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An Extension of Baum–Fulton–MacPherson's Riemann–Roch Theorem for Singular Varieties

By

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Introduction

In [1] Baum, Fulton and MacPherson extend the Grothendieck–Riemann–Roch theorem (abbr. GRR) for non–singular projective varieties to (possibly singular) quasi–projective varieties over any field and proper morphisms. More precisely, in [1, Chapter 1] they prove the Riemann–Roch (abbr. RR) in the case of (possibly singular) complex projective varieties with the ordinary singular homology theory with rational coefficients, by deformation to the normal bundle, and in [1, Chapter 2], in more general, they prove the RR in the case of (possibly singular) quasi–projective varieties over any field and proper morphisms with the Chow homology theory with rational coefficeints [3], by the "Grassmannian graph construction". If the field is the complex numbers C, then the above Chow homology theory can be replaced by the singular homology theory (and Borel–Moore homology theory for non–compact varieties). Thus [1, Chap. 2] generalizes [1, Chap. 1].

In this paper our varieties are complex projective varieties and the homology is the singular homology theory with rational coefficeints.

Let K_0 be the covariant functor of Grothendieck groups of algebraic coherent sheaves. For a morphism $f\colon X \to Y$ the pushforward $f_*: K_0(X) \to K_0(Y)$ is defined to be the alternating sum of higher direct images $R^i f_*$. Let $H_{2*}(;Q)$ be the even part of the Q–homology covariant functor. The GRR–theorem says ("homologically") that *if* $\tau: K_0(X) \to H_{2*}(X;Q)$ *is defined by* $\tau(\mathscr{F}) := [td(T_X)ch(\mathscr{F})] \cap [X] \,(= td(T_X) \cap (ch(\mathscr{F}) \cap [X]))$, *where td is the Todd class and ch is the Chern character, then* τ *becomes a natural transformation.*

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When it comes to singular varieties, the above definition of $\tau(\mathscr{F})$ does not make any sense since the tangent bundle T_X cannot be defined. However, by introducing the *localized Chern character* $ch_X^M(\mathscr{F})$ of the coherent sheaf $\mathscr{F}$ (which is a homology class of X) with X embedded into a non–singular quasi–projective variety M, as a substitute of $(ch(\mathscr{F}) \cap [X])$, Baum–Fulton–MacPherson proved in [1] that *there exists a unique natural transformation* $Td_* : K_0 \to H_{2*}(\ ; \boldsymbol{Q})$ *satisfying the extra condition that for a smooth variety* X $Td_*(\mathcal{O}_X)$ *is equal to the Poincaré dual of the usual total Todd cohomology class* $td(T_X)$ *of the tangent bundle* T_X*, where* $\mathcal{O}_X$ *is the structure sheaf of* X. Here for a variety X $Td_*(\mathscr{F})$ is defined by $i^* td(T_M) \cap ch_X^M(\mathscr{F})$, which turns out to be independent of the choice of the embedding $i: X \to M$. For a smooth variety X $i^* td(T_M) \cap ch_X^M(\mathscr{F})$ is equal to $td(T_X) \cap ch(\mathscr{F}) \cap [X]$, thus in the category of smooth varieties Td_* is nothing but the GRR–transformation τ above.

For a variety X $Td_*(X) := Td_*(\mathcal{O}_X)$ is called the *homology Todd class* of X, the 0–th component of which is the arithmetic genus $\chi(X, \mathcal{O}_X)$ of X. And Td_* is multiplicative, i. e., $Td_*(X \times Y) = Td_*(X) \times Td_*(Y)$, which is a generalization of $\chi(X \times Y, \mathcal{O}_{X \times Y}) = \chi(X, \mathcal{O}_X)\chi(Y, \mathcal{O}_Y)$. Thus the homology Todd class is a generalization of the arithmetic genus to higher dimensional homology classes. The above Baum–Fulton–MacPherson's Riemann–Roch theorem (abbr. BFM–RR) has another aspect, besides being an extension of GRR, that the natural transformation $Td_* : K_0 \to H_{2*}(\ ; \boldsymbol{Q})$ is a "singular" version of the usual Todd cohomology class of non–singular varieties (cf. MacPherson's survey article [10]). As commented in the introduction of [1], establishing BFM–RR was clearly motivated by the "singular" version of Chern cohomology class, i. e., the Chern–MacPherson homology class [9] : Let F be the constructible function covariant functor. Then *there exists a unique natural transformation* $C_* : F \to H_{2*}(\ ; \boldsymbol{Z})$ *satisfying the extra condition that for a smooth variety* X $C_*(\mathbb{1}_X)$ *is equal to the Poincaré dual of the usual total Chern cohomology class* $c(T_X)$ *of the tangent bundle* T_X*, where* $\mathbb{1}_X$ *is the characteristic function on* X. For a variety X, $C_*(X) := C_*(\mathbb{1}_X)$ is called the *Chern–MacPherson class* of X, the 0–th component of which is the topological Euler–Poincaré characteristic $\chi(X)$ of X. Recently Kwieciński [7] has proved that C_* is multipliciative, i. e., $C_*(X \times Y) = C_*(X) \times C_*(Y)$, which is a generalization of $\chi(X \times Y) = \chi(X)\chi(Y)$ (see also [8]).

In [12] the author extends C_* so that the Chern class c in the above extra condition can be replaced by the Chern polynomial $c_{(q)} = \sum_{i \geq 0} q^i c_i$. In this paper, in the same spirit as that in [12], we shall extend the BFM–RR–transformation Td_* so that the Todd class td in the above extra condition can be replaced by the "Todd polynomial" $td_{(q)} = \sum_{i \geq 0} q^i td_i$.

Theorem A. *Let* $K_0^{(q)}(X) := K_0(X) \otimes Q[q, q^{-1}]$ *and for a morphism* $f: X \to Y$ *the twisted pushforward* $f_*^{(q)} : K_0^{(q)}(X) \to K_0^{(q)}(Y)$ *is defined simply by* $f_*^{(q)} := q^{\dim X - \dim Y} f_*$, *where* $f_* : K_0(X) \to K_0(Y)$ *is the original one. And let* $H_{2*}^{(q)}(; Q) := H_{2*}(; Q[q, q^{-1}])$. *Then there exists a unique natural transformation* $Td_{(q)*} : K_0^{(q)} \to H_{2*}^{(q)}(; Q)$ *satisfying the extra condition that*

$$Td_{(q)*}(\mathcal{O}_X) = td_{(q)}(T_X) \cap [X] \quad \textit{for smooth } X,$$

and such that if we "evaluate" $Td_{(q)*}$ *at* $q=1$, *then we recover BFM–RR* $Td_{(1)*} = Td_*$.

To obtain the above "twisted" natural transformation in a similar way to that of the BFM–RR Td_*, we consider how to modify the basic ingredients $td(T_M)$ and $ch_X^M(\mathscr{F})$ in the BFM–RR–theorem. In BFM–RR $ch_X^M(\mathscr{F})$ is defined by $ch_X^M(\mathscr{F}) := ch_X^M(\mathscr{E}.)$, where $\mathscr{E}.$ *is a resolution of the sheaf* $i_*\mathscr{F}$ for an embedding $i: X \to M$. In our case we define $ch_{(q)}{}_X^M(\mathscr{F}) := q^{\dim X - \dim M} ch_{(q)}{}_X^M(\mathscr{E}.)$, by introducing some "twisting coefficient", where $ch_{(q)} = \sum_{i \geq 0} q^i ch_i$ is the "Chern character polynomial". Then, in a similar manner as in [1], we can show that for a coherent sheaf $\mathscr{F}$ and for an embedding $i: X \to M$, $i^* td_{(q)}(T_M) \cap ch_{(q)}{}_X^M(\mathscr{F})$ is actually independent of the embedding $i: X \to M$ and that the transformation $Td_{(q)*} : K_0^{(q)} \to H_{2*}^{(q)}(; Q)$ defined by $Td_{(q)*}(\mathscr{F}) = i^* td_{(q)}(T_M) \cap ch_{(q)}{}_X^M(\mathscr{F})$ is nothing but the natural transformation in the above theorem.

It turns out that there is a simple relationship between the "twisted" natural transformation $Td_{(q)*}$ and BFM–RR Td_* as follows: for a variety X and a coheret sheaf $\mathscr{F}$ on X,

$$Td_{(q)*}(\mathscr{F}) = q^{\dim X} \sum_{i \geq 0} q^{-i} Td_{*i}(\mathscr{F}), \tag{0. 1}$$

where the natural transformation $Td_{*i}: K_0 \to H_{2i}(; Q)$ is the projection of $Td_* : K_0 \to H_{2*}(; Q)$ to the $2i$–dimensional component. If, from the very beginning, we define the "twisted" natural transformation $Td_{(q)*} : K_0^{(q)} \to H_{2*}^{(q)}(; Q)$ by (0. 1) and by extending it linearly with respect to the Laurent polynomial ring $Q[q, q^{-1}]$, then it is not hard to see Theorem A itself. However, the aim of this paper is to prove Theorem A in an analogous manner to that of BFM–RR Theorem, as a natural extension or generalization of it.

As a corollary of the above theorem, we can get the following "twisted" version of GRR :

Corollary B. *For a non–singular projective variety X, if we define the homomorphism* $\tau^{(q)} : K_0^{(q)}(X) \to H^{2*(q)}(X; Q) (:= H^{2*}(X; Q[q, q^{-1}]))$ *by* $\tau^{(q)}(\mathscr{F}) :=$

$ch_{(q)}(\mathscr{F})td_{(q)}(T_X)$ and extending linearly with respect to $Q[q, q^{-1}]$, then $\tau^{(q)}$ becomes a natural transformation, i. e., for a morphism $f: X \to Y$ the following equality holds:

$$ch_{(q)}(f_*^{(q)}\mathscr{F})td_{(q)}(T_Y) = f_*(ch_{(q)}(\mathscr{F})td_{(q)}(T_X))$$

where $f_*: H^{2*(q)}(X; Q) \to H^{2*(q)}(Y; Q)$ is the Gysin homomorphism.

If we consider $'K_0^{(q)}$ to be just a linear extension of the functor K_0 with respect to the Laurent polynomial ring $Q[q, q^{-1}]$, then by considering the map from a smooth X to a point we can see that one cannot get such a natural transformation $\tau: {}'K_0^{(q)} \to H_{2*}^{(q)}(\ ; Q)$ satisfying the condition that $\tau(\mathcal{O}_X) = td_{(q)}(T_X) \cap [X]$ for smooth X. In general we can show

Theorem C. *Let $cl: K \to H^{2*}(\ ; Q[q, q^{-1}])$ be a total characteristic class of complex vector bundles. Then $\tau: {}'K_0^{(q)} \to H_{2*}^{(q)}(\ ; Q)$ is a natural transformation satisying that $\tau(\mathcal{O}_X) = cl(T_X) \cap [X]$ for smooth X if and only if $cl = \lambda \circ td$ and $\tau = \lambda \circ Td_*$ for some $\lambda \in Q[q, q^{-1}]$.*

The organization of the paper is as follows. In § 1 we recall some basic facts and results about characteristic cohomology classes. In § 2 we show the universality theorem of BFM–RR transformation Td_* and Theorem C. In § 3 we prove Theorem A and give a characterization of the "twisted" BFM–RR transformation $Td_{(q)*}$.

§ 1. Preliminaries

In this section we introduce some notation and recall some basic results on characteristic classes of complex vector bundles, which we will use in the rest of the paper.

A usual characteristic class cl of complex vector bundles is a rule assigning to any complex vector bundle E over any topological space X an element $cl(E)$ of the cohomology group of X such that it satisfies the naturality condition, i. e., $cl(f^*E) = f^*cl(E)$ for any map $f: Y \to X$. To paraphrase this more fashionably, let $\mathscr{V}ect: \mathscr{T}op \to \mathscr{E}ns$ be the contravariant functor from the category $\mathscr{T}op$ of topological spaces to the category $\mathscr{E}ns$ of sets, such that $\mathscr{V}ect(X)$ = the set of isomorphism classes of vector bundles over X, and let $H^*(\ ; \Lambda) := H^*(\ ; \mathbf{Z}) \otimes_{\mathbf{Z}} \Lambda: \mathscr{T}op \to \mathscr{E}ns$ be the usual cohomology contravariant functor, where we ignore the algebraic structures of cohomology rings. Then the above characteristic class cl is a natural transformation $cl: \mathscr{V}ect \to H^*(\ ; \Lambda)$. If cl is multiplicative, i. e.,

$cl(E \oplus F) = cl(E)\,cl(F)$, then cl can be strengthened to be a natural transformation from the Grothendieck contravariant functor K^0 to $H^*(\ ;\Lambda)$, where K^0 and $H^*(\ ;\Lambda)$ are now functors from $\mathcal{T}op$ to the category $\mathcal{R}ing$ of rings.

Let n be a non–negative integer and a partition of n be a non–decreasing sequence $\{r_1, r_2, \ldots, r_k\}$ of positive integers whose sum is exactly equal to n. The (only) partition of zero is conventionally zero itself. Let $I_k(n)$ denote such a partition $\{r_1, r_2, \ldots, r_k\}$ of n and let $I(n)$ denote the set of all partitions of n and $p(n)$ denote the cardinality $|I(n)|$ of the set $I(n)$. Let $P_n(c_1, c_2, \ldots, c_n)$ be a homogeneous polynomial of degree n with the weight of c_i being i. Thus $P_n(c_1, c_2, \ldots, c_n)$ is a linear combination of $I_k(n)$–Chern class $c_{I_k(n)} := c_{r_1} c_{r_2} \ldots c_{r_k}$, i. e.,

$$P_n(c_1, c_2, \ldots, c_n) = \sum_{I_k(n) \in I(n)} \lambda_{I_k(n)} c_{I_k(n)}.$$

Any characteristic class $cl\colon \mathcal{V}ect \to H^*(\ ;\Lambda)$ can be expressed as a formal power series of Chern classes, i. e., $cl = \lambda_0 c_0 + \sum_{n \geq 1} P_n(c_1, c_2, \ldots, c_n)$. A fundamental result about Chern classes is that *the total Chern class* $c\colon K^0 \to H^{2*}(\ ;\Lambda)^\wedge := 1 + \sum_{i \geq 1} H^{2i}(\ ;\Lambda)$ *is universal for all multiplicative characteistic classes.* I. e., given any multiplicative characteristic class $cl\colon K^0 \to H^{2*}(\ ;\Lambda)^\wedge$, there exists an endomorphism $\Phi_{cl}\colon H^{2*}(\ ;\Lambda)^\wedge \to H^{2*}(\ ;\Lambda)^\wedge$ such that the following diagram commutes :

$$\begin{array}{ccc} & \overset{c}{\nearrow} & H^{2*}(\ ;\Lambda)^\wedge \\ K^0 & & \downarrow \Phi_{cl} \\ & \underset{cl}{\searrow} & H^{2*}(\ ;\Lambda)^\wedge. \end{array}$$

Indeed there exists a *unique multiplicative sequence* $M := \{1, M_1(x_1), M_2(x_1, x_2), \ldots, M_i(x_1, x_2, \ldots, x_i), \ldots\}$ such that $cl = 1 + \sum_{n \geq 1} M_n(c_1, c_2, \ldots, c_n)$. Thus the multiplicative sequence M gives rise to the endomorphism $\Phi_{cl}\colon H^{2*}(\ ;\Lambda)^\wedge \to H^{2*}(\ ;\Lambda)^\wedge$ defined by

$$\Phi_{cl}(1 + \sum_{i \geq 1} x_i) := 1 + \sum_{i \geq 1} M_i(x_1, x_2, \ldots, x_i),$$

and cl is the composite of the total Chern class $c = 1 + \sum_{i \geq 1} c_i$ and the endomorphism Φ_{cl}.

Let X be a compact complex manifold of dimension $m \geq n$ and then as usual we define the $2n$–dimensional characteristic cohomology class $c_{I_k(n)}(X)$ of the manifold X as follows :

$$c_{I_k(n)}(X) := c_{I_k(n)}(T_X) = c_{r_1}(T_X)\, c_{r_2}(T_X) \ldots c_{r_k}(T_X),$$

where T_X is the tangent bundle of X. And we denote the $2(m-n)$-dimensional homology class $c_{I_k(n)}(X)\cap[X]$ by simply $c_{I_k(n)}[X]$. In particular, if $m=n$, then $c_{I_k(n)}[X]$ is an integer and called the $I_k(n)$*-Chern number* of X (see [11, § 16]).

For a partition $I_j(n)=\{r_1, r_2, \ldots, r_j\}$ of n we define the $I_j(n)$-projective space $\boldsymbol{P}^{I_j(n)}$ by :

$$\boldsymbol{P}^{I_j(n)}:=\boldsymbol{P}^{r_1}\times\boldsymbol{P}^{r_2}\times\ldots\times\boldsymbol{P}^{r_j}.$$

With these definitions above, we have the following fundamental result :

Theorem (1.2) ([11, Theorem 16.7 and a remark right after it]). *The* $p(n)\times p(n)$ *matrix* M_n *whose entries are* $I_k(n)$*-Chern numbers of* $I_j(n)$*-projective spaces* $\boldsymbol{P}^{I_j(n)}$:

$$M_n:=(c_{I_k(n)}[\boldsymbol{P}^{I_j(n)}])$$

is non-singular.

Corollary (1.3) (The linear independence of Chern numbers [11, § 16]). *If* $(P_n(c_1, c_2, \ldots, c_n))[\boldsymbol{P}^{I_j(n)}]=0$ *for all* $I_j(n)\in I(n)$, *then* $P_n(c_1, c_2, \ldots, c_n)=0$ *as a polynomial. Hence, if* $(P_n(c_1, c_2, \ldots, c_n))[X]=0$ *for any compact complex manifold of dimension n, equivalently* $(P_n(c_1, c_2, \ldots, c_n))[T_X]=0$ *by the Poincaré duality, then* $P_n(c_1, c_2, \ldots, c_n)=0$ *as a polynomial..*

Corollary (1.4). *If r is a rational number and* $(P_n(c_1, c_2, \ldots, c_n))[\boldsymbol{P}^{I_j(n)}]=r$ *for all* $I_j(n)\in I(n)$, *then* $P_n(c_1, c_2, \ldots, c_n)=r\circ td_n(c_1, c_2, \ldots, c_n)$, *where* $td_n(c_1, c_2, \ldots, c_n)$ *is the n-th Todd class.*

Proof. Use the fact (see [6]) that $(td_n(c_1, c_2, \ldots, c^n))[\boldsymbol{P}^{I_j(n)}]=1$ for any $I_j(n)\in I(n)$ and Corollary (1.3).

Proposition (1.5). ([14]) *Let* $m\geq n$ *and let X and Y be compact complex manifolds of dimensions m–n and n, respectively, and let* $\pi: X\times Y\to X$ *be the projection. Then*

$$\pi_*(c_{I_k(n)}[X\times Y])=(c_{I_k(n)}[Y])[X].$$

Corollary (1.6). *Let* $m\geq n$ *and let X and Y be compact complex manifolds of dimensions* $m-n$ *and n, respectively, and let* $\pi: X\times Y\to X$ *be the projection. Then*

$$\pi_*(P_n(c_1, c_2, \ldots, c_n)[X\times Y]) = (P_n(c_1, c_2, \ldots, c_n)[Y])[X].$$

We introduce more terminology. The natural transformation $cl = \sum_{i\geq 0} cl_i$: $\mathcal{V}ect \to H^*(\ ;\Lambda)$ shall be called a *total characteristic class*, instead of just a characteristic class. Here $cl_0 = \lambda_0 c_0$ and $cl_i = P_i(c_1, c_2, \ldots, c_i)$. If we are arbitrarily given isobaric polynomials $cl_0 = \lambda_0 c_0$ and $cl_i = P_i(c_1, c_2, \ldots, c_i)$ $(1\leq i\leq n)$, then $cl^{(n)} := \sum_{0\leq i\leq n} cl_i$ is called a *degree-n characteristic class.*

Using Corollary (1. 6) we can show the following proposition, which is a generalization of the above "linear independence of Chern numbers" :

Proposition (1. 7) ([14]). *Let $cl^{(n)}$ be a degree-n characteristic class. If $cl^{(n)}(T_X) = 0$ for any compact complex manifold of dimension n, then $cl^{(n)} = 0$ as a polynomial.*

§ 2. The Universality of BFM–Riemann–Roch Transformation Td_*

The uniqueness of the BFM–RR–transformation Td_* : $K_0 \to H_{2*}(\ ;\mathbb{Q})$ satisfying the extra condition that $\tau(\mathcal{O}_X) = td(T_X)\cap[X]$ for any smooth variety X follows from the following lemma, which can be shown by resolution of singularities.

Lemma (2. 1) (cf. [2, §4. 2]). *Let τ, τ' : $K_0 \to H_{2*}(\ ;\mathbb{Q})$ be two natural transformations. Then $\tau = \tau'$ if and only if $\tau(\mathcal{O}_W) = \tau'(\mathcal{O}_W)$ for any smooth variety W.*

Proof. It suffices to note that for any variety X $K_0(X)$ is generated by the structure sheaves $\mathcal{O}_V$ for all subvarieties V of X, and furthermore that for any structure sheaf $\mathcal{O}_V$ there exist a finitely many smooth varietes W_i, morphisms $\pi_i: W_i \to V$ and some integers m_i's such that

$$\mathcal{O}_V = \sum_i m_i \pi_{i*}\mathcal{O}_{W_i}.$$

This lemma is an analogue of : *For two natural transformations τ, τ' : $F \to H_{2*}(\ ;\mathbb{Z})$ $\tau = \tau'$ if and only if $\tau(1_X) = \tau'(1_X)$ for any smooth variety X.* The proof of this is similar to that of the above [5, 9].

In [1], however, the uniqueness of Td_* is not shown by using resolution of singularities at all, but by the following strengthened "uniqueness theorem" of Td_* :

"Uniqueness theorem" ([1]). *BFM–RR–transformation* $Td_* : K_0 \to H_{2*}(; Q)$ *is the only natural transformation* τ *satisfying the property that*

$$\tau(\mathcal{O}_{P^n}) = [P^n] + \text{homology classes of lower degrees}$$

for each projective space P^n, $n=0, 1, 2, \dots$.

This "Uniqueness theorem" follows from the fact [1, Chap. III, § 1] that for each variety X $Td_* \otimes Q$ induces an isomorphism $K_0(X) \otimes Q \xrightarrow{\sim} H_{2*}(X; Q)$ and the following "Identity theorem" ([3, § 5]):

"Identity theorem" ([3, § 5]). *If* $\alpha : H_{2*}(; Q) \to H_{2*}(; Q)$ *is a natural transformation such that for each projective space* P^i, $i=0, 1, 2, 3, \dots,$

$$\alpha([P^i]) = [P^i] + \text{homology classes of lower degrees,}$$

then α *must be the identity.*

It turns out that "Uniqueness theorem" tells us more about BFM–RR–transformation Td_*:

Theorem (2.2) (The universality of BFM–RR–transformation Td_* [15, Theorem 1]). *If* $\tau : K_0 \to H_{2*}(; Q)$ *is a natural transformation, then there exists a unique sequence* $\{r_i\}_{i \geq 0}$ *of rational numbers such that* $\tau = \sum_{i \geq 0} r_i Td_{*i}$, *where* $Td_{*i} : H_{2*}(; Q) \to H_{2i}(; Q)$ *is the projection of* Td_* *to the* $2i$*–dimensional component. Thus BFM–RR–transformation* $Td_* : K_0 \to H_{2*}(; Q)$ *is universal in the sense that for any natural transformation* $\tau : K_0 \to H_{2*}(; Q)$ *there exists a unique natural transformation* $\Phi_\tau : H_{2*}(; Q) \to H_{2*}(; Q)$ *such that* $\tau = \Phi_\tau \circ Td_*$, *where* Φ_τ *is defined by* $\Phi_\tau(\sum_{i \geq 0} x_i) = \sum_{i \geq 0} r_i x_i$.

$$\begin{array}{ccc} & \overset{Td_*}{\nearrow} & H_{2*}(; Q) \\ K_0 & & \downarrow \Phi_\tau \\ & \underset{\tau}{\searrow} & H_{2*}(; Q) \end{array}$$

By the same argument (see [15, Proof of Theorem 1]) and using "Identity theorem", we can show the following

Theorem (2.3) (The linearity of endomorphism $H_{2*}(; Q) \to H_{2*}(; Q)$). *If* $\tau : H_{2*}(; Q) \to H_{2*}(; Q)$ *is a natural transformation, then there exists a unique sequence* $\{r_i\}_{i \geq 0}$ *of rational numbers such that* $\tau = \sum_{i \geq 0} r_i \pi_i$, *where* $\pi_i : H_{2*}(; Q) \to H_{2i}(; Q)$ *is the projection to the* $2i$*–dimensional component. Namely,* $\tau(\sum_{i \geq 0} x_i) = \sum_{i \geq 0} r_i x_i$.

Remark (2.4). It is easy to see that "Uniqueness theorem" and Theorem (2.2) are equivalent and that "Identity theorem" and Theorem (2.3) are equivalent.

Remark (2.5). Establishing Theorem (2.2) has been motivated by Kennedy's conjecture that *any natural transformation* $\tau: F \to H_{2*}(\ ; Z)$ *can be uniquely expressed as* $\tau = \sum_{i\geq 0} m_i C_{*i}$, *where* $C_{*i}: H_{2*}(\ ; Z) \to H_{2i}(\ ; Z)$ *is the projection of the natural transformation* C_* *to the* $2i$-*dimensional component and each* m_i *is an integer.* This conjecture is still unsolved. Note that Kennedy's conjecture is equivalent to claiming that $C_*: F \to H_{2*}(\ ; Z)$ *is the unique natural transformation satisfying the property that*

$$\tau(1_{P^n}) = [P^n] + \textit{homology classes of lower degrees}$$

for each projective spdce P^n, $n=0, 1, 2, \ldots$

The "linear" natural transformation $\sum_{i\geq 0} r_i Td_{*i}$ has the following "characterization":

Theorem (2.6). *Let* $cl^{(n)}: \mathcal{V}ect \to H^{2*}(\ ; Q)$ *be a degree-n characteristic class of complex vector bundles, and let* $\{cl^{(n)}\}_{n\geq 0}$ *be a sequence of degree-n characteristic classes. Then* $\tau: K_0 \to H_{2*}(\ ; Q)$ *is a natural transformation satisfying the "dimension-wise universal smooth condition" that* $\tau(\mathcal{O}_X) = cl^{(\dim X)}(T_X) \cap [X]$ *for any smooth variety X, if and only if there exists a unique sequence* $\{r_i\}_{i\geq 0}$ *of rational numbers such that*

(i) $$\tau = \sum_{i\geq 0} r_i Td_{*i}, \text{ and}$$

(ii) $$cl^{(n)} = \sum_{0\leq i\leq n} r_i td_{n-i}.$$

This theorem follows from Theorem (2.2) and Proposition (1.7), or without using "Uniqueness theorem", we can also show this theorem by Lemma (2.1), Corollaries (1.4) and (1.6).

Proof I (Using Theorem (2.2)). By Theorem (2.2) there exists a unique sequence $\{r_i\}_{i\geq 0}$ of rational numbers such that $\tau = \sum_{i\geq 0} r_i Td_{*i}$. So it remains to prove that each degree-n characteristic class $cl^{(n)} = \sum_{0\leq i\leq n} r_i td_{n-i}$. Since $(\sum_{i\geq 0} r_i Td_{*i})(\mathcal{O}_X) = (\sum_{0\leq i\leq n} r_i td_{n-i})(T_X)$ for any smooth variety X, we have the following equality:

$$cl^{(n)}(T_X) \cap [X] = (\sum_{0\leq i\leq n} r_i td_{n-i})(T_X) \cap [X]$$

for any smooth variety of dimension n, i. e., by the Poincaré duality, we have

$$cl^{(n)}(T_X)=(\sum_{0\leq i\leq n} r_i td_{n-i})(T_X)$$

for any smooth variety of dimension n. i. e.,

$$(cl^{(n)}-\sum_{0\leq i\leq n} r_i td_{n-i})(T_X)=0$$

for any smooth variety of dimension n. Thus from Proposition (1. 7) we can conclude that $cl^{(n)}-\sum_{0\leq i\leq n} r_i td_{n-i}=0$, i. e.,

$$cl^{(n)}=\sum_{0\leq i\leq n} r_i td_{n-i}.$$

Q. E. D.

Proof II (Using Lemma (2. 1)). Suppose that $\tau: K_0\to H_{2*}(\,;Q)$ is a natural transformation satisfying the "dimension-wise universal smooth condition" that $\tau(\mathcal{O}_X)=cl^{(\dim X)}(T_X)\cap[X]$ for any smooth variety X, with $cl^{(0)}=\lambda_0^0 c_0$ and $cl^{(n)}=\lambda_0^n+\sum_{1\leq i\leq n} P_i^n(c_1, c_2, \ldots, c_i)$. For each partition $I_j(i)$ of i, we consider the projection $\pi: P^{n-i}\times P^{I_j(i)}\to P^{n-i}$. Then by the naturality of our transformation τ, since $\pi_*(\mathcal{O}_{P^{n-i}\times P^{I_j(i)}})=\chi(P^{I_j(i)}, \mathcal{O}_{P^{I_j(i)}})\mathcal{O}_{P^{n-i}}=\mathcal{O}_{P^{n-i}}$ (because $\chi(P^{I_j(i)}, \mathcal{O}_{P^{I_j(i)}})=1$, see [6]), we have

$$(*)\qquad \pi_*\tau(\mathcal{O}_{P^{n-i}\times P^{I_j(i)}})=\tau(\mathcal{O}_{P^{n-i}}).$$

Hence by the "dimension-wise universal smooth condition", we have

$$(**)\qquad \pi_*(cl^{(n)}[P^{n-i}\times P^{I_j(i)}])=cl^{(n-i)}[P^{n-i}].$$

Therefore, by Corollary (1. 6) we get

$$\begin{aligned}\text{LHS. of }(**)&=\pi_*(\cdots+P_i^n(c_1, c_2, \ldots, c_i)[P^{n-i}\times P^{I_j(i)}]+\cdots)\\&=(P_i^n(c_1, c_2, \ldots, c_i)[P^{I_j(i)}])[P^{n-i}]\\&\quad+\text{homology classes of degree}<2(n-i).\end{aligned}$$

$$\text{RHS. of }(**)=\lambda_0^{n-i}[P^{n-i}]+\text{homology classes of degree}<2(n-i).$$

Hence by looking at the $2(n-i)$-dimensional part of $(**)$, we have the following equality :

$$(P_i^n(c_1, c_2, \ldots, c_i)[P^{I_j(i)}])[P^{n-i}]=\lambda_0^{n-i}[P^{n-i}], \text{ i. e.,}$$

$$P_i^n(c_1, c_2, \ldots, c_i)[P^{I_j(i)}] = \lambda_0^{n-i} \text{ for each } I_j(i) \in I(i).$$

Hence by Corollary (1. 4) we can conclude that

$$P_i^n(c_1, c_2, \ldots, c_i) = \lambda_0^{n-i} td_i(c_1, c_2, \ldots, c_i).$$

Thus, letting $r_i = \lambda_0^i$, we can see that there exists a unique seqence $\{r_i\}_{i \geq 0}$ of rational numbers such that $cl^{(n)} = \sum_{0 \leq i \leq n} r_i td_{n-i}$ for each n. Then it remains to show that $\tau = \sum_{i \geq 0} r_i Td_{*i}$. Since

$$\tau(\mathcal{O}_X) = cl^{(\dim X)}(T_X) \cap [X] = (\sum_{0 \leq i \leq n} r_i td_{n-i})(T_X) \cap [X]$$

$$= (\sum_{i \geq 0} r_i Td_{*i})(\mathcal{O}_X),$$

it follows from Lemma (2. 1) that $\tau = \sum_{i \geq 0} r_i Td_{*i}$. Q. E. D.

Let Λ be a commutative integral domain with unit and let $'K_0^\Lambda := K_0(\ \ ) \otimes \Lambda$ be just the linear extension of the functor K_0 with respect to Λ. Then as a corollary of Proof II, we can show the following

Corollary (2. 7). *Let $cl: \mathcal{V}ect \to H^{2*}(\ ; \Lambda)$ be a total characteristic class of complex vector bundles. Then $\tau: {'K_0^\Lambda} \to H_{2*}(\ ; \Lambda)$ is a natural transformation satisfying the extra condition that for a smooth X $\tau(\mathcal{O}_X) = cl(T_X) \cap [X]$, if and only if there exists an element λ of Λ such that $cl = \lambda \circ td$ and $\tau = \lambda \circ Td_*$.*

Thus it follows from Corollary (2. 7) that there is no natural transformation $\tau: {'K_0^{(q)}} \to H_*^{(q)}(\ ; Q)$ satisfying the extra condition that $\tau(\mathcal{O}_X) = td_{(q)}(T_X) \cap [X]$ for any smooth X. A similar situation occurs in C_* and in [12] we extended C_* to the Chern polynomial $c_{(q)} = 1 + \sum_{i \geq 1} q^i c_1$, introducing the "twisted" constructible function functor. In the following section we will discuss the extension of BFM–RR Td_* to the Todd polynomial $td_{(q)} = 1 + \sum_{i \geq 1} q^i td_i$.

§ 3. A "Twisted" Version $Td_{(q)*}$ of Baum–Fulton–MacPherson–Riemann–Roch Td_*

First we recall the Chern character polynomial $ch_{(q)}$ (this notation and definition are borrowed from Hirzebruch's book [6, § 12]) and the Todd polynomial $td_{(q)}$.

Definition (3. 1). Let q be an indeterminant.

(i) **Chern character polynomial** $ch_{(q)}$ is defined by :

$$ch_{(q)} := ch_0 + \Sigma_{i\geq 1} q^i ch_i,$$

where $ch = ch_0 + \Sigma_{i\geq 1} ch_i$ is the total Chern character. To be more precise, if we let α_i's be Chern roots of E,

$$ch_{(q)}(E) := \Sigma_{1\leq i\leq \mathrm{rank}E} \exp(q\alpha_i).$$

(ii) **Todd polynomial** $td_{(q)}$ is defined by :

$$td_{(q)} := 1 + \Sigma_{i\geq 1} q^i td_i,$$

where $td = 1 + \Sigma_{i\geq 1} td_i$ is the total Todd class. To be more precise,

$$td_{(q)}(E) := \prod_{1\leq i\leq \mathrm{rank}E} \{q\alpha_i/(1-\exp(-q\alpha_i))\}.$$

It is well-known (e. g., see [4] or [6]) that an important connection between the usual Chern character $ch = ch_{(1)}$ and Todd class $td = td_{(1)}$ is the following formula, which plays a key role in the formulation of GRR and BFM-RR :

Formula (3. 2). $\Sigma_{0\leq p\leq n}(-1)^p ch(\Lambda^p E^\vee) = c_n(E)\, td(E)^{-1}$, *where E is a complex vector bundle of rank n and $E^\vee$ is the dual of E.*

The proof of Formula (3. 2) is well-known or standard, but we recall it here for the sake of *Formula* $(3.2)^{(q)}$ below : Let α_i's be Chern roots of E $(1\leq i\leq n)$. Then the proof is as follows.

$$\begin{aligned}\Sigma_{0\leq p\leq n}(-1)^p ch(\Lambda^p E^\vee) &= \Sigma_{0\leq p\leq n}(-1)^p \Sigma_{i_1<\dots<i_p} \exp(-\alpha_{i_1} - \cdots - \alpha_{i_p})\\ &= \prod_{1\leq i\leq n}(1-\exp(-\alpha_i))\\ &= \alpha_1 \dots \alpha_n \prod_{1\leq i\leq n}((1-\exp(-\alpha_i))/\alpha_i)\\ &= c_n(E)\, td(E)^{-1}.\end{aligned}$$

If we follow this proof, i. e., if we replace α_i by $q\alpha_i$ in the above proof, then we get the following

Formula $(3.2)^{(q)}$. $\Sigma_{0\leq p\leq n}(-1)^p ch_{(q)}(\Lambda^p E^\vee) = q^n c_n(E)\, td_{(q)}(E)^{-1}$

(or $\Sigma_{0\leq p\leq n}(-1)^p ch_{(q)}(\Lambda^p E^\vee) = \{c_{(q)}\}_n(E)\, td_{(q)}(E)^{-1}$, if we use the Chern polynomial $c_{(q)}$.)

In this paper the appearance of the twisting "q^n" in this **Formula (3. 2)**$^{(q)}$ plays a key role.

Remark (3. 3). In [6] Hirzebruch defined a generalized Todd genus or T_y–genus as follows.

$$T_y(E) := \prod_{1\le i\le \operatorname{rank} E}\{((y+1)\alpha_i/(1-\exp(-(y+1)\alpha_i))-y\alpha_i\}.$$

For $E=T_M$, the tangent bundle of a complex manifold M, if we "evaluate" $T_y(T_M)$ at some special values, then we get some known invariants of M; e. g., $T_0(T_M)$ is the Todd class of M, $T_{-1}(T_M)$ is the Euler–Poincaré characteristic of M, and $T_1(T_M)$ is the signature of M. In our definition we drop the additional term "$y\alpha_i$". If we use Hirzebruch's T_y–genus, then we have

$$\sum_{0\le p\le n}(-1)^p ch_{(q)}(\Lambda^p E^\vee) = c_n(E)\, T_{(q-1)}(E)^{-1} \sum_{0\le p\le n}(q-1)^p ch_{(q)}(\Lambda^p E^\vee).$$

Indeed,

$$\begin{aligned}
&\sum_{0\le p\le n}(-1)^p ch_{(q)}(\Lambda^p E^\vee)\\
&\qquad = \prod_{1\le i\le n}(1-\exp(-q\alpha_i))\\
&\qquad = \prod_{1\le i\le n}\{(1-\exp(-q\alpha_i))/(\alpha_i+\alpha_i(q-1)\exp(-q\alpha_i))\}\\
&\qquad\qquad \times \prod_{1\le i\le n}(\alpha_i+\alpha_i(q-1)\exp(-q\alpha_i))\\
&\qquad = \alpha_1\ldots\alpha_n\Big[\prod_{1\le i\le n}\{(q\alpha_i/(1-\exp(-q\alpha_i)))-\alpha_i(q-1)\}\Big]^{-1}\\
&\qquad\qquad \times \prod_{1\le i\le n}(1+(q-1)\exp(-q\alpha_i))\\
&\qquad = c_n(E)\, T_{(q-1)}(E)^{-1}\sum_{0\le p\le n}(q-1)^p ch_{(q)}(\Lambda^p E^\vee).
\end{aligned}$$

Thus this formula becomes sort of "redundant", this is why we take up the simpler form $td_{(q)}$. We do not know whether one can formulate a GRR type theorem for T_y–genus or not, which remains to be seen.

Definition (3. 4) (*A "twisted" version $K_0^{(q)}$ of the Grothendieck covariant functor K_0*). Let $K_0^{(q)}(X) := K_0(X)\otimes \mathbf{Q}[q, q^{-1}]$ for any variety X and for a morphism $f\colon X\to Y$, the pushforward $f_*^{(q)}$ is defined by :

$$f_*^{(q)} = q^{\operatorname{reldim}(f)} f_*,$$

where f_* is the pushforward for the original functor K_0 and $\mathrm{reldim}(f):=\dim X-\dim Y$ is the *relative dimension* of *f*. (Then it is obvious that $K_0^{(q)}$ is a covaiant functor.)

There are several proofs of BFM–RR Td_*, i. e., by deformation to the normal bundle [1, Chap. 1], by Grassmannian–graph [1, Chap. 2], by using topological K–theory [2], and by Fulton–MacPherson's bivariant theory [4, Chap. 18]. In this paper we follow the proof by deformation to the normal bundle, since it is least hard to give an analogous proof of our "twisted" version $Td_{(q)*}$. Instead of writing down all ingredients and details developed in [1, Chap. 1], we just write down necessary points and key formulae to get our "twisted" version.

The main ingredient in proving BFM–RR–theorem is the *localized Chern character* $ch_X^M(E.)$, living in the Q–homology $H_{2*}(X;Q)$, where X is a compact complex subspace of a complex manifold M, and $E.$ is a complex of topological vector bundles on M which is exact off X. To be more precise (for more details see [1, Chap. 1]), $ch_X^M(E.)$ is defined as follows: Let $d(E.)\in K^0(M, M-X)$ be the difference–bundle of the complex $E.$, $ch: K^0(M, M-X)\to H^{2*}(M, M-X; Q)$ be the Chern character and $L: H^{2*}(M, M-X; Q)\to H_{2*}(X;Q)$ the Lefschetz duality isomorphism. Then $ch_X^M(E.)$ is defined to be the value of $d(E.)$ by the composite $L\circ ch$, i. e., $ch_X^M(E.):=L\circ ch(d(E.))$. The "twisted" version of the localized Chern character is obtained by replacing ch by the Chern character polynomial $ch_{(q)}$. To be more precise, let $L_{(q)}:=L\otimes Q[q, q^{-1}]: H^{2*}(M, M-X; Q[q, q^{-1}])\to H_{2*}(X; Q[q, q^{-1}])$ be the extension of the Lefschetz duality isomorphism L with respect to the Laurent polynomial ring $Q[q, q^{-1}]$, and let $ch_{(q)}: K^0(M, M-X)\to H^{2*}(M, M-X; Q[q, q^{-1}])$ be the "Chern character polynomial". Then the *localized Chern character polynomial* (*or localized "twisted" Chern character*) $ch_{(q)}{}_X^M(E.)$ of a complex $E.$ is defined by:

Definition (3. 5). $ch_{(q)}{}_X^M(E.):=L_{(q)}\circ ch_{(q)}(d(E.))$.

Then it is not hard to see that by definition the localized "twisted" Chern character $ch_{(q)}{}_X^M(E.)$ satisfy all the properties (listed in [1, Chap. 1, §§2–3]) enjoyed by the localized Chern character $ch_X^M(E.)$, except [1, Chap. 1, Proposition (3. 4)]. Namely, in the formulae described in [1, Chap. 1, §§2–3] symbol ch is just replaced by symbol $ch_{(q)}$. Our "twisted" vesion of [1, Chap. 1, Proposition (3. 4)] is as follows:

Proposition (3. 6). *Let $E.$ be a complex of bundles on M exact off X, and let $\pi: N\to M$ be a vector bundle over M with M regarded as a subspace of N by the zero–section. Let $\Lambda^{\bullet}\pi^*N^\vee$ be the Koszul–Thom complex on N. Then $\Lambda^{\bullet}\pi^*N^\vee\otimes$*

$\pi^* E.$ *is exact on* $N-X$ *and*

$$ch_{(q)}{}_X^M(\Lambda^{\cdot}\pi^* N^{\vee}\otimes\pi^* E.)=q^{\operatorname{rank}N}(td_{(q)}(N)^{-1}\cap ch_{(q)}{}_X^M(E.)).$$

(Note the appearance of the "twisting" coefficeint $q^{\operatorname{rank}N}$, which plays an important role later. Also note that $td_{(q)}(N)^{-1}$ actually means $i^* td_{(q)}(N)^{-1}$, where i: $X\to M$ is the inclusion map. However, we sometimes omit the pull–back symbol, unless some confusion occurs.)

Following the proof of [1, Chap. 1, Proposition (3. 4)], we can see that the above formula follows from **Formula (3. 2)**$^{(q)}$ in the previous section, so its proof is omitted.

The localized Chern character $ch_X^M(\mathscr{F})$ of a coherent sheaf $\mathscr{F}$ on X is defined via a resolution $E.$ of the coherent sheaf $i_*\mathscr{F}$ on M, i. e., $ch_X^M(\mathscr{F}):=ch_X^M(E.)$. In [1, Chap. 1] it is shown that it is independent of the choice of resolution $E.$ of $i_*\mathscr{F}$. When it comes to the twisted verision, one might be tempted to define the "twisted" localized Chen character $ch_{(q)}{}_X^M(\mathscr{F})$ of a coherent sheaf $\mathscr{F}$ on X simply by $ch_{(q)}{}_X^M(E.)$. But it is not the case, and our "twisted" version is taken up to be the following :

Definition (3. 7). $\qquad ch_{(q)}{}_X^M(\mathscr{F}):=q^{\dim X-\dim M}ch_{(q)}{}_X^M(E.)$

$$(\textit{i.e.}, :=q^{\operatorname{reldim}(i)}ch_{(q)}{}_X^M(E.)).$$

Of course, this "twisted" localized Chern character of a coherent sheaf $\mathscr{F}$ is independent of the choice of resolution $E.$ and if $0\to\mathscr{F}'\to\mathscr{F}\to\mathscr{F}''\to 0$ is a short exact sequence of coherent sheaves on X, then $ch_{(q)}{}_X^M(\mathscr{F})=ch_{(q)}{}_X^M(\mathscr{F}')+ch_{(q)}{}_X^M(\mathscr{F}'')$.

With this definition, it turns out that our "twisted" version of [1, Chap. 1, Proposition (5. 3), p. 115] is of the same form ; simply "*ch*" and "*td*" being replaced by "$ch_{(q)}$" and "$td_{(q)}$". Namely, we have

Proposition (3. 8). *Let* $X\subset M\subset P$ *with* M *and* P *non–singular quasi–projective varieties. Let* N *be the normal bundle of* M *in* P. *Then for any coherent sheaf* $\mathscr{F}$ *on* X,

$$ch_{(q)}{}_X^P(\mathscr{F})=td_{(q)}(N)^{-1}\cap ch_{(q)}{}_X^M(\mathscr{F}).$$

Proof. Let i: $X\to M$ and j: $M\to P$ be the inclusion maps. Then by the definition we have

$$ch_{(q)}{}^{P}_{X}(\mathscr{F}) = q^{\dim X - \dim P}\, ch_{(q)}{}^{P}_{X}(E.^{\sim}) \tag{3.8.1}$$

where it should be noted that $E.^{\sim}$ is a (or any) resolution of the coherent sheaf $j_*(i_*\mathscr{F})$. Then by [1, Chap. 1, Homotopy Property (2.5) and Lemma (5.2)], we can see

$$ch_{(q)}{}^{P}_{X}(E.^{\sim}) = ch_{(q)}{}^{N}_{X}(E.^{\#}). \tag{3.8.2}$$

Here $E.^{\#}$ is a resolution of $s_*(i_*\mathscr{F})$, where $s: M \to N$ is the zero-section of the normal bundle N. Noticing the fact that the definition of the localized ("twisted") Chern character of a coherent sheaf is independent of the choice of the resolution, and by observing the fact that if we let $E.$ be a resolution of $i_*\mathscr{F}$, then $\Lambda^{\cdot}\pi^* N^{\vee} \otimes \pi^* E.$ is a resolution of $s_*(i_*\mathscr{F})$, we can see

$$ch_{(q)}{}^{N}_{X}(E.^{\#}) = ch_{(q)}{}^{N}_{X}(\Lambda^{\cdot}\pi^* N^{\vee} \otimes \pi^* E.). \tag{3.8.3}$$

Hence, by (3.8.1–3), we get

$$\begin{aligned} ch_{(q)}{}^{P}_{X}(\mathscr{F}) &= q^{\dim X - \dim P}\, ch_{(q)}{}^{N}_{X}(\Lambda^{\cdot}\pi^* N^{\vee} \otimes \pi^* E.). \\ &= q^{\dim X - \dim P}(q^{\operatorname{rank} N}\, td_{(q)}(N)^{-1} \cap ch_{(q)}{}^{M}_{X}(E.)) \\ &\qquad \text{(by Proposition (3.6))}. \end{aligned} \tag{3.8.4}$$

By noticing that dim P = dim M + rank N, from (3.8.4) we get

$$ch_{(q)}{}^{P}_{X}(\mathscr{F}) = q^{\dim X - \dim M}(td_{(q)}(N)^{-1} \cap ch_{(q)}{}^{M}_{X}(E.)). \tag{3.8.5}$$

Thus by the definition of $ch_{(q)}{}^{M}_{X}(\mathscr{F})$ we get

$$ch_{(q)}{}^{P}_{X}(\mathscr{F}) = td_{(q)}(N)^{-1} \cap ch_{(q)}{}^{M}_{X}(\mathscr{F}). \tag{3.8.6}$$

Q. E. D.

Now we are ready to define our "twisted" version $Td_{(q)*}$ of BFM–Riemann–Roch transformation. For each variety X with $X \subset M$, M non-singular, the homomorphism $Td_{(q)}{}^{M}_{*}: K_0^{(q)}(X) \to H_{2*}^{(q)}(X; \mathbf{Q})$ is defined as follows: for each coherent sheaf $\mathscr{F}$ on X

$$Td_{(q)}{}^{M}_{*}(\mathscr{F}) := td_{(q)}(T_M) \cap ch_{(q)}{}^{M}_{X}(\mathscr{F})$$

and extend this linearly with respect to $\mathbf{Q}[q, q^{-1}]$.

This definition seems to be dependent on the choice of an embedding $i: X \to$

M, but it turns out to be independent of it :

Proposition (3.9). *Let $X\subset P$ and $X\subset Q$ be two embeddings of X into non-singular quasi-projective varieties P and Q. Then we have*

$$Td_{(q)}{}^{P}_{*}=Td_{(q)}{}^{Q}_{*}.$$

i. e., $Td_{(q)}{}^{P}_{*}(\mathscr{F})=Td_{(q)}{}^{Q}_{*}(\mathscr{F})$ *for each* $\mathscr{F}$.

Proof. First we show the following lemma

Lemma (3.9.1). *Let $X\subset M\subset P$, where M and P are non-singular. Then*

$$Td_{(q)}{}^{P}_{*}=Td_{(q)}{}^{M}_{*},$$

i. e., $Td_{(q)}{}^{P}_{*}(\mathscr{F})=Td_{(q)}{}^{M}_{*}(\mathscr{F})$ *for each* $\mathscr{F}$.

Proof of Lemma (3.9.1). Let $i: X\to M$ and $j: M\to P$ be the inclusion maps. Then by the definition of $Td_{(q)}{}^{P}_{*}(\mathscr{F})$, we have

$$\begin{aligned}Td_{(q)}{}^{P}_{*}(\mathscr{F})&=i^*j^*td_{(q)}(T_P)\cap ch_{(q)}{}^{P}_{X}(\mathscr{F})\\&=i^*(j^*td_{(q)}(T_P))\cap i^*td_{(q)}(N)^{-1}\cap ch_{(q)}{}^{M}_{X}(\mathscr{F})\\&\qquad\text{(by Proposition (3.8))}\\&=i^*(j^*td_{(q)}(T_P)td_{(q)}(N)^{-1})\cap ch_{(q)}{}^{M}_{X}(\mathscr{F})\\&=i^*(td_{(q)}(T_P|_M)td_{(q)}(N)^{-1})\cap ch_{(q)}{}^{M}_{X}(\mathscr{F}).\end{aligned}$$

Since $0\to T_M\to T_P|_M\to N\to 0$ is exact and $td_{(q)}$ is multiplicative, we get

$$td_{(q)}(T_P|_M)td_{(q)}(N)^{-1}=td_{(q)}(T_M).$$

Thus we have

$$Td_{(q)}{}^{P}_{*}(\mathscr{F})=i^*td_{(q)}(T_M)\cap ch_{(q)}{}^{M}_{X}(\mathscr{F})=Td_{(q)}{}^{M}_{*}(\mathscr{F}).$$

Q. E. D.

To finish the proof of the proposition, we consider the embedding $P\subset P\times Q$, by means of the mapping $x\mapsto(x, q)$ for some point q of Q, fixed. Similarly we consider the embedding $Q\subset P\times Q$ by means of the mapping $x\mapsto(p, x)$ for some point p of P, fixed. Then we have the situation where $X\subset P\subset P\times Q$ and $X\subset Q\subset$

$P\times Q$. Hence by Lemma (3. 9. 1) we have

$$Td_{(q)*}^{P}(\mathscr{F}) = Td_{(q)*}^{P\times Q}(\mathscr{F}) = Td_{(q)*}^{Q}(\mathscr{F}).$$

Q. E. D.

Thus, for each complex projective variety X the above homomorphism $Td_{(q)*}^{M}: K_0^{(q)}(X)\to H_{2*}^{(q)}(X; Q)$ is independent of the choice of an embedding of X into a non–singular quasi–projective variety. (Hence, it suffices to consider the ambient projective space where a given projective varitiey X lies in.) So we can just denote $Td_{(q)*}$ without the superscript.

Now we are ready to state our main theorem :

Theorem (3. 10). *The transformation* $Td_{(q)*}: K_0^{(q)}\to H_{2*}^{(q)}(; Q)$ *is the unique natural transformation satisfying the extra condition that if* X *is smooth then* $Td_{(q)*}(\mathcal{O}_X) = td_{(q)}(T_X)\cap[X]$. *If we "evaluate"* $Td_{(q)*}$ *at* $q=1$, *then we get BFM–RR transformation* $Td_{(1)*} = Td_*$.

The second statement is clear by the construction of $Td_{(q)*}$ and it is also easy to see (i) that by the construction of $Td_{(q)*}$ the transformation $Td_{(q)*}$ satisfies the above extra condition and (ii) that by Lemma (2. 1) $Td_{(q)*}$ is the unique transformation satisfying the above extra condition. Thus it remains to show the naturality of transformation $Td_{(q)*}$, i. e., for any morphism $f: X\to Y$ the following diagram is commutative :

$$\begin{array}{ccc} K_0^{(q)}(X) & \xrightarrow{Td_{(q)*}} & H_{2*}^{(q)}(X) \\ \downarrow f_*^{(q)} & & \downarrow f_* \\ K_0^{(q)}(Y) & \xrightarrow[Td_{(q)*}]{} & H_{2*}^{(q)}(Y). \end{array}$$

Since any morphism $f: X\to Y$ is the composite of an imbedding $j: X\to Y\times P^N$ (for some projective space P^N) and the projection $\pi: Y\times P^N\to Y$, i. e., $f=\pi\circ j$, it suffices to show the commutativity of the above diagram for the embedding case when $f: X\to Y$ is assumed to be an embedding and the projection $\pi: Y\times P^N\to Y$.

Proposition (3. 11). *If* $f: X\to Y$ *is an inclusion map, then the above diagram is commutative.*

Proof. Let $k: Y\to M$ be an inclusion map, where M is non–singular. It suffices to show that for each coherent sheaf $\mathscr{F}$

$$f_*(Td_{(q)}{}_*^M(\mathscr{F})) = Td_{(q)}{}_*^M(f_*^{(q)}(\mathscr{F})).$$

$$f_*(Td_{(q)}{}_*^M(\mathscr{F})) = f_*(f^*k^*td_{(q)}(T_M) \cap ch_{(q)}{}_X^M(\mathscr{F}))$$
(by the definition of $Td_{(q)}{}_*^M(\mathscr{F})$)

$$= k^*td_{(q)}(T_M) \cap f_*ch_{(q)}{}_X^M(\mathscr{F}) \text{ (by the projection formula)}$$

$$= q^{\dim X - \dim M}(k^*td_{(q)}(T_M) \cap f_*ch_{(q)}{}_X^M(E.))$$
(by the definition of $ch_{(q)}{}_X^M(\mathscr{F})$)

$$= q^{\dim X - \dim M}(k^*td_{(q)}(T_M) \cap ch_{(q)}{}_Y^M(E.))$$
(by [1, Property (2. 1)(a), p. 109])

$$= q^{\dim X - \dim Y}(k^*td_{(q)}(T_M) \cap q^{\dim Y - \dim M}\, ch_{(q)}{}_Y^M(E.)).$$

Here we observe that since $E.$ is a resolution of $k_*f_*\mathscr{F} = k_*(f_*\mathscr{F})$, we have

$$ch_{(q)}{}_Y^M(f_*\mathscr{F}) = q^{\dim Y - \dim M}\, ch_{(q)}{}_Y^M(E.).$$

Therefore we get that

$$f_*(Td_{(q)}{}_*^M(\mathscr{F})) = q^{\dim X - \dim Y}(k^*td_{(q)}(T_M) \cap ch_{(q)}{}_Y^M(f_*\mathscr{F})).$$

Since $f_*^{(q)}(\mathscr{F}) = q^{\mathrm{reldim}(f)}f_*(\mathscr{F}) = q^{\dim X - \dim Y}f_*(\mathscr{F})$ by the definition of our "twisted" pushforward $f_*^{(q)}$, we get

$$f_*(Td_{(q)}{}_*^M(\mathscr{F})) = k^*td_{(q)}(T_M) \cap ch_{(q)}{}_Y^M(f_*^{(q)}\mathscr{F}))$$

$$= Td_{(q)}{}_*^M(f_*^{(q)}(\mathscr{F})).$$

Q. E. D.

Proposition (3. 12). *Let π: $Y \times P^N \to Y$ be projection. Then the following diagram is commutative:*

$$\begin{array}{ccc} K_0^{(q)}(Y \times P^N) & \xrightarrow{\quad Td_{(q)*} \quad} & H_{2*}^{(q)}(Y \times P^N) \\ \downarrow \pi_*^{(q)} & & \downarrow \pi_* \\ K_0^{(q)}(Y) & \xrightarrow[\quad Td_{(q)*} \quad]{} & H_{2*}^{(q)}(Y). \end{array}$$

Proof. This "projection" case is reduced to showing the commutativity of the above diagram for $Y =$ a point $\{pt\}$ (see [4, p. 287] for the explanation for this.) Let $\pi: P^N \to pt$. Then, since $K_0^{(q)}(P^N)$ is generated by $\mathcal{O}(n)$ and $Td_{(q)*}(\mathcal{F}) = i^* td_{(q)}(T_M) \cap ch_{(q)}{}_X^M(\mathcal{F}) = td_{(q)}(T_X) \cap ch_{(q)}(\mathcal{F}) \cap [X]$ if X is smooth, what we want to show is:

$$(3.\ 12.\ 1)\qquad [ch_{(q)}(\pi_*^{(q)}\mathcal{O}(n))\, td_{(q)}(T_{\{pt\}})] \cap [pt]$$

$$= \pi_*([ch_{(q)}(\mathcal{O}(n))\, td_{(q)}(T_{P^N})] \cap [P^N]),$$

i. e., by the definition of the twisted pushforward $\pi_*^{(q)}$

$$(3.\ 12.\ 2)\qquad q^N ch_{(q)}(\pi_*\mathcal{O}(n)) = \int_{P^N}(ch_{(q)}(\mathcal{O}(n))\, td_{(q)}(T_{P^N})).$$

Here, following Fulton's book [4, Remark 3. 2. 2.] $\int_X \alpha$ denotes the 0–dimensional component of $\alpha \cap [X]$. Since $\pi_*\mathcal{O}(n) = \sum(-1)^i H^i(P^N, \mathcal{O}(n))$ and $ch_{(q)} = ch_0 + \sum_{i\geq 1} q^i ch_i$,

$$q^N ch_{(q)}(\pi_*\mathcal{O}(n)) = q^N ch_0(\pi_*\mathcal{O}(n))$$

$$= q^N \sum(-1)^i \dim H^i(P^N, \mathcal{O}(n))$$

$$= q^N \chi(P^N, \mathcal{O}(n)).$$

On the other hand, (see [4, Example 15. 1. 4])

$$\int_{P^N}(ch_{(q)}(\mathcal{O}(n))\, td_{(q)}(T_{P^N}))$$

$$= \int_{P^N}\{e^{qxN}(qx/1 - e^{-qx})^{N+1}\}, \text{ where } x = c_1(\mathcal{O}(P^N)(1)).$$

$$= [q^N \chi(P^N, \mathcal{O}(n)) x^N] \cap [P^N]$$

$$= q^N \chi(P^N, \mathcal{O}(n)).$$

Thus (3. 12. 1) holds. Q. E. D.

As a corollary of Theorem (3. 10), we can get the twisted version of GRR:

Corollary (3. 13) (*A "twisted" version of (GRR). For a non–singular projective variety X, if we define the homomorphism* $\tau^{(q)}: K_0^{(q)}(X) \to H^{2*(q)}(X; Q)$ $(:= H^{2*}(X; Q[q, q^{-1}]))$ *by* $\tau^{(q)}(\mathcal{F}) := ch_{(q)}(\mathcal{F})\, td_{(q)}(T_X)$ *and extending linearly with respect to* $Q[q, q^{-1}]$, *then* $\tau^{(q)}$ *becomes a natural transformation, i. e., for a morphism* $f: X \to Y$ *the following equality holds*:

$$ch_{(q)}(f_*^{(q)}\mathscr{F})\,td_{(q)}(T_Y)=f_*(ch_{(q)}(\mathscr{F})\,td_{(q)}(T_X))$$

where $f_*: H^{2*(q)}(X; Q)\to H^{2*(q)}(Y; Q)$ *is the Gysin homomorphism. Namely, the following diagram commutes*:

$$\begin{array}{ccc} K_0^{(q)}(X) & \xrightarrow{\tau^{(q)}} & H^{2*(q)}(X; Q) \\ \downarrow f_*^{(q)} & & \downarrow f_* \\ K_0^{(q)}(Y) & \xrightarrow[\tau^{(q)}]{} & H^{2*(q)}(Y; Q). \end{array}$$

Remark (3. 14). Originally we had this theorem first, the proof of which was a "twisted" version of the proof of GRR (see [4, pp. 287–288]), which is left as an exercise for readers. The key point is to see how the twisting coefficient "$q^{\text{some power}}$" fits in the proof. After we had this twisted version of GRR, we could see how to modify the proof of BFM–RR to get our twisted BFM–RR $Td_{(q)*}$, and then we noticed a relationship between $Td_{(q)*}$ and Td_* that *for a variety X, the homomorphism* $Td_{(q)*}: K_0^{(q)}(X)\to H_{2*}^{(q)}(X; Q)$ *can be simply described as*

(3. 14. 1) $$Td_{(q)*}=q^{\dim X}\sum_{i\geq 0}q^{-i}Td_{*i}.$$

There are at least two ways to see this. Firstly, if we take a closer look at the definitions $Td_{(q)*}(\mathscr{F})=i^*td_{(q)}(T_M)\cap ch_{(q)}{}_X^M(\mathscr{F})$ and $Td_*(\mathscr{F})=i^*td(T_M)\cap ch_X^M(\mathscr{F})$, in particular $ch_{(q)}{}_X^M(\mathscr{F})$ and $ch_X^M(\mathscr{F})$, then by some combinatorial computation (left as an exercise) we can see that

$$Td_{(q)*}(\mathscr{F})=\sum_{i\geq 0}q^{\dim X-i}Td_{*i}(\mathscr{F}).$$

Secondly, it suffices to show that the transformation $T: K_0^{(q)}\to H_{2*}^{(q)}(\ ; Q)$ defined by the righthand side of (3. 14. 1) is a natural transformation satisfying the extra condition that $T(\mathscr{O}_X)=td_{(q)}(T_X)\cap[X]$ for a smooth X. It is easy to see that T satisfies the extra condition. And by the definition of the twisted pushforward $f_*^{(q)}$ and the naturality of Td_*, hence the naturality of Td_{*i}, we can see the naturality of T. In this sense, Theorem (3. 10) itself can be proved very quickly using the natural transformation T defined by the right–hand–side of (3. 14. 1), but, as stated in the introduction, the aim of our proof of Theorem (3. 10) is to show that even if "td" and "ch" are replaced by "$td_{(q)}$" and "$ch_{(q)}$", the whole proof of BFM–RR–theorem works by necessarily introducing twisting coefficients "$q^{\text{some power}}$" in suitable places.

Finally, let us call $Td_{(q)*}(X):=Td_{(q)*}(\mathscr{O}_X)$ the *twisted homology Todd* *class* of X. The 0–th component of $Td_{(q)*}(X)$, denoted by $\chi^{(q)}(X, \mathcal{O}_X)$, is just $q^{\dim X}\chi(X, \mathcal{O}_X)$. Obviously $\chi^{(q)}$ is multiplicative. In fact, $Td_{(q)*}$ is multiplicative. Indeed, using the multiplicativity of the localized twisted Chern character, we can show the following

Proposition (3. 15). *The followoing diagram is commutative:*

$$\begin{array}{ccc} K_0^{(q)}(X)\otimes K_0^{(q)}(Y) & \xrightarrow{\quad Td_{(q)*}\times Td_{(q)*}\quad} & H_{2*}^{(q)}(X;\mathbf{Q})\otimes H_{2*}^{(q)}(Y;\mathbf{Q}) \\ \downarrow\times & & \downarrow\times \\ K_0^{(q)}(X\times Y) & \xrightarrow[\quad Td_{(q)*}\quad]{} & H_{2*}^{(q)}(X\times Y;\mathbf{Q}), \end{array}$$

where the vertical homomorphisms are Künneth homomorphisms. In particular,

$$Td_{(q)*}(X\times Y)=Td_{(q)*}(X)\times Td_{(q)*}(Y).$$

This is an analogue of the multiplicativity of the twisted MacPherson classes [8].

Before finishing this section, we give a certain characterization of our twisted version $Td_{(q)*}$ of BFM–RR theorem. Let Λ be a field of characteristic zero. Let $cl\colon \mathcal{V}ect\to H^{2*}(\ ;\Lambda)$ be a total characteristic cohomology class of complex vector bundles. Let $K_0^{\Lambda}(X):=K_0(X)\otimes\Lambda$.

Definition (3. 16). For a morphism $f\colon X\to Y$, the pushforward $f_*^{cl}\colon K_0^{\Lambda}(X)\to K_0^{\Lambda}(Y)$ is defined by

$$f_*^{cl}(\mathcal{F}):=\alpha(f)f_*(\mathcal{F}),$$

where the twisting coefficient $\alpha(f)\in\Lambda$ depends only on the characteristic class cl dim X and dim Y, and f_* is the usual pushforward.

If the "twisting" operator α, assigning an element $\alpha(f)$ of Λ to each morhism f, satisfies the property that $\alpha(f\circ g)=\alpha(f)\circ\alpha(g)$, then the above "twisted" pushforward f_*^{cl} is functorial. If we consider the above twisted pushforward in the Riemann–Roch type situation, then although we cannot characterize the "twisting" operator α we can characterize the characteristic class cl, i. e., we can get the following theorem, which is a sort of characterization of the twisted version $Td_{(q)*}$ of BFM–RR :

Theorem (3. 17). *The twisted pushforward f_*^{cl} becomes a covariant functor with a certain twisting operator α so that there exists a (unique) natural transformation $\tau^{cl}: K_0^{\Lambda}(X) \to H_{2*}(\ ;\Lambda)$ satisfying the extra condition ("smooth cl-condition") that $\tau^{cl}(\mathcal{O}_X) = cl(T_X) \cap [X]$ for any smooth X, if and only if $cl = \eta \sum_{i\geq 0} \lambda^i td_i$ for some η and $\lambda \in \Lambda$ and $\tau^{cl} = \eta Td_{(\lambda)*}$.*

Proof. (If part) Suppose that $cl = \eta \sum_{i\geq 0} \lambda^i td_i$ for some η and $\lambda \in \Lambda$ and let the transformation $\tau^{cl} := \eta Td_{(\lambda)*}$. Then, as in the twisted version $Td_{(q)*}$, we just define $f_*^{cl}(\mathscr{F}) := f_*^{(\lambda)}(\mathscr{F}) = \lambda^{\mathrm{reldim}(f)} f_*(\mathscr{F})$, then τ^{cl} is natural. Indeed

$$\begin{aligned} \tau^{cl} \circ f_*^{cl}(\mathscr{F}) &= \eta Td_{(\lambda)*} \circ f_*^{(\lambda)}(\mathscr{F}) && \text{(by the definition)} \\ &= \eta f_* \circ Td_{(\lambda)*}(\mathscr{F}) && \text{(by the twisted version } Td_{(q)*}) \\ &= f_* \circ (\eta Td_{(\lambda)*}(\mathscr{F})) \\ &= f_* \circ \tau^{cl}(\mathscr{F}). \end{aligned}$$

(Only if part) First, by a similar manner to that of Proof II of Theorem (2. 6) (or see [12, Proof of Lemma (2. 3)]) we can show that the total characteristic class cl must be a linear form of individual Todd classes, i. e., $cl = \sum_{i\geq 0} \lambda_i td_i$ for some $\lambda_i \in \Lambda$. From this we can claim in a similar manner to that of [12, Theorem (2. 2)] that $cl = \eta \sum_{i\geq 0} \lambda^i td_i$ for some η and $\lambda \in \Lambda$. Hence, by the "smooth cl-condition" we get $\tau^{cl} = \eta Td_{(\lambda)*}$. Q. E. D.

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