $\tau(a^*a) - \tau(a^*)\tau(a) \geq 0$, we have

$$\begin{aligned} \|\tau(a)\Omega\|^2 &= \omega(\tau(a^*)\tau(a)) \leq \omega(\tau(a^*a)) \\ &= \omega(a^*a) = \|a\Omega\|^2 : \quad a \in \mathcal{M}. \end{aligned}$$

Then the linear operator T defined on $\mathcal{M}\Omega$ by (2.14) extends to a contraction on \mathcal{K} . For $a \in \mathcal{M}$, $a' \in \mathcal{M}'$ we have

$$\langle a'\Omega, T(a\Omega) \rangle = \langle a'\Omega, \tau(a)\Omega \rangle = \langle \tau'(a')\Omega, a\Omega \rangle$$

so that (2.15) holds.

Lemma 2.6. *In the situation of Lemma 2.5, the following are equivalent (for real $\gamma > 0$):*

i) $\|\tau(a)\Omega\| \leq e^{-\gamma} \|a\Omega\| \quad \text{for all } a \text{ in } \mathcal{M} \text{ with } \omega(a) = 0; \tag{2.16}$

ii) $\|T\Psi\| \leq e^{-\gamma} \|\Psi\| \quad \text{for all } \Psi \text{ in } \mathcal{K} \text{ with } \langle \Omega, \Psi \rangle = 0; \tag{2.17}$

iii) $\|\tau'(a')\Omega\| \leq e^{-\gamma} \|a'\Omega\| \quad \text{for all } a' \text{ in } \mathcal{M}' \text{ with } \omega'(a') = 0; \tag{2.18}$

iv) $\|T^*\Phi\| \leq e^{-\gamma} \|\Phi\| \quad \text{for all } \Phi \text{ in } \mathcal{K} \text{ with } \langle \Omega, \Phi \rangle = 0; \tag{2.19}$

v) $\|T^*T\Phi\| \leq e^{-2\gamma} \|\Phi\| \quad \text{for all } \Phi \text{ in } \mathcal{K} \text{ with } \langle \Omega, \Phi \rangle = 0; \tag{2.20}$

vi) $\|TT^*\Phi\| \leq e^{-2\gamma} \|\Phi\| \quad \text{for all } \Phi \text{ in } \mathcal{K} \text{ with } \langle \Omega, \Phi \rangle = 0. \tag{2.21}$

Proof. Let \mathcal{K} be the orthogonal complement of Ω in \mathcal{K} . Since $T\Omega = T^*\Omega = \Omega$, T and T^* map \mathcal{K} into itself, and $(T|_{\mathcal{K}})^* = T^*|_{\mathcal{K}}$. Then ii), iv), v) and vi) are equivalent. Clearly i) is a special case of ii) and iii) is a special case of iv). Conversely, $T|_{\mathcal{K}}$ is the closure of the map $a\Omega \mapsto \tau(a)\Omega$ with $\omega(a) = \langle \Omega, a\Omega \rangle = 0$, so that i) implies ii), and $T^*|_{\mathcal{K}}$ is the closure of the map $a'\Omega \mapsto \tau'(a')\Omega$ with $\omega'(a') = \langle \Omega, a'\Omega \rangle = 0$, so that ii) implies iv).

In the following, we shall refer to the equivalent conditions of Lemma 2.6 with $\gamma > 0$ as to the *spectral gap* condition. Indeed, if one takes the largest γ for which the above conditions hold, $1 - e^{-2\gamma}$ is the gap between the eigenvalue $\lambda = 1$ of the positive self-adjoint contraction T^*T and the rest of its spectrum.

The same holds for TT^* . Obviously, when the contraction T is self-adjoint, $e^{-\gamma}$ is the spectral radius of its restriction to K .

In the commutative case, there is a considerable amount of literature concerning estimation of $e^{-\gamma}$ for self-adjoint T [10, 14], which has been extended to the non self-adjoint case by Fill [23]. Conditions v), vi) above are essentially a generalization of the multiplicative reversibilization of Fill.

Remark. The equivalent conditions of Lemma 2.6 imply that

$$\lim_{k \rightarrow \infty} \varphi \circ \tau^k(a) = \omega(a) \quad \forall a \in \mathcal{M}, \varphi \in S(\mathcal{M}). \tag{2.22}$$

The converse implication is not true in general.

§ 3. Main Result

Let \mathcal{M} be a von Neumann algebra of operators on a separable Hilbert space \mathcal{H} , and let $\{\tau_n : n=1, 2, \dots\}$ be a sequence of dynamical maps on \mathcal{M} . Assume that each τ_n has an invariant faithful normal state ω_n with representative vector $\Omega_n \in \mathcal{H}$ which is cyclic and separating for \mathcal{M} . Thus, by Lemma 2.4, there exists a sequence $\{\tau'_n : n=1, 2, \dots\}$ of dynamical maps on \mathcal{M}' such that

$$\langle \tau'_n(a')\Omega_n, a\Omega_n \rangle = \langle a'\Omega_n, \tau_n(a)\Omega_n \rangle : \quad a \in \mathcal{M}, a' \in \mathcal{M}'. \tag{3.1}$$

Assume that each τ_n has a spectral gap, in the sense that there exist strictly positive constants $\gamma_n : n=1, 2, \dots$ such that, for all $n=1, 2, \dots$,

$$\|\tau_n(a)\Omega_n\| \leq e^{-\gamma_n} \|a\Omega_n\| \quad \text{for all } a \text{ in } \mathcal{M} \text{ with } \omega_n(a)=0. \tag{3.2}$$

By Lemma 2.6, a similar spectral gap holds also for τ'_n .

Assume also that there exists a sequence $R_n : n=1, 2, \dots$ of elements of \mathcal{M}' such that

$$R_n \Omega_n = \Omega_{n-1} : \quad n=2, 3, \dots \tag{3.3}$$

Equivalently (see Lemma 2.1), for $n=2, 3, \dots$, ω_{n-1} is majorized by a scalar multiple $\lambda_n \omega_n$ of ω_n , and $R_n \in \mathcal{M}'$ is such that

$$R_n^* R_n = x_{\omega_{n-1}, n}, \tag{3.4}$$

where $x_{\omega_{n-1}, n}$ is the unique positive element of \mathcal{M}' such that

$$\omega_{n-1}(a) = \langle x_{\omega_{n-1}, n} \Omega_n, a \Omega_n \rangle : \quad a \in \mathcal{M}. \tag{3.5}$$

Our problem is to find conditions on $\{\gamma_n\}$ and on $\{R_n\}$ ensuring that, for any initial state $\varphi_0 \in S(\mathcal{M})$, letting $\varphi_n = \varphi_{n-1} \circ \tau_n : n=1, 2, \dots$, one has

$$\lim_{n \rightarrow \infty} \|\varphi_n - \omega_n\| = 0. \tag{3.6}$$

By Corollary 2.2, it suffices to prove (3.6) for φ_0 in the dense set $S_{\Omega_1}(\mathcal{M})$. Then

$$\varphi_0(a) = \langle x_{\varphi_0,1} \Omega_1, a \Omega_1 \rangle : \quad a \in \mathcal{M} \tag{3.7}$$

for a suitable positive element $x_{\varphi_0,1}$ of \mathcal{M}' , and

$$\begin{aligned} \varphi_1(a) &= \varphi_0(\tau_1(a)) = \langle x_{\varphi_0,1} \Omega_1, \tau_1(a) \Omega_1 \rangle \\ &= \langle \tau'_1(x_{\varphi_0,1}) \Omega_1, a \Omega_1 \rangle = \langle x_1 \Omega_1, a \Omega_1 \rangle = \langle \Psi_1, a \Omega_1 \rangle : \quad a \in \mathcal{M}, \end{aligned} \tag{3.8}$$

where

$$x_1 = \tau'_1(x_{\varphi_0,1}) \in \mathcal{M}'_+ ; \quad \Psi_1 = x_1 \Omega_1 = T_1^* x_{\varphi_0,1} \Omega_1 . \tag{3.9}$$

Lemma 3.1. *Let φ_1 be given by (3.8), (3.9). Under the above conditions, for each $n=2, 3, \dots$, φ_n is a normal state on \mathcal{M} (actually, $\varphi_n \in S_{\omega_n}(\mathcal{M})$), which can be represented in the form*

$$\varphi_n(a) = \langle x_n \Omega_n, a \Omega_n \rangle = \langle \Psi_n, a \Omega_n \rangle : \quad a \in \mathcal{M}, \tag{3.10}$$

where

$$x_n = \tau'_n(R_n^* x_{n-1} R_n) \in \mathcal{M}'_+ ; \quad \Psi_n = x_n \Omega_n = T_n^* R_n^* \Psi_{n-1} . \tag{3.11}$$

Proof. Since (3.10) holds for $n=1$, it suffices to prove that it holds for n if it holds for $n-1$. Indeed, for all $a \in \mathcal{M}$, we have

$$\begin{aligned} \varphi_n(a) &= \varphi_{n-1}(\tau_n(a)) = \langle \Psi_{n-1}, \tau_n(a) \Omega_{n-1} \rangle \\ &= \langle \Psi_{n-1}, \tau_n(a) R_n \Omega_n \rangle = \langle R_n^* \Psi_{n-1}, \tau_n(a) \Omega_n \rangle \\ &= \langle R_n^* \Psi_{n-1}, T_n a \Omega_n \rangle = \langle T_n^* R_n^* \Psi_{n-1}, a \Omega_n \rangle = \langle \Psi_n, a \Omega_n \rangle, \end{aligned}$$

with Ψ_n given by (3.11). However, $\Psi_{n-1} = x_{n-1} \Omega_{n-1} = x_{n-1} R_n \Omega_n$, so that also

$$\begin{aligned} \varphi_n(a) &= \langle R_n^* x_{n-1} R_n \Omega_n, \tau_n(a) \Omega_n \rangle \\ &= \langle \tau'_n(R_n^* x_{n-1} R_n) \Omega_n, a \Omega_n \rangle = \langle x_n \Omega_n, a \Omega_n \rangle, \end{aligned}$$

with x_n given by (3.11), since R_n and x_{n-1} are in \mathcal{M}' .

Lemma 3.2. *Under the above assumptions, let*

$$\alpha_n = \gamma_n - \log \|R_n\|, \tag{3.12}$$

$$\beta_n = e^{-\gamma_n} \|(R_n^* R_n - 1) \Omega_n\|. \tag{3.13}$$

Then, for all $n=2, 3, \dots$,

$$\|\Psi_n - \Omega_n\| \leq e^{-\alpha_n} \|\Psi_{n-1} - \Omega_{n-1}\| + \beta_n . \tag{3.14}$$

Proof. By the preceding Lemma, and by the remarks following Lemma 2.4, we have

$$\Psi_n - \Omega_n = T_n^* R_n^* \Psi_{n-1} - \Omega_n = T_n^* (R_n^* \Psi_{n-1} - \Omega_n).$$

In addition, since

$$\begin{aligned} \langle R_n^* \Psi_{n-1} - \Omega_n, \Omega_n \rangle &= \langle \Psi_{n-1}, \Omega_{n-1} \rangle - \langle \Omega_n, \Omega_n \rangle \\ &= \varphi_{n-1}(\mathbf{1}) - \omega_n(\mathbf{1}) = 0, \end{aligned}$$

the spectral gap assumption and Lemma 2.6 imply that

$$\| \Psi_n - \Omega_n \| \leq e^{-\tau n} \| R_n^* \Psi_{n-1} - \Omega_n \|.$$

Finally, note that

$$R_n^* \Psi_{n-1} - \Omega_n = R_n^* (\Psi_{n-1} - \Omega_{n-1}) + (R_n^* R_n - 1) \Omega_n.$$

Theorem 3.3. *Under the above assumptions, suppose also that there exist real constants $\alpha > 0$, $\beta \geq 0$, $1 > \delta > \varepsilon \geq 0$ such that*

$$\alpha_n \geq \alpha n^{\delta-1}, \quad \beta_n \leq \beta n^{\varepsilon-1}; \quad n=1, 2, \dots \tag{3.15}$$

Then there is a constant C (depending on φ_0), such that

$$| \varphi_n(a) - \omega_n(a) | \leq C \| a \| n^{\varepsilon-\delta} \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.16}$$

Proof. See [13], Proposition 2.4.

Remark. The results of the present section have been proved in [13] under the additional assumption that, for each $n=1, 2, \dots$, one has

$$\omega_n(a \tau_n(b)) = \omega_n(\tau_n(a) b): \quad a, b \in \mathcal{M} \text{ (detailed balance)}. \tag{3.17}$$

In that situation, one simply has

$$\tau'_n(a') = J_n \tau_n(J_n a' J_n) J_n,$$

where J_n is the modular involution associated with the pair (\mathcal{M}, Ω_n) in \mathcal{A} (it may be the case that J_n is independent of n , as happens when the Ω_n are in the same natural positive cone). The new result here, contained in Lemmas 3.1, 3.2, is that detailed balance is not really needed (cf. [9] for the case of the algebra of functions on a finite space), provided one can prove a spectral gap condition without it (for instance, using the reversibilization argument of Fill [23]).

§ 4. Relative Hamiltonians

Here we assume that the sequence of states $\{\omega_n : n=1, 2, \dots\}$ is constructed starting from ω_1 and from a sequence $\{h_n : n=2, 3, \dots\}$ of relative Hamiltonians in such a way that

$$\omega_n = (\omega_{n-1})^{h_n}: \quad n=2, 3, \dots, \tag{4.1}$$

in the sense of eq. (2.10), and we estimate the quantities $\|R_n\|$ and $\|(R_n^* R_n - 1)\Omega_n\|$ in terms of $h_n : n=2, 3, \dots$

We restrict to bounded $h_n = h_n^* \in \mathcal{M}$, although ω^h can be defined also for self-adjoint h which is only bounded from below, since we need $\omega_{n-1} \in S_{\omega_n}(\mathcal{M})$ in order to have the operators $R_n \in \mathcal{M}'$ on which our analysis is based, and this in turn implies that $-h_n$ is bounded from below, so that h_n is bounded.

We are able to prove that $\omega_{n-1} \in S_{\omega_n}(\mathcal{M})$ under the assumption that the function $t \rightarrow \sigma_t^{(n-1)}(h_n) = \Delta_{\mathfrak{b}_{n-1}}^t h_n \Delta_{\mathfrak{a}_{n-1}}^{-t}$ extends to an analytic function on the strip $\{z \in \mathbb{C} : -1/2 < \text{Im } z < 0\}$, continuous on the boundaries, with values in \mathcal{M} ; we believe that this is only a sufficient condition. In order to avoid excessive notational burdens, we give a proof of the following statement:

Lemma 4.1. *Let ω be a faithful normal state in \mathcal{M} , with $\omega(a) = \langle \Omega, a\Omega \rangle : a \in \mathcal{M}$, and with associated modular automorphism group $\sigma_t = \Delta_{\mathfrak{b}}^t \cdot \Delta_{\mathfrak{a}}^{-t} : t \in \mathbb{R}$. Let $h = h^* \in \mathcal{M}$ be such that the function $t \rightarrow \sigma_t(h)$ extends to an analytic function on the strip $\{z \in \mathbb{C} : -1/2 < \text{Im } z < 0\}$, continuous on the boundaries, with values in \mathcal{M} , and let ω^h be defined by eq. (2.10). Denote by Φ the normalized vector $\Omega(h) / \|\Omega(h)\|$. Then there exists a unique R in \mathcal{M}' such that*

$$\Omega = R\Phi ; \tag{4.2}$$

R is invertible, and

$$\|R\|, \|R^{-1}\| \leq \exp \{ \| \|h\| \| \} , \tag{4.3}$$

where

$$\| \|h\| \| = \sup \{ \|\sigma_{-is}(h)\| : 0 \leq s \leq 1/2 \} .$$

Proof. Consider the differential equations

$$\begin{cases} \frac{d}{ds} V(s) = -V(s)\sigma_{-is}(h) : & 0 \leq s \leq \frac{1}{2} \\ V(0) = \mathbf{1} \end{cases} \tag{4.4}$$

and

$$\begin{cases} \frac{d}{ds} \tilde{V}(s) = \sigma_{-is}(h)\tilde{V}(s) : & 0 \leq s \leq \frac{1}{2} \\ \tilde{V}(0) = \mathbf{1} . \end{cases} \tag{4.5}$$

Both equations have unique solutions in \mathcal{M} satisfying the bounds

$$\|V(s)\|, \|\tilde{V}(s)\| \leq \exp \{ s \| \|h\| \| \} : \quad 0 \leq s \leq \frac{1}{2} .$$

Moreover, $\tilde{V}(s) = V(s)^{-1}$ for all $s \in [0, 1/2]$. Indeed,

$$\frac{d}{ds} [V(s)\tilde{V}(s)] = V(s)[- \sigma_{-is}(h) + \sigma_{-is}(h)]\tilde{V}(s) = 0$$

so that $V(s)\tilde{V}(s) = \mathbf{1}$ for all s ; and in addition the constant $\mathbf{1}$ solves the differential equation for $\tilde{V}(s)V(s)$, which reads

$$\begin{cases} \frac{d}{ds}[\tilde{V}(s)V(s)] = \sigma_{-is}(h)[\tilde{V}(s)V(s)] - [\tilde{V}(s)V(s)]\sigma_{-is}(h) : 0 \leq s \leq \frac{1}{2} \\ \tilde{V}(0)V(0) = 1. \end{cases}$$

By the uniform boundedness of $\sigma_{-is}(h)$ on $[0, 1/2]$, the solution to the latter equation is unique, so that $\tilde{V}(s)V(s) = 1$ for all s .

Now we have the iterated series

$$V(s) = \sum_{k=0}^{\infty} (-1)^k \int_0^s ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{k-1}} ds_k \sigma_{-is_k}(h) \cdots \sigma_{-is_2}(h) \sigma_{-is_1}(h), \tag{4.6}$$

and it is clear that

$$\begin{aligned} \Omega(h) &= V(1/2)\Omega = J \Delta_B^{1/2} V(1/2)^* \Omega \\ &= J \Delta_B^{1/2} V(1/2)^* \Delta_Q^{1/2} J \Omega = J V(1/2) J \Omega, \end{aligned}$$

where the last equality follows from the explicit expression of $V(s)$. Hence

$$\Phi = \|V(1/2)\Omega\|^{-1} J V(1/2) J \Omega$$

or

$$\Omega = \|V(1/2)\Omega\| J \tilde{V}(1/2) J \Phi = R \Phi.$$

Since Φ is cyclic and separating for \mathcal{M} and for \mathcal{M}' as Ω is, it follows that $R \in \mathcal{M}'$ is uniquely determined to be

$$R = \|V(1/2)\Omega\| J \tilde{V}(1/2) J. \tag{4.7}$$

An obvious estimate gives

$$\begin{aligned} \|R\| &\leq \|V(1/2)\| \|\tilde{V}(1/2)\| \\ &\leq \exp\left\{\frac{1}{2}\|h\|\right\} \exp\left\{\frac{1}{2}\|h\|\right\} = \exp\{\|h\|\}. \end{aligned}$$

Moreover,

$$R^{-1} = \|V(1/2)\Omega\|^{-1} J V(1/2) J.$$

We have

$$1 = \|\Omega\| = \|\tilde{V}(1/2)V(1/2)\Omega\| \leq \|\tilde{V}(1/2)\| \|V(1/2)\Omega\|,$$

so that

$$\|V(1/2)\Omega\|^{-1} \leq \|\tilde{V}(1/2)\|.$$

Hence

$$\|R^{-1}\| \leq \|\tilde{V}(1/2)\| \|V(1/2)\| \leq \exp\{\|h\|\}.$$

Proposition 4.2. *Suppose that the sequence $\{h_n : n=2, 3, \dots\}$ of relative Hamiltonians is such that the functions $t \rightarrow \sigma_t^{(n-1)}(h_n)$ extend to analytic functions on the strip $\{z \in \mathbb{C} : -1/2 < \text{Im } z < 0\}$, continuous on the boundaries, with values in \mathcal{M} ; let $\|h_n\|_{n-1} = \sup\{\|\sigma_{is}^{(n-1)}(h_n)\| : 0 \leq s \leq 1/2\}$. Then*

$$\|R_n\| \leq \exp\{\|h_n\|_{n-1}\}, \tag{4.8}$$

$$\|(R_n^*R_n - 1)\Omega_n\| \leq (\exp\{2\|h_n\|_{n-1}\} - 1)^{1/2}. \tag{4.9}$$

Proof. (4.8) is proved in Lemma 4.1. Next,

$$\begin{aligned} \|(R_n^*R_n - 1)\Omega_n\|^2 &= \|R_n^*\Omega_{n-1} - \Omega_n\|^2 \\ &= \|R_n^*\Omega_{n-1}\|^2 + \|\Omega_n\|^2 - 2\operatorname{Re}\langle R_n^*\Omega_{n-1}, \Omega_n \rangle \\ &= \|R_n^*\Omega_{n-1}\|^2 + \|\Omega_n\|^2 - 2\|\Omega_{n-1}\|^2 = \|R_n^*\Omega_{n-1}\|^2 - 1 \\ &\leq \|R_n^*\|\|\Omega_{n-1}\| - 1 \leq \exp\{2\|h_n\|_{n-1}\} - 1, \end{aligned}$$

which is (4.9).

Remark. In applications where the spectral gap γ_n tends to 0 as $n \rightarrow \infty$, one needs $\|h_n\|_{n-1} \rightarrow 0$ faster than γ_n (at least). This implies that $\|\omega_n - \omega_{n-1}\| \rightarrow 0$ as $n \rightarrow \infty$, but by no means does it necessarily follow that ω_n converges to a limit as $n \rightarrow \infty$.

Remark. In the most classical applications (simulated annealing on a compact state space X with time-independent energy function $U: X \rightarrow [0, \infty)$ and with a sequence β_n of inverse temperatures increasing to $+\infty$), one has simply $h_n = (\beta_n - \beta_{n-1})U \geq 0$. The case of non-compact X and unbounded U can be handled as in [24].

§ 5. Generalizations

In the main application of the above results, simulated annealing, the states ω_n represent thermal states at different temperatures of a fictitious finite (but large) physical system. For infinite physical systems thermal states at different temperatures are typically *disjoint* states on a C^* -algebra \mathcal{A} , meaning that for each n there is a GNS triple $(\mathcal{H}_n, \pi_n, \Omega_n)$ associated with the pair (\mathcal{A}, ω_n) , the cyclic vector Ω_n is also separating for the von Neumann algebra $\pi_n(\mathcal{A})''$, but no subrepresentation of π_n is unitarily equivalent to a subrepresentation of π_m for $n \neq m$. Unfortunately, there is no simple generalization of the above techniques to this new situation, in view of the following

Lemma 5.1. *Let ω_1, ω_2 be states on a C^* -algebra \mathcal{A} , with GNS triples $(\mathcal{H}_1, \pi_1, \Omega_1), (\mathcal{H}_2, \pi_2, \Omega_2)$ such that Ω_i is also separating for the bicommutant $\pi_i(\mathcal{A})''$ of $\pi_i(\mathcal{A})$ in $\mathcal{B}(\mathcal{H}_i): i=1, 2$. If the operator $R: \pi_2(\mathcal{A})\Omega_2 \subseteq \mathcal{H}_2 \rightarrow \mathcal{H}_1$ defined by*

$$R\pi_2(a)\Omega_2 = \pi_1(a)\Omega_1: \quad a \in \mathcal{A} \tag{5.1}$$

is closable, then π_1 is unitarily equivalent to a subrepresentation of π_2 .

Proof. Let q be the densely defined quadratic form on $\pi_2(\mathcal{A})\Omega_2 \subseteq \mathcal{H}_2$ given by

$$q(\pi_2(a)\mathcal{Q}_2)=\|\pi_1(a)\mathcal{Q}_1\|^2=\|R\pi_2(a)\mathcal{Q}_2\|^2: \quad a \in \mathcal{A}. \tag{5.2}$$

Since R is closable, q is closable. Let x be the positive self-adjoint operator in \mathcal{H}_2 associated with the closure \bar{q} of q : then $\pi_2(\mathcal{A})\mathcal{Q}_2 \subseteq \mathcal{D}(x^{1/2})$ and

$$\|x^{1/2}\pi_2(a)\mathcal{Q}_2\|^2=\|\pi_1(a)\mathcal{Q}_1\|^2. \tag{5.3}$$

For unitary $u \in \mathcal{A}$, one has

$$\|x^{1/2}\pi_2(u)\pi_2(a)\mathcal{Q}_2\|^2=\|\pi_1(u)\pi_1(a)\mathcal{Q}_1\|^2=\|\pi_1(a)\mathcal{Q}_1\|^2=\|x^{1/2}\pi_2(a)\mathcal{Q}_2\|^2$$

so that $\pi_2(u)^*x\pi_2(u)=x$ in the sense of quadratic forms, and the spectral projections of x commute with $\pi_2(u)$. Since a Banach *-algebra is generated as a linear space by its unitary elements, x is affiliated with the commutant $\pi_2(\mathcal{A})'$, and

$$\|\pi_2(a)x^{1/2}\mathcal{Q}_2\|^2=\|x^{1/2}\pi_2(a)\mathcal{Q}_2\|^2=\|\pi_1(a)\mathcal{Q}_1\|^2: \quad a \in \mathcal{A}. \tag{5.4}$$

Let \mathcal{K} be the closed subspace of \mathcal{H}_2 given by $\overline{\pi_2(\mathcal{A})x^{1/2}\mathcal{Q}_2}$; \mathcal{K} is stable under $\pi_2(\mathcal{A})$. Let U be the linear operator mapping $\pi_1(\mathcal{A})\mathcal{Q}_1 \subseteq \mathcal{H}_1$ into \mathcal{K} defined by

$$U\pi_1(a)\mathcal{Q}_1=\pi_2(a)x^{1/2}\mathcal{Q}_2: \quad a \in \mathcal{A}. \tag{5.5}$$

By (5.4), U extends to an isometry of \mathcal{H}_1 into \mathcal{K} . Moreover, U is actually unitary from \mathcal{H}_1 onto \mathcal{K} . Indeed, for all a, b in \mathcal{A} , one has

$$\begin{aligned} \langle U\pi_1(a)\mathcal{Q}_1, \pi_2(b)x^{1/2}\mathcal{Q}_2 \rangle &= \langle \pi_2(a)x^{1/2}\mathcal{Q}_2, \pi_2(b)x^{1/2}\mathcal{Q}_2 \rangle \\ &= \langle \pi_2(a)\mathcal{Q}_2, x\pi_2(b)\mathcal{Q}_2 \rangle = \langle \pi_1(a)\mathcal{Q}_1, \pi_1(b)\mathcal{Q}_1 \rangle \end{aligned}$$

where the last two equalities follow from the fact that x is affiliated with $\pi_2(\mathcal{A})'$ and by polarization from (5.3), respectively. Hence $U^*\pi_2(b)x^{1/2}\mathcal{Q}_2=\pi_1(b)\mathcal{Q}_1$ and $UU^*\pi_2(b)x^{1/2}\mathcal{Q}_2=\pi_2(b)x^{1/2}\mathcal{Q}_2$. By density, $UU^*=1$ on \mathcal{K} .

Now it is an easy exercise to prove that

$$U\pi_1(a)U^*=\pi_2(a)|_{\mathcal{K}} \quad \forall a \in \mathcal{M} \tag{5.6}$$

which proves that π_1 is unitarily equivalent to a subrepresentation of π_2 .

For this reason, the only generalization running on the same lines as the arguments of § 3 can be obtained by assuming the following: we have dynamical maps $\tau_n: n=1, 2, \dots$, all defined on the same von Neumann algebra \mathcal{M} , each map with an invariant faithful normal state $\omega_n=\langle \mathcal{Q}_n, \cdot \mathcal{Q}_n \rangle$, and there exist closed operators $R_n: n=2, 3, \dots$, affiliated with \mathcal{M}' , such that

$$\mathcal{Q}_n \in \mathcal{D}(R_n), \quad R_n\mathcal{Q}_n=\mathcal{Q}_{n-1}: \quad n=2, 3, \dots \tag{5.7}$$

Conditions equivalent to (5.7) with closed unbounded R_n are discussed in Kosaki [25]. In particular, it is *not* true that, if ω_n is a faithful normal state on \mathcal{M} ,

each normal state on \mathcal{M} can be represented in this form.

In order to make sense of the formulas in Lemma 3.1 in this more general situation, it suffices to assume that

$$T_{n-1}^* \text{ maps } \mathcal{H} \text{ into } \mathcal{D}(R_n^*) \quad \forall n=2, 3, \dots \tag{5.8}$$

However, something more is needed to imitate the estimates in Lemma 3.2 and Theorem 3.3. To be specific, we assume the following.

Assumption 5.2. *Each τ_n can be written as the product of two dynamical maps $\tilde{\tau}_n$ and $\hat{\tau}_n$*

$$\tau_n = \tilde{\tau}_n \hat{\tau}_n \tag{5.9}$$

with similar properties: i. e. $\tilde{\tau}_n$ and $\hat{\tau}_n$ leave ω_n invariant, so that they are associated with contractions \tilde{T}_n and \hat{T}_n on \mathcal{H} such that

$$\tilde{T}_n(a\Omega_n) = \tilde{\tau}_n(a)\Omega_n; \quad \hat{T}_n(a\Omega_n) = \hat{\tau}_n(a)\Omega_n; \tag{5.10}$$

moreover $\tilde{\tau}_n$ satisfies a spectral gap condition with a constant e^{-i_n} , so that

$$\|\tilde{T}_n^* \Psi\| \leq e^{-i_n} \|\Psi\| \quad \forall \Psi \in \mathcal{H} \text{ with } \langle \Omega_n, \Psi \rangle = 0, \tag{5.11}$$

and finally

$$\hat{T}_{n-1}^* \text{ maps } \mathcal{H} \text{ into } \mathcal{D}(R_n^*): \quad n=2, 3, \dots \tag{5.12}$$

The above conditions are rather natural if

$$\tau_n = \exp [(t_n - t_{n-1})\mathcal{L}_n] \tag{5.13}$$

with $t_n > 0$, \mathcal{L}_n being the generator of a semigroup of dynamical maps: one can take

$$\tilde{\tau}_n = \exp [(t_n - t_{n-1})(1 - \zeta_n)\mathcal{L}_n], \quad \hat{\tau}_n = \exp [(t_n - t_{n-1})\zeta_n\mathcal{L}_n] \tag{5.14}$$

with $0 < \zeta_n < 1$. In the case of classical Langevin diffusion on \mathbf{R}^n (cf. [26]), in which \mathcal{L}_n is a differential operator of the form $-\Delta + \beta_n \nabla U \cdot \nabla$, a condition of the form (5.12) follows from suitable intrinsic hypercontractivity properties of the semigroup generated by \mathcal{L}_n provided that U grows at infinity fast (typically, faster than $(\text{const.})|x|^2$, cf. [27]).

As a consequence of (5.9), we have

$$T_n^* = \hat{T}_n^* \tilde{T}_n^*: \quad n=1, 2, \dots \tag{5.15}$$

As a consequence of (5.12) and of the closed graph theorem, the operators \tilde{R}_n^* defined by

$$\tilde{R}_n^* = R_n^* \hat{T}_{n-1}^*: \quad n=2, 3, \dots \tag{5.16}$$

are everywhere defined and bounded.

Now define a sequence $\tilde{\Psi}_n$ of vectors in \mathcal{H} by

$$\begin{cases} \tilde{\Psi}_1 = \tilde{T}_1^* x_{\varphi_0, 1} \Omega_1 \\ \tilde{\Psi}_n = \tilde{T}_n^* \tilde{R}_n^* \tilde{\Psi}_{n-1}; \quad n=2, 3, \dots \end{cases} \tag{5.17}$$

Then the vectors Ψ_n such that $\varphi_n(a) = \langle \Psi_n, a \Omega_n \rangle : a \in \mathcal{M}$ are given by

$$\Psi_n = \hat{T}_n^* \tilde{\Psi}_n : \quad n=1, 2, \dots \tag{5.18}$$

Lemma 5.3. *Under the above assumptions, let*

$$\tilde{\alpha}_n = \tilde{\gamma}_n - \log \|\hat{R}_n\|, \tag{5.19}$$

$$\tilde{\beta}_n = e^{-\tilde{\gamma}_n} \|(\hat{R}_n^* \hat{R}_n - 1) \Omega_n\|. \tag{5.20}$$

Then, for all $n=2, 3, \dots$.

$$\|\tilde{\Psi}_n - \Omega_n\| \leq e^{-\tilde{\alpha}_n} \|\tilde{\Psi}_{n-1} - \Omega_{n-1}\| + \tilde{\beta}_n. \tag{5.21}$$

Proof. We have

$$\tilde{\Psi}_n - \Omega_n = \tilde{T}_n^* \tilde{R}_n^* \tilde{\Psi}_{n-1} - \Omega_n = \tilde{T}_n^* (\tilde{R}_n^* \tilde{\Psi}_{n-1} - \Omega_n).$$

Moreover, $\tilde{R}_n^* \tilde{\Psi}_{n-1} - \Omega_n$ is orthogonal to Ω_n since

$$\begin{aligned} \langle \tilde{R}_n^* \tilde{\Psi}_{n-1} - \Omega_n, \Omega_n \rangle &= \langle \tilde{R}_n^* \tilde{T}_{n-1}^* \tilde{\Psi}_{n-1}, \Omega_n \rangle - \langle \Omega_n, \Omega_n \rangle \\ &= \langle \Psi_{n-1}, \Omega_{n-1} \rangle - \langle \Omega_n, \Omega_n \rangle = \varphi_{n-1}(1) - \omega_n(1) = 0. \end{aligned}$$

Then

$$\|\tilde{\Psi}_n - \Omega_n\| \leq e^{-\tilde{\gamma}_n} \|\tilde{R}_n^* \tilde{\Psi}_{n-1} - \Omega_n\|.$$

Note that

$$\tilde{R}_n^* \tilde{\Psi}_{n-1} - \Omega_n = \tilde{R}_n^* (\tilde{\Psi}_{n-1} - \Omega_{n-1}) + \tilde{R}_n^* \Omega_{n-1} - \Omega_n$$

and

$$\tilde{R}_n^* \Omega_{n-1} = \tilde{R}_n^* \hat{T}_{n-1} \Omega_{n-1} = \tilde{R}_n^* \hat{T}_{n-1} R_n \Omega_n = \tilde{R}_n^* \hat{R}_n \Omega_n.$$

Theorem 5.4. *Under the above assumptions, suppose also that there exist real constants $\alpha > 0, \beta \geq 0, 1 > \delta > \varepsilon \geq 0$ such that*

$$\tilde{\alpha}_n \geq \alpha n^{\delta-1}, \quad \tilde{\beta}_n \leq \beta n^{\varepsilon-1} : \quad n=1, 2, \dots \tag{5.22}$$

Then there is a constant C (depending on φ_0), such that

$$|\varphi_n(a) - \omega_n(a)| \leq C \|a\| n^{\varepsilon-\delta} \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{5.23}$$

Proof. We have

$$\begin{aligned} |\varphi_n(a) - \omega_n(a)| &= |\langle \Psi_n - \Omega_n, a \Omega_n \rangle| \\ &\leq \|\Psi_n - \Omega_n\| \|a \Omega_n\| = \|\hat{T}_n (\tilde{\Psi}_n - \Omega_n)\| \|a \Omega_n\| \leq \|\tilde{\Psi}_n - \Omega_n\| \|a\|. \end{aligned}$$

It suffices to prove that

$$\|\tilde{\Psi}_n - \Omega_n\| \leq C n^{\varepsilon-\delta},$$

and this is accomplished exactly as in Theorem 3.3, taking advantage of Lemma 5.3.

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