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Riemann–Hilbert Factorizations and Inverse Scattering for the AKNS–Equation with L^1–Potentials I

By

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Abstract

A detailed group theoretic approach is developed for the 2×2 AKNS system with potentials supported on the half–line. This approach uses consistently Riemann–Hilbert splittings interpreted as factorizations in certain Banach Lie groups. In particular, it is shown how meromorphic splittings introduced by Beals and Coifman can be replaced with regular Riemann–Hilbert splittings leading to the group theoretic notion of a scattering data.

Key words. inverse scattering, Riemann–Hilbert splitting, Banach Lie groups, scattering map

Introduction

Through the work of Sato [9] and Segal and Wilson [11], it is known that certain classes of solutions to the Korteweg–de Vries (KdV) or the Kadomtsev–Petviashivilli (KP) equation form interesting infinite dimensional manifolds. The appearance of an infinite dimensional Grassmannian in the theory of the KdV equation can be traced back to the factorization problem of Riemann–Hilbert type on the unit circle. Moreover, similar factorizations can be formulated for the AKNS systems, and in particular for, the modified KdV equation. The solutions obtained along these lines include all solitons, all rational and all quasiperiodic solutions ("Krichever's method"). In contrast, the solutions obtained from the inverse scattering method on the real line $\boldsymbol{R}$ are generically not in the Sato–Segal–Wilson Grassmannian.

Our goal is to clarify this situation by studying the role of the Riemann–

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Hilbert factorizations in the inverse scattering method. We restrict our attention to 2×2 matrices for the sake of simplicity of presentation. The main results of this paper should carry over to the $n\times n$ case without much change. In this paper we study the 2×2–AKNS–equation

$$\partial_x M(x, z) = [z\mathscr{I}, M(x, z)] + Q(x) M(x, z),$$

where $\mathscr{I} = \operatorname{diag}(-i, i)$ and Q is an off–diagonal matrix with Lebesgue integrable coefficient functions. Our approach uses Riemann–Hilbert splittings, interpreted as factorizations in the Banach Lie group

$$G = \left\{ \begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix};\ a, b, c, d \in \mathscr{A},\ (1+a)(1+d) - cb = 1 \right\}, \text{ where } \mathscr{A}$$

denotes the Banach algebra of Fourier transforms of $L^1(\mathbb{R})$–functions.

Using extensively the work of Beals–Coifman [1] and Bar Yaakov [4] we find that two different decompositions of G are of importance. At one hand, the groups G_- and G_+ of elements $g(z)$ of G with holomorphic extensions to $\mathbb{C}_-$ and $\mathbb{C}_+$ and $\lim_{|z|\to\infty} g(z) = I$ define the Riemann–Hilbert splitting. On the other hand, it turns out that for an open and dense subset P_0^- of the potentials P^- under consideration, the AKNS–equation and the Riemann–Hilbert splitting are related by the equation $g(z)^x = g_-(x, z)^{-1} g_+(x, z)$, $x \leq 0$, where $g_\varepsilon \in G_\varepsilon$ and $g(z)^x = e^{xz\mathscr{I}} g(z) e^{-xz\mathscr{I}}$. Moreover, $g(z) = W_-(z)^{-1} W_+(z)$, where W_- is a lower triangular matrix with diagonal $\mathbb{I}$, i. e. $W_- \in \mathscr{L}_1$, and W_+ is an upper triangular matrix with diagonal $\mathbb{I}$, i. e. $W_+ \in \mathscr{U}_1$. In the proof of the above splitting result we use work of Beals–Coifman, who show that a splitting exists with matrices that are meromorphic in z, and of Sattinger–Zurkowski, who show that the matrices of Beals–Coifman are a product of a holomorphic factor and a very specific meromorphic factor. Note that our result is equivalent with the statement that the holomorphic factors are in G_ε. In addition we show that W_- can be chosen to lie in $\mathscr{L}_1 \cap G_+$ and W_+ can be chosen to be in $\mathscr{U}_1 \cap G_-$. Furthermore, we show that there exists an analytic injective (scattering) map from P_0^- to $(\mathscr{L}_1 \cap G_+) \times (\mathscr{U}_1 \times G_-)$ that has an open and dense image $\mathscr{W}_0$. Conversely, we prove that for $g \in \mathscr{W}_0$ we have $g^x \in G_- G_+$ for $x \leq 0$; therefore the corresponding Riemann–Hilbert splitting yields a solution of the AKNS–equation with a potential $Q = Q_g$. We show that the (inverse scattering) map $g \to Q_g$ maps $\mathscr{W}_0$ onto P_0^- and that it is the inverse of the scattering map $P_0^- \to \mathscr{W}_0$. In the final section we interpret this in terms of the natural image of $\mathscr{W}_0$ in the quotient $\mathscr{M} = \mathscr{L}^- \backslash G / \mathscr{U}^+$, where $\mathscr{L}^-$ and $\mathscr{U}^+$ are the lower triangular and upper triangular matrices in G_- and G_+ respectively.

It is an interesting open question whether the scattering map can be extended to an injective map from all potentials to $\mathscr{M}$.

This paper is divided up into five chapters. The first chapter recalls some basic definitions and results concerning the Banach Lie group G predominantly used in this paper. This includes facts about the Riemann–Hilbert splitting in $G_-G_+\subset G$. In § 2 triangular decompositions of G are investigated. We note that each $g\in G$ can be written in the form $g=uldu$, where u and l are upper and lower triangular respectively, and where d is diagonal. The last part of this chapter investigates the elements of type $ldu\in G_+$, a technical preparation for Chapter 5.

In § 3 we refine results of [1], [4], and [10]. Though this is of very technical character, it gives the basis for the ensuing investigations. In particular, we show how the "meromorphic splitting" of [1] can be reinterpreted (using [10]) as a Riemann–Hilbert splitting. Also, (3. 7. 10) will provide the basis for further investigations dealing with scattering and inverse scattering on the whole real line.

Chapter 4 describes in detail scattering and inverse scattering for the AKNS –equation if the corresponding potential has support in the negative half–line. Here we use in detail the analysis of [1] and [4].

Finally, in Chapter 5 we interpret the previous results in terms of (continuous) quotients of G. In particular we discuss how one can imbed an open dense set P_0^- of potentials into the quotient $\mathscr{M}=\mathscr{L}^-\backslash G/\mathscr{U}^+$.

In a forthcoming publication we plan to extend the results of this paper to potentials integrable on the whole real line. Our starting point will be that for such a potential Q scattering can be defined separately for the restrictions of Q to the left and the right half–line respectively. Equation (3. 7. 10) shows how these two scattering problems are related.

Hence scattering for Q is determined by the scattering of its restriction to the negative half–line together with the transition matrices. It turns out that these transition matrices are closely related to the "discrete scattering data" used in [4]. Finally, we would like to remark that the study of factorization problems appearing in scattering theory was pioneered by Shabat in [12] for potentials with no discrete scattering data. Our goal, on the other hand, is to give a unified theory for all (reasonable) potentials in $L^1(\boldsymbol{R})$.

§ 1. Notation and Basic Results

1. 1. In this chapter we mainly collect notation and some basic results. For proofs we refer to [5] or more original literature [8] etc.

Let

$$(1.1.1)\qquad L^1(\boldsymbol{R})=\{f\colon \boldsymbol{R}\to\boldsymbol{C};\ f \text{ is measurable}, \int_{-\infty}^{+\infty}|f(x)|\,dx<\infty\},$$

where dx denotes the Lebesgue measure. Similarly we use $L^1(0, \infty)$, $L^1(-\infty, 0)$ etc.

By $\mathscr{F}(f)$ we denote the Fourier transformation

(1. 1. 2) $$\mathscr{F}(f)(z)=\hat{f}(z)=\int_{-\infty}^{\infty} e^{izp}f(p)\ dp.$$

The inverse Fourier transform is then given by

(1. 1. 3) $$\mathscr{F}^{-1}(g)(z)=\check{g}(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-izp}g(z)\ dz.$$

The following spaces will be considered throughout this paper

(1. 1. 4) $$\mathscr{A}\ =\mathscr{F}(L^1(\boldsymbol{R})),$$

(1. 1. 5) $$\mathscr{A}^+=\mathscr{F}(L^1(0, \infty)),$$

(1. 1. 6) $$\mathscr{A}^-=\mathscr{F}(L^1(-\infty, 0)).$$

It is known that $\mathscr{A}$ consists of continuous functions vanishing at $\pm\infty$ and that functions in $\mathscr{A}^+$ and $\mathscr{A}^-$ have holomorphic extensions to the upper half–plane $\boldsymbol{C}_+$ and the lower half–plane $\boldsymbol{C}_-$ respectively. Moreover, $f(z)\to 0$ as $|z|\to\infty$ in the corresponding half–plane.

We note that the bounded projections on $\mathscr{A}^+$ and $\mathscr{A}^-$ are given by :

(1. 1. 7) $$[f]^+(z)=\lim_{\varepsilon\to 0^+}\int_{\boldsymbol{R}}\frac{f(\xi)}{\xi-(z+i\varepsilon)}\frac{d\xi}{2\pi i},$$

(1. 1. 8) $$[f]^-(z)=\lim_{\varepsilon\to 0^+}\int_{\boldsymbol{R}}\frac{-f(\xi)}{\xi-(z-i\varepsilon)}\frac{d\xi}{2\pi i}.$$

We will use the norm

(1. 1. 9) $$\|f\| = \|\check{f}\|_1, f\in\mathscr{A},$$

where $\|\circ\|_1$ denotes the L^1–norm.

With this norm, $\mathscr{A}$, $\mathscr{A}^+$, and $\mathscr{A}^-$ are Banach algebras. The rational functions in $\mathscr{A}$, $\mathscr{A}^+$, and $\mathscr{A}^-$ are dense in the corresponding space. In particular, $\mathscr{A}$ is decomposing in the sense of [7].

1. 2. In order to obtain invertible elements it is necessary to extend the algebra $\mathscr{A}$. We set

(1. 2. 1) $$\mathscr{W}=\boldsymbol{C}1+\mathscr{A}.$$

This is called the *Wiener algebra.* Its norm is given by

$$\|\alpha+f\| = |\alpha| + \|f\| . \tag{1.2.2}$$

It is well known that $\mathscr{W}$ is a Banach algebra and its rational elements $\mathscr{W}_{rat}$ are dense. A similar statement holds for the closed subalgebras $\mathscr{W}^{\varepsilon}=\boldsymbol{C}1+\mathscr{A}^{\varepsilon}$.

By $\mathscr{W}^*$ we denote the invertible elements of $\mathscr{W}$. By Wiener's Theorem it is known that $\alpha+f\in\mathscr{W}^*$ if and only if $\alpha\neq 0$ and $\alpha+f(x)\neq 0$ for all $x\in\boldsymbol{R}$.

For this paper it is important to note

(1.2.3) *$\mathscr{W}^*$ is open and dense in $\mathscr{W}$.*

(1.2.4) *Every $f\in\mathscr{W}^*$ has a representation of the form $f=f^- d_0^r f^+$, where*

$$f^-\in(\mathscr{W}^-)^*,\ f^+\in(\mathscr{W}^+)^*,\ r\in\boldsymbol{Z} \text{ and } d_0(z)=\frac{z-i}{z+i}.$$

We also note that r parametrizes the connected components of $\mathscr{W}^*$.

Finally, we recall that for a continuous function f: $\boldsymbol{R}\to\boldsymbol{C}\setminus\{0\}$ satisfying $f(x)\to a$, $0\neq a\in\boldsymbol{C}$, as $x\to\pm\infty$, we define a winding number by

$$\#(f)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d(\arg f)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f'(x)}{f(x)}\,dx. \tag{1.2.5}$$

The winding number satisfies

$$\#(f^{-1})=-\#(f)=\#(\bar{f}), \tag{1.2.6}$$

$$\#(fg)=\#(f)+\#(g). \tag{1.2.7}$$

(1.2.8) $\#:\mathscr{W}^*\to\boldsymbol{Z}$ *is continuous.*

Moreover, if $f\in(\mathscr{W}_{rat})^*$, then

$$\#(f)=\#\text{zeroes}-\#\text{poles of } f \text{ in } \boldsymbol{C}_+. \tag{1.2.9}$$

1.3. In the following chapters we are using essentially the group

$$G=\left\{g=\begin{pmatrix}1+a & b\\ c & 1+d\end{pmatrix},\ a,\ b,\ c,\ d\in\mathscr{A},\ \det g=1\right\}. \tag{1.3.1}$$

From [5; **2.9**] we know that G is a connected Banach Lie group with Lie algebra

(1. 3. 2) $$\mathfrak{g}=Lie\ G=\left\{h=\begin{pmatrix} a & b \\ c & -a\end{pmatrix},\ a,\ b,\ c\in\mathscr{A}\right\}.$$

Similarly, we consider the closed and connected Lie subgroups

(1. 3. 3) $$G_-=\{g\in G;\ a,\ b,\ c,\ d\in\mathscr{A}_-\},$$

(1. 3. 4) $$G_+=\{g\in G;\ a,\ b,\ c,\ d\in\mathscr{A}_+\}.$$

The Lie algebras $\mathfrak{g}_-=\text{Lie } G_-$ and $\mathfrak{g}_+=\text{Lie } G_+$ have an analogous description.

We will use frequently

(1. 3. 5) $$\mathfrak{g}=\mathfrak{g}_-+\mathfrak{g}_+$$

(1. 3. 6) *G_-G_+ is open and dense in G.*

We say that $g\in G$ has a Riemann–Hilbert splitting if and only if $g\in G_-G_+$. The factorization $g=g_-g_+$ is then called a Riemann–Hilbert splitting of g. More precisely, every $g\in G$ can be written in the form

(1. 3. 7) $$g=g_-AD^rA^{-1}g_+,\quad g_\varepsilon\in G_\varepsilon,\ A\in Gl\ (2,\ \mathbb{C}).$$

Here we use $D=\text{diag}\left(\frac{z-i}{z+i},\ \frac{z+i}{z-i}\right)$. Note that we can always assume $r\geq 0$ in (1. 3. 7). Also note that $G_-D^rG_+$ has no interior points if $r>0$.

We will frequently use the following fact (see [5 ; 2. 11])

(1. 3. 8) *The canonical map $G_-\times G_+\to G_-G_+$ is an analytic equivalence.*

In what follows we will sometimes use the projection

(1. 3. 9) $$\textstyle\prod_\varepsilon : G_-G_+\to G_\varepsilon.$$

From (1. 3. 8) it follows that $\prod_\varepsilon$ is well defined and analytic.

Finally, from [5 ; 2. 1. 11] we recall the following characterization of G_ε

(1. 3. 10) *Let $g\in G$. If g has a holomorphic extension to C_ε and if $g(z)\to I$ as $z\in C_\varepsilon$, $|z|\to\infty$, then $g\in G_\varepsilon$.*

§ 2. Triangular Decompositions of G

2. 1. The Lie group G defined in (1. 3. 1) has the important decomposition (1. 3. 7) and allows a Riemann–Hilbert splitting for an open dense subset.

For the purposes of scattering and inverse scattering of the AKNS–equation another decomposition is of particular importance.

For this we introduce the subgroups

(2. 1. 1) $$\mathscr{U}_1=\left\{\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix};\ a\in\mathscr{A}\right\},$$

(2. 1. 2) $$\mathscr{L}_1=\left\{\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix};\ a\in\mathscr{A}\right\},$$

(2. 1. 3) $$\mathscr{D}\ =\left\{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix};\ a\in(1+\mathscr{A})^*\right\}.$$

If we allow the diagonal to be arbitrary, then we drop the subscript " 1 ", i. e. we have

(2. 1. 4) $$\mathscr{U}=\mathscr{D}\mathscr{U}_1 \ \textit{and}\ \mathscr{L}=\mathscr{D}\mathscr{L}_1.$$

The intersections of $\mathscr{U}$, $\mathscr{U}_1$, $\mathscr{L}$, $\mathscr{L}_1$ and $\mathscr{D}$ with G_ε will be denoted by $\mathscr{U}^\varepsilon$, $\mathscr{U}_1^\varepsilon$, $\mathscr{L}^\varepsilon$, $\mathscr{L}_1^\varepsilon$, and $\mathscr{D}^\varepsilon$ respectively.

From [5 ; Theorem 3. 3] we recall

(2. 1. 5) *Every* $\mathscr{A}\in$G admits a decomposition $\mathscr{A}=ULHU'$

where U, $U'\in\mathscr{U}_1$, $L\in\mathscr{L}_1$ and $H\in\mathscr{D}$. Moreover, from **3. 3** we know

(2. 1. 6) $\Omega_l=\mathscr{L}\mathscr{D}\mathscr{U}$ is open and dense in G.

Similarly, $\Omega_u=\mathscr{U}\mathscr{D}\mathscr{L}$ is open and dense in G and also $\Omega=\Omega_u\cap\Omega_l$ is open and dense in G. Similar to the Riemann–Hilbert splitting we have for the canonical map

(2. 1. 7) $\mathscr{L}_1\times\mathscr{D}\times\mathscr{U}_1\rightarrow\Omega_l$ *is an analytic equivalence.*

Using (1. 2. 4) we see

(2. 1. 8) $$\mathscr{D}=\bigcup_{r\geq 0}\mathscr{D}^-D^r\mathscr{D}^+,$$

where $D=\mathrm{diag}\left(\frac{z+i}{z-i}, \frac{z-i}{z+i}\right)$. As a matter of fact, abbreviating $\mathscr{D}_r=\mathscr{D}^- D^r \mathscr{D}^+$ we see that $\Omega_{l,r}=\mathscr{L}_1 \mathscr{D}_r \mathscr{U}_1$ is a connected component of Ω_l.

2. 2. In this section we start the investigation of triangular decompositions of G_+ and G_-. We will have occasion to use this in the description of scattering and inverse scattering of the AKNS–equation.

We set

(2. 2. 1) $$\Omega_l^\varepsilon=\Omega_l \cap G_\varepsilon,$$

(2. 2. 2) $$\Omega_u^\varepsilon=\Omega_u \cap G_\varepsilon,$$

(2. 2. 3) $$\Omega^\varepsilon=\Omega \cap G_\varepsilon.$$

First we prove

Lemma. *The sets Ω^ε, Ω_u^ε, Ω_l^ε, $\varepsilon=\pm$, are open and dense in G_ε.*

Proof. It clearly suffices to carry out the proof for Ω^+. We note that Ω^+ is open in G_+ and non–empty. Now let G_+^{rat} denote the rational elements of G_+. We know from [7] that G_+^{rat} is dense in G_+. It therefore suffices to show that every $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_+^{rat}$ can be approximated arbitrarily well by an element of $G_+^{rat} \cap \Omega$. But this is achieved by moving the zeroes of a and d off the real axis. □

2. 3. The goal of this section is to investigate the sets Ω_u^ε and Ω_l^ε more closely.

Proposition. *Let $g \in \Omega_l^+$ and assume that there exist $l \in \mathscr{L}_1^-$, $u \in \mathscr{U}_1^-$ and d $\in \mathscr{D}$ such that g=ldu. Then l and u are rational and d is meromorphic in C_+. Moreover, d, d^{-1}, l and u have poles in C_+ at the same points.*

Proof. Set $u=\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, $l=\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$, and $d=\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. Then we have

$$g=ldu=\begin{pmatrix} \alpha & \alpha y \\ \alpha x & \beta+\alpha x y \end{pmatrix}.$$

Clearly, this implies

(1) $$\alpha, \beta+\alpha x y \in 1+\mathscr{A}_+,$$

$$(2) \qquad \alpha x,\ \alpha y \in \mathscr{A}_{+}.$$

Splitting $\alpha = \alpha_{+} d_0^r \alpha_{-}$, where $d_0 = \frac{z-i}{z+i}$, we thus see that $\alpha^{*} = \alpha_{+}^{-1}\alpha$ is holomorphic in C_{+} and satisfies $\lim_{z\to\infty} \alpha^{*}(z) = 1$ in C_{+}. But α^{*} is clearly meromorphic in C_{-} and satisfies $\lim_{z\to\infty} \alpha^{*}(z) = 1$ in C_{-}. This implies that α^{*} is rational, whence $\alpha = \alpha_{+}\alpha^{*}$ has zeros in C_{+} precisely at those points where $\alpha^{*} = 0$; consequently, $\beta = \alpha^{-1}$ has poles there. An argument similar to the one just given shows that also $\alpha^{*} y$ and $\alpha^{*} x$ are rational. Therefore x and y are rational. Next we consider $q = \beta + \alpha x y$. Assume that for $z_0 \in C_{+}$ we have $\alpha(z_0) = 0$. Then at this point $\alpha q = 1 + \alpha x \alpha y$ vanishes. Thus

$$(3) \qquad 0 = 1 + \lim_{z\to z_0} (\alpha x\, \alpha y) = 1 + \lim_{z\to z_0} (\alpha x) \lim_{z\to z_0} (\alpha y)$$

where we used (2). We see therefore that in C_{+}, x and y have poles where α has zeros and in fact $\operatorname{ord}_{z_0} \alpha = \operatorname{ord}_{z_0} \alpha^{*} = -\operatorname{ord}_{z_0} x = -\operatorname{ord}_{z_0} y$, for each $z_0 \in C_{+}$. □

If in the above proof $\alpha_{+} = 1$, then also d is rational. We state this as

Corollary 1. *If $g \in \Omega_l^{+}$, $g = ldu$ with $l \in \mathscr{L}_1^{-}$, $u \in \mathscr{U}_1^{-}$ and $d = d_0^r d_{-}$ for some $d_{-} \in (1 + \mathscr{A}_{-})^{*}$. Then l, d, and u are rational.*

Corollary 2. *Let $g \in \Omega_l^{+}$, then there exist $u \in \mathscr{U}_1^{+}$, $l \in \mathscr{L}_1^{+}$, $d \in \mathscr{D}^{+}$ and $g' \in G_{+}^{rat}$ such that*

$$g = l g' u d.$$

Moreover, one can obtain $g' = l' d' u'$ where u', d', l' are rational, $u' \in \mathscr{U}_1^{-}$, $l' \in \mathscr{L}_1^{-}$, and $d' \in D^r d_{-}$ for some $d_{-} \in \mathscr{D}^{-}$.

Proof. Since $g \in \Omega_l^{+}$, there exist $\tilde{l} \in \mathscr{L}_1$, $\tilde{u} \in U_1$, $\tilde{d} \in \mathscr{D}$ such that $g = \tilde{l}\tilde{d}\tilde{u}$. Factoring $\tilde{d} = d_{-} D^r d_{+}$ with $d_{+} \in \mathscr{D}^{+}$, $d_{-} \in \mathscr{D}^{-}$ we obtain $g = \tilde{l} d_{-} D^r d_{+} \tilde{u} = \tilde{l} D^r d_{-} u' d_{+}$. Next we split $\tilde{l} = l^{+} l^{-}$ and $u' = u^{-} u^{+}$. Then $g = l^{+}(l^{-} D^r d_{-} u^{-}) u^{+} d_{+}$. To $g' = l^{-} D^r d_{-} u^{-}$ we can apply Corollary 1 and obtain the assertion. □

Remark. 1) From the proof it is clear that one can also obtain a representation of g in the form $g = d l g' u$.

2) Mutatis mutandis the lemma above also holds for Ω_l^{-} and Ω_u^{+}, Ω_u^{-}.

3) The rational functions x and y have poles only in C_{+}.

2. 4. In this section we prove essentially the converse of the lemma above. We recall from (1. 3. 1) that for $g \in G$ we always have $\det g = 1$.

Proposition. *Let* $x=\frac{N_+N_-}{P_+}\in\mathscr{A}_-$ *be rational such that* N_ε *has roots only in* $C_\varepsilon\cup R$ *and* P_+ *has roots only in* C_+. *Then there exists* $\tau_+=ldu\in G_+$ *satisfying*

$$l=\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix},\quad d=\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix},\quad u=\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}$$

where y *is rational,* $y\in\mathscr{A}_-$ *and* $\beta=\frac{P_+}{(z+i)^r}$.

Proof. A straightforward computation shows

$$ldu=\begin{pmatrix}\beta & \beta y\\ \beta x & \beta^{-1}+\beta xy\end{pmatrix}. \tag{1}$$

Since $\det(ldu)=1$ it suffices to show

(2a) β holomorphic in C_+

(2b) βx holomorphic in C_+

(2c) βy holomorphic in C_+

(2d) $\beta^{-1}+\beta xy$ holomorphic in C_+.

We can assume $\beta=d_0^r b_-$, $y\in\mathscr{A}_-$. Hence

$$x=\frac{N_+N_-}{P_+} \tag{3}$$

$$y=\frac{R_+R_-}{T_+} \tag{4}$$

$$\beta=\frac{B_+B_-}{C_+C_-}. \tag{5}$$

Writing $b_-=\frac{U_-U_+}{L_+}$, we obtain from (2a) that L_+ divides $(z-i)^r$. Thus $C_+=1$ and $C_-=(z+i)^r$, $r\geq 0$. We therefore have $\beta=\frac{B_+B_-}{(z+i)^r}$. Since β is invertible, $\deg(B_+B_-)=r$. Combining B_- and $s=\deg B_-$ factors of $z+i$, we see $\frac{B_-}{(z+i)^s}\in(1+\mathscr{A}_+)^*$. It is easy to see that in this case we have a solution of our problem if and only if we have a solution with $B_-=1$. Hence we can assume

$$\beta=\frac{B_+}{C_-},\quad \text{where}\quad C_-=(z+i)^r,\ r\geq 0. \tag{5$'$}$$

Next we evaluate (2b) :

$$\frac{B_+N_+N_-}{C_-P_+} \quad \text{is holomorphic in} \quad C_+.$$

Since N_+ and P_+ have no common divisors, P_+ divides B_+, i. e.

(6) $$B_+=P_+B_+'.$$

A similar argument applies to (2c) :

$$\frac{B_+R_+R_-}{C_-T_+} \quad \text{is holomorphic in} \quad C_+,$$

hence T_+ divides B_+, i. e.

(7) $$P_+B_+'=T_+Q_+.$$

Finally, we evaluate (2d) :

(8) $$\frac{C_-}{P_+B_+'}+\frac{P_+B_+'}{C_-}\cdot\frac{N_+N_-}{P_+}\cdot\frac{R_+Q_+R_-}{P_+B_+'} \quad \text{is holomorphic in} \quad C_+.$$

Note that in the second summand we can cancel P_+B_+', whereas in the first summand nothing can be cancelled. Thus the first summand has a pole at the roots of B_+', but the second summand it holomorphic there. This implies $B_+'=1$. Altogether we obtain

(9) $$x=\frac{N_+N_-}{P_+}$$

(10) $$y=\frac{R_+Q_+P_-}{P_+}$$

(11) $$\beta=\frac{P_+}{C_-}, \quad \text{where} \quad C_-=(z+i)^r,\ r\geq 0.$$

With this, (2a), (2b), and (2c) are automatically satisfied. This equation (2d) is now equivalent to

(12) $$P_+ \quad \text{divides} \quad C_-^2+(R_+Q_+R_-)N_+N_-.$$

Form (7) we know that Q_+ is a factor of P_+(since $B_+'=1$), therefore (12) implies $Q_+=1$. Thus we have to show that we can satisfy

(13) $$C_-^2+(R_+R_-)N_+N_-=P_+H_+H_-$$

with $C_- = (z+i)^r$, $r = \deg P_+$, $\deg R_+R_- < \deg P_+$.

To show that (13) has a solution we proceed as follows. We know that P_+ and N_+N_- have no common divisors. Hence there exist polynomials $\bar{R}$ and $\bar{H}$ such that

$$P_+\bar{H} - \bar{R}N_+N_- = 1. \tag{14}$$

Multiplying (14) by C_-^2 gives a solution of (13) where only the last condition is not yet satisfied :

$$P_+(\bar{H}C_-^2) - (\bar{R}C_-^2)N_+N_- = C_-^2. \tag{15}$$

It is clear that we obtain another solution of

$$H = \bar{H}C_-^2 + N_+N_-K \tag{16}$$

$$R = \bar{R}C_-^2 + P_+K.$$

To obtain (13) it now suffices to choose K so that $\deg R < \deg P_+$. But dividing $\bar{R}C_-^2$ by P_+ yields a remainder R satisfying this requirement. □

§ 3. Matrix Decomposition and Solutions to the AKNS System

3. 1. In this section we start to discuss the AKNS–equation. As will become clear from the text below we draw extensively from [1] and [4]. Another relevant reference is [2]. Let us consider the system of ordinary differential equations

$$\partial_x M(x, z) = [zJ, M(x, z)] + Q(x)M(x, z), \tag{3. 1. 1}$$

where

$$M(x, z) \in Gl(2; \mathbf{C}) \text{ for all } z \in \mathbf{R},\ x \in \mathbf{R}. \tag{3. 1. 2}$$

$$J = \operatorname{diag}(-i, i), \tag{3. 1. 3}$$

$$Q(x) = \begin{pmatrix} 0 & q_1(x) \\ q_2(x) & 0 \end{pmatrix}. \tag{3. 1. 4}$$

We are mainly interested in the case $Q \in \mathbf{P}$, where

(3. 1. 5) $$P=\{Q;\ q_1,\ q_2\in L^1(\boldsymbol{R})\}.$$

We would like to point out that the derivative in (3. 1. 1) is taken in the distributional sense.

It is easy to see that for two solutions M_1 and M_2 of (3. 1. 1) we have

(3. 1. 6) $$M_1(x,\ z)=M_2(x,\ z)e^{xzJ}A(z)e^{-xzJ}$$

with some matrix $A(z)$ independent of x.

It will be convenient to use the following abbreviation

(3. 1. 7) $$A^x=A(z)^x=e^{xzJ}A(z)e^{-xzJ}.$$

We have stated in the introduction that we want to describe P from a group theoretical point of view using a Riemann–Hilbert problem for the group G defined in 1. 3. To do this we will show $M(x,\ \cdot\ )\in G$ for all $x\in\boldsymbol{R}$.

To obtain $M\in G$ we impose the additional condition

(3. 1. 8) $$\lim_{x\to-\infty} M(x,\ z)=I \text{ for all } z\in\boldsymbol{R}.$$

(3. 1. 9) $$M(x,\ z) \text{ is bounded in } x \text{ for all fixed } z\in\boldsymbol{R}.$$

It has been shown in [4; Lemma 1. 2] that for $Q\in P$ the equation (3. 1. 1) together with the conditions (3. 1. 8) and (3. 1. 9) has a unique continuous solution $M=M^Q$.

We show

Proposition. *Let $Q\in P$, then the unique solution $M=M^Q$ of* (3. 1. 1), (3. 1. 8), *and* (3. 1. 9) *also satisfies*

(3. 1. 10) $$M(x,\ \cdot\ )\in I+\mathrm{Mat}(2,\ 2\ ;\ \mathscr{A}) \textit{ for all } x\in\boldsymbol{R},$$

(3. 1. 11) $$\det M(x,\ z)=1 \textit{ for all } x,\ z\in\boldsymbol{R}.$$

Moreover, if $Q\in P$ has compact support, then $M(x,\ \cdot\ )$ extends to an entire function and $M(\ \cdot\ ,\ z)$ is absolutely continuous.

Proof. As pointed out in [1; §2] M is the unique solution of the Volterra equation

(3. 1. 12) $$M(x,\ z)=I+\int_{-\infty}^{x} e^{(x-y)zJ}Q(y)M(y,\ z)e^{(y-x)zJ}dy.$$

Note that for fixed $x_0 \in \boldsymbol{R}$ and all $x \leq x_0$ we have $M(x, z) = M^{Q'}(x, z)$, where

$$Q'(y) = \begin{cases} Q(y) & \text{for} \quad y \leq x_0 \\ 0 & \text{for} \quad x_0 < y. \end{cases} \tag{3.1.13}$$

It was shown in [4; Theorem (1. 13)] that $Q \in \boldsymbol{P}$ implies

$$S^Q - I \in \mathrm{Mat}(2, 2; \mathscr{A}),$$

where

$$S^Q(z) = I + \int_{-\infty}^{+\infty} e^{-yzJ} Q(y) M(y, z) e^{yzJ} dy. \tag{3.1.14}$$

It is easy to see that with x_0 and Q' as above

$$M^{Q'}(x, z) = e^{xzJ} S^{Q'}(z) e^{-xzJ}, \qquad x \geq x_0 \tag{3.1.15}$$

holds.

This implies that $M(x_0, \circ) = M^{Q'}(x_0, \circ)$ satisfies (3. 1. 10).

The statement (3. 1. 11) can be derived from [1]. But it can also easily be seen directly:

One notes that $\partial_x \phi = A\phi$ implies $\partial_x \det \phi = (\operatorname{trace} A) \det \phi$. In our situation we set $\phi = M^Q e^{xzJ}$, then $\partial_x \phi = (zJ + Q)\phi$, whence $\partial_x \det \phi = (\operatorname{trace}(zJ + Q)) \det \phi = 0$. Thus $\det \phi = \det M^Q$ does not depend on x. But $\lim_{x \to -\infty} M^Q = I$ by (3. 1. 8) and the claim (3. 1. 11) follows. □

Corollary. *For $Q \in \boldsymbol{P}$ the unique solution of* (3. 1. 1), (3. 1. 8), *and* (3. 1. 9) *is contained in G.*

3. 2. From [4; Theorem (1. 13)] we know $S^Q - I \in \mathrm{Mat}(2, 2; \mathscr{A})$. In the last section we have used this to show that $M^Q - I \in \mathrm{Mat}(2, 2; \mathscr{A})$ holds. Later (see Theorem 3. 5) we will need a stronger result.

For $Q \in \boldsymbol{P}$ and $x \in \boldsymbol{R}$ we set

$$B^Q(x, z) = I + \int_{-\infty}^{x} e^{-yzJ} Q(y) M^Q(y, z) e^{yzJ} dy. \tag{3.2.1}$$

Theorem. *For $Q \in \boldsymbol{P}$ we have*

a) $B^Q(x, \cdot) \in I + \mathrm{Mat}(2, 2; \mathscr{A})$

b) $B^{Q}(x,\ \cdot)-S^{Q}\to 0$ *in* $\mathrm{Mat}(2,2;\mathscr{A})$ *as* $x\to\infty$.

Proof. We will follow closely the proof of Theorem (1.13) of [4]. We have

$$B^{Q}(x,z)=I+\sum_{k=1}^{\infty}B^{k}(x,z), \tag{1}$$

where

$$B^{k}(x,z)=\int\cdots\int_{y_k<y_{k-1}<\cdots<y_1<x} e^{-y_1 zJ}Q(y_1)e^{y_1 zJ}e^{-y_2 zJ}Q(y_2)\cdots e^{-y_k zJ}Q(y_k)e^{y_k zJ}dy_1\cdots dy_k. \tag{2}$$

For each $k>1$ the scalar coefficients $B_{ij}^{k}(x,z)$ of $B^{k}(x,z)$ are given as a sum

$$B_{ij}^{k}(x,z)=\sum_{I}R_{i}^{k}(x,z) \tag{3}$$

where the summation is taken over all sequences of the form $I=(i_1, i_2, \ldots, i_{k-1})$ with $1\le i_1,\ldots,i_{k-1}\le \mathrm{n}=2$ and $i_l\ne i_{l+1}$ and

$$R_{I}^{k}(x,z)=\int_{-\infty}^{x}e^{-iy_1 z(\lambda_i-\lambda_{i_1})}q_{ii_1}(y_1)\int_{-\infty}^{y_1}e^{-iy_2 z(\lambda_{i_1}-\lambda_{i_2})}q_{i_1i_2}(y_2)\cdots\int_{-\infty}^{y_{k-1}}e^{-iy_k z(\lambda_{i_{k-1}}-\lambda_j)}q_{i_{k-1}j}(y_k)\,dy_k\cdots dy_1. \tag{4}$$

Here $\lambda_s=-1$ if $s=1$ and $\lambda_s=1$ if $s=2$. Rewriting (4) we obtain

$$R_{I}^{k}(x,z)=\int\cdots\int_{y_k<y_{k-1}<\cdots<y_1<x} e^{iz[y_1(\lambda_i-\lambda_{i_1})+\cdots+y_k(\lambda_{i_{k-1}}-\lambda_{i_k})]}q_{ii_1}(y_1)q_{i_1i_2}(y_2)\cdots q_{i_{k-1}j}\,dy_k\cdots dy_1. \tag{5}$$

To prove our claim we replace q_{ii_1} by $q_{ii_1}\chi_x$ where χ_x denotes the characteristic function of $(-\infty, x)$ and make the change of variables $y_1\to u=y_1(\lambda_i-\lambda_{i_1})+\cdots+y_k(\lambda_{i_{k-1}}-\lambda_j)$. Then

$$R_{I}^{k}(x,z)=\int_{-\infty}^{\infty}e^{-izu}q_I(x,u)\,du, \tag{6}$$

where

$$q_I(x,u)=\int_B(\lambda_i-\lambda_{i_1})^{-1}(q_{ii_1}\chi_x)((\lambda_i-\lambda_{i_1})^{-1}(u-y_2(\lambda_{i_1}-\lambda_{i_2})\cdots-y_k(\lambda_{i_{k-1}}-\lambda_j)))q_{i_1i_2}(y_2)\cdots q_{i_{k-1}j}(y_k)\,dy_k\cdots dy_1 \tag{7}$$

and the region of integration B is described in $(y_2, \dots, y_k)$ space by

$$(8)\qquad \begin{aligned} &y_k < \cdots < y_2 \\ &y_2 < (\lambda_i - \lambda_{i_1})^{-1}(u - (\lambda_{i_1} - \lambda_{i_2})y_2 - \cdots - (\lambda_{i_{k-1}} - \lambda_j)y_k). \end{aligned}$$

Setting $\delta(k, n) = \left[\frac{k-1}{n^2}\right]$, $n=2$, one shows as in [4]

$$(9)\qquad q_I(x, u) = \frac{1}{\delta(k, n)!} \int_{\cup \sigma B} * \, dy_k \cdots dy_2,$$

where $*$ denotes the same integrand as in (7) and where σ denotes permutations of $y_2, \dots, y_k$ that do not change $*$. Then

$$\begin{aligned} (10)\qquad &\int_{-\infty}^{\infty} |q_I(x, u)| du \\ &\leq \frac{|\lambda_i - \lambda_{i_n}|^{-1}}{\delta(k, n)!} \int_{R^k} |q_{ii_1} \chi_x| ((\lambda_i - \lambda_{i_1})^{-1}(u - y_2(\lambda_{i_1} - \lambda_2) \\ &\qquad \cdots y_k(\lambda_{i_{k-1}} - \lambda_j)))| \, |q_{i_1 i_2}(y_2)| \cdots |q_{i_{k-1} j}(y_k)| \, dy_k \cdots dy_2 \\ &\leq \frac{C}{\delta(k, n)!} |q_{ii_1}|_1 \cdots |q_{i_{k-1} j}|_1 \leq \frac{C}{\delta(k, n)!} |Q|_1^k. \end{aligned}$$

In view of (6) this shows

$$(11)\qquad |R_I^k(x, \bullet)|_{\mathscr{A}} = \int_{-\infty}^{\infty} |q_I(x, u)| \, du \leq \frac{C}{\delta(k, n)!} |Q|_1^k.$$

But as in [4] this implies $|B^k(x, \bullet) - I| \leq C \sum_{k=1}^{\infty} \frac{n^{k+1}}{\delta(k, n)!} |Q|_1^k < \infty$. This proves part (a) of our claim. To verify (b) we consider $S^Q(z) - B^Q(x, z)$. Clearly, this is given like in (2) but with $y_1 \in (x, \infty)$. Thus replacing χ_x by $1 - \chi_x$ we can derive (6) and (9) as before. Now (10) gives $\int_{-\infty}^{\infty} |q_I'(x, u)| \, du \leq \frac{C}{\delta(k, n)!} |Q(1-\chi_x)|_1 \bullet |Q|_1^{k-1}$, whence $|R_I'^k(x, \bullet)|_{\mathscr{A}} = \int_{-\infty}^{\infty} |q_I'(x, u)| \, du \leq \frac{C}{\delta(k, n)!} |Q(1-\chi_x)|_1 |Q|_1^k$. As a consequence we have $|S^Q(z) - B^Q(x, z)| \leq C |Q(1-\chi_x)|_1 \sum_{k=1}^{\infty} \frac{n^{k+1}}{\delta(k, n)!} |Q|_1^{k-1} \leq CA(Q) \bullet |Q(1-\chi_x)|_1$. Clearly now, this last term tends to zero as $x \to \infty$, proving the claim.

Part (b) of the above theorem can be rephrased as

Corollary 1. *For $Q \in P$ we have $M^Q(x, \bullet)^{-x} \to S^Q$ in $I + \mathrm{Mat}(2, 2; \mathscr{A})$ as $x \to \infty$.*

Corollary 2. *For $Q\in P$ the* (1, 1)*-coefficient $M^Q(x, \cdot)_{11}$ of $M^Q(x, \cdot)$ converges in $1+\mathscr{A}$ to $(S^Q)_{11}$ as $x\to\infty$.*

Proof. It suffices to note that convergence in $CI+\mathrm{Mat}(2, 2; \mathscr{A})$ occurs by definition (1. 3. 1) entrywise and that $(M^Q(x, z)^{-x})_{11}=M^Q(x, z)_{11}$.

3. 3. In this section we recall some more definitions and results of [1] which are of importance for the considerations of this paper.

Let $Q\in P$. We introduce functions $M^+(x, z)=(M^Q)^+(x, z)$, $x\in R$, $z\in C_+$ and $M^-(x, z)=(M^Q)^-(x, z)$, $x\in R$, $z\in C_-$ by the conditions

(3. 3. 1) $\qquad M^\varepsilon(x, z)$ *satisfies* (3. 1. 1), (3. 1. 8), *and* (3. 1. 9) *for* $z\in C_\varepsilon$.

The following results have been shown in [1 ; Theorem A]

Theorem 1 ([1]). *Let $Q\in P$. Then the following holds*
a) *There exist bounded discrete subsets $Z_\varepsilon\subset C_\varepsilon$ such that* (3. 3. 1) *has for every $z\in C_\varepsilon\backslash Z_\varepsilon$ a unique solution $M^\varepsilon(x, z)$ and such that for every $x\in R$ the function $M^\varepsilon(x, z)$ is meromorphic in C_ε with poles precisely at the points of Z_ε, and $\lim_{x\to\infty} M^\varepsilon(x, z)=I$.*
b) *There exists a dense and open subset $P_0\subset P$ such that for $Q\in P_0$ we actually have*

(3. 3. 2) $\qquad Z_+$ *and* Z_- *are finite*

(3. 3. 3) $\qquad$ *the poles of $M^+(x, z)$ and $M^-(x, z)$ are simple,*

(3. 3. 4) $\qquad$ *distinct columns of M^+ and M^- have distinct poles*

(3. 3. 5) $\qquad M^\varepsilon(x, \cdot)$ *extends continuously to R from C_ε, $\varepsilon=\pm$.*

Remark. a) From [1] it actually follows that P_0 is defined by (3. 3. 2) to (3. 3. 5).

b) The elements of P_0 are sometimes called generic potentials.

c) From Lemma 2. 22 and Theorem 3. 32 in [4] it follows that $M(x, z)$ is absolutely continuous in x if $Q\in P_0$.

d) The proof of Proposition 3. 1 shows that det $M^\varepsilon(x, z)=1$ for $z\in\bar{C}_\varepsilon$.

e) Since M^ε extends to R for $Q\in P_0$ and satisfies (3. 1. 1) we know that $M^\varepsilon=MW_\varepsilon^x$ for some matrix function $W_\varepsilon(z)$, independent of x. This formula also appears in [4, 2. 11]. Note that for a comparison with [4] one has to replace W_- by $(W_-)^{-1}$.

More precisely on has (recall the definition of $\mathscr{U}_1$ and $\mathscr{L}_1$ from 2.1).

Theorem 2 ([1]). *Let $Q\in P_0$. Then there exist $W_+\in\mathscr{U}_1$ and $W_-\in\mathscr{L}_1$ such that for $\varepsilon=\pm$.*

$$M^\varepsilon(x, z)=M(x, z)W_\varepsilon(z)^x \text{ for } x, z\in \boldsymbol{R}. \tag{3.3.6}$$

In the 2×2 case, the poles of M^+ occur only in the second column and the poles of M^- occur only in the first column. Moreover, for $z_0\in Z_\varepsilon$,

$$\operatorname{Res} M^\varepsilon(x, z_0)=\lim_{z\to z_0}[M^\varepsilon(x, z)W_\varepsilon(z_0)^x], \tag{3.3.7}$$

where $W_+(z_0)$ is an upper triangular matrix (with diagonal 0) and $W_-(z_0)$ is a lower triangular matrix (with diagonal 0).

Remark. To avoid confusion we point out that for $z\in \boldsymbol{R}$ the matrix $W_\varepsilon(z)$ has diagonal I, whereas for $z_0\in \boldsymbol{C}_+\cup \boldsymbol{C}_-$ the matrix $W_\varepsilon(z_0)$ has diagonal 0.

The set

$$\mathcal{S}=\{(W_-, W_+, Z=Z_-\cup Z_+, \{W(z_i); z_i\in Z\})\} \tag{3.3.8}$$

is called the set of *scattering data.*

More precisely, we will refer to W_+, W_- as the *continuous scattering data* and to the remaining parameters as *discrete scattering data.*

3.4. From Theorem 2 of the last section one would suspect that one can remove all poles from M^ε by a matrix of type W_ε. This can actually be done. For $Q\in P_0$ and for some more general Q's this is contained in [9; Theorem 1.4].

It is our goal to refine this result to obtain a new type of forward scattering.

The Theorem below generalizes [9; Theorem 1.4].

Let $Q\in P_0$ and let Z_+, Z_- be associated with Q as in (3.3.2) and $W_\varepsilon(z_i)$ as in (3.3.7). Then we set for $\varepsilon=\pm$

$$V^{-\varepsilon}(z)=(V^Q)^{-\varepsilon}(z)=I+\sum_{z_i\in Z_\varepsilon}\frac{1}{z-z_i}W_\varepsilon(z_i). \tag{3.4.1}$$

Theorem. *Let $Q\in P_0$, then there exist $\eta^\varepsilon\in G$ such that with $V^{-\varepsilon}$ as above*

$$M^\varepsilon(x, z)=\eta^\varepsilon(x, z)V^{-\varepsilon}(z)^x \text{ for } z\in \boldsymbol{C}_\varepsilon\cup \boldsymbol{R}, \tag{3.4.2}$$

$$\eta^\varepsilon(x, z) \text{ has a holomorphic extension in } z \text{ to } \boldsymbol{C}_\varepsilon, \tag{3.4.3}$$

$$\eta^\varepsilon(x, \circ)\in G_\varepsilon \text{ if } x\leq 0, \tag{3.4.4}$$

(3. 4. 5) $$\lim_{x\to-\infty} \eta^{\varepsilon}(x, z)=I \text{ for } z\in C_{\varepsilon}.$$

Moreover, $V^{-\varepsilon}\in G_{-\varepsilon}$ *and the decomposition* (3. 4. 2) *is unique.*

Proof. From [9 ; Theorem 1. 4] we know

(1) $$M^{\varepsilon}(x, z)=\eta^{\varepsilon}(x, z)\, U^{-\varepsilon}(z)^{x}, \quad \varepsilon=\pm.$$

Here (3. 4. 5) holds and U^{-} is upper triangular with diagonal I and off–diagonal element a finite sum $u(z)=\sum_{z_0\in Z_+}\sum_{k}\frac{q_k(z_0)}{(z-z_0)^k}$. For U^{+} the analogous statement holds. From Theorem 3. 3. 2 we know that the poles of M^{+} occur in the second column only and since $Q\in P_0$, these poles are all simple. From (1) we see that the first column of η^{+} is holomorphic in C^{+} and we obtain for the matrix coefficients

(2) $$(M^{+})_{12}=(\eta^{+})_{11}u \text{ and } (M^{+})_{22}=(\eta^{+})_{21}u.$$

We recalled above that the poles of M^{+} are simple; moreover, since $\det M^{+}=\det \eta^{+}=1$, $(\eta^{+})_{11}$ and $(\eta^{+})_{21}$ do not vanish simultaneously at any $z_0\in Z_{+}$. Therefore, the poles in u are simple. Now (3. 3. 7) implies that U^{-} is given by (3. 4. 1). Similarly one proves that U^{+} is of the form stated. This implies that $V^{-\varepsilon}$ is in $G_{-\varepsilon}$. It is also easy to see that the decomposition (3. 4. 2) is unique. Next we note that $\eta^{\varepsilon}\in G$ since M^{ε}, $V^{-\varepsilon}\in G$. We want to show that actually $\eta^{\varepsilon}(x, \bullet)\in G_{\varepsilon}$ if $x\leq 0$. To verify this we will apply (1. 3. 10). It suffices to show

(3) $$\lim_{\substack{z\to\infty\\ z\in C_{\varepsilon}}} \eta^{\varepsilon}(x, z)=I.$$

But from Theorem 3. 3. 1 we know $\lim_{z\to\infty} M^{\varepsilon}(x, z)=I$ and (3. 4. 1) implies $\lim_{z\to\infty} V^{-\varepsilon}(z)=I$, whence also $\lim_{z\to\infty} V^{-\varepsilon}(z)^{x}=I$ if $x\leq 0$. Note that we use here the form of J and the form of $V^{-\varepsilon}$. Now (3. 4. 2) implies (3) and (1. 3. 10) implies the claim. □

3. 5. In the last section we decomposed $M^{\varepsilon}=\eta^{\varepsilon}(V^{-\varepsilon})^{x}$, where η^{ε} was satisfying several conditions for $x\leq 0$. In this section we want to discuss decompositions $M^{+}=\varphi^{+}l^{x}$ and $M^{-}=\varphi^{-}u^{x}$ where φ^{ε} satisfies conditions for $x\geq 0$. In the last section we had $V^{\varepsilon}\in G_{\varepsilon}$; in this section l and u are not (necessarily) contained in G_{+} or in G_{-}.

Theorem. *Let* $Q\in P_0$, *then there exist* $\varphi^{\varepsilon}\in G$ *and* $l\in\mathscr{L}$, $u\in U$ *such that*

(3. 5. 1) $$M^+(x, z)=\varphi^+(x, z)\, l(z)^x \text{ for } z\in C_+\cup R,$$

(3. 5. 2) $$M^-(x, z)=\varphi^-(x, z)\, u(z)^x \text{ for } z\in C_-\cup R,$$

(3. 5. 3) $$\varphi^\varepsilon \text{ has a holomorphic extension to } C_\varepsilon,$$

(3. 5. 4) $$\varphi^\varepsilon(x, \bullet)\in G_\varepsilon \text{ for } x\geq 0,$$

(3. 5. 5) $$\lim_{x\to\infty} \varphi^\varepsilon(x, z)=I \text{ for all } z\in C_\varepsilon.$$

Proof. Since $Q\in P_0$ we know from [1 ; 5. 9] that $\lim_{x\to\infty} M^+(x, z)=L_0(z)$ for $z\in C_+$, where $L_0(z)=\mathrm{diag}\{\delta(z), \delta(z)^{-1}\}$ is meromorphic in C_+. Moreover, [4 ; 2. 18] and [4 ; 2. 66c] imply $L_0\in G$, and that δ has only simple zeroes at $z_i \in Z_+$ and no additional poles or zeroes in C_+. We want to show that actually $\delta\in 1+\mathscr{A}_+$ holds. To verify this we recall first from Corollary 2 of section 3. 2 that $M(x, \bullet)_{11}\to S_{11}$ in $1+\mathscr{A}_+$. Next we recall from (3. 3. 6) that for $z\in R$ we have $M^+(x, z)=M(x, z)W_+^x$, where W_+ is upper triangular with diagonal I. Hence $M(x, \bullet)_{11}=M^+(x, \bullet)_{11}$. This shows that $\lim_{x\to\infty} M^+(x, z)_{11}=S_{11}$ exists in $1+\mathscr{A}$, actually then it exists in $1+\mathscr{A}_+$. From [4 ; 2. 18] we thus conclude $\lim_{x\to\infty} M^+(x, \bullet)_{11}=\delta$ in $1+\mathscr{A}_+$. This shows in particular (since δ has only finitely many zeroes)

(0) $$\delta\in 1+\mathscr{A}_+ \text{ and } \lim_{z\to\infty} L_0(z)=I.$$

Now we set

(1) $$\bar{M}^+(x, z)=M^+(x, z)L_0(z)^{-1}.$$

It $M^+(x, z)=\begin{pmatrix} a(x, z) & b(x, z) \\ c(x, z) & d(x, z)\end{pmatrix}$, then $\bar{M}^+(x, z)=\begin{pmatrix} a\delta^{-1} & b\delta \\ c\delta^{-1} & d\delta\end{pmatrix}$, where we have suppressed the variables. We want to find a matrix $h=\begin{pmatrix} 1 & 0 \\ \gamma & 1\end{pmatrix}$ such that $h\in G$ and $\varphi^+=\bar{M}^+h$ satisfies the requirements of the theorem. First we note that $\bar{M}^+h$ is of the form

$$\bar{M}^+h=\begin{pmatrix} a\delta^{-1}+\gamma b\delta & b\delta \\ c\delta^{-1}+\gamma d\delta & d\delta\end{pmatrix}.$$

Here the second column is analytic in C_+ and has a continuous extension to R since b and d have simple poles at z_j whereas δ is zero there. Therefore one needs to show that the first column can be made analytic by a suitable choice of γ. Thus we want

$$a\delta^{-1}+\gamma b\delta \text{ and } c\delta^{-1}+\gamma d\delta \text{ analytic.} \tag{2}$$

To this end we choose

$$\gamma(x, z)=e^{2ixz}\sum_{z_i\in Z_+}\frac{\gamma_i}{z-z_i}. \tag{3}$$

Then $h\in G$ and $\lim_{x\to\infty} h(x, z)=I$ for $z\in C_+$. It suffices to choose γ_i so that corresponding residues in (2) cancel out. This yields the conditions

$$a(z_j)\delta_j+e^{2ixz_j}\gamma_j(b\delta)(z_j)=0, \tag{4}$$

$$c(z_j)\delta_j+e^{2ixz_j}\gamma_j(d\delta)(z_j)=0. \tag{5}$$

Here $\delta_j=\text{Res}(\delta^{-1}, z_j)$ and $(b\delta)(z_j)$ and $(c\delta)(z_j)$ make sense since b and d have simple poles at z_j and δ has a simple zero at z_j(see Theorem 3. 3. 2 and the remarks on δ above).

Next we evaluate (3. 3. 7) at $z_0=z_j$. We obtain

$$b_j=\text{Res}(b, z_j)=e^{-2ixz_j}a(z_j)w_j, \tag{6}$$

$$d_j=\text{Res}(d, z_j)=e^{-2ixz_i}c(z_j)w_j. \tag{7}$$

Moreover, $(b\delta)(z_j)=b_j\delta_j^{-1}$ and $(d\delta)(z_j)=d_j\delta_j^{-1}$. Inserting this into (4) and (5) we see that (4) and (5) follow from (6) and (7) if we choose

$$\gamma_j=-\delta_j^2 w_j^{-1}. \tag{8}$$

With this choice of γ_j (whence h) we have shown

$$\varphi^+(x, z)=\tilde{M}(x, z)h(x, z)\in G, \tag{9}$$

$$\varphi^+(x, \cdot) \quad \text{has a holomorphic extension to} \quad C_+ \tag{10}$$

$$\lim_{x\to\infty}\varphi^+(x, z)=I \quad \text{for all} \quad z\in C_+. \tag{11}$$

$$M^+(x, z)=\varphi^+(x, z)h(x, z)^{-1}L_0(z), \tag{12}$$

thus (3. 5. 1) is satisfied with

$$l(z)=h_0(z)L_0(z)\in\mathscr{L}, \tag{13}$$

where $h_0(z)=\begin{pmatrix} 1 & 0 \\ -\gamma_0(z) & 1 \end{pmatrix}$

(14) $$\gamma_0(z)=\sum_{z_i\in Z_+}\frac{\gamma_i}{z-z_i}.$$

Finally we show $\varphi^+(x,\ \circ)\in G_+$ for $x\geq 0$. But we have seen in (0) that $L_0(z)\to I$ as $z\to\infty$ and we also know $h(x, z)\to I$, $M^+(x, z)\to I$ as $z\to\infty$ for $z\in C_+$. Therefore $\varphi^+(x, z)\to I$ as $z\to\infty$, whence (1. 3. 10) implies $\varphi^+\in G_+$. Mutatis mutandis the same argument applies to φ^- and the theorem is proven. □

Corollary 1. *For $Q\in P_0$ there exist $q_-\in\mathscr{L}_1$ and $q_+\in\mathscr{U}_1$ such that for all $z\in R$*

(3. 5. 6) $$\lim_{x\to\infty}\varphi^+(x, z)^{-x}=q_-(z),$$

(3. 5. 7) $$\lim_{x\to\infty}\varphi^-(x, z)^{-x}=q_+(z).$$

Proof. From [4 ; Lemma 2. 66] we know that $\lim_{x\to\infty} M^+(x, z)^{-x}=L(z)$ holds, $z\in R$, $L\in\mathscr{L}$. From (3. 5. 1) we thus obtain $L(z)\,l(z)^{-1}=\lim_{x\to\infty}\varphi^+(x, z)^{-x}$. From the proof of the theorem above we know that L and l are both in $\mathscr{L}$ and have the same diagonal factor L_0. This finishes the proof of (3. 5. 6). The remaining part is seen analogously. □

Corollary 2. *Under the assumptions above we have*

(3. 5. 8) $$q_-=Ll^{-1}\ \textit{and}\ q_+=Uu^{-1}.$$

From (13) and (14) of the proof of the above theorem we also obtain

Corollary 3. $l\in\mathscr{L}_1^-\mathscr{D}$.

3. 6. We consider the continuity condition at $x=0$. We have

(3. 6. 1) $$\eta^+(0, z)\,V^-(z)=\varphi^+(0, z)\,l(z),$$

(3. 6. 2) $$\eta^-(0, z)\,V^+(z)=\varphi^-(0, z)\,u(z).$$

For our considerations later the following functions τ^+ and τ^- will be of special importance.

(3. 6. 3) $$\tau^+(z)=l(z)\,V^-(z)^{-1},$$

(3. 6. 4) $$\tau^{-}(z)=u(z)V^{+}(z)^{-1}.$$

Proposition. a) $\tau^{\varepsilon}\in G_{\varepsilon}$,
b) $\tau^{+}\in\mathscr{L}_{1}\mathscr{D}\mathscr{U}_{1}$,
c) $\tau^{-}\in\mathscr{U}_{1}\mathscr{D}\mathscr{L}_{1}$,
d) $l=l^{+}l^{-}$, $l^{\varepsilon}\in G_{\varepsilon}$, $l^{+}=\tau^{+}$, $l^{-}=V^{-}$,
e) $u=u^{-}u^{+}$, $u^{\varepsilon}\in G_{\varepsilon}$, $u^{-}=\tau^{-}$, $u^{+}=V^{+}$.

Proof. Part (a) follows from (3. 6. 1) and (3. 6. 2) in view of the fact that $\eta^{\varepsilon}(0, z)$, $\varphi^{\varepsilon}(0, z)\in G_{\varepsilon}$ holds by Theorem 3. 4 and Theorem 3. 5. The remaining parts follow easily. □

3. 7. For easier reference we write down various (mostly obvious) relations between the many different matrix functions considered so far

(3. 7. 1) $$\eta^{+}(0, z)=\varphi^{+}(0, z)\tau^{+}(z),$$

(3. 7. 2) $$\eta^{-}(0, z)=\varphi^{-}(0, z)\tau^{-}(z),$$

(3. 7. 3) $$M^{-}(x, z)^{-1}M^{+}(x, z)=(W_{-}(z)^{-1}W_{+}(z))^{x},$$

whence in view of (3. 4. 2), (3. 5. 1), and (3. 5. 2)

(3. 7. 4) $$\eta^{-}(x, z)^{-1}\eta^{+}(x, z)=(V^{+}(z)W_{-}(z)^{-1}W_{+}(z)V^{-}(z)^{-1})^{x},$$

(3. 7. 5) $$\varphi^{-}(x, z)^{-1}\varphi^{+}(x, z)=(u(z)W_{-}(z)^{-1}W_{+}(z)l(z)^{-1})^{x}.$$

For $x=0$ we evaluate (3. 7. 4) and apply (3. 7. 1). This yields

(3. 7. 6) $$\tau^{-}(z)^{-1}\varphi^{-}(0, z)^{-1}\varphi^{+}(0, z)\tau^{+}(z)=V^{+}(z)W_{-}(z)^{-1}W_{+}(z)^{-1}V^{-}(z)^{-1}.$$

A similar treatment of (3. 7. 5) gives

(3. 7. 7) $$\tau^{-}(z)\eta^{-}(0, z)^{-1}\eta^{+}(0, z)\tau^{+}(z)^{-1}=u(z)W_{-}(z)^{-1}W_{+}(z)l(z)^{-1}.$$

Comparing (3. 7. 4) at $x=0$ with (3. 7. 7) shows

(3. 7. 8) $$\begin{aligned}&V^{+}(z)W_{-}(z)^{-1}W_{+}(z)V^{-1}(z)^{-1}\\&\quad=\tau^{-}(z)^{-1}u(z)W_{-}(z)^{-1}W_{+}(z)l(z)^{-1}\tau^{+}(z).\end{aligned}$$

The analogous procedure with (3. 7. 5) and (3. 7. 6) yields the same equation.

Finally, recall (see e. g. [4; 2. 18]) that $S(z)=L(z)W_{+}(z)^{-1}$ and $S(z)=$

$U(z)W_-(z)^{-1}$, where L is lower triangular with diagonal term L_0 and U is upper triangular with diagonal term U_0. Thus we have

$$W_-(z)^{-1}W_+(z)=W_-(z)^{-1}S(z)^{-1}S(z)W_+(z)=U^{-1}(z)L(z).$$

We state this and insert it into (3. 7. 8). Thus we obtain

$$(3.7.9)\qquad W_-(z)^{-1}W_+(z)=U^{-1}(z)L(z)$$

$$(3.7.10)\qquad \begin{aligned}&V^+(z)W_-(z)^{-1}W_+(z)V^-(z)^{-1}\\&=\tau^-(z)^{-1}(u(z)U^{-1}(z)L(z)l(z)^{-1})\tau^+(z).\end{aligned}$$

The last formula will be of particular importance in a later publication. We would also like to point out that (3. 7. 10) shows that the τ's change an "upper–lower decomposition" into a "lower–upper decomposition".

3. 8. In this section we show that P_0 is invariant under a natural involution of P. For $Q\in P$ we set $Q^*(x)=-\overline{Q(-x)}$.

Theorem. $P_0^*=P_0$.

Proof. Let $Q\in P_0$. It is straight forward to verify that the functions

$$(1)\qquad F(x,z)=\overline{M^-(-x,\bar{z})},\quad x\in R,\quad z\in C_+,$$

$$(2)\qquad G(x,z)=\overline{M^+(-x,\bar{z})},\quad x\in R,\quad z\in C_-,$$

satisfy the AKNS–equation with Q^*. Since $Q^*\in P$ we thus know

$$(3)\qquad \overline{M^-(-x,\bar{z})}=M^{*+}(x,z)C(z)^x,\quad z\in C_+$$

$$(4)\qquad \overline{M^+(-x,\bar{z})}=M^{*+}(x,z)D(z)^x,\quad z\in C_+.$$

Here $C(z)$ and $D(z)$ are matrices independent of x. Since M^- and M^{*+} both satisfy (3. 1. 8) and (3. 1. 9), the matrix function $C(z)^x$ stays bounded as $x\to-\infty$ and also as $x\to+\infty$. This implies that $C(z)$ is a diagonal matrix, whence $C(z)^x=C(z)$. Similarly one proves that $D(z)$ is diagonal, i. e. $D(z)^x=D(z)$ holds. Now we use (3. 1. 8) and $\lim_{x\to\infty}M^+(x,z)=L_0(z)$ for $z\in C_+$ ([1 ; 5. 9]) in (4) and obtain

$$(5)\qquad D(z)=\overline{L_0(\bar{z})}.$$

Similarly we see

$$C(z)=\overline{U_0(\bar{z})}. \tag{6}$$

Hence

$$\overline{M^-(-x,\bar{z})}=M^{*+}(x,z)\overline{U_0(\bar{z})},\quad z\in C_+ \tag{7}$$

$$\overline{M^+(-x,\bar{z})}=M^{*+}(x,z)\overline{L_0(\bar{z})},\quad z\in C_-. \tag{8}$$

From the last two equations it is clear that $M^{*\varepsilon}(x,\ \bullet\ )$ has a continuous extension to $\bar{C}_\varepsilon$. Moreover, since $M^\varepsilon(x,\ \bullet\ )$ has only finitely many poles and $L_0(\ \bullet\ )$ and $U_0(\ \bullet\ )$ have only finitely many poles and zeroes in C_ε, also $M^{*+}(x,\ \bullet\ )$ has only finitely many poles in C_ε.

Solving (7) and (8) for $M^{*\varepsilon}$ yields

$$M^{*+}(x,z)=\overline{M^-(-x,\bar{z})U_0(\bar{z})^{-1}} \tag{9}$$

$$M^{*-}(x,z)=\overline{M^+(-x,\bar{z})L_0(\bar{z})^{-1}}. \tag{10}$$

Since $L_0=\mathrm{diag}(\delta,\delta^{-1})$ where δ has only simple zeroes and δ^{-1} has only simple poles at the points of Z_+, we see that $M^{*-}(x,\ \bullet\ )$ has only simple poles at $Z^*_-=\bar{Z}_+$. Similarly one sees that $M^{*+}(x,\ \bullet\ )$ has only simple poles at $Z^*_+=\bar{Z}_-$. Moreover, the second column of M^{*-} and the first column of M^{*+} are analytic. This shows that (3. 3. 2), $\cdots$, (3. 3. 5) are satisfied, whence $Q^*\in P_0$.

Corollary. *Let $Q\in P_0$. Then for the finite exceptional sets Z for Q and Z^* for Q^* we have $Z^*_+=\bar{Z}_-$ and $Z^*_-=\bar{Z}_+$.*

§ 4. Scattering and Inverse Scattering on a Half–line

4. 1. In this chapter we shall study the space P_0^- of potentials defined below. On this space the map $Q\mapsto S^Q$ is injective. In a later publication we will exploit the fact that analogous statements also hold for P_0^+ to obtain a description of scattering and inverse scattering for potentials on the whole real line.

We set

$$P^+=\{Q\in P;\ \mathrm{support}(Q)\subset\{x;\ x\geq 0\}\}, \tag{4. 1. 1}$$

$$P^-=\{Q\in P;\ \mathrm{support}(Q)\subset\{x,\ x\leq 0\}\}, \tag{4. 1. 2}$$

(4. 1. 3) $$P_0^\varepsilon = \{P_0 \cap P^\varepsilon\}.$$

Theorem. *P_0^ε is open and dense in P^ε.*

Proof. The proof of Proposition 2. 30 in [1] and the remark before Lemma 4. 7 loc. cit. show that P_0^+ is dense in P^+; it is trivially open, since P_0 is. Theorem 3. 8 implies that also P_0^- is open and dense in P^-. □

4. 2. If $Q \in P^-$, then we know from [1; § 5] that the scattering matrix $S = S^Q$ can be decomposed in two ways (see also [4; 2. 18])

(4. 2. 1) $$S(z) = L(z) W_+(z)^{-1},$$

(4. 2. 2) $$S(z) = U(z) W_-(z)^{-1}.$$

On the other hand, since $Q(x) = 0$ for $x \geq 0$, we have $M(x, z) = S(z)^x$ for $x \geq 0$. Comparing this to (3. 3. 6) yields

(4. 2. 3) $$M^+(x, z) = L(z)^x \text{ for } x \geq 0, \quad z \in \boldsymbol{R},$$

(4. 2. 4) $$M^-(x, z) = U(z)^x \text{ for } x \geq 0, \quad z \in \boldsymbol{R}.$$

From these two formulas we obtain in view of (3. 4. 2)

Corollary. *If $Q \in P_0^-$, then*

(4. 2. 5) $$L(z) = \eta^+(0, z) V^-(z),$$

(4. 2. 6) $$U(z) = \eta^-(0, z) V^+(z).$$

Note that these formulas describe the decomposition of L and U in $G_- G_+$.

4. 3. Let $Q \in P_0^-$ and W_-, W_+ be as in Theorem 3. 3. 2. Then rewriting (3. 7. 4) we obtain

(4. 3. 1) $$\eta^-(x, z)^{-1} \eta^+(x, z) = [(W_-(z) V^+(z)^{-1})^{-1} W_+(z) V^-(z)^{-1}]^x.$$

It is clear that the first factor is in $\mathscr{L}_1$ and the second one is in U_1. We set (using (1. 3. 9))

(4. 3. 2) $$W_-'^{+}(z) = \Pi_+(W_-(z) V^+(z)^{-1}),$$

(4. 3. 3) $$W_+^{'+}(z)=\Pi_+(W_+(z)V^-(z)^{-1}).$$

Setting also $W_-^{'-}=\Pi_-(W_-(z)V^+(z)^{-1})$ and $W_+^{'-}(z)=\Pi_-(W_+(z)V^-(z)^{-1})$ we obtain

(4. 3. 4) $$[[\eta^-(x,z)(W_-^{'-}(z)^x)^{-1}]^{-1}[\eta^+(x,z)(W_+^{'+}(z)^x)^{-1}] = [W_-^{'+}(z)^{-1}W_+^{'-}(z)]^x.$$

It is easy to see that $W_\varepsilon^{'\varepsilon}(z)^x \in G_\varepsilon$ for $x \leq 0$. We also know from (3. 4. 4) that $\eta^\varepsilon(x, \bullet) \in G_\varepsilon$ if $x \leq 0$. Setting $\sigma^\varepsilon(x,z)=\eta^\varepsilon(x,z)(W_\varepsilon^{'\varepsilon}(z)^x)^{-1}$ we thus have shown

Proposition. *Let $Q \in P_0^-$; then there exists $\sigma^\varepsilon(x,z) \in G$ such that*

a) $\sigma^-(x,z)^{-1}\sigma^+(x,z)=(W_-^{'+}(z)^{-1}W_+^{'-}(z))^x$

b) $\sigma^\varepsilon(x,z) \in G_\varepsilon$ *for* $x \leq 0$.

4. 4. From the last Proposition it is easy to derive that σ^ε satisfies the AKNS–equation (3. 1. 1) for the same potential Q as M^ε does, *provided* $x \leq 0$. Since here we are mainly interested in $L^1(R^-)=L^1(-\infty, 0)$, σ^ε and M^ε are in a sense equivalent. Consequently, we can replace W_ε by $W_\varepsilon^{'-\varepsilon}$. We note

Lemma. *The map $P_0^- \to \mathcal{U}_1^-$, $Q \mapsto W_\varepsilon^{'-\varepsilon}$ is analytic.*

Proof. The map $Q \mapsto S^Q$ is analytic by [1; Corollary 1. 54]. Since $S=LW_+^{-1}=UW_-^{-1}$, $Q \mapsto W_+, W_-, L, U$ is analytic by (2. 1. 7). From Corollary 4. 2 we obtain now that also $Q \mapsto V^\varepsilon$, is analytic, whence $Q \mapsto W_\varepsilon^{'-\varepsilon}$ is analytic. □

4. 5. In this section we obtain a different description of the scattering matrix $S=S^Q$ for $Q \in P_0^-$. The results of this section are particularly important for this paper.

Recall that $S=LW_+^{-1}=UW_-^{-1}$ and from **3. 6** we know $\tau^+=l(V^-)^{-1}$, $\tau^-=u(V^+)^{-1}$. This shows

(4. 5. 1) $$S=(Ll^{-1})\tau^+(W_+(V^-)^{-1})^{-1}$$

(4. 5. 2) $$S=(Uu^{-1})\tau^-(W_-(V^+)^{-1})^{-1}.$$

In view of Corollary 4. 2 we also know

(4. 5. 3) $$S=\eta^+(0,z)(W_+(V^-)^{-1})^{-1}$$

(4. 5. 4) $$S=\eta^{-}(0, z)(W_{-}(V^{+})^{-1})^{-1}.$$

Using the notation introduced in **4. 3** we now obtain

(4. 5. 5) $$S=\sigma^{+}(0, z)(W_{+}'^{-})^{-1}$$

(4. 5. 6) $$S=\sigma^{-}(0, z)(W_{-}'^{+})^{-1}$$

Note that altogether we have a fairly precise description of $\sigma^{\varepsilon}(0, z)$. The following relation will be of particular importance for our description of P_0^-.

(4. 5. 7) $$\sigma^{-}(0, z)^{-1}\sigma^{+}(0, z)=(W_{-}'^{+})^{-1}W_{+}'^{-}.$$

From (3. 5. 8) we recall

(4. 5. 8) $$q_{-}=Ll^{-1},$$

(4. 5. 9) $$q_{+}=Uu^{-1}.$$

Then combining Corollary 4. 2 with (3. 6. 3) and (3. 6. 4) we have

(4. 5. 10) $$\sigma^{+}(0, z)=q_{-}\tau^{+}(W_{+}'^{+})^{-1},$$

(4. 5. 11) $$\sigma^{-}(0, z)=q_{+}\tau^{-}(W_{-}'^{-})^{-1}.$$

Proposition. *Let* $Q\in P_0^-$ *; then*

a) $q_{-}=\varphi^{+}(0, \bullet)\in\mathscr{L}_1^+$ *and* $q_{+}=\varphi^{-}(0, \bullet)\in\mathscr{U}_1^-$,

b) $\tau^{+}\in\mathscr{L}_1^-\mathscr{D}\mathscr{U}_1^-$ *and* $\tau^{-}\in\mathscr{U}_1^+\mathscr{D}\mathscr{L}_1^+$

Moreover, the decompositions (4. 5. 10) *and* (4. 5. 11) *are unique if the occurring factors satisfy* (a) *and* (b) *and* $W_{+}'^{+}\in\mathscr{U}_1^+$ *and* $W_{-}'^{-}\in\mathscr{L}_1^-$.

Proof. From **3. 5** we know $q_{-}=\lim_{x\to\infty}\varphi^{+}(x, z)^{-x}$. Since $Q\in P_0^-$, we know that φ^{+} solves the AKNS–system (3. 4. 1) for $Q=0$ and $x\geq 0$. Thus $\varphi^{+}(x, z)=\varphi^{+}(0, z)^{x}$ for $x\geq 0$. But (3. 5. 4) shows that $\varphi^{+}(0, z)\in G_{+}$ holds; whence $q_{-}=\varphi(0, z)\in G_{+}$. Similarly one shows $q_{+}=\varphi^{-}(0, z)\in G_{-}$. To verify (b) we note $\tau^{+}=l(V^{-})^{-1}$ by (3. 6. 3) and $l\in\mathscr{L}_1^-\mathscr{D}$ by Corollary 3 of **3. 5**. This shows $\tau^{+}\in\mathscr{L}_1^-\mathscr{D}\mathscr{U}_1^-$; similarly one checks $\tau^{-}\in U_1^+\mathscr{D}\mathscr{U}_1^+$. Assume now $\sigma^{+}(0, z)=q_{-}\tau^{+}(W_{+}'^{+})^{-1}=\tilde{q}_{-}\tilde{\tau}^{+}(\tilde{W}_{+}'^{+})^{-1}$ where the occurring factors are in the required sets. Then decomposing $\tau^{+}=l^{-}du^{-}$ and $\tilde{\tau}^{+}=\tilde{l}^{-}\tilde{d}\tilde{u}^{-}$ we obtain $d=\tilde{d}$ and $q_{-}l^{-}$

$=\tilde{q}_-\tilde{l}^-$. Since q_-, $\tilde{q}_-\in G_+$ and l^-, $\tilde{l}^-\in G_-$, $q_-=\tilde{q}_-$ and $l^-=\tilde{l}$ follows. Similarly one shows $u^-=\tilde{u}^-$ and $W_+^{'+}=\tilde{W}_+^{'+}$. This finishes the proof. □

Corollary. *Let* $Q\in P_0^-$, *then*

a) $\sigma^+(0, \bullet)\in\mathcal{L}_1\mathcal{D}\mathcal{U}_1$,

b) $\sigma^-(0, \bullet)\in\mathcal{U}_1\mathcal{D}\mathcal{L}_1$.

4. 6. Using the decomposition (4. 5. 5) and (4. 5. 6) we show

Proposition. *The map* $Q\to(\sigma^+(0, z), \sigma^-(0, z))$ *from* P_0^- *to* $G_+\times G_-$ *is analytic and injective.*

Proof. The map is analytic since $Q\to S^Q$ is analytic, moreover, factorizing in G_-G_+ and G_+G_- is analytic. Therefore it only remains to show that the map is injective. But once we know $\sigma^+(0, z)$ and $\sigma^-(0, z)$, then (4. 5. 7) shows that we also know $W'=(W_-^{'+})^{-1}W_+^{'-}$. Since we started from $Q\in P_0^-$ we know that $W'^x=\sigma^-(x, z)^{-1}\sigma^+(x, z)$ is solvable for all $x\leq 0$ (see Proposition 4. 3) and that σ^ε satisfies the AKNS–equation (3. 1. 1) with the originally given Q. (See remark at the beginning of Section **4. 4.**) This proves the claim. □

4. 7. In this section we start to develop our description of scattering and inverse scattering on the half–line $\boldsymbol{R}$.

We consider the set $\mathcal{W}$ of pairs $(W_-^{'+}, W_+^{'-})\in\mathcal{L}_1^+\times\mathcal{U}_1^-$ satisfying (4. 7. 1) and (4. 7. 2).

$$[(W_-^{'+})^{-1}W_+^{'-}]^x\in G_-G_+ \text{ for all } x\leq 0. \tag{4.7.1}$$

Define $\sigma^\varepsilon(x, z)$ by $[(W_-^{'+})^{-1}W_+^{'-}]^x=\sigma^-(x, z)^{-1}\sigma^+(x, z)$. Then we require in view of Corollary 4. 5

$$\sigma^+(0, z)\in\mathcal{L}_1\mathcal{D}\mathcal{U}_1 \text{ and } \sigma^-(0, z)\in\mathcal{U}_1\mathcal{D}\mathcal{L}_1. \tag{4.7.2}$$

Before investigating $\mathcal{W}$ in more detail we show

Proposition. *The map* Σ *given by* $Q\to(W_-^{'+}, W_+^{'-})$ *from* P_0^- *to* $\mathcal{L}_1^+\times\mathcal{U}_1^-$ *is analytic and injective and its image is contained in* $\mathcal{W}$.

Proof. Since $Q\to S^Q$ is analytic and the projection from G_-G_+ and G_+G_- to G_- and G_+ are analytic we see that the map under consideration is analytic. Now we note that (4. 7. 1) follows from Proposition 4. 3 and (4. 7. 2) is a consequence of Corollary 4. 5. The injectivity of the map follows from Proposition 4. 6. $\square$

4. 8. Eventually (see Corollary 4. 12) we want to show that $\mathscr{W}$ is open and dense in $\mathscr{L}_1^+ \times \mathscr{U}_1^-$ and that there exists a subset $\mathscr{W}_0$ of $\mathscr{W}$ that is homeomorphic with P_0^-.

We start by introducing for $Q \in P_0^-$

(4. 8. 1) $$\sigma(x, z) = \sigma^+(x, z)(W_+'^-(z)^x)^{-1} = \sigma^-(x, z)(W_-'^+(z)^x)^{-1}$$

Note that the two expressions on the right side actually are the same by Proposition 4. 3.

Next we claim

Lemma. *Let $Q \in P_0^-$, then σ satisfies the equation*

(4. 8. 2) $$\sigma(x, z) = I - [\sigma(W_+'^- - I)^x]^- - [\sigma(W_-'^+ - I)^x]^+, \quad x \leq 0,\ z \in \boldsymbol{R},$$

(4. 8. 3) $$\sigma^+(x, z) = I + \int_{-\infty}^{\infty} \frac{\sigma(W_+'^- - W_-'^+)^x}{\eta - z} \frac{d\eta}{2\pi i}, \quad x \leq 0,\ z \in \boldsymbol{C}_+,$$

(4. 8. 4) $$\sigma^-(x, z) = I - \int_{-\infty}^{\infty} \frac{\sigma(W_+'^- - W_-'^+)^x}{\eta - z} \frac{d\eta}{2\pi i}, \quad x \leq 0,\ z \in \boldsymbol{C}_-.$$

Proof. First we note $I - [\sigma(W_+'^- - I)^x]^- - [\sigma(W_-'^+ - I)^x]^+ = I - [\sigma^+ - \sigma]^- - [\sigma^- - \sigma]^+ = I + [\sigma - I]^- + [\sigma - I]^+ = I + \sigma - I = \sigma$. The formula (4. 8. 3) follows from (1. 1. 7) and the fact that

$$\sigma^+ - I = [\sigma^+ - \sigma^-]^+ = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{\sigma(W_+'^-)^x - \sigma(W_-'^+)^x}{\eta - (z + i\varepsilon)} \frac{d\eta}{2\pi i}$$

since for $z \in \boldsymbol{C}_+$ we can carry out the limit in ε. The formula (4. 8. 4) follows similarly. $\square$

4. 9. In this section we consider the operator C_x acting on $\boldsymbol{C}I + \mathrm{Mat}(2, \mathscr{A})$ by the formula

(4. 9. 1) $$C_x(A) = [A(W_+'^- - I)^x]^- + [A(W_-'^+ - I)^x]^+$$

The operator C_x has been considered in [1 ; § 7] and in [4].

Clearly, C_x depends on $(W_-'^+, W_+'^-) \in \mathscr{L}_1^+ \times \mathscr{U}_1^-$. Moreover, the formula (4. 8. 2) can be rewritten as

(4. 9. 2) $$\sigma - I = -C_x(I) - C_x(\sigma - I).$$

Therefore, if $I+C_x$ is invertible, then we can compute $\sigma-I$ from

$$\sigma-I=-(I+C_x)^{-1}C_x(I). \tag{4.9.3}$$

Here we show

Proposition. *Let* $(W_-^{\prime +}, W_+^{\prime -})\in\mathscr{L}_1^+\times\mathscr{U}_1^-$. *Then* $I+C_x$ *is a Fredholm operator of index* 0. *The number* $\kappa=\frac{1}{2}\dim\operatorname{Ker}(I+C_x)$ *is the partial index of* $[(W_-^{\prime +})^{-1}W_+^{\prime -}]^x$. *In particular,* $[(W_-^{\prime +})^{-1}W_+^{\prime -}]^x\in G_-G_+$ *for all* $x\le 0$ *if and only if* $\operatorname{Ker}(I+C_x)=0$ *for all* $x\le 0$ *and this holds if and only if* $I+C_x$ *is invertible for all* $x\le 0$.

Remark. If A is a 2×2 matrix such that det A$=1$ then its partial indices κ_1 and κ_2 satisfy $\kappa_1+\kappa_2=0$. In the Proposition above the partial index is the one that is nonnegative.

Proof. We define the elements $w_1, w_2\in\mathscr{A}$ by the equations

$$(W_-^{\prime +})^x-I=\begin{pmatrix}0 & 0\\ w_2 & 0\end{pmatrix} \quad\text{and}\quad (W_+^{\prime -})^x-I=\begin{pmatrix}0 & w_1\\ 0 & 0\end{pmatrix}. \tag{4.9.4}$$

We set $H=\left\{\begin{pmatrix}a\\ b\end{pmatrix};\ a,\ b\in\mathscr{A}\right\}$ and equip H with the product norm $\left\|\begin{pmatrix}a\\ b\end{pmatrix}\right\|=\|a\|+\|b\|$. Next we define a linear operator $Q: H\oplus H\to H\oplus H$ by

$$Q=\begin{pmatrix}I & \pi_+L(w_2)\\ \pi_-L(w_1) & I\end{pmatrix} \tag{1}$$

where $\pi_\varepsilon: \mathscr{A}\to\mathscr{A}_\varepsilon$ denotes the canonical projection and $L(a)$, $a\in\mathscr{A}$, denotes "left multiplication" in H, $L(a)\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}au\\ av\end{pmatrix}$.

A straightforward computation shows

$$I+C_x=\varphi^{-1}Q\varphi, \tag{2}$$

where $\varphi: \operatorname{Mat}(2,\mathscr{A})\to H\oplus H$ is given by $\varphi\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a\\ c\end{pmatrix}\oplus\begin{pmatrix}b\\ d\end{pmatrix}$. Since φ is an isomorphism, it suffices to show that Q is a Fredholm operator of index 0. To see this we note first $Q=R^{-1}Q_0S^{-1}$ where

(3) $$R=\begin{pmatrix} I & -\pi_+L(w_2) \\ 0 & I \end{pmatrix}, \qquad S=\begin{pmatrix} I & 0 \\ -\pi_-L(w_1) & I \end{pmatrix}$$

and $Q_0=\mathrm{diag}(I-\pi_+L(w_2)\pi_-L(w_1), I)$. Since R and S are invertible operators, it suffices to show that $D=I-\pi_+L(w_2)\pi_-L(w_1)$ is a Fredholm operator of index 0. To verify this we note that $L(w_1)$ is a bounded operator on H and $\pi_+L(w_2)\pi_-$ is compact (see e. g. [7; p. 225]). From (2) and the decomposition of Q it is clear that

$$\kappa=\frac{1}{2}\dim\mathrm{Ker}(I+C_x)=\frac{1}{2}\dim\mathrm{Ker}(I-\pi_+L(w_2)\pi_-L(w_1)).$$

It is also straightforward to see that $\mathrm{Ker}(I-T)\subset H_+=\left\{\begin{pmatrix} a \\ b\end{pmatrix};\ a,\ b\in\mathscr{A}_+\right\}$; here we have set $T=\pi_+L(w_2)\pi_-L(w_1)$. Since T acts diagonally on H we see that $\dim\mathrm{Ker}(I-T)=2\dim\mathrm{Ker}(I-T_0)$, where T_0 acts like T on $\mathscr{A}$ (and not H). To finish the proof it suffices now to show that the partial index of $[(W_-'^{+})^{-1}W_+'^{-}]^x=\begin{pmatrix} I & w_1 \\ -w_2 & 1-w_1w_2\end{pmatrix}$ is κ. To determine the partial index of this matrix we have to compute the dimension of the subspace of elements of H_+ that are mapped into H_-. Thus we have to consider the two conditions

(4) $$a_++w_1b_+\in\mathscr{A}.$$

(5) $$-w_2a_++b_+-w_1w_2b_+\in\mathscr{A}_-.$$

From (4) we obtain

(6) $$a_+=-(w_1b_+)_+.$$

Inserting this into (5) yields $w_2(w_1b_+)_++b_+-w_2(w_1b_+)\in\mathscr{A}_-$. This is equivalent with $b_+-w_2(w_1b_+)_-\in\mathscr{A}_-$, which in turn is equivalent with $b_+-(w_2(w_1b_+)_-)_+=0$, i. e. with b_+ being in the kernel of the operator $I-\pi_+L(w_2)\pi_-L(w_1)$. □

Remark. We would like to note that the above proof works for arbitrary $w_1,\ w_2\in\mathscr{A}$. Hence the Proposition actually holds for arbitrary $(W_-',\ W_+')\in\mathscr{L}_1\times\mathscr{U}_1$.

In the following section we show that κ vanishes if x is close enough to $-\infty$.

4. 10. In this section we investigate the operators C_x in more detail. From the proof of the last theorem it follows that we have to look essentially only at the operator

(4. 10. 1) $$D(w_2, w_1)=\pi_+L(w_2)\pi_-L(w_1), \qquad w_1, w_2\in\mathscr{A}.$$

Lemma. *For all* $b_+\in\mathscr{A}_+$ and all $w_1, w_1', w_2, w_2'\in\mathscr{A}$ *we have*

(4. 10. 2) $$\begin{aligned}&\| D(w_2, w_1)b_+-D(w_2', w_1')b_+ \|\\ &\leq \| w_2-w_2' \| \; \| \pi_-L(w_1)b_+ \| + \| w_2' \| \; \| \pi_-L(w_1-w_1')b_+ \|\end{aligned}$$

Proof. First we note

$$\begin{aligned}&\| D(w_2, w_1)b_+-D(w_2', w_1')b_+ \|\\ &\quad\leq \| D(w_2, w_1)b_+-D(w_2', w_1)b_+ \|\\ &\qquad + \| D(w_2', w_1)b_+-D(w_2', w_1')b_+ \|\\ &\quad= \| \pi_+L(w_2-w_2')\pi_-L(w_1)\, b_+ \|\\ &\qquad + \| \pi_+L(w_2')\pi_-L(w_1-w_1')b_+ \| .\end{aligned}$$

From this the claim follows. □

We recall that in the proof of Proposition 4. 9 actually $w_1(z, x)=w_1(z)e^{-2ixz}$ and $w_2(z, x)=e^{2ixz}w_2(z)$ occur. We note (using the notation (1. 1. 2))

$$\begin{aligned}&\| \pi_-L(w_1(z, x))b_+ \|\\ &\quad\leq\int_{-\infty}^{0} \Big| \int_0^{\infty}\hat{w}_1(p+2x-t)\hat{b}_+(t)\,dt\Big|\,dp\\ &\quad\leq\int_0^{\infty} | \hat{b}_+(t) |\Big(\int_{-\infty}^{2x-t} | \hat{w}_1(p)|\, dp\Big)dt\leq \| b_+ \| \int_{-\infty}^{2x} | \hat{w}_1(p)|\, dp.\end{aligned}$$

and record it as

(4. 10. 3) $$\| \pi_-L(w_1(z, x)\,)\, b_+ \| \leq \| b_+ \| \int_{-\infty}^{2x} | \hat{w}_1(p)|\, dp.$$

Proposition. *The map* $(x, w_1, w_2, b_+)\to D(w_2(z, x), w_1(z, x))b_+$ *is uniformly continuous.*

Proof. Using (4. 10. 2) we obtain

$$\begin{aligned}&\| D(w_2(z, x), w_1(z, x))b_+-D(w_2'(z, x'), w_1'(z, x'))b_+' \|\\ &\quad\leq \| D(w_2(z, x), w_1(z, x)\,)(b_+-b_+') \|\\ &\qquad + \| D(w_2(z, x), w_1(z, x))b_+'-D(w_2'(z, x'), w_1(z, x'))b_+' \|\end{aligned}$$

$$\leq \| w_2(z, x) \| \ \| w_1(z, x) \| \ \| b_+ - b'_+ \|$$
$$+ \| w_2(z, x) - w'_2(z, x)' \| \ \| \pi_- L(w_1(z, x)) b_+ \|$$
$$+ \| w'_2(z, x') \| \ \| \pi_- L(w_1(z, x) - w'_1(z, x')) b'_+ \| .$$

It is easy to see that $\| w(z, x) \| = \| w \|$ is independent of x. Hence the first term above is equal to $\| w_2 \| \ \| w_1 \| \ \| b_+ - b'_+ \|$ and the first factor of the third term is $\| w'_2 \|$. Next we consider $\| w_2(z, x) - w'_2(z, x') \| \leq \| w_2(z, x) - w_2(z, x') \| + \| w_2(z, x') - w'_2(z, x') \|$. The special form of $w(z, y)$ implies $\| w_2(z, x) - w_2(z, x') \| = \int_{-\infty}^{\infty} | \hat{w}_2(p-2x) - \hat{w}_2(p-2x') | \, dp \leq C \| w_2 \| \, | x-x' |$ for some $C>0$. Altogether this shows that the second summand above can be estimated against $(C \| w_2 \| \, | x-x' | + \| w_2 - w'_2 \|) \| w_1 \| \ \| b_+ \|$.

Finally, we consider the second factor in the third summand above. Using (4. 10. 3) we obtain

$$\| \pi_- L(w_1(z, x) - w'_1(z, x')) b'_+ \|$$
$$\leq \| \pi_- L(w_1(z, x) - w_1(z, x')) b'_+ \|$$
$$+ \| \pi_- L(w_1(z, x') - w'_1(z, x')) b'_+ \|$$
$$\leq \int_{-\infty}^{0} \left| \int_0^{\infty} [\hat{w}_1(p+2x-t) - \hat{w}_1(p+2x'-t)] \, \hat{b}'_+(t) \, dt \right| dp$$
$$+ \| b'_+ \| \int_{-\infty}^{2x'} | \hat{w}_1(p) - \hat{w}'_1(p) | \, dp.$$

Here the last term is $\leq \| b'_+ \| \ \| w_1 - w'_1 \|$, while the first term is $\leq \| w_1(z, x) - w_1(z, x') \| \ \| b'_+ \| \leq C \| w_1 \| \, | x-x' | \, \| b'_+ \|$, where the first estimate just rephrases an estimate for the convolution and the last estimate has been shown before. Summing up we have shown

$$\text{(4. 10. 4)} \quad \| D(w_2(z, x), w_1(z, x)) b_+ - D(w'_2(z, x'), w'_1(z, x')) b'_+ \|$$
$$\leq \| w_2 \| \ \| w_1 \| \ \| b_+ - b'_+ \| + (C \| w_2 \| \, | x-x' |$$
$$+ \| w_2 - w'_2 \|) \| w_1 \| \ \| b_+ \| + \| w'_2 \| \, (C \| w_1 \| \, | x-x' |$$
$$+ \| w_1 - w'_1 \|) \| b_+ \| .$$

From this the claim follows. □

Finally, we prove

Theorem. *Let* $w_1, w_2 \in \mathscr{A}$. *Then there exist* $N>0$ *and* $\varepsilon>0$ *such that for* $x < -N$ *and* $w'_1, w'_2 \in \mathscr{A}$ *satisfying* $\| w_1 - w'_1 \| < \varepsilon$, $\| w_2 - w'_2 \| < \varepsilon$, *the operator* I

$-\pi_+L(w_2'(z, x))\pi_-L(w_1'(z, x))$ *is invertible.*

Proof. Set $\| w_2 \| = \| w_2(z, x) \| = A$. Choose $N>0$, $0<\varepsilon<1$ and such that $\int_{-\infty}^{-2N} |\hat{w}_1'(p)|\, dp < \frac{1}{A+1}$ for all $\| w_1 - w_1' \| < \varepsilon$. Let now $x<-N$. We want to show that the equation

$$D(w_2'(z, x), w_1'(z, x))b_+ = b_+ \tag{1}$$

has only the trivial solution. To this end we consider

$$\begin{aligned}
\| D(w_2'(z, x), w_1'(z, x))b_+ \| &\leq \| w_2' \| \ \| \Pi_-L(w_1'(z, x))b_+ \| \\
&\leq (\| w_2' - w_2 \| + \| w_2 \|) \ \| \Pi_-L(w_1'(z, x)b_+ \| \\
&\leq (\varepsilon + A) \ \| \Pi_-L(w_1'(z, x))b_+ \| \\
&\leq (A+\varepsilon)\int_0^\infty |\hat{b}_+(t)| \left(\int_{-\infty}^{2x-t} |\hat{w}_1'(p)|\, dp\right)dt \\
&\leq (A+\varepsilon)\left(\int_0^\infty |\hat{b}_+(t)|\, dt\right)\left(\int_{-\infty}^{2x} |\hat{w}_1'(p)|\, dp\right) \\
&\leq (A+\varepsilon)\frac{1}{A+1} \| b_+ \| < \| b_+ \| .
\end{aligned}$$

This shows that (1) has only the trivial solution. Since $I-\pi_+L(w_2'(z, x))\pi_-L(w_1'(z, x))$ is a Fredholm operator of index zero as shown in Proposition 4.9 and since $\mathrm{Ker}(I-\pi_+L(w_2'(z, x))\pi_-L(w_1'(z, x)))=0$ for $x<N$ as shown just above, the assertion follows. □

4.11. We want to see that the set $\mathscr{W}$ defined in **4.7** is open. First we consider

$$\mathscr{W}' = \{(W_-'^{+}, W_+'^{-}) \in \mathscr{L}_1^+ \times \mathscr{U}_1^- \,;\, [(W_-'^{+})^{-1} W_+'^{-}]^x \in G_-G_+ \text{ for all } x \leq 0\} \tag{4.11.1}$$

Proposition. *$\mathscr{W}'$ is open in $\mathscr{L}_1^+ \times \mathscr{U}_1^-$.*

Proof. We have seen in Theorem 4.10 that for each $W_0 = (W_-'^{+}, W_+'^{-})$ there exists $N>0$ such that $I+C_x$ is invertible for $x\leq -N$. Moreover, one can choose N and a neighborhood Ξ of W_0 such that $I+C_x$ is invertible for $x\leq -N$ and all $W\in \Xi$. Assume now that $I+C_x$ is invertible for $x\leq 0$ and W_0. Then we note that $I+C_x$ depends uniformly continuously on W by Proposition 4.10; thus we can choose a neighborhood $\Xi' \subset \Xi$ of W_0 such that $I+C_x$ is invertible for all $-N\leq x\leq 0$. □

Corollary. *$\mathscr{W}$ is open in $\mathscr{L}_1^+ \times \mathscr{U}_1^-$.*

4.12. We want to show that $\mathscr{W}$ is dense in $\mathscr{L}_1^+ \times \mathscr{U}_1^-$. This will follow from an even stronger statement proved in Section **4.13**. With the definition of Σ introduced in **4.7** we set

$$\mathscr{W}_0 = \Sigma(P_0^-). \tag{4.12.1}$$

From Proposition 4.7 we know $\mathscr{W}_0 \subset \mathscr{W}$. We will characterize $\mathscr{W}_0$ inside $\mathscr{W}$ intrinsically below. We will also show later that $\Sigma: P_0^- \to \mathscr{W}_0$ is a homeomorphism. First we find part of the inverse map for Σ.

Let $W = (W_-^{'+}, W_+^{'-}) \in \mathscr{W}$. Then we know from the definition of $\mathscr{W}$ that we have for $x \leq 0$, $z \in R$

$$[(W_-^{'+})^{-1} W_+^{'-}]^x = \sigma^-(x, z)^{-1} \sigma^+(x, z). \tag{4.12.2}$$

Moreover, for $x=0$ we know $\sigma^-(0, z) \in \mathscr{U}_1 \mathscr{D} \mathscr{L}_1$ and $\sigma^+(0, z) \in \mathscr{L}_1 \mathscr{D} \mathscr{U}_1$. Thus we can write

$$\sigma^-(0, z) = q_+^- \tau^- (W_-^{'-})^{-1}, \tag{4.12.3}$$

$$\sigma^+(0, z) = q_-^+ \tau^+ (W_+^{'+})^{-1}, \tag{4.12.4}$$

where $q_+^- \in \mathscr{U}_1^-$, $W_-^{'-} \in \mathscr{L}_1^-$, $q_-^+ \in \mathscr{L}_1^+$, $W_+^{'+} \in \mathscr{U}_1^+$, $\tau^- \in G_-$, $\tau^+ \in G_+$ and where

$$\tau^- = u^+ d^- (V^+)^{-1}, \tag{4.12.5}$$

$$\tau^+ = l^- d^+ (V^-)^{-1}, \tag{4.12.6}$$

with $u^+ \in \mathscr{U}_1^+$, $V^+ \in \mathscr{L}_1^+$, $l^- \in \mathscr{L}_1^-$, d^-, $d^+ \in \mathscr{D}$, and $V^- \in \mathscr{U}_1^-$.

Note that the map which assigns to $W \in \mathscr{W}$ any of the factors occurring in (4.12.3), (4.12.4), (4.12.5), or (4.12.6) is analytic.

From the proof of Lemma 2.3 it follows that *we can assume that V^+ and V^- are rational functions.*

Theorem. *$\mathscr{W}_0 = \{W \in \mathscr{W}$; V^+ and V^- have only simple poles$\}$*

Proof. We set $W_+' = W_+^{'-} W_+^{'+}$ and $W_+' = W_-^{'+} W_-^{'-}$. Then we have for $x \leq 0$

$$[(W_-')^{-1} W_+']^x = \eta^-(x, z)^{-1} \eta^+(x, z) \tag{4.12.7}$$

where

$$\eta^{+}(x, z)=\sigma^{+}(x, z)(W_{+}^{'+})^{x}, \tag{4.12.8}$$

$$\eta^{-}(x, z)=\sigma^{-}(x, z)(W_{-}^{'-})^{x}. \tag{4.12.9}$$

We would like to point out that $\eta^{+}\in G_{+}$ and $\eta^{-}\in G_{-}$ for $x\leq 0$. Finally we set

$$W_{+}=W_{+}^{'}V^{-}, \tag{4.12.10}$$

$$W_{-}=W_{-}^{'}V^{+}. \tag{4.12.11}$$

From this we obtain for $x\leq 0$

$$[(W_{-})^{-1}W_{+}]^{x}=M^{-}(x, z)^{-1}M^{+}(x, z), \tag{4.12.12}$$

where

$$M^{+}(x, z)=\eta^{+}(x, z)V^{-}(z)^{x}, \tag{4.12.13}$$

$$M^{-}(x, z)=\eta^{-}(x, z)V^{+}(z)^{x}. \tag{4.12.14}$$

Since $\eta^{\varepsilon}\in G_{\varepsilon}$ and V^{ε} is rational with only simple poles in $C_{-\varepsilon}$, we know that M^{ε} is meromorphic in C_{ε} with only finitely many simple poles off the real axis.

We claim that W_{+}, W_{-}, M^{+}, M^{-} satisfy the equations [4; 2.22] for $x\leq 0$.

$$M(x, z)=I-[M(W_{+}-I)^{x}]^{-}-[M(W_{-}-I)^{x}]^{+} + \sum_{r\in Z}\frac{1}{r-z}\operatorname{Res} M(x, r) \tag{4.12.15}$$

$$\operatorname{Res} M(x, r)=\Big(I+\int_{-\infty}^{\infty}\frac{M(x, \eta)(W_{+}(\eta)-W_{-}(\eta))^{x}}{\eta-r}\frac{d\eta}{2\pi i} + \sum_{s\neq r}\frac{\operatorname{Res} M(x, s)}{s-r}\Big)\operatorname{Res}[V^{-}(r)+V^{+}(r)]^{x} \tag{4.12.16}$$

where we set

$$M(x, z)=M^{+}(x, z)[W_{+}(z)^{x}]^{-1}=M^{-}(x, z)[W_{-}(z)^{x}]^{-1} \tag{4.12.17}$$

$$Z=Z_{-}\cup Z_{+}=(\text{poles of } V^{+})\cup(\text{poles of } V^{+}) \tag{4.12.18}$$

$$\operatorname{Res} M(x, r)=\operatorname{Res} M_{\varepsilon}(x, r) \text{ if } r\in Z_{\varepsilon}. \tag{4.12.19}$$

We would like to note that our notation is consistent with the notation of the previous chapters and sections and also with the notation used in [4], except that we write W_ε and not $I+W_\varepsilon$ and that we use for W_- the inverse matrix.

The proof of (4. 12. 15) and (4. 12. 16) can be taken–mutatis mutandis– from the proof of [4; Lemma 2. 22]. Next we consider the case $x\geq 0$. We note first that (4. 12. 2) at $x=0$ and (4. 12. 3) and (4. 12. 4) imply

$$(W_-'^{+})^{-1}W_+' = (\tau^-)^{-1}[(q_+^-)^{-1}q_-^+]\tau^+. \tag{4. 12. 20}$$

From this we obtain

$$(W_-')^{-1}W_+' = (\tau^-)^{-1}[(q_+^-)^{-1}q_-^+]\tau^+. \tag{4. 12. 21}$$

Using (4. 12. 5) and (4. 12. 6) and the definition of W_+, W_- we have

$$W_-^{-1}W_+ = (d^-)^{-1}[(q_+^- u^+)^{-1}(q_-^+ l_-)]d^+. \tag{4. 12. 22}$$

From this it is clear that we have

$$(W_-^{-1}W_+)^x = (H^x)^{-1}N^x, \tag{4. 12. 23}$$

where $H=q_+^- u^+ d^-$ is an upper triangular matrix with diagonal d^- and $N=q_-^+ l^- d^+$ is a lower triangular matrix with diagonal d^+. From (4. 12. 5) and (4. 12. 6) and Proposition 2. 3. 1 we obtain H^x is meromorphic in C_- with finitely many simple poles at Z_- and that N^x is meromorphic in C_+ with finitely many simple poles at Z_+. Setting $M^-(x, z)=H(z)^x$, $M^+(x, z)=N(z)^x$ and $M(x, z)=M^+(x, z)[W_+(z)^{-1}]^x=M^-(x, z)[W_-(z)^{-1}]^x$ for $x\geq 0$ it is not hard to verify that (4. 12. 15) and (4. 12. 16) hold for $x\geq 0$.

We want to apply [4; Theorem 3. 32] to see that the potential Q associated with M^+, M^- (and thus with σ^+, σ^-) is integrable over R. Before we can do this we need to define scattering data ω in the sense of [1] or [4]. We set

$$\omega = (W_-(z),\ W_+(z),\ Z,\ \mathrm{Res}\ V^-(r),\ \mathrm{Res}\ V^+(s)). \tag{4. 12. 24}$$

It remains to show that ω satisfies the "winding number condition" [4; 2. 19]. Using $\#(f)$ to denote the winding number of a function f we thus have to show

$$\#[W_-^{-1}W_+]_{22} = -\#(Z_+)+\#(Z_-). \tag{4. 12. 25}$$

To verify this we recall that $q_+^- u^+$ is upper triangular and $q_-^+ l^-$ is lower triangular, whence $[(q_+^- u^+)^{-1}(q_-^+ l^-)]_{22}=1$. This shows that (4. 12. 22) implies

(4. 12. 26) $$(W_-^{-1}W_+)_{22}=(d_{22}^-)^{-1}d_{22}^+.$$

From this we obtain $\#(W_-^{-1}W_+)_{22}=\#(d_{22}^+)-\#(d_{22}^-)$. We know from the proof of Proposition 2. 3. 1 that $d_{22}^+=\beta_+\beta^*$ and $d_{22}^-=\gamma_-\gamma^*$ with β^*, γ^* rational, β_+ without zeroes and poles in C_+ and γ_- without zeros and poles in C_-. Hence $\#(W_-^{-1}W_+)_{22}=\#(\beta^*)-\#(\gamma^*)$. Note that the rational functions occurring here have no zeroes or poles on the real axis or at ∞. Hence we know $\#\beta^*=\#$(zeroes of β^* in C_+)$-\#$(poles of β^* in C_+) and $\#\gamma^*=-\#$(zeroes of γ^* in C_-)$+\#$(poles of γ^* in C_-). From (4. 12. 5) we see that $d_{11}^+=(d_{22}^+)^{-1}$ has no poles in C_+; similarly, (4. 12. 4) shows that d_{22}^- has no poles in C_-. Therefore $\#(W_-^{-1}W_+)_{22}=-\#$(zeroes of d_{11}^+ in C_+)$+\#$(zeroes of d_{22}^- in C_-). But the proof of Proposition 2. 3 shows that the number of zeroes of d_{11}^+ in C_+ is exactly the number of poles of V^- in C_+, i. e. this number is $\#Z_+$. Similarly, the number of zeroes of d_{22}^- in C_- is exactly the number of poles of V^+ in C_-, i. e. this number is $\#Z_-$, finishing the proof of (4. 12. 25).

Now we can apply [4; Theorem 3. 32] to see that the potential Q associated with M^+ and M^-(and thus with σ^+ and σ^-) is integrable over $\boldsymbol{R}$, i. e. $Q\in\boldsymbol{P}$.

Since M^- and M^+ are meromorphic solutions satisfying all requirements of [1] or [4], we see that $Q\in\boldsymbol{P}_0$. But $M^-=H^x$ and $M^+=N^x$ for $x\geq 0$, therefore we actually have $Q\in\boldsymbol{P}_0^-$. □

Corollary. *The set $\mathscr{W}_0$ is open and the mapping Σ: $\boldsymbol{P}_0^-\to\mathscr{W}_0$ is an analytic bijection.*

Proof. We have seen in Proposition 4. 7 that Σ is analytic and injective; moreover, it is surjective by the definition (4. 12. 1) of $\mathscr{W}_0$. The theorem above shows that $\mathscr{W}_0$ is open. □

4. 13. Next we want to show

Proposition. *$\mathscr{W}_0$ is dense in $\mathscr{L}_1^+\times\mathscr{U}_1^-$.*

Proof. Let $W=(W_-'^{+}, W_+'^{-})\in\mathscr{L}_1^+\times\mathscr{U}_1^-$ such that $((W_-'^{+})^{-1}W_+'^{-})_{22}$ is invertible. Let $n=\text{index}((W_-'^{+})^{-1}W_+'^{-})_{22}$ and let $\delta>0$; then choose rational functions $V^-\in\mathscr{U}_1^-$ and $V^+\in\mathscr{L}_1^+$ such that

(1) $$|V^\varepsilon-I|<\delta \text{ and}$$

(2) $$\text{index}([W_-'^{-}V^+]^{-1}[W_+'^{-}V^-])_{22}=\text{index}((W_-'^{+})^{-1}W_+'^{-})_{22}$$

(3) $$\omega=(W_-'^{+}V^+,\ W_+'^{-}V^-,\ Z=Z_-\cup Z_+,\ \text{Res } V^\varepsilon(Z_i))$$

are scattering data in the sense of [4] ;thus, in particular, ω satisfies the "winding number condition" [4 ; 2. 19]. From [4 ; Theorem 3. 12] it therefore follows that there exists some $\tilde{\omega}=(\tilde{W}_-, \tilde{W}_+, \tilde{Z}=\tilde{Z}_-\cup\tilde{Z}_+, \text{Res } \tilde{V}^{\varepsilon}(\tilde{Z}))$ corresponding to a potential $Q\in P_0$ and satisfying $|\omega-\tilde{\omega}|<\delta$, where the norm here denotes the norm in the space of scattering data $\mathscr{I}$ in the sense of [4] . In particular we have

$$(4) \qquad |\tilde{W}_{\varepsilon}-W_{\varepsilon}^{'-\varepsilon}V^{-\varepsilon}|<\delta,$$

$$(5) \qquad |V^{\varepsilon}-\tilde{V}^{\varepsilon}|<\delta.$$

From this we show

$$(6) \qquad |\tilde{W}_{\varepsilon}(\tilde{V}^{-\varepsilon})^{-1}-W_{\varepsilon}^{'-\varepsilon}|<C\delta,$$

where C does not depend on $\delta\leq 1$. To verify this we note

$$|\tilde{W}_{\varepsilon}(\tilde{V}^{-\varepsilon})^{-1}-W_{\varepsilon}^{'-\varepsilon}|=|(\tilde{W}_{\varepsilon}-W_{\varepsilon}^{'-\varepsilon}\tilde{V}^{-\varepsilon})(\tilde{V}^{-\varepsilon})^{-1}|\leq A|\tilde{W}_{\varepsilon}-W_{\varepsilon}^{'-\varepsilon}\tilde{V}^{-\varepsilon}|$$

$$\leq A(|\tilde{W}-W_{\varepsilon}^{'-\varepsilon}V^{-\varepsilon}|+|W_{\varepsilon}^{'-\varepsilon}(V^{-\varepsilon}-\tilde{V}^{-\varepsilon})|)\leq C\delta.$$

Note that C can be chosen independent of $\delta\leq 1$ since $|\tilde{V}^{-\varepsilon}-I|\leq 2\delta$. From (6) it also follows that for $\tilde{W}_{\varepsilon}^{'-\varepsilon}=[\tilde{W}_{\varepsilon}(\tilde{V}^{-\varepsilon})^{-1}]^{-\varepsilon}$ we have

$$(7) \qquad |\tilde{W}_{\varepsilon}^{'-\varepsilon}-W_{\varepsilon}^{'-\varepsilon}|<C\delta.$$

In fact, by Theorem 4. 1 we can choose $\tilde{\omega}$ so that $Q\in P_0\cap(P_0^-+P_0^+)$ holds. Note that $P_0\cap(P_0^-+P_0^+)$ is open and dense in P since P_0 and $P_0^-+P_0^+$ are open and dense in P. Since $\tilde{\omega}\in\mathscr{I}_0=\{w\in\mathscr{I}\,;\,\omega \text{ corresponds to } Q\in P_0\}$ we know that $\tilde{W}=(\tilde{W}_-^{'+}, \tilde{W}_+^{'-})$ satisfies (4. 7. 1). We claim that since $Q\in P_0\cap(P_0^-+P_0^+)$, for the corresponding $\tilde{W}$ also (4. 7. 2) holds. Moreover we will show that $\tilde{W}$ is in fact in $\mathscr{W}_0$. Since $Q\in P_0\cap(P_0^-+P_0^+)$ we have $Q=Q^-+Q^+$ with $Q^{\varepsilon}\in P_0^{\varepsilon}$. We will denote all the quantities associated with Q^- with a " ^ " and all quantities associated with Q with a " ~ ". Then for $x=0$

$$(8) \qquad \tilde{M}^+(0, z)=\hat{M}^+(0, z)\,U(z)$$

where $U\in\mathscr{U}_1$. We know from Theorem 3. 4 that (8) can be written in terms of V's and η's as

$$(9) \qquad \tilde{\eta}^+(0, z)\,\tilde{V}^-(z)=\hat{\eta}^+(0, z)\,\hat{V}^-(z)\,U(z).$$

A similar computation with $\tilde{M}^-(0, z)$ gives

$$\tilde{\eta}^-(0, z)\tilde{V}^+(z) = \hat{\eta}^-(0, z)\hat{V}^+(z)L(z) \tag{10}$$

where $L(z) = \mathscr{L}_1$.

Thus in view of (3. 7. 4), we get

$$\hat{V}^+L\tilde{W}_-^{-1}\tilde{W}_+(\hat{V}^-U)^{-1} = (\hat{\eta}^-(0, \bullet))^{-1}\hat{\eta}^+(0, \bullet). \tag{11}$$

Note, however, that (9) and (10) imply that there exist $u^+ \in \mathscr{U}_1^+$ and $l^- \in \mathscr{L}_1^-$ such that $\hat{V}^-U = \tilde{V}^-u^+$, and $\hat{V}^+L = \tilde{V}^+l^-$.

Thus $\tilde{W}_+(\hat{V}^-U)^{-1} = \tilde{W}_+(\tilde{V}^-)^{-1}(u^+)^{-1} = \tilde{W}_+^{\prime -}\tilde{W}_+^{\prime +}(u^+)^{-1}$ where we use notation introduced in **4. 3**. Similarly one can rewrite $\tilde{W}_-(\hat{V}^+L)^{-1}$. Thus we have

$$\tilde{W}_-^{\prime -}l^-(\tilde{W}_-^{\prime +})^{-1}\tilde{W}_+^{\prime -}\tilde{W}_+^{\prime +}(u^+)^{-1} = (\hat{\eta}^-(0, \bullet))^{-1}\hat{\eta}^+(0, \bullet). \tag{12}$$

Hence also

$$(\tilde{W}_-^{\prime +})^{-1}\tilde{W}_+^{\prime -} = (\tilde{W}_-^{\prime -})^{-1}(\hat{\eta}^-(0, \bullet))^{-1}\hat{\eta}^+(0, \bullet)u^+(\tilde{W}_+^{\prime +})^{-1} \tag{13}$$

holds.

Recall that we want to show $\tilde{W} \in \mathscr{W}_0$. First we note that $Q \in \boldsymbol{P}_0$, whence (4. 7. 1) is satisfied. Thus the left hand side of (13) determines $\tilde{\sigma}^+(0, z)$ and $\tilde{\sigma}^-(0, z)$ by

$$(\tilde{W}_-^{\prime +})^{-1}\tilde{W}_+^{\prime -} = \tilde{\sigma}^-(0, z)^{-1}\tilde{\sigma}^+(0, z). \tag{14}$$

Note that here $\tilde{\sigma}^\varepsilon(0, \bullet) \in \boldsymbol{G}_\varepsilon$. Hence

$$\tilde{\sigma}^+(0, z) = \hat{\eta}^+(0, z)u^+(z)(\tilde{W}_+^{\prime +}(z))^{-1}, \tag{15}$$

$$\tilde{\sigma}^-(0, z) = \hat{\eta}^-(0, z)l^-(z)\tilde{W}_-^{\prime -}(z). \tag{16}$$

On the other hand we know $Q^- \in \boldsymbol{P}_0^-$. Therefore Section **4. 3** defines $\hat{\sigma}^\varepsilon(0, z)$ such that $\hat{\sigma}^\varepsilon(0, z) = \hat{\eta}^\varepsilon(0, z)(\hat{W}_\varepsilon^{\prime \varepsilon}(z))^{-1}$. Moreover, Corollary 4. 5 shows $\hat{\sigma}^+(0, \bullet) \in \mathscr{L}_1\mathscr{D}\mathscr{U}_1$ and $\hat{\sigma}^-(0, \bullet) \in \mathscr{U}_1\mathscr{D}\mathscr{L}_1$. This together with (15) and (16) implies $\tilde{\sigma}^+(0, \bullet) \in \mathscr{L}_1\mathscr{D}\mathscr{U}_1$ and $\tilde{\sigma}^-(0, \bullet) \in \mathscr{U}_1\mathscr{D}\mathscr{L}_1$. Thus we have shown that $\tilde{W} \in \mathscr{W}$.

To finish the proof of the claim we have to determine the $\tilde{V}^\varepsilon$ associated with $\tilde{\sigma}^\varepsilon(0, \bullet)$ and to show that they have simple poles. But (15) and the formula relating $\hat{\sigma}^+$ and $\hat{\eta}^+$ above shows that the U_1^- part of $\tilde{\sigma}^+$ is the same as the one of $\hat{\eta}^+$. Comparing (4. 12. 3), (4. 12. 5), and (4. 12. 7) now implies that the U_1^- part of $\tilde{\sigma}^+$ is $(\hat{V}^-)^{-1}$. And similarly the $\mathscr{L}_1^+$ part of $\tilde{\sigma}^-$ is $(\hat{V}^+)^{-1}$. By construction $Q^- \in P_0^-$, whence $\hat{V}^\varepsilon$ has only simple poles and $\tilde{W} \in \mathscr{W}_0$ follows. □

Corollary. *The sets* $\mathscr{W}$ and $\mathscr{W}_0$ *are open and dense in* $\mathscr{L}_1^+ \times U_1^-$.

§ 5. Geometric Scattering and Inverse Scattering

5. 1. In the last chapters we have discussed a map associating with certain $Q \in P$ elements in the Banach Lie group G. Recall from (1. 3)

(5. 1. 1) $$G = \left\{ g = \begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix}; \ a, b, c, d \in \mathscr{A}, \ \det g = 1 \right\}$$

In this chapter we are interested in a geometric interpretation of this map and a geometric extension of its inverse.

Recall from (1. 3. 6) that G_-G_+ is open and dense in G.

Let $g \in G_-G_+$ and set as before

(5. 1. 2) $$g(z)^x = e^{xzJ} g(z) e^{-xzJ}.$$

Since the actions $g \to g^x$ of R on G is continuous, there exists an open interval I around 0 such that $g^x \in G_-G_+$ for all $x \in I$. Hence

(5. 1. 3) $$g(z)^x = g^-(x, z)^{-1} g^+(x, z) \quad \text{for } z \in R, \ x \in I.$$

We want to "differentiate" the equation (5. 1. 3) for x. Since the action of R on G is continuous and since the splitting $G_-G_+ \to G_- \times G_+$ is analytic, the coefficients of $g^\varepsilon(x, z)$ are continuous functions of x. Therefore we can differentiate $g^\varepsilon(x, z)$ for x in the distributional sense.

We obtain $[zJ, (g^-)^{-1} g^+] = -(g^-)^{-1} \partial_x g^- (g^-)^{-1} g^+ + (g^-)^{-1} \partial_x g^+$. Multiplying by g^- from the left and by $(g^+)^{-1}$ from the right we thus obtain

(5. 1. 4) $$\partial_x g^- (g^-)^{-1} + g^- zJ (g^-)^{-1} = \partial_x g^+ (g^+)^{-1} + g^+ zJ (g^+)^{-1}.$$

Note that this implies that both sides are entire functions of z. Next recall that $g^\varepsilon(x, z) \to 1$ as $|z| \to \infty$, $z \in C_\varepsilon$. Hence, subtracting zJ from both sides of (5. 1. 4), dividing by z and taking the limit as $|z| \to \infty$ we see that both sides now are equal to some $Q = Q(x)$ independent of z. This shows

(5. 1. 5) $$\partial_x g^- (g^-)^{-1} + g^- zJ (g^-)^{-1} = zJ + Q(x),$$

(5. 1. 6) $$\partial_x g^+ (g^+)^{-1} + g^+ zJ (g^+)^{-1} = zJ + Q(x).$$

Note that the off–diagonal terms in g^ε vanish as $|z| \to \infty$. Hence

(5. 1. 7) $$Q(x) = \begin{pmatrix} 0 & q_1(x) \\ q_2(x) & 0 \end{pmatrix}.$$

As a consequence we obtain

Proposition. *Let $g \in G_- G_+$ and I be an open interval such that $g^x \in G_- G_+$ for all $x \in I$. Then the functions $g^\varepsilon(x, z) \in G_\varepsilon$, given by $g(z)^x = g^-(x, z)^{-1} g^+(x, z)$ satisfy the AKNS–equation* (3. 1. 1) *for all $z \in \boldsymbol{R}$ and $x \in I$.*

Remark. The "potential" Q obviously is determined by $g \in G_- G_+$. To indicate this dependence we will sometimes write $Q = Q_g$.

5. 2. It is clear that the map $g \to Q_g$ is in some sense an "inverse" of the map Σ defined in Proposition 4. 7. More precisely we have

Theorem. *Set*

(5. 2. 1) $$\mathscr{G}_0' = \{(W_-'^{+})^{-1} W_+'^{-}) \in \mathscr{W}_0\},$$

then

(5. 2. 2) $$g(z)^x \in G_- G_+ \textit{ for all } x \leq 0,\ g \in \mathscr{G}_0'.$$

Defining $Q_g(x)$ by (5. 1. 5) *and* (5. 1. 6) *for $x \leq 0$ and by $Q_g(x) = 0$ for $x > 0$ we also have*

(5. 2. 3) $$Q_g \in P_0^- \textit{ for all } g \in \mathscr{G}_0',$$

(5. 2. 4) $$\Sigma(Q_g) = g \textit{ for all } g \in \mathscr{G}_0'.$$

Proof. Let $g \in \mathscr{G}_0'$. Then $g = (W_-'^{+})^{-1} W_+'^{-}$, where $W = (W_-'^{+}, W_+'^{-}) \in \mathscr{W}_0$. Hence (4. 7. 1) implies (5. 2. 2). Moreover, (5. 1. 5) and (5. 1. 6) define a function Q_g for $x \leq 0$. Setting $Q_g(x) = 0$ for $x > 0$ then defines Q_g on $\boldsymbol{R}$. We have seen in Corollary 4. 12 that $\Sigma: P_0^- \to \mathscr{W}_0$ is bijective. Let $Q' \in P_0^-$ be such that $\Sigma(Q') = W$. The Proposition 4. 3 shows $g^\varepsilon(x, z) = \sigma^\varepsilon(x, z)$ for all $x \leq 0$. Moreover, tracing back the definitions for σ^ε and η^ε we see that σ^ε differs from M^ε only by a factor of type $A(z)^x$. Hence M^ε and σ^ε satisfy the same differential equation, whence $Q' = Q_g$ a.e. This proves (b) and (c). □

5. 3. We would like to imbed P_0^- into a natural quotient $\mathscr{M}$ of G. The quotient $\mathscr{M}$ is suggested by the Proposition below.

Recall that we use $\mathscr{L}^- = \mathscr{L} \cap G_-$ and $\mathscr{U}^+ = \mathscr{U} \cap G_+$.

Proposition. *Let $g \in \mathscr{G}_0'$. Then for every $h \in \mathscr{L}^- g \mathscr{U}^+$ we have* (5. 2. 2) *and* (5. 2. 3) *for the same potential $Q = Q_g$.*

Proof. Let $h = L^- g U^+$. Then $h^x = (L^-)^x g^x (U^+)^x = (L^-)^x g^-(x, \circ)^{-1} g^+(x, \circ)^{-1} (U^+)^x$ for all $x \leq 0$. Since $(L^-)^x \in G_-$ and $(U^+)^x \in G_+$ for $x \leq 0$, the statement (5. 2. 2) follows. Moreover, since $g^\varepsilon(x, z)$ and $g^\varepsilon(x, z) A(z)^x$ both solve the AKNS-equation with the same potential Q, the statement (5. 2. 3) follows as well. □

5. 4. In the rest of this chapter we investigate the quotient $\mathscr{M} = \mathscr{L}^- \backslash G / \mathscr{U}^+$ of G.

In this section we collect a few facts about quotients. For this we consider a topological space E and a topological group H acting continuously from the left on E. We set

$$E/H = \{H.\, x\,;\ x \in E\}. \tag{5. 4. 1}$$

Let $\pi : E \to E/H$ denote the canonical projection and give E/H the quotient topology, i. e., $U \subset E/H$ is open in E/H if and only if $\pi^{-1}(U)$ is open in E.

If $H = H_1 \times H_2$, then H_1 acts on E/H_2 ([3 ; Top ; Chap. III ; § 2, Proposition 11]). The "transitivity of quotients" [3 ; Chap. I, § 3, Proposition 7] then implies

$$E/H \cong (E/H_2)/H_1 \cong E/H_2)/H_2. \tag{5. 4. 2}$$

We will apply the above remark to $E = G$ and $H = \mathscr{L}^- \times \mathscr{U}^+$, acting on G by $(l^-, u^+).\ g = l^- g (u_+)^{-1}$.

Finally we note that the canonical projection $\pi : E \to E/H$ is an open (and surjective) map [3 ; Chap. III ; § 2, Lemma 2]. Therefore in view of [3 ; Chap. I ; § 8 ; Proposition 8] we know

$$E/H \text{ is Hausdorff} \Longleftrightarrow \{(x\,;\, h.\, x)\,;\ x \in E,\ h \in H\} \text{ is closed in } E \times E. \tag{5. 4. 3}$$

5. 5. In this section we consider the action

$$(l, u).\ g = l g u^{-1} \tag{5. 5. 1}$$

of $\mathscr{L}^- \times \mathscr{U}^+$ on G in more detail. We recall that we use $\mathscr{L}^- = \mathscr{L} \cap G_-$ and $\mathscr{U}^+$

$=\mathscr{U}\cap G_+$. We introduced the abbreviations

$$(5.5.2)\qquad \mathscr{M}=\mathscr{L}^-\backslash G/\mathscr{U}^+=G/(\mathscr{L}^-\times\mathscr{U}^+).$$

Using (5.4.2) we see that $\mathscr{M}$ is a quotient space of either Banach manifold $\mathscr{L}^-\backslash G$ or $G/\mathscr{U}^+$. To make sure that $\mathscr{M}$ is Hausdorff we show in view of (5.4.3)

Proposition. *The set* $\mathscr{R}=\{(g, l^-gu^+)\ ;\ g\in G,\ l^-\in\mathscr{L}^-,\ u^+\in\mathscr{U}^+\}$ *is closed in* $G\times G$.

Proof. Assume $g_n\to g$, $l_n^-g_nu_n^+\to h$, $l_n^-\in\mathscr{L}^-$, $u_n^+\in\mathscr{U}^+$. Consider first the special case: $g=ldu$ and $h=l'd'u'$, where l, l' and u, u' have diagonal I. Then, since Ω_l is open in G, without loss of generality we can assume $g_n=l_nd_nu_n$ and $l_n\to l$, $d_n\to d$, $u_n\to u$. Therefore,

$$l_n^-g_nu_n^+=\tilde{l}_n^-d_n^-l_nd_nu_nd_n^+\tilde{u}_n^+\to l'd'u',$$

whence

$$\tilde{l}_n^-d_n^-l_n(d_n^-)^{-1}\to l',$$

$$d_n^-d_nd_n^+\to d',$$

$$(d_n^+)^{-1}u_nd_n^+\tilde{u}_n^+\to u'.$$

If also $d, d'\in(C+\mathscr{A}_-)^*(C+\mathscr{A}_+)^*$, then $d_n^\varepsilon\to d^\varepsilon$, whence $\tilde{u}_n^+\to\tilde{u}^+$ and $\tilde{l}_n^-\to\tilde{l}^-$. As a consequence we obtain $(g, h)\in\mathscr{R}$.

Now we consider the general case. We define as before $\mathscr{L}_1$ (resp. $\mathscr{U}_1$) to be the set of elements in $\mathscr{L}$ (resp. $\mathscr{U}$) with diagonal I. From [5 ; Corollary 3.3] we know that after multiplying g and h by some $l_0\in\mathscr{L}_1$ we can assume l_0g, $l_0h\in\mathscr{L}_1\mathscr{D}\mathscr{U}_1$. Moreover, multiplying on the right by some $d_0\in\mathscr{D}$, we can assume

$$(1)\qquad l_0gd_0\in\mathscr{L}_1\mathscr{D}\mathscr{U}_1,\ l_0gd_0=ldu,\ d=d_-d_+,$$

$$(2)\qquad l_0hd_0\in\mathscr{L}_1\mathscr{D}\mathscr{U}_1,\ l_0hd_0=l'd'u',\ d'=d_-'d_+'.$$

Since $l_0g_nd_0\to l_0gd_0$, we obtain

$$(3)\qquad l_0l_n^-g_nu_n^+d_0=(l_0l_n^-l_0^{-1})(l_0g_nd_0)(d_0^{-1}u_n^+d_0)\to l_0hd_0=l'd'u'.$$

Moreover, from (1) we know $l_0g_nd_0=l_nd_nu_n$, $d_n=d_n^-d_n^+$. Hence $l_0g_nd_0=$

$l_n d_n u_n \to l d_u = l_0 g d_0$ converges componentwise. We set

$$l_0 l_n^- l_0^{-1} = q_n a_n^-, \qquad a_n^- \quad \text{diagonal}, \ q_n \in \mathscr{L}_1,$$

$$d_0^{-1} u_n^+ d_0 = b_n^+ p_n, \qquad b_n^+ \quad \text{diagonal}, \ p_n \in \mathscr{U}_1.$$

Then

$$l_0 l_n^- g_n u_n^+ d_0 = q_n a_n^- l_n d_n u_n b_n^+ p_n = (q_n a_n^- l_n (a_n^-)^{-1})(a_n^- d_n b_n^+)((b_n^+)^{-1} u_n b_n^+ p_n)$$

converges to $l' \, d' \, u'$. But this implies

(4) $$q_n a_n^- l_n (a_n^-)^{-1} \to l'$$

(5) $$a_n^- d_n b_n^+ \to d' = d'_- d'_+$$

(6) $$(b_n^+)^{-1} u_n b_n^+ p_n \to u'.$$

In (5) we know $d_n = d_n^- d_n^+ \to d = d^- d^+$ componentwise. Therefore $a_n^- d_n^- \to d'_-$, $d_n^+ b_n^+ \to d'_+$, whence a_n^- and b_n^+ converge. Since we noticed already above that u_n and l_n converge, we conclude from (4) and (6) that p_n and q_n converge. But now from the definition of q_n and p_n we see that actually l_n^- and u_n^+ converge. Therefore $(g, h) \in \mathscr{R}$ and the Proposition is proven. □

Corollary. *$\mathscr{M} = \mathscr{L}_- \backslash G / \mathscr{U}^+$ is Hausdorff.*

5.6. We want to find a natural open and dense subset of $\mathscr{M} = \mathscr{L}^- \backslash G / \mathscr{U}^+$. To this end we consider the natural map $\mathscr{L}_1^+ \mathscr{D}_0 \mathscr{U}_1^- \mapsto G \to \mathscr{M}$, where $\mathscr{D}_0 = \left\{ \operatorname{diag}\left\{ \left(\frac{z-i}{z+i}\right)^r, \left(\frac{z-i}{z+i}\right)^{-r} \right\}, \ r \in \mathbf{Z} \right\}$. From [5; Theorem 2.3] we recall that Ω_l is analytically isomorphic with $\mathscr{L}_1 \times \mathscr{D} \times \mathscr{U}_1$.

Proposition. *The natural map $\mathscr{L}_1^+ \mathscr{D}_0 \mathscr{U}_1^- \mapsto G \to \mathscr{M}$ is a homeomorphism from $\mathscr{L}_1^+ \times \mathscr{D}_0 \times \mathscr{U}_1^- \cong \mathscr{L}_1^+ \mathscr{D}_0 \mathscr{U}_1^-$ onto the open dense subset $\mathscr{L}_1^- \backslash \Omega_l / \mathscr{U}_1^+ \cong \mathscr{L}_1^+ \times \mathscr{D}_0 \times \mathscr{U}_1^-$ of $\mathscr{M}$.*

Proof. Clearly, the map above (which we will denote again by π) is continuous. Next we show that it is injective. Assume $\pi(l^+ D^r u^-) = \pi(\tilde{l}^+ D^s \tilde{u}^-)$. Then there exist $l^- \in \mathscr{L}^-$ and $u^+ \in \mathscr{U}^+$ such that $l^+ D_0^r u^- = l^- \tilde{l}^+ D^s \tilde{u}^- u^+$ holds. Collecting terms we see that $l D_0^r = D_0^s u$ holds where $l = (l^- \tilde{l}^+)^{-1} l^+$ and $u = \tilde{u}^- u^+ (u^-)^{-1}$. Note that here the diagonal in l is in $\mathscr{D}^-$ and the diagonal in u is in $\mathscr{D}^+$.

A straightforward computation shows that the above equation is equivalent to $l\in\mathscr{D}^-$, $u\in\mathscr{D}^+$ and $lD^r=D^su$. But this is equivalent to $r=s$, $l=I$ and $u=I$. Hence $l^+=l^-\tilde{l}^+$ with l^-, l^+, $\tilde{l}^+\in\mathscr{L}_1$ and $u^-=\tilde{u}^-u^+$ with u^-, $\tilde{u}^-$, $u^+\in\mathscr{U}_1$. But then $l^+(\tilde{l}^+)^{-1}=l^-\in\mathscr{L}_1^-\cap\mathscr{L}_1^+=\{I\}$ and $(\tilde{u}^-)^{-1}u^-=u^+\in\mathscr{U}_1^+\cap\mathscr{U}_1^-=\{I\}$, whence $l^-=I$, $u^+=I$, and $l^+D^ru^-=\tilde{l}^+D^s\tilde{u}^-$ follows. It remains to show that π $(\mathscr{L}_1^+\mathscr{D}_0\mathscr{U}_1^-)$ is open and dense in $\mathscr{M}$ and that $\pi|\mathscr{L}_1^+\mathscr{D}_0\mathscr{U}_1^-$ is an open map. To verify this we recall that $\pi: G\to\mathscr{M}$ is open and we note

$$\Omega_l=\mathscr{L}\mathscr{D}\mathscr{U}=\mathscr{L}D^-D_0D^+\mathscr{U}=\mathscr{D}^-\mathscr{L}\mathscr{D}_0\mathscr{U}\mathscr{D}^+$$
$$=D^\cdot L_1^-\mathscr{L}_1^+\mathscr{D}_0\mathscr{U}_1^-\mathscr{U}_1^+\mathscr{D}^+=\mathscr{L}^-\mathscr{L}^+\mathscr{D}_0\mathscr{U}_1^-\mathscr{U}^+,$$

whence $\pi(\Omega_l)=\pi(\mathscr{L}_1^+\mathscr{D}_0\mathscr{U}_1^-)$ is open and dense in $\mathscr{M}$. Moreover, assume $K\subset\mathscr{L}_1^+\mathscr{D}_0\mathscr{U}_1^-$ is open in $\mathscr{L}_1^+\mathscr{D}_0\mathscr{U}_1^-$. Then $\pi^{-1}(K)=\mathscr{L}^-K\mathscr{U}^+$ is open in G, hence $\pi(K)=\pi(\mathscr{L}^-K\mathscr{U}^+)$ is open in $\mathscr{M}$. □

5. 7. In this section we relate the scattering map Σ: $P_0^-\to\mathscr{L}_1^+\times\mathscr{U}_1^-$ defined in **4. 7** to $\mathscr{M}$.

First we note

$$\mathscr{M}_0=\pi(\mathscr{L}_1^+\mathscr{D}_0\mathscr{U}_1^-) \tag{5. 7. 1}$$

has the connected components

$$\mathscr{M}_0^r=\Pi\Big(\mathscr{L}_1^+\operatorname{diag}\Big(\Big(\frac{z-i}{z+i}\Big)^r,\Big(\frac{z-i}{z+i}\Big)^{-r}\Big)\mathscr{U}_1^-\Big). \tag{5. 7. 2}$$

Hence, combining Proposition 4. 7 with (4. 12. 1), Theorem 4. 12 and Corollary 4. 13 we obtain

Theorem 1. *The scattering map* $P_0^-\xrightarrow{\Sigma}\mathscr{W}_0\hookrightarrow\mathscr{L}_1^+\times\mathscr{U}_1^-\xrightarrow{\pi}\mathscr{M}_0^0$ *is injective and continuous and has as image the dense open subset* $\pi(\mathscr{G}_0')$ *of* $\mathscr{M}_0^0$.

This implies that P_0^- can be considered as an open and dense subset of $\mathscr{M}_0^0$. Moreover, Proposition 5. 3 shows

Theorem 2. *For every* $g\in\pi(\mathscr{G}_0')\subset\mathscr{M}_0^0$ *we have* $g^x\in G_-G_+$ *for all* $x\leq 0$. *Hence the inverse scattering map* $\pi(\mathscr{G}_0')\to P_0^-$, $g\to Q_g$ *is well defined and it is the inverse to the scattering map* $P_0^-\mapsto\mathscr{M}_0^0$.

Remark. As outlined above the image of the embedding $P_0^-\mapsto\mathscr{M}$ is only open and dense in the connected component $\mathscr{M}_0^0$ of $\mathscr{M}_0$. One could perhaps obtain an open and dense image if one would replace $\mathscr{M}$ by some "quotient" of

$\mathcal{M}$ by $\mathcal{D}_0$. We have not pursued this at this point since a) $\mathcal{M}$ is a natural object from the point of view of Riemann–Hilbert splitting, and b) the extension of the map $P_0^- \mapsto \mathcal{M}$ to a map form $P^- \to \mathcal{M}$ has not yet been clarified.

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