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Local Derivations, Automorphisms and Commutativity Preserving Maps on $\mathscr{L}^+(\mathscr{D})$

By

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§ 1. Introduction

In this paper we study linear operators $\Phi: \mathscr{L}^+(\mathscr{D}) \longrightarrow \mathscr{L}^+(\mathscr{D})$, where $\mathscr{D}$ is a dense linear subspace in a Hilbert–space $\mathscr{H}$. $\mathscr{L}^+(\mathscr{D})$ is a *–algebra which in general contains unbounded operators. Such *–algebras of unbounded operators have been studied for more than 20 years (see e. g. [7]). Much results of this theory state analogies between *–algebras of unbounded operators and algebras of bounded operators. That will be the case also in this paper. We want to generalize results of LARSON, SOUROUR and OMLADIČ (see [5], [6]) concerning maps $\Phi: \mathscr{B}(X) \longrightarrow \mathscr{B}(X)$ on the algebra of all bounded linear operators on a Banach space X. The linearity together with further assumptions on the map Φ which seem to be rather mild leads to strong conclusions about the structure of Φ.

Before we come to the results we want to collect some definitions, notations and introductory results. Let $\mathscr{D}$ be a dense linear subspace of a Hilbert–space $\mathscr{H}$, then $\mathscr{L}^+(\mathscr{D}) = \{A \mid A\mathscr{D} \subseteq \mathscr{D},\ A^*\mathscr{D} \subseteq \mathscr{D}\}$ is a *–algebra with respect to the usual operations and the involution $A \longmapsto A^+ := A^* \mid \mathscr{D}$. A unital subalgebra $\mathscr{A}$ of $\mathscr{L}^+(\mathscr{D})$ is called an Op*–algebra.

By the system of seminorms

$$\varphi \longmapsto \| \varphi \|_A := \langle A\varphi, A\varphi \rangle^{1/2} \quad ; \qquad A \in \mathscr{L}^+(\mathscr{D})$$

a topology t on $\boldsymbol{D}$ is generated. The seminorms

$$A \longmapsto \| A \|_{\mathscr{M}} := \sup_{\varphi, \psi \in \mathscr{M}} |\langle \varphi, A\psi \rangle| \quad ; \qquad \mathscr{M} \subset \mathscr{D} \text{ t–bounded}$$

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resp. $\{\|\cdot\|_{\mathcal{N}} \mid \mathcal{N}\subset\mathcal{D}$ t–relatively compact$\}$ generate the topologies $\tau_{\mathcal{D}}$ resp. $\tau_{\mathcal{D}}^{c}$ on $\mathcal{L}^+(\mathcal{D})$. We always want to assume that $\mathcal{D}[\mathrm{t}]$ is complete, this is equivalent with $\mathcal{D}=\bigcap_{A\in\mathcal{L}^+(\mathcal{D})} D(\overline{A})$.

In the sequel we make an extensive use of rank–one operators in $\mathcal{L}^+(\mathcal{D})$. They are of the form $\varphi\longmapsto\langle\phi, \varphi\rangle\chi$ with some ϕ, $\chi\in\mathcal{D}$, we use the notation $\chi\otimes\phi$. It is easy to check that $(\chi\otimes\phi)^+=\phi\otimes\chi$. The linear hull of all rank–one operators, i. e. the set of all finite–rank operators, is denoted by $\mathcal{F}(\mathcal{D})$. $\mathcal{F}(\mathcal{D})$ is a two–sided *–ideal in $\mathcal{L}^+(\mathcal{D})$. If $\mathcal{D}[\mathrm{t}]$ is an (F)–space then $\overline{\mathcal{F}(\mathcal{D})}^{\tau_{\mathcal{D}}^{c}}=\mathcal{L}^+(\mathcal{D})$ (see [8]).

If $\mathcal{A}$ is an algebra then a linear map $\Phi\colon \mathcal{A}\longrightarrow\mathcal{A}$ is called a local derivation (resp. local automorphism) if for every $a\in\mathcal{A}$ there exists a derivation (resp. automorphism) $\Psi^{(a)}$, depending on a, such that $\Phi(a)=\Psi^{(a)}(a)$. Note that every derivation on $\mathcal{L}^+(\mathcal{D})$ is inner (see [4]). If $\mathcal{A}$ is unital one can define generalized derivations as linear maps Ψ on $\mathcal{A}$ for which $\Psi(ab)=\Psi(a)b+a\Psi(b)-a\Psi(I)b$ for all a, $b\in\mathcal{A}$. Ψ is an inner generalized derivation if there are x, $y\in\mathcal{A}$ such that $\Psi(a)=xa+ay$ for all $a\in\mathcal{A}$. If all derivations on $\mathcal{A}$ are inner then also all generalized derivations and vice versa. Now one can define also local generalized derivations similar to the local derivations. Furthermore on *–algebras $\mathcal{A}$ local *–automorphisms may be defined.

In the second section of our paper we show that local derivations, local generalized derivations or local automorphisms Φ on $\mathcal{L}^+(\mathcal{D})$ are even "global". That's a bit surprising because the only other assumptions concerning the topological structure of $\mathcal{D}[\mathrm{t}]$ seem to be of technical nature. In section 3 we generalize a result of OMLADIČ concerning commutativity preserving maps. Roughly speaking one can say that such maps are nearly automorphisms or anti–automorphisms.

§ 2. Derivations and Automorphisms

The following theorem generalizes a result of LARSON/SOUROUR about local (generalized) derivations

Theorem 2.1. *Let $\Phi\colon \mathcal{L}^+(\mathcal{D})\longrightarrow\mathcal{L}^+(\mathcal{D})$ be a local generalized derivation, i.e. $\Phi(A)=X_A A+AY_A$ with some X_A, $Y_A\in\mathcal{L}^+(\mathcal{D})$ depending on A. Then there are X, $Y\in\mathcal{L}^+(\mathcal{D})$ such that $\Phi(A)=XA+AY$. In particular, if $X_A=-Y_A$ for all $A\in\mathcal{L}^+(\mathcal{D})$ then $X=-Y$ and Φ is a derivation.*

Proof. The idea of the proof is adapted from [5]. The new point is that we have to get X, Y in $\mathcal{L}^+(\mathcal{D})$. For rank–one operators $\phi\otimes\chi$

$$\Phi(\phi\otimes\chi)=X_{\phi,\chi}(\phi\otimes\chi)+(\phi\otimes\chi)Y_{\phi,\chi}=\xi_{\phi,\chi}\otimes\chi+\phi\otimes\eta_{\phi,\chi}.$$

As in [5] p. 189–191 we can prove in some stages that it is possible to take $\xi_{\phi,\chi} = X\phi$, $\eta_{\phi,\chi} = \tilde{Y}\chi$ with some linear maps $X, \tilde{Y}: \mathscr{D} \longrightarrow \mathscr{D}$. Now we have to prove that $X, \tilde{Y} \in \mathscr{L}^+(\mathscr{D})$. Assume that $\langle \chi, \phi \rangle = 1$, i.e. $P = \phi \otimes \chi$ is a projection. We have $\Phi(P) = X\phi \otimes \chi + \phi \otimes \tilde{Y}\chi$. Since Φ is a local generalized derivation we get $P \cdot \Phi(I-P) \cdot P = 0$ and therefore $P \cdot \Phi(I) \cdot P = P \cdot \Phi(P) \cdot P$.
That means $\langle \chi, \Phi(I)\phi \rangle P = \{\langle \chi, X\phi \rangle + \langle \tilde{Y}\chi, \phi \rangle\} P$. Hence

$$\langle \chi, \Phi(I)\phi \rangle = \langle \chi, X\phi \rangle + \langle \tilde{Y}\chi, \phi \rangle \quad \text{for all } \chi, \phi \in \mathscr{D} \text{ with } \langle \chi, \phi \rangle = 1.$$

Linearity immediately implies

$$\langle \chi, \Phi(I)\phi \rangle = \langle \chi, X\phi \rangle + \langle \tilde{Y}\chi, \phi \rangle \qquad \text{for all } \chi, \phi \in \mathscr{D}. \tag{1}$$

The adjoint $\Phi(I)^+$ exists in $\mathscr{L}^+(\mathscr{D})$, therefore

$$|\langle \chi, X\phi \rangle| \leq \| \Phi(I)^+\chi - \tilde{Y}\chi \| \cdot \| \phi \| \qquad \forall \chi, \phi \in \mathscr{D}.$$

Hence $\chi \in D(X^*)$, now (1) implies

$$X^*\chi = \Phi(I)^+\chi - \tilde{Y}\chi \in \mathscr{D} \qquad \forall \chi \in \mathscr{D}$$

and so $X \in \mathscr{L}^+(\mathscr{D})$. In the same way one can show that $\tilde{Y} \in \mathscr{L}^+(\mathscr{D})$. Now (1) becomes an equation in $\mathscr{L}^+(\mathscr{D})$, namely : $\Phi(I) = X + \tilde{Y}^+$.

If we define $Y := \tilde{Y}^+$ we have

$$\Phi(F) = XF + FY \tag{2}$$

at first for all rank–one operators, therefore for all finite–rank operators F. In addition (2) is true also for $F = I$. Now take $\Phi_0(A) := XA + AY$ and $\Psi := \Phi - \Phi_0$. With Φ also Ψ is a local generalized derivation. Assume that there is a $T \in \mathscr{L}^+(\mathscr{D})$ with $S = \Psi(T) \neq 0$. Then a $\rho \in \mathscr{D}$ exists with $S\rho = \sigma \neq 0$. Let P be the orthogonal projection from $\mathscr{D}$ onto $lin\{\rho, \sigma\}$. P is in $\mathscr{F}(\mathscr{D})$ and $P \cdot \Psi((I-P)T(I-P)) \cdot P = 0$. Therefore $T - (I-P)T(I-P) \in \mathscr{F}(\mathscr{D})$ and $P\Psi(T)P = 0$. From the last equation we get a contradiction with $P\Psi(T)P\rho = \sigma \neq 0$. Hence the map Ψ must be identically zero, i.e. $\Phi(A) = XA + AY$ for all $A \in \mathscr{L}^+(\mathscr{D})$. q. e. d.

Now we want to consider (local) automorphisms on $\mathscr{L}^+(\mathscr{D})$. The following proposition is an essential tool for studying automorphisms.

Proposition 2. 2. *Let* $\mathscr{A}$ be a *–subalgebra of $\mathscr{L}^+(\mathscr{D})$ *with* $\mathscr{A} \supset \mathscr{F}(\mathscr{D})$.
(i) *Suppose that* $\Phi: \mathscr{A} \longrightarrow \mathscr{A}$ *is bijective, linear and preserves rank–one operators in both directions (i.e.* $\Phi(F)$ *is a rank–one operator if and only if* F *is a*

rank–one operator). Then either

(a) *there are bijective linear operators* $U_1,\ U_2 : \mathscr{D} \longrightarrow \mathscr{D}$ *such that*

$$\Phi(\varphi\otimes\phi) = U_1\varphi\otimes U_2\phi \quad \textit{for all} \quad \varphi,\ \phi\in\mathscr{D}$$

or

(b) *there are bijective antilinear operators* $V_1,\ V_2 : \mathscr{D} \longrightarrow \mathscr{D}$ *such that*

$$\Phi(\varphi\otimes\phi) = V_1\phi\otimes V_2\varphi \quad \textit{for all} \quad \varphi,\ \phi\in\mathscr{D}.$$

(ii) *If* $\Phi : \mathscr{A} \longrightarrow \mathscr{A}$ *preserves rank–one projections in both directions then either*

(a) *there is a linear operator* $U\in\mathscr{L}^+(\mathscr{D})$, *invertible in* $\mathscr{L}^+(\mathscr{D})$, *such that*

$$\Phi(F) = UFU^{-1} \qquad \forall F\in\mathscr{F}(\mathscr{D})$$

or

(b) *there is a bijective antilinear operator* $V : \mathscr{D} \longrightarrow \mathscr{D}$ *with* $V^*\mathscr{D}\subset\mathscr{D}$ *such that*

$$\Phi(F) = VF^+V^{-1} \qquad \forall F\in\mathscr{F}(\mathscr{D}).$$

(iii) *If* Φ *in addition is *–invariant one can choose* U *resp.* V *such that* $U^+ = U^{-1}$ resp. $V^+ = V^{-1}$.

Proof. Let $\mathscr{F}^1(\mathscr{D})$ be the set of all rank–one operators in $\mathscr{L}^+(\mathscr{D})$. Every maximal linear space in $\mathscr{F}^1(\mathscr{D}) \cup \{0\}$ is of the form

$$[\varphi]_l = \{\varphi\otimes\phi \mid \phi\in\mathscr{D}\} \text{ resp. } [\varphi]_r = \{\phi\otimes\varphi \mid \phi\in\mathscr{D}\} \text{ with some } \varphi\neq o.$$

Because Φ is bijective and rank–one preserving maximal linear spaces in $\mathscr{F}^1(\mathscr{D}) \cup \{0\}$ are mapped onto maximal linear spaces in $\mathscr{F}^1(\mathscr{D}) \cup \{0\}$. Hence for every $\varphi\in\mathscr{D}$

1. $\Phi([\varphi]_l) = [G_l(\varphi)]_l$ or $\Phi([\varphi]_l) = [G_l(\varphi)]_r$ with some $G_l : \mathscr{D} \longrightarrow \mathscr{D}$

and

2. $\Phi([\varphi]_r) = [G_r(\varphi)]_r$ or $\Phi([\varphi]_r) = [G_r(\varphi)]_l$ with some $G_r : \mathscr{D} \longrightarrow \mathscr{D}$.

Note that $G_l(\varphi) = o$ resp. $G_r(\varphi) = o$ if and only if $\varphi = o$. If $\varphi,\ \phi \neq o$ are linearly independent then

$$dim([\varphi]_l \cap [\phi]_l) = dim([\varphi]_r \cap [\phi]_r) = 0$$

otherwise

$$dim([\varphi]_l \cap [\psi]_l) = dim([\varphi]_r \cap [\psi]_r) = dim\ \mathscr{D} = \infty.$$

Furthermore $dim([\varphi]_l \cap [\psi]_r) = 1$ for all φ, $\psi \in \mathscr{D} \setminus \{o\}$. Together with the bijectivity of Φ that implies either

(I) $\Phi([\varphi]_l) = [G_l(\varphi)]_l$ and $\Phi([\varphi]_r) = [G_r(\varphi)]_r$ for all $\varphi \in \mathscr{D}$

or

(II) $\Phi([\varphi]_l) = [G_l(\varphi)]_r$ and $\Phi([\varphi]_r) = [G_r(\varphi)]_l$ for all $\varphi \in \mathscr{D}$.

Assume that (I) is true. For a fixed $\varphi_0 \neq o$ and $\psi \in \mathscr{D}$ we get then

$$\Phi(\varphi_0 \otimes \psi) = G_l(\varphi_0) \otimes H_0(\psi) \qquad \text{with some } H_0 : \mathscr{D} \longrightarrow \mathscr{D}.$$

Now

$$H_0(\psi) = \Phi(\varphi_0 \otimes \psi)^+ G_l(\varphi_0) / \| G_l(\varphi_0) \|^2$$

so that H_0 linearly depends on ψ.

With Φ also H_0 is injective. Together with Φ also Φ^{-1} fulfils (I), therefore H_0 is surjective. Hence $\Phi(\varphi_0 \otimes \psi) = G_l(\varphi_0) \otimes U_2\psi$. Variation of φ with fixed ψ in the same way leads to $\Phi(\varphi \otimes \psi) = U_1^{(\psi)}(\varphi) \otimes U_2\psi$.

Remark that $U_1^{(\psi)}$ may depend on ψ, it remains to prove that it doesn't. If ψ_1, ψ_2 are linearly dependent it is easy to check that $U_1^{(\psi_1)} = U_1^{(\psi_2)}$. Now let ψ_1, ψ_2 linearly independent. Then $U_2\psi_1$, $U_2\psi_2$ are linearly independent and

$$\Phi(\varphi \otimes \psi_1) = U_1^{(\psi_1)}\varphi \otimes U_2\psi_1$$

$$\Phi(\varphi \otimes \psi_2) = U_1^{(\psi_2)}\varphi \otimes U_2\psi_2$$

$$\Rightarrow \qquad \Phi(\varphi \otimes \psi_1 + \psi_2) = U_1^{(\psi_3)}\varphi \otimes U_2(\psi_1 + \psi_2)$$

$$= \Phi(\varphi \otimes \psi_1) + \Phi(\varphi \otimes \psi_2)$$

$$= U_1^{(\psi_1)}\varphi \otimes U_2\psi_1 + U_1^{(\psi_2)}\varphi \otimes U_2\psi_2$$

$$\Rightarrow \qquad (U_1^{(\psi_3)}\varphi - U_1^{(\psi_1)}\varphi) \otimes U_2\psi_1 = (U_1^{(\psi_2)}\varphi - U_1^{(\psi_3)}\varphi) \otimes U_2\psi_2$$

$$\Rightarrow \qquad U_1^{(\psi_1)}\varphi = U_1^{(\psi_3)}\varphi = U_1^{(\psi_2)}\varphi \quad \text{for all } \varphi \in \mathscr{D}$$

hence $U_1^{(\psi)}$ doesn't depend on ψ. So we have proved that (I) implies (i) (a).

The proof that (II) implies (i) (b) goes along the same way.

Now suppose that Φ even preserves rank–one projections and (i) (a) is fulfilled. Then $\langle\phi, \varphi\rangle = 1$ implies $\langle U_2\phi, U_1\varphi\rangle = 1$. With some linearity arguments we get :

$$\langle U_2\phi, U_1\varphi\rangle = \langle\phi, \varphi\rangle \qquad \forall\phi, \varphi\in\mathscr{D}$$

$$\Rightarrow \quad \mathscr{D} = Ran\ U_2 \subseteq D(U_1^*) \quad \text{and} \quad U_1^* U_2\phi = \phi \quad \forall\phi\in\mathscr{D}$$

$$\mathscr{D} = Ran\ U_1 \subseteq D(U_2^*) \quad \text{and} \quad U_2^* U_1\varphi = \varphi \quad \forall\varphi\in\mathscr{D}$$

$$\Rightarrow \quad U_1, U_2 \in \mathscr{L}^+(\mathscr{D}) \qquad \text{and} \quad U_2 = (U_1^+)^{-1}.$$

With $U := U_1$ one gets

$$\Phi(\varphi\otimes\phi) = (U\varphi)\otimes[(U^{-1})^+\phi] = U(\varphi\otimes\phi)U^{-1}.$$

Because of linearity $\Phi(F) = UFU^{-1}$ follows for all $F\in\mathscr{F}(\mathscr{D})$.

If (i) (b) was true then (ii) (b) can be proved analogously.

Now suppose that (ii) (a) is true and Φ in addition is *–invariant. From $\Phi(F^+) = \Phi(F)^+$ we get then

$$UF^+U^{-1} = (U^{-1})^+F^+U^+$$

$$\Rightarrow \qquad U^+UF^+ = F^+U^+U$$

$$\Rightarrow \qquad U^+U = \lambda I \qquad (\lambda > 0).$$

Therefore we can choose $\tilde{U} = \frac{1}{\sqrt{\lambda}}U$. In case (ii) (b) the *–invariance of Φ implies $V^+V = \lambda I$ $(\lambda > 0)$. Hence one can take $\tilde{V} = \frac{1}{\sqrt{\lambda}}V$. That proves (iii).

q. e. d.

Remarks.

• Part (ii) of the proposition has been proved for $\mathscr{A} = \mathscr{B}(X)$, X a Banach–space, in [6], [10]. In our proof we use ideas of these two papers. Our proof of (i) with obvious modifications also works in the Banach–space situation.

• (ii) includes some regularity statements about the operators U, V; namely $U \in\mathscr{L}^+(\mathscr{D})$ resp. $V^*\mathscr{D}\subseteq\mathscr{D}$. We couldn't prove similar things for the operators U_1, U_2, V_1, V_2 in (i). If one modifies (i) for the Banach–space situation one can prove that if Φ is continuous then U_1, U_2, V_1, V_2 are bounded.

• In [10] Synnatzschke proved a version of (ii) for abstract algebras. It is not difficult to reformulate and to prove also (i) in the language of Synnatzschke.

• If $\mathscr{D}[t]$ is an (F)–space and Φ is τ_{D}^{c}–continuous then in the cases (ii), (iii) the representation of Φ can be transfered from $\mathscr{F}(\mathscr{D})$ onto the whole algebra $\mathscr{A}$ because of $\overline{\mathscr{F}(\mathscr{D})}^{\tau_{\mathscr{D}}^{c}}=\mathscr{L}^{+}(\mathscr{D})$.

Now we get the following results.

Theorem 2. 3.
(i) *Let Φ be an automorphism (resp. antiautomorphism) on $\mathscr{L}^{+}(\mathscr{D})$. Then there exists an operator U with U, $U^{-1}\in\mathscr{L}^{+}(\mathscr{D})$ such that $\Phi(X)=UXU^{-1}$ for all $X\in\mathscr{L}^{+}(\mathscr{D})$ (resp. there exists an antilinear operator V with V, V^{-1}, V^{*}, V^{-1*} : $\mathscr{D}\longrightarrow\mathscr{D}$ such that $\Phi(X)=VX^{+}V^{-1}$ for all $X\in\mathscr{L}^{+}(\mathscr{D})$).*
(ii) *If Φ is a *–automorphism (resp. *–antiautomorphism) then in addition U can be chosen unitary (resp. V can be chosen antiunitary).*

Proof. (i): One–dimensional elements S (in $\mathscr{L}^{+}(\mathscr{D})$) are characterized by the existence of a non–trivial linear functional g_S on $\mathscr{L}^{+}(\mathscr{D})$ with $SXS=g_S(X)\cdot S\ \ \forall X\in\mathscr{L}^{+}(\mathscr{D})$. Hence automorphisms and antiautomorphisms preserve rank–one operators in both directions. Furthermore P is a projection iff $\Phi(P)$ is a projection. According to Proposition 2. 2(ii) there are the possibilities

(a) $\quad \Phi(F)=UFU^{-1} \qquad \forall F\in\mathscr{F}(\mathscr{D})$

or

(b) $\quad \Phi(F)=VF^{+}V^{-1} \qquad \forall F\in\mathscr{F}(\mathscr{D})$

U resp. V already have the desired properties. Assume that Φ is an automorphism and (b) is true. Then the following equation would hold true

$$\Phi(F)\,\Phi(G)=VF^{+}V^{-1}VG^{+}V^{-1}=\Phi(GF)=\Phi(G)\,\Phi(F) \qquad \forall F,\ G\in\mathscr{F}(\mathscr{D}).$$

But this is impossible because $\mathscr{F}(\mathscr{D})$ is not commutative. Therefore if Φ is an automorphism then (a) must be true. In the same way one shows that (b) is true iff Φ is an antiautomorphism. It remains to prove that (a) resp. (b) is true for all $X\in\mathscr{L}^{+}(\mathscr{D})$. We demonstrate this for case (a).

Let X be arbitrary and P a projection in $\mathscr{F}(\mathscr{D})$. Then

$$U[X-(I-P)X(I-P)]\,U^{-1}=\Phi(X-(I-P)X(I-P))$$
$$=\Phi(X)-(I-UPU^{-1})\,\Phi(X)\,(I-UPU^{-1}).$$

Hence $UXP=\Phi(X)\,UP$. In particular, if $\|\varphi\|=1$ then $UX\varphi\otimes\varphi=\Phi(X)\,U\varphi\otimes\varphi$.

Therefore $UX\varphi = \Phi(X)\, U\varphi \quad \forall \varphi \in \mathscr{D}$, that implies $\Phi(X) = UXU^{-1} \quad \forall X \in \mathscr{L}^+(\mathscr{D})$.

The proof for the antiautomorphisms is analogous. Assertion (ii) now follows from Proposition 2. 2(iii). q. e. d.

Remark. The representation of *–automorphisms in $\mathscr{L}^+(\mathscr{D})$ has already been shown in [12].

Theorem 2. 4.

(i) *Let Φ be a bijective local automorphism on $\mathscr{L}^+(\mathscr{D})$.* Then either

(a) *Φ is an automorphism*

or

(b) *Φ is an antiautomorphism*

(ii) *If in addition Φ is *–invariant then either*

(a) *Φ is a *–automorphism*

or

(b) *Φ is a *–antiautomorphism*

Proof. (i): Φ is a local automorphism, therefore

$$\Phi(\varphi\otimes\psi) = U_{\varphi,\psi}\varphi \otimes ((U_{\varphi,\psi}^{-1})^+\psi).$$

Φ preserves rank–one projections. On the other hand if $\Phi(X) = U_X X U_X^{-1} = \varphi\otimes\psi$ then $X = U_X^{-1}\varphi \otimes U_X^+\psi$ so that Φ preserves rank–one projections in both directions. According to Proposition 2. 2(ii) either

(a) $\Phi(F) = UFU^{-1} \qquad \forall F \in \mathscr{F}(\mathscr{D})$

or

(b) $\Phi(F) = VF^+V^{-1} \qquad \forall F \in \mathscr{F}(\mathscr{D})$.

We demonstrate for (a) that then also

$$\Phi(X) = UXU^{-1} \qquad \forall X \in \mathscr{L}^+(\mathscr{D}).$$

Let X be arbitrary and P a projection in $\mathscr{F}(\mathscr{D})$. Then

$$U(X-(I-P)X(I-P))\,U^{-1} = \Phi(X-(I-P)X(I-P))$$

$$= \Phi(X) - \Psi_{P,X}((I-P)X(I-P))$$

with an automorphism $\Psi_{P,X}$. From Theorem 2. 3 we get the representation $\Psi_{P,X}(Y)=\hat{U}Y\hat{U}^{-1}$ with some $\hat{U}$. Hence

$$U(X-(I-P)X(I-P))U^{-1}=\Phi(X)-(I-\hat{U}P\hat{U}^{-1})\Psi_{P,X}(X)(I-\hat{U}P\hat{U}^{-1})$$

$$\Rightarrow \qquad PXU^{-1}\hat{U}P=PU^{-1}\Phi(X)\hat{U}P.$$

Take $P=\varphi\otimes\phi$ with $\langle\phi,\varphi\rangle=1$ then we get

$$\langle\phi, XU^{-1}\hat{U}\varphi\rangle=\langle\phi, U^{-1}\Phi(X)\hat{U}\varphi\rangle.$$

Because of linearity this equation is valid for all ϕ, $\varphi\in\mathscr{D}$. Hence $XU^{-1}\hat{U}=U^{-1}\Phi(X)\hat{U}$, that implies $UXU^{-1}=\Phi(X)$.

Assertion (ii) now follows from Proposition 3. 2(iii) q. e. d

Now the question appears if one can exclude the possibility that a local automorphism is a global antiautomorphism. This is possible for a wide variety of domains $\mathscr{D}$. We need the following lemma.

Lemma 2. 5. *Assume that D is one of the following domains:*

(0) $\mathscr{D}=\boldsymbol{d}$, (*$\boldsymbol{d}$ the space of finite sequences*)

(i) $\mathscr{D}[\mathrm{t}]$ *is an (F)-space*

(ii) $\mathscr{D}[\mathrm{t}]$ *is a (QF)-space, for which a* t*-bounded subset* $\mathscr{M}$ exists with $\dim(\overline{lin\,\mathscr{M}}^{\,t})=\infty$

(iii) $\mathscr{D}[\mathrm{t}]$ *is the (QF)-space, constructed in* [2] (*which doesn't belong to case* (ii)).

(iv) $\mathscr{D}[\mathrm{t}]$ *is one of the (DF)-spaces, constructed in* [3], [1].

Then there are an orthonormal system $(\varphi_n)_{n=1}^{\infty}\subset\mathscr{D}$ *and a generalized shift-operator R on this system, i.e.* $R\varphi_n:=\lambda_n\varphi_{n+1}$ $(\lambda_n>0)$ *such that* $R\in\mathscr{L}^+(\mathscr{D})$.

Proof. The problem is to find φ_n, λ_n in such a way that R is in $\mathscr{L}^+(\mathscr{D})$.

(0): Take the usual shift-operator on the canonical orthonormal system $\varphi_n=\boldsymbol{e}_n$.

(i): If $\mathscr{D}$ is an (F)-space then the existence of suitable λ_n has been proved for any orthonormal system $(\varphi_n)\subset\mathscr{D}$ in [9] Lemma 3. 4.

(ii): $\overline{lin\,\mathscr{M}}^{\,t}$ is an infinite-dimensional (F)-space therefore (i) applies.

(iii): Let $\boldsymbol{d}$ be the space of finite sequences, h some fixed vector from $\boldsymbol{l}_2\backslash\boldsymbol{d}$ and $\boldsymbol{d}_h=lin\{\boldsymbol{d}, h\}$. The construction of the (QF)-space $\mathscr{D}$ in [2] uses the space $\boldsymbol{d}_h$. For our purpose it is enough to find a weighted shift-operator in $\mathscr{L}^+(\boldsymbol{d}_h)$ (see [2] Lemma 2). But that is obviously possible because $\boldsymbol{d}_h$ and $\boldsymbol{d}$ are unitary equivalent.

(iv): In [1] (DF)-spaces are constructed as follows. Let (α_n) be an increasing sequence of positive reals with $\lim_{n\to\infty}\frac{\ln n}{\alpha_n}=0$ and

$$X:=\{a=(a_n)\mid \sum_n |\rho^{\alpha_n}a_n|<\infty \quad \forall 0<\rho<1\}$$
$$=\{a=(a_n)\mid \sup_n |\rho^{\alpha_n}a_n|<\infty \quad \forall 0<\rho<1\}.$$

Now let $\mathscr{D}\subset l_2$ be given by

$$\mathscr{D}:=\{f=(f_n)\mid \sum_n |f_n a_n|<\infty \quad \forall a\in X\}$$
$$=\{f=(f_n)\mid \sup_n |f_n a_n|<\infty \quad \forall a\in X\}.$$

We want to use the canonical orthonormal system (e_n) and have to find a sequence (λ_n) such that

$$\sup_n |\lambda_n f_{n+1} a_n|<\infty \text{ and } \sup_n |\lambda_n f_n a_{n+1}|<\infty \quad \text{for all} \quad f\in\mathscr{D},\ a\in X.$$

It is enough to have

$$\sup_n |\lambda_n a_n|<\infty \quad \text{and} \quad \sup_n |\lambda_n a_{n+1}|<\infty \quad \text{for all} \quad a\in X.$$

We can fix some $\rho\in(0, 1)$ and choose then $\lambda_n=\min(\rho^{\alpha_n}, \rho^{\alpha_{n+1}})$ to meet these requirements.

In [3] (DF)-spaces are constructed as follows. Take a matrix $(\alpha_{k,n})_{k,n=1}^{\infty}$ with

(a) $0\leq\alpha_{k,n}\leq 1$

(b) $\alpha_{k,n}=1$ if $n\leq k$

(c) $(\alpha_{k,n})^{1/2}\leq\alpha_{k+1,n}$.

Then define

$$X=\{a=(a_n)\mid \sup_n |a_n|\alpha_{k,n}<\infty \quad \forall k\in N\}$$

and

$$\mathscr{D}=\{f=(f_n)\mid \Sigma_n |f_n a_n|^2<\infty \quad \forall a\in X\}.$$

Now there are two possibilities

1. There is no row in $(\alpha_{k,n})$ with an infinite number of non–zero elements. Then X is the space of all sequences and $\mathscr{D}=d$. Every weighted shift on the canonical orthonormal system is in $\mathscr{L}^+(\mathscr{D})$.
2. There is a row in $(\alpha_{k,n})$ with an infinite number of non–zero elements α_{k,n_j} $(j=1, 2, \ldots)$. Then all elements $\alpha_{k+l,n_j}(l=0, 1, \ldots)$ are non–zero because of (c). We want to choose in $\mathscr{D}$ the orthonormal system $(e_{n_j})_{j=1}^{\infty}$. Then it is enough to consider the matrix $(\alpha_{k+l,n_j})_{j=1,2,\ldots}^{i=0,i,\ldots}$. So we can suppose without loss of generality that all $\alpha_{k,n}$ are non–zero. Similar to the other type of (DF)–spaces we have to construct a sequence (λ_n) with

$$\sum_n |\lambda_n f_{n+1} a_n|^2 < \infty \text{ and } \sum_n |\lambda_n f_n a_{n+1}|^2 < \infty \quad \text{for all } f\in\mathscr{D},\ a\in X.$$

Again it is enough to have

$$\sup_n |\lambda_n a_n| < \infty \text{ and } \sup_n |\lambda_n a_{n+1}| < \infty \quad \text{for all } a\in X$$

because $\sum |f_n|^2 < \infty$. We can choose $\lambda_n = \min(\alpha_{1,n}, \alpha_{1,n+1})$. q. e. d.

Theorem 2. 6. *Assume that there is a weighted shift operator R in $\mathscr{L}^+(\mathscr{D})$. Then every bijective local automorphism Φ is an automorphism. If in addition Φ is *–invariant then Φ is a *–automorphism.*

Proof. Because of Theorem 2. 4 we only have to find a contradiction if we assume $\Phi(X) = VX^+V^{-1}\ \forall X\in\mathscr{L}^+(\mathscr{D})$. For the weighted shift operator R the relations $ker\ R^2 = ker\ R$ and $ker(R^+)^2 \neq ker\ R^+$ are true. Hence there is no invertible linear or antilinear map $W: \mathscr{D}\longrightarrow\mathscr{D}$ with $R^+ = WRW^{-1}$. However we assumed $U_R R U_R^{-1} = \Phi(R) = VR^+V^{-1}$ and that leads to the desired contradiction.
q. e. d.

Remark. The theorems 2. 1, 2. 6 show that the sets $\mathscr{S}$ of generalized inner derivations resp. of automorphisms on $\mathscr{L}^+(\mathscr{D})$ are algebraically reflexive in the following sense :

$$\text{ref}\ \mathscr{S} := \{T\in\mathscr{L}(\mathscr{L}^+(\mathscr{D}))\mid T(X)\in\mathscr{S}(X) \quad \forall X\in\mathscr{L}^+(\mathscr{D})\} = \mathscr{S}.$$

Both sets are certain subsets of the set of elementary operators on $\mathscr{L}^+(\mathscr{D})$. Note that the set of all elementary operators on $\mathscr{L}^+(\mathscr{D})$ is not algebraically reflexive. This can be shown as in [5] section 3.

The referee suggested to consider also maps Φ on subalgebras $\mathscr{A}\subset\mathscr{L}^+(\mathscr{D})$. A check of the proofs shows that for unital algebras $\mathscr{A}\supset\mathscr{F}(\mathscr{D})$ there are necessary only few modifications to obtain results similar to the Theorems 2. 1/

2. 4.

Proposition 2. 7. *Let $\mathscr{A} \supset \mathscr{F}(\mathscr{D})$ be an Op^*-algebra*
(i) *For every derivation Φ on $\mathscr{A}$* there exists an $A \in \mathscr{L}^+(\mathscr{D})$ *with*

$$\Phi(X) = AX - XA \qquad \forall X \in \mathscr{A}.$$

(ii) *Let Φ be an automorphism (resp. antiautomorphism) on $\mathscr{A}$. Then there exists an operator U with U, $U^{-1} \in \mathscr{L}^+(\mathscr{D})$ such that $\Phi(X) = UXU^{-1}$ for all $X \in \mathscr{A}$ (resp. there exists an antilinear operator V with V, V^{-1}, V^*, $V^{-1*} : \mathscr{D} \longrightarrow \mathscr{D}$ such that $\Phi(X) = VXV^{-1}$ for all $X \in \mathscr{A}$).*
*If Φ is a *-automorphism (resp. *-antiautomorphism) then in addition U can be chosen unitary (resp. V can be chosen antiunitary).*

Remark. For some Op^*-algebras the representation of *-automorphisms has also been obtained by Takesue in [11], see also [7], Theorem 6. 3. 6. Takesue used other assumptions instead of $\mathscr{A} \supset \mathscr{F}(\mathscr{D})$, so that also some algebras with $\mathscr{A} \not\supset \mathscr{F}(\mathscr{D})$ can be considered.

Proof. (ii) can be proved by obvious modifications of the proof of Theorem 2. 3. For the proof of (i) we refer to [7], p. 167. q. e. d.

For a generalized derivation Φ the map $\Psi(x) := \Phi(x) - \Phi(I)x$ is a derivation. Hence we can note the following corollary.

Corollary 2. 8. *For every generalized derivation Φ on an Op^*-algebra $\mathscr{A} \supset \mathscr{F}(\mathscr{D})$ there exist operators A, $B \in \mathscr{L}^+(\mathscr{D})$ with $\Phi(X) = AX + XB$ for all $X \in \mathscr{A}$.*

Now we can generalize the theorems 2. 1/2. 4.

Theorem 2. 9. *Let $\mathscr{A} \supset \mathscr{F}(\mathscr{D})$ be an Op^*-algebra.*
(i) *Every local (generalized) derivation on $\mathscr{A}$ is a (generalized) derivation.*
(ii) *Every bijective local automorphism on $\mathscr{A}$ is either an auto- or an antiautomorphism.*

The proof is left to the reader. There are only needed some obvious modifications of the proofs of Theorem 2. 1 resp. 2. 4.

Remark. For other algebras $\mathscr{A} \not\supset \mathscr{F}(\mathscr{D})$ (resp. $\mathscr{A} \not\supset \mathscr{F}(\mathscr{H})$ in the bounded situation) it seems to be an open question if there are local derivations/automorphisms which are not global ones.

§ 3. Commutativity Preserving Maps on $\mathscr{L}^+(\mathscr{D})$

In this section we want to generalize a result of Omladič [6] for the algebra $\mathscr{L}^+(\mathscr{D})$. Some ideas of the proof are as in [6] but note that Omladič's proof is not complete and correct in all steps. The theorem is the following one.

Theorem 3. 1. *Let Φ: $\mathscr{L}^+(\mathscr{D}) \longrightarrow \mathscr{L}^+(\mathscr{D})$ be linear, bijective and commutativity preserving (i. e. $[\Phi(A), \Phi(B)]=0 \Longleftrightarrow [A, B]=0$). Furthermore assume that $\sigma(A) \neq \phi \Longleftrightarrow \sigma(\Phi(A)) \neq \phi$. Then either*

(a) $\Phi(X)=p(X) \cdot I+\kappa UXU^{-1} \qquad \forall X \in \mathscr{L}^+(\mathscr{D})$

or

(b) $\Phi(X)=p(X) \cdot I+\kappa VXV^{-1} \qquad \forall X \in \mathscr{L}^+(\mathscr{D})$

with a linear functional p on $\mathscr{L}^+(\mathscr{D})$, $\kappa \neq 0$, $U \in \mathscr{L}^+(\mathscr{D})$ resp. V: $\mathscr{D} \longrightarrow \mathscr{D}$ invertible and antilinear with $V^+\mathscr{D} \subseteq \mathscr{D}$.

Remarks.

• It is easy to check the reversed conclusion that if Φ has the form (a) or (b) then it preserves commutativity and $\sigma(\Phi(X))=\sigma(X)+p(X) \neq \phi$ iff $\sigma(X) \neq \phi$.

• In Omladič's theorem for maps Φ: $\mathscr{L}(X) \longrightarrow \mathscr{L}(X)$ the space X may be finite–dimensional, but then *dim* $X \geq 3$ is needed. As already mentioned some corrections of Omladič's proof are necessary. They can be done as in our following proof where we have only at one point a restriction concerning the dimension of $\mathscr{D}$, namely again *dim* $\mathscr{D} \geq 3$.

• If Φ acts on a Banach–algebra $\mathscr{L}(X)$ then the assumption about the spectra is unnecessary because all operators in $\mathscr{L}(X)$ have non–empty spectrum.

We will give the proof step by step. Note that we always want the assumptions of the theorem to be fulfilled.

Lemma 3. 2. *For a subset $\mathscr{R} \subseteq \mathscr{L}^+(\mathscr{D})$ let $\mathscr{R}'=\{X \in \mathscr{L}^+(\mathscr{D}) \mid XR=RX \ \forall R \in \mathscr{R}\}$ be the commutant of $\mathscr{R}$. Then $\Phi(\mathscr{R}')=[\Phi(\mathscr{R})]'$ and $\Phi(\mathscr{R}'')=[\Phi(\mathscr{R})]''$. Furthermore $\Phi(I)=\omega \cdot I$ with some $\omega \neq 0$.*

The proof is easy and therefore omitted.

Lemma 3. 3 (see [6] Lemma 3. 1, [10] Theorem 3. 1). *Assume that for $A \in \mathscr{L}^+(\mathscr{D})$ the spectrum $\sigma(A)$ contains more than one element. Then*

$$\dim\{A\}''=2 \quad \Longleftrightarrow \quad A=\alpha P+\beta I$$

with a nontrivial projection $P\in\mathscr{L}^+(\mathscr{D})$ *and* $\alpha\neq 0$.

Lemma 3.4 (see [6] Lemma 3.2, [10] Theorem 3.2). *Let* $P\neq Q$ *be two non-trivial commuting projections in* $\mathscr{L}^+(\mathscr{D})$.
(i) *Either* P *or* $(I-P)$ *and either* Q *or* $(I-Q)$ *are rank-one projections if and only if* $\mathscr{C}=\{P\}'+\{Q\}'$ *has codimension* 2 *in* $\mathscr{L}^+(\mathscr{D})$.
(ii) *If* P, Q *are of rank one then there are nilpotent rank-one operators* U, V $(U^2=V^2=0)$ *with* $P=UV$, $Q=VU$ *and* $\mathscr{L}^+(\mathscr{D})=\mathscr{C}\oplus lin(U, V)$.

The next lemma will be used in our proof instead of Omladič's Lemma 3.5 which is false. We will give a counterexample later.

Lemma 3.5. *Let* P, $R\in\mathscr{L}^+(\mathscr{D})$ *be rank-one projections. Then there is a chain* $Q_1, Q_2, \dots, Q_n$ *such that* (P, Q_1), (Q_1, Q_2), $\dots$, (Q_n, R) *are pairs of disjoint rank-one projections.*

In our context $\mathscr{D}$ is always infinite-dimensional, but for the proof of this lemma we need only $dim\ \mathscr{D}\geq 3$.

Proof. Take $P=\varphi_P\otimes\phi_P$, $R=\varphi_R\otimes\phi_R$ with $\langle\phi_P, \varphi_P\rangle=\langle\phi_R, \varphi_R\rangle=1$ (φ_P, φ_R, ϕ_P, $\phi_R\in\mathscr{D}$). As usual we sometimes interpret the vectors from $\mathscr{D}$ as linear functionals on $\mathscr{D}$.

1. Assume that $ker\ \phi_P\cap ker\ \phi_R\not\subseteq lin\{\varphi_P, \varphi_R\}$.
Then choose $\varphi_1\in(ker\ \phi_P\cap ker\ \phi_R)\backslash lin\{\varphi_P, \varphi_R\}$ and ϕ_1 with $\langle\phi_1, \varphi_1\rangle=1$, $\langle\phi_1, \varphi_P\rangle=\langle\phi_1, \varphi_R\rangle=0$. Now we can take $n=1$, $Q_1=\varphi_1\otimes\phi_1$.

2. Now assume that $ker\ \phi_P\cap ker\ \phi_R\subseteq lin\{\varphi_P, \varphi_R\}$.
$ker\ \phi_P\cap ker\ \phi_R$ is nontrivial because of $dim\ \mathscr{D}\geq 3$. If (φ_P, φ_R) would be linearly dependent we would get the contradiction $\langle\phi_P, \varphi_P\rangle=\langle\phi_R, \varphi_R\rangle=0$.
If (ϕ_P, ϕ_R) would be linearly dependent then $dim\ \mathscr{D}\geq 3$ would imply $ker\ \phi_P\cap ker\ \phi_R=lin\{\varphi_P, \varphi_R\}$ and again we would get the contradiction $\langle\phi_P, \varphi_P\rangle=\langle\phi_R, \varphi_R\rangle=0$. Hence (φ_P, φ_R) are linearly independent as well as (ϕ_P, ϕ_R). There is a nontrivial $\varphi\in lin\{\varphi_P, \varphi_R\}\cap(ker\ \phi_P\cap ker\ \phi_R)$. The ansatz $\varphi=\alpha\varphi_P+\beta\varphi_R$ implies $\langle\phi_P, \varphi_R\rangle\cdot\langle\phi_R, \varphi_P\rangle=1$, in particular $\langle\phi_P, \varphi_R\rangle, \langle\phi_R, \varphi_P\rangle\neq 0$.
Now choose φ_1 such that $\varphi_1\in ker\ \phi_P$, $\varphi_1\notin ker\ \phi_R$, $(\varphi_P, \varphi_R, \varphi_1)$ linearly independent. (This is possible because $ker\ \phi_P\cap lin\{\varphi_P, \varphi_R\}=ker\ \phi_R\cap lin\{\varphi_P, \varphi_R\}$ but $ker\ \phi_P\neq ker\ \phi_R$.) Take $\phi_1=\frac{1}{\langle\phi_R, \varphi_1\rangle}(\phi_R-\langle\phi_R, \varphi_P\rangle\phi_P)$ and $\varphi_2=\langle\phi_R, \varphi_1\rangle(\varphi_R-\langle\phi_P, \varphi_R\rangle\varphi_P)$. (φ_2, φ_R) are linearly independent because of $\langle\phi_P, \varphi_R\rangle\neq 0$, hence also $(\varphi_1, \varphi_2, \varphi_R)$ are linearly independent. Now choose ϕ_2 with $\langle\phi_2, \varphi_1\rangle=\langle\phi_2, \varphi\rangle=0$, $\langle\phi_2, \varphi_2\rangle=1$. Then we can take $n=2$, $Q_1=\varphi_1\otimes\phi_1$, $Q_2=\varphi_2\otimes\phi_2$. q. e. d.

Now we go the first big step towards the proof of Theorem 3.1.

Proposition 3. 6 (see [6] Proposition 3. 4).
Let P, Q be disjoint rank–one projections. Then

$$\Phi(P)=\kappa R+\lambda_P I, \quad \Phi(Q)=\kappa S+\lambda_Q I$$

with disjoint rank–one projections R, S and $\kappa\neq 0$.

Proof. The steps of the proof are as in [6]. We will remark only some modifications and completions. Write $P=\varphi_P\otimes\psi_P$, $Q=\varphi_Q\otimes\psi_Q$ and set $U=\varphi_P\otimes\psi_Q$, $V=\varphi_Q\otimes\psi_P$. Remark that $\langle\psi_P, \varphi_P\rangle=\langle\psi_Q, \varphi_Q\rangle=1$, $\langle\psi_P, \varphi_Q\rangle=\langle\psi_Q, \varphi_P\rangle=0$. Every rank–one operator $T=\varphi\otimes\psi$ has the non–empty spectrum $\sigma(T)=\{0, \langle\psi, \varphi\rangle\}$. Hence, according to the spectral assumption of Theorem 3. 1, the spectra $\sigma(\Phi(P))$, $\sigma(\Phi(Q))$, $\sigma(\Phi(U))$, $\sigma(\Phi(V))$ are non–empty. With $A=\Phi(P)$, $B=\Phi(U)$, $C=\Phi(V)$, $D=\Phi(Q)$ assume $a\in\sigma(A)$, $b\in\sigma(B)$, $c\in\sigma(C)$, $d\in\sigma(D)$.
(I) *$A-a$, $D-d$ cannot both be nilpotent*
This can be proved as in [6].
(II) *$A-a$, $D-d$ are scalar multiples of projections*
Without loss of generality assume $a=d=0$. The Lemmata 3. 2/3. 3 imply $A^2=\alpha A+\beta I$, $D^2=\gamma D+\delta I$. Because of $0\in\sigma(A)$, $0\in\sigma(D)$ it follows that $\beta=\delta=0$. Assume $\alpha=0$, then $\gamma\neq 0$ according to (I).
Take $G=A+D=\Phi(P+Q)$, then

$$G^2\cdot(G-\gamma)^2=0 \tag{3}$$

$P+Q$ is a projection, with Lemma 3. 2 and Lemma 3. 3 we get $dim\{G\}''=2$. Hence $G^2=\mu G+\nu I$. From (3) we get now three possibilities :

(a) $G^2=0$
(b) $G(G-\gamma)=0$
(c) $(G-\gamma)^2=0$

Now we deal with these three cases. Remark that A and D commute because P and Q do so.

(a) $$0=(A+D)^2=2AD+D^2$$
$$\Rightarrow\ 0=2A^2D+AD^2=\gamma AD.$$

$\gamma=0$ contradicts (I), but with $\gamma\neq 0$ we get $AD=0=D^2$. This is also a contradiction.

(b) $$0=(A+D)(A+D-\gamma)=2AD-\gamma A$$
$$\Rightarrow\ 0=2AD^2-\gamma AD=\gamma AD.$$

Again $\gamma=0$ contradicts (I) and $\gamma\neq 0$ leads to $AD=0=A$ what is impossible.

(c) $$0=(A+D-\gamma)^2=2A(D-\gamma)+(D-\gamma)^2$$
$$\Rightarrow\ 0=2A^2(D-\gamma)+A(D-\gamma)^2=-\gamma A(D-\gamma).$$

Here $\gamma=0$ contradicts (I) and $\gamma\neq 0$ leads to

$$(D-\gamma)^2=-\gamma(D-\gamma)=0\,;\ D=\gamma I \quad \text{what is impossible.}$$

So we can exclude $\alpha=0$, $\gamma=0$ can be excluded in the same way.

(III) $R=(A-a)/\alpha$ and $(D-d)/\gamma=S$ are projections. Because Φ is bijective we get with Lemma 3. 4 that either R or $(I-R)$ and either S or $(I-S)$ are rank –one projections, assume that R and S are such ones. Then the operators

$$\Phi(P)=A=\alpha R+aI, \quad \Phi(Q)=D=\gamma S+dI$$

get the desired structure. Also in the other cases A resp. D is a linear combination of I and R resp. S. It is not difficult to check now $RS=SR=0$ and $\alpha=\gamma$. This is left to the reader. q. e. d.

Corollary 3. 7. *For all rank–one projections P one has $\Phi(P)=\kappa R_P+\lambda_P I$ with a rank–one projection R_P. $\kappa\neq 0$ is independent of P.*

Proof. Combine Proposition 3. 6 and Lemma 3. 5. q. e. d.

The next step of the proof of Theorem 3. 1 is the definition of the linear functional p. Fix some $P_0=\varphi_P\otimes\phi_P$ with $\langle\phi_P,\varphi_P\rangle=1$, then $\Phi(P_0)=\kappa(\xi\otimes\eta)+\lambda I$ with $\kappa\neq 0$, $\langle\eta,\xi\rangle=1$. Define

$$p(A)\ :=\langle\eta,\Phi(A)\xi\rangle-\kappa\langle\phi_P,A\varphi_P\rangle \tag{4}$$

$$\Psi(A)\ :=\frac{1}{\kappa}(\Phi(A)-p(A)I). \tag{5}$$

In particular, $p(P_0)=\lambda$, $\Psi(P_0)=\xi\otimes\eta$. Remark that p and Ψ depend on the choice of P_0.

Proposition 3. 8.
(i) *Ψ fulfils the assumptions of Theorem* 3. 1
(ii) $\Psi(I)=I$
(iii) $\Psi(P_0AP_0)=\Psi(P_0)\,\Psi(A)\,\Psi(P_0) \qquad \forall A\in\mathscr{L}^+(\mathscr{D})$.

Proof. Linearity, the property of commutativity preserving and the spectral condition follow from the construction of Ψ. That Ψ is bijective can be proved as in [6] Proposition 4. 1. Since $\Phi(I)=\omega I$ we get $p(I)=\omega-\kappa$, $\Psi(I)=I$. (iii) can also be shown as in [6]. q. e. d.

Now some efforts are necessary to prove the following proposition. Omladič tried to use his Lemma 3. 5 from [6], but his lemma is false in some special cases. We had to find some modifications. Nevertheless ideas of Omladič are essentially used also in our proof.

Proposition 3. 9. *$\Psi(R_0)$ is a rank–one projection for every rank–one projection R_0.*

For the proof we distinguish the following cases for $R_0=\varphi_R\otimes\phi_R$ (Remark : $P_0=\varphi_P\otimes\phi_P$)

(a) $\langle\phi_P, \varphi_R\rangle=\langle\phi_R, \varphi_P\rangle=0$
(b) $\langle\phi_P, \varphi_R\rangle\cdot\langle\phi_R, \varphi_P\rangle\neq 1$
(c) $\langle\phi_P, \varphi_R\rangle\cdot\langle\phi_R, \varphi_P\rangle=1$ and (φ_P, φ_R) or (ϕ_P, ϕ_R) linearly dependent
(d) $\langle\phi_P, \varphi_R\rangle\cdot\langle\phi_R, \varphi_P\rangle=1$, (φ_P, φ_R) and (ϕ_P, ϕ_R) linearly independent.

Before we come to the proof of proposition 3. 9 we note the following lemma.

Lemma 3. 10. *Let $P_0=\varphi_P\otimes\phi_P$, $R_0=\varphi_R\otimes\phi_R$ be projections in $\mathscr{L}^+(\mathscr{D})$. If not (d) then there are rank–one nilpotents U, V for which $P_0=UV$. Furthermore $Q_0=VU$ is a projection disjoint with P_0 and $R_0\in lin\{P_0, Q_0, U, V\}$.*

This lemma is the "right part" of Lemma 3. 5 in [6], the proof is omitted. The assertion of Lemma 3. 10 is false in case (d). This can be seen as follows. U resp. V must have the structure $U=\varphi_P\otimes\phi$ resp. $V=\varphi\otimes\phi_P$. For φ, ϕ one gets the requirements $\langle\phi, \varphi\rangle=1$, $\phi_R\in lin\{\phi, \phi_P\}$, $\varphi_R\in lin\{\varphi, \varphi_P\}$. They cannot be fulfilled in case (d). An example for rank–one projections P, R which are in relation (d) can be given already in a three–dimensional space $\mathscr{D}$, namely take

$$\varphi_P=(100)^T,\quad \varphi_R=(010)^T,\quad \phi_P=(111)^T,\quad \phi_R=(11-1)^T.$$

In the following we mainly distinguish between (d) and the other cases where Lemma 3. 10 works.

Proof of Proposition 3. 9. If we are not in case (d) the proof goes exactly along the line of Omladič's proof for Proposition 4. 1 in [6]. In the case (d) take $\varphi\in ker\ \phi_R\backslash ker\ \phi_P$ and $R_\lambda=(\varphi_R+\lambda_\varphi)\otimes\phi_R$. R_λ is a rank–one projection for all λ. Furthermore the pair (P_0, R_λ) belongs to one of the cases (a)–(c) for all $\lambda\neq 0$. Hence $\Psi(R_\lambda)$ is a rank–one projection for all $\lambda\neq 0$. Now it is easy to verify that $\Psi(R_0)$ is also a rank–one projection. q. e. d.

The same can be done for the map Ψ^{-1}, hence Ψ preserves rank–one projections in both directions. According to Proposition 2. 2 (ii) we have either

- $\Psi(F)=UFU^{-1} \qquad \forall F\in\mathscr{F}(\mathscr{D})$

or

- $\Psi(F)=VF^+V^{-1} \qquad \forall F\in\mathscr{F}(\mathscr{D})$

where U resp. V are maps with the properties asserted in Theorem 3. 1.

Proof of Theorem 3. 1. p, Ψ are constructed in such a way that $\Phi(X)=\kappa\Psi(X)+p(X)I$ (see (4), (5)). It remains to prove that the possible representations of Ψ given above are valid not only for $F\in\mathscr{F}(\mathscr{D})$ but for all $X\in\mathscr{L}^+(\mathscr{D})$. At first we want to prove that for all rank–one projections R_0

$$\Psi(R_0AR_0)=\Psi(R_0)\Psi(A)\Psi(R_0) \qquad \forall A\in\mathscr{L}^+(\mathscr{D}).$$

Again we have the cases (a)–(d) for the relation between P_0 and R_0. If not (d) then the proof goes as in [6]. Otherwise we consider R_λ as in the proof of Proposition 3. 9, then it is $R_\lambda=R_0+\lambda S_0$ with some rank–one operator S_0. For all $\lambda\neq 0$ we have $\Psi(R_\lambda AR_\lambda)=\Psi(R_\lambda)\Psi(A)\Psi(R_\lambda)$. *Hence*

$$\begin{aligned}
&\Psi(R_0AR_0)-\Psi(R_0)\Psi(A)\Psi(R_0)\\
&\quad=\lambda\cdot[\Psi(S_0)\Psi(A)\Psi(R_0)+\Psi(R_0)\Psi(A)\Psi(S_0)\\
&\qquad-\Psi(S_0AR_0)-\Psi(R_0AS_0)]+\lambda^2[\Psi(S_0)\Psi(A)\Psi(S_0)-\Psi(S_0AS_0)]\\
&\quad=:\lambda M_A+\lambda^2 N_A \qquad \text{for all } \lambda\neq 0.
\end{aligned}$$

That implies $M_A=N_A=0$, therefore

$$\Psi(R_0AR_0)-\Psi(R_0)\Psi(A)\Psi(R_0)=0 \qquad \text{for all } A\in\mathscr{L}^+(\mathscr{D}).$$

Now we assume that $\Psi(F)=UFU^{-1}\quad\forall F\in\mathscr{F}(\mathscr{D})$. For $R_0=\varphi_R\otimes\phi_R$ and arbitrary $X\in\mathscr{L}^+(\mathscr{D})$ we have

$$\Psi(R_0XR_0)=\Psi(R_0)\Psi(X)\Psi(R_0)$$

$$U(R_0XR_0)U^{-1}=UR_0U^{-1}\Psi(X)UR_0U^{-1}$$

$$R_0XR_0=R_0U^{-1}\Psi(X)UR_0$$

$$\langle\phi_R,X\varphi_R\rangle\varphi_R\otimes\phi_R=\langle\phi_R,U^{-1}\Psi(X)U\varphi_R\rangle\varphi_R\otimes\phi_R.$$

Hence, if $\langle\phi_R,\varphi_R\rangle=1$ then $\langle\phi_R,X\varphi_R\rangle=\langle\phi_R,U^{-1}\Psi(X)U\varphi_R\rangle$. But linearity then implies, that this equation is true for all $\phi_R,\varphi_R\in\mathscr{D}$. Therefore $X=U^{-1}\Psi(X)U$ resp. $UXU^{-1}=\Psi(X)$.

In the case $\Psi(F)=VF^+V^{-1}$ $\forall F\in\mathscr{F}(\mathscr{D})$ the proof of $\Psi(X)=VX^+V^{-1}$ $\forall X\in\mathscr{L}^+(\mathscr{D})$ is analogous. q. e. d.

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