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Global Existence for Systems of Nonlinear Wave Equations in Two Space Dimensions

By

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§ 1. Introduction

We consider the Cauchy problem for systems of fully nonlinear wave equations of the type

$$\square u_i = F_i(u, u', u'') \quad \text{in } \overline{\boldsymbol{R}}_+ \times \boldsymbol{R}^n, \quad i = 1, \cdots, N, \tag{1.1}$$

$$u_i(0, x) = \varepsilon f_i(x), \ \partial_t u_i(0, x) = \varepsilon g_i(x), \quad x \in \boldsymbol{R}^n, \ i = 1, \cdots, N, \tag{1.2}$$

where $\partial_0 = \partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $j = 1, \cdots, N$, $\square = \partial_t^2 - \sum_{i=1}^n \partial_i^2$ is the D'Alembertian, $F = (F_j)$, $u = (u_j)$, $u' = (u_{j,a}) = (\partial_a u_j)$, $u'' = (u_{j,ab}) = (\partial_a \partial_b u_j)$ with $j = 1, \cdots, N$ and $a, b = 0, 1, \cdots, n$. Let $f_i, g_i \in C_0^\infty(\boldsymbol{R}^n)$ for $i = 1, \cdots, N$ and $\varepsilon > 0$ be a small parameter.

We assume that F is a smooth function in its arguments satisfying $F = O(|u|^p + |u'|^p + |u''|^p)$ near $(u, u', u'') = 0$ with some positive integer p and

$$\text{(A1)} \qquad \frac{\partial F_i}{\partial u_{j,ab}} = 0 \quad \text{for all } j \neq i \text{ and } a, b = 0, \cdots, n.$$

Let T_ε be the life-span of the classical solution to (1.1)-(1.2). When F does not depend on u explicitly, namely $F = F(u', u'')$, the following results are known : When $n = 3$, $T_\varepsilon \geq \exp\{c\varepsilon^{-1}\}$ for $p = 2$, $T_\varepsilon = +\infty$ for $p \geq 3$, and when $n = 2$, $T_\varepsilon \geq c\varepsilon^{-2}$ for $p = 2$, $T_\varepsilon \geq \exp\{c\varepsilon^{-2}\}$ for $p = 3$ and $T_\varepsilon = +\infty$ for $p \geq 4$, if ε is sufficiently small. (See Klainerman [7], Kovalyov [9], Li-Yu [12].)

In general cases it is known that when $n = 3$, $T_\varepsilon \geq c\varepsilon^{-2}$ for $p = 2$ and $T_\varepsilon = +\infty$ for $p \geq 3$. Klainerman [8] showed that when $n = 3$ and $p = 2$, if the quadratic part of nonlinear terms satisfies the null condition (which will be

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stated later), then $T_\varepsilon = +\infty$ for sufficiently small ε. See also Christodoulou [1] and John [5].

When $n=2$, it was shown that $T_\varepsilon \geq c\varepsilon^{-6}$ for $p=3$, and $T_\varepsilon = +\infty$ for $p \geq 4$. (See Li–Yu–Zhou [13]. In fact they are concerning single equations, but these results also hold for systems.)

Recently Godin [2] showed that for single semilinear wave equations in two space dimensions of the type

$$\square u = F(u') \quad \text{in } \overline{R}_+ \times R^2,\ F(u') = O(|u'|^2) \quad \text{near } u' = 0,$$

$T_\varepsilon \geq \exp\{c/\varepsilon^{-2}\}$ when the quadratic part of F satisfies the null condition, and $T_\varepsilon = +\infty$ when the quadratic and the cubic part of F satisfy the null condition.

His proof is using some transformation associated with the null condition to treat the quadratic term. It works well for single and semilinear equations, but it is not applicable to the system (1. 1). So we only treat the case $n=2$, $p=3$ for the system in this paper, and we prove the global existence under certain assumptions, following the method used in Klainerman [8].

§ 2. Klainerman's Null Condition and the Main Theorem

In this section we state about Klainerman's null condition (see Klainerman [8] and Christodoulou [1]) and our main theorem.

Definition 2. 1. *Let F be a smooth function of $u=(u_i)$, $v=(v_{i,a})$ and $w=(w_{i,ab})$ with $i=1, \cdots, N$ and $a, b = 0, 1, \cdots, n$. We say that F satisfies the null condition when*

$$F(\lambda_i, \mu_i X_a, \nu_i X_a X_b) = 0$$

for all $\lambda, \mu, \nu \in R^N$ and all $X=(X_0, X_1, \cdots, X_n) \in R^{n+1}$ such that $X_0^2 - X_1^2 - \cdots - X_n^2 = 0$.

We assume that F in (1. 1) satisfies the following:

(A2) $\qquad F = O(|u|^3 + |u'|^3 + |u''|^3) \quad$ near $(u, u', u'') = 0$,

(A3) $\qquad F(u, 0, 0) = O(|u|^5)$,

(A4) $\qquad F_i(u, u', u'') = G_i(u, u', u'') + H_i(u, u', u''),\ i = 1, \cdots, N,$

where G_i, the cubic term of F_i, satisfies the null condition, and $H_i = O(|u|^4 + |u'|^4 + |u''|^4)$ near $(u, u', u'') = 0$.

Without loss of generalities we can also assume that

(A5) (1. 1) is quasi–linear, i. e., F_i is linear in $(u_{i,ab})$ for $i = 1, \cdots, N$.

Now we can state our main theorem.

Theorem 1. *Let $n = 2$. Assume F satisfies* (A1)–(A4), *then there exists a positive constant ε_0(depending on f, g, F) such that for any $\varepsilon \leq \varepsilon_0$, a smooth solution $u(t, x)$ to* (1. 1)–(1. 2) *exists for* $0 \leq t < +\infty$.

The proof of this theorem will be given in section 4.

Remark 1. If we only assume (A1)–(A3), it was proved in Li–Yu–Zhou [13] that $T_\varepsilon \geq \exp\{c\varepsilon^{-2}\}$. But it is not known whether this result is sharp or not.

Remark 2. For simplicity we assumed that the initial conditions depend linearly on the parameter ε, but we can show all the results mentioned above, if we replace (1. 2) by the following:

$$(2.1) \qquad u_i(0, x) = f_i(x; \varepsilon),\ \partial_t u_i(0, x) = g_i(x; \varepsilon), \quad i = 1, \cdots, N,$$

where f_i, $g_i \in C^\infty(\boldsymbol{R}^2 \times \bar{\boldsymbol{R}}_+)$, $f_i(x; 0) = g_i(x; 0) = 0$ and $f_i(x; \varepsilon) = g_i(x; \varepsilon) = 0$ for $|x| > R$ with some $R > 0$.

Following Klainerman [7], we introduce $\Omega_{0j} = t\partial_j + x_j\partial_t$ for $j = 1, 2$, $\Omega_{12} = x_1\partial_2 - x_2\partial_1$, $\Omega_{ba} = -\Omega_{ab}$ for $0 \leq a < b \leq 2$, and $\Gamma_0 = t\partial_t + \sum_{j=1}^2 x_j\partial_j$. Then it follows that

$$(2.2) \qquad [\Gamma_0, \Box] = -2\Box,\ [\Omega_{ab}, \Box] = 0 \quad \text{for any } a, b = 0, 1, 2.$$

Let $\eta = (\eta_{ab})_{a,b=0,1,2} = \mathrm{diag}(-1, 1, 1)$, then one can easily verify that

$$(2.3) \qquad [\Omega_{ab}, \partial_c] = \eta_{ac}\partial_b - \eta_{bc}\partial_a,$$

$$(2.4) \qquad [\Omega_{ab}, \Omega_{cd}] = \eta_{ac}\Omega_{bd} - \eta_{bc}\Omega_{ad} + \eta_{ad}\Omega_{bc} - \eta_{bd}\Omega_{ac},$$

$$(2.5) \qquad [\Omega_{ab}, \Gamma_0] = 0,$$

(2.6) $$[\Gamma_0, \partial_a] = -\partial_a.$$

Let $\Gamma_1 = \Omega_{01}$, $\Gamma_2 = \Omega_{02}$, $\Gamma_3 = \Omega_{12}$, $\Gamma_4 = \partial_t$, $\Gamma_5 = \partial_1$, $\Gamma_6 = \partial_2$. We write $\Gamma^\alpha = \Gamma_0^{\alpha_0}\Gamma_1^{\alpha_1} \cdots \Gamma_6^{\alpha_6}$ for any multi-index $\alpha = (\alpha_0, \cdots, \alpha_6)$. From (2.3)–(2.6) we find that

(2.7) $$\Gamma^\alpha \Gamma^\beta u = \Gamma^{\alpha+\beta}u + \sum_{|\gamma| \leq k+l-1} C_\gamma^{\alpha\beta}\Gamma^\gamma u$$

for $|\alpha| = k$, $|\beta| = l$, and that

(2.8) $$\Gamma^\alpha \partial_a u = \partial_a \Gamma^\alpha u + \sum_{b=0}^{2} \sum_{|\beta| \leq k-1} C_{\beta b}^{\alpha a} \partial_b \Gamma^\beta u(t, x)$$

for $|\alpha| = k$, $a = 0, 1, 2$, where $C_\gamma^{\alpha\beta}$ and $C_{\beta b}^{\alpha a}$ are appropriate constants.

If $u(t, x)$ satisfies $\Box u(t, x) = f(t, x)$, then (2.2) leads to

(2.9) $$\Box(\Gamma^\alpha u) = \Gamma^\alpha f(t, x) + \sum_{|\beta| \leq k-1} C_\beta^\alpha \Gamma^\beta f(t, x)$$

for $|\alpha| = k$.

For any integer k and for any scalar or vector valued function $u(t, x)$, define

$$|u(t, x)|_k = \sum_{|\alpha| \leq k} |\Gamma^\alpha u(t, x)|,$$

$$|\partial u(t, x)|_k = \sum_{|\alpha| \leq k} \sum_{a=0}^{2} |\Gamma^\alpha \partial_a u(t, x)|,$$

$$\|u(t)\|_p = \|u(t, \cdot)\|_p = \left(\int_{R^2} |u(t, x)|^p dx\right)^{1/p} \text{ for } 1 \leq p < +\infty,$$

$$\|u(t)\|_\infty = \|u(t, \cdot)\|_\infty = \sup_{x \in R^2} |u(t, x)|,$$

$$\|u(t)\|_{p,k} = \|u(t, \cdot)\|_{p,k} = \left(\int_{R^2} |u(t, x)|_k^p dx\right)^{1/p} \text{ for } 1 \leq p < +\infty,$$

$$\|u(t)\|_{\infty,k} = \|u(t, \cdot)\|_{\infty,k} = \sup_{x \in R^2} |u(t, x)|_k,$$

$$\|\partial u(t)\|_{2,k} = \|\partial u(t, \cdot)\|_{2,k} = \left(\int_{R^2} |\partial u(t, x)|_k^2 dx\right)^{1/2}.$$

In the remainder of this section we state some results concerning the null condition. For any smooth functions f, g, define

(2.10) $$Q_{ab}(f, g) = \partial_a f \partial_b g - \partial_b f \partial_a g \quad \text{for } a, b = 0, 1, 2,$$

and

$$Q(f, g) = \partial_t f \partial_t g - \sum_{j=1}^{2} \partial_j f \partial_j g. \tag{2.11}$$

These functions are closely connected with the null condition. (See Klainerman [8].)

In fact, suppose $F = \{F_i(u, u', u'')\}_{i=1,\cdots,N}$ satisfies the assumptions (A1)-(A5). Then one can write G_i, which is the cubic part of F_i, as

$$\begin{aligned} G_i(u, u', u'') = & \sum_{j,k} P_i^{jk}(u, u', u'') Q(u_j, u_k) \\ & + \sum_{a,b,j,k} P_i^{abjk}(u, u', u'') Q_{ab}(u_j, u_k) \\ & + \sum_{a,j} P_{i,a}^{j}(u, u') Q(u_j, \partial_a u_i) \\ & + \sum_{a,b,c,j} P_{i,c}^{abj}(u, u') Q_{ab}(u_j, \partial_c u_i), \end{aligned} \tag{2.12}$$

where P_i^{jk}, P_i^{abjk}, $P_{i,a}^{j}$ and $P_{i,c}^{abj}$ are linear combinations of their arguments and

$$\frac{\partial P_i^{jk}}{\partial u_{l,cd}} = \frac{\partial P_i^{abjk}}{\partial u_{l,cd}} = 0 \quad \text{for all } l \neq i.$$

The following lemma due to Klainerman [8] is essential for our proof of the theorem.

Lemma 2.2. *For any integer $k \geq 0$, there exists a constant C_k such that*

$$\text{(i)} \quad |\Gamma^\alpha Q_{ab}(f, g)| \leq C_k(1+t)^{-1}(|f(t,x)|_{[k/2]+1}|g(t,x)|_{k+1} + |g(t,x)|_{[k/2]+1}|f(t,x)|_{k+1}), \quad a, b = 0, 1, 2, \tag{2.13}$$

$$\text{(ii)} \quad |\Gamma^\alpha Q(f, g)| \leq C_k(1+t)^{-1}(|f(t,x)|_{[k/2]+1}|g(t,x)|_{k+1} + |g(t,x)|_{[k/2]+1}|f(t,x)|_{k+1}) \tag{2.14}$$

for any $|\alpha| = k$ and any smooth functions f, g.

Proof. For $|\alpha| = k$, we can show that

$$\Gamma^\alpha Q_{ab}(f, g) = Q_{ab}(\Gamma^\alpha f, g) + Q_{ab}(f, \Gamma^\alpha g) + \sum_{c,d=0}^{2} \sum_{|\beta|+|\gamma| \leq k-1} C_{\beta\gamma cd}^{\alpha} Q_{cd}(\Gamma^\beta f, \Gamma^\gamma g)$$

and similarly,

$$\Gamma^{\alpha}Q(f,g)=Q(\Gamma^{\alpha}f,g)+Q(f,\Gamma^{\alpha}g)+\sum_{|\beta|+|\gamma|\leq k-1}C_{\beta\gamma}^{\alpha}Q(\Gamma^{\beta}f,\Gamma^{\gamma}g).$$

Hence it suffices to prove the assertions for $k=0$. When $t\leq 1$, it is clear that (i) and (ii) hold. From the definition of Γ's, we can write

$$\begin{aligned}Q_{12}(f,g)&=\frac{1}{t}(\Omega_{01}f\partial_{2}g-\Omega_{02}\partial_{1}g-\partial_{t}f\,\Omega_{12}g),\\ Q_{0j}(f,g)&=\frac{1}{t}(\partial_{t}f\,\Omega_{0j}g-\Omega_{0j}f\partial_{t}g),\\ Q(f,g)&=\frac{1}{t}(\partial_{t}f\,\Gamma_{0}g-\sum_{j=1}^{2}\Omega_{0j}\partial_{j}g).\end{aligned}$$

Therefore the assertion holds for $t\geq 1$ and this completes the proof.

§ 3. Preliminary Results for Linear Wave Equations

In this section we state some results for linear wave equations

(3.1) $$\square u(t,x)=f(t,x),\quad (t,x)\in\overline{\boldsymbol{R}}_{+}\times\boldsymbol{R}^{2},$$

where f is a smooth function satisfying

(3.2) $$f(t,x)=0\quad\text{for } |x|>t+R$$

with some constant $R>0$.

Lemma 3.1. *Let $u(t,x)$ be a smooth solution of (3.1) with initial data 0. Suppose $0\leq\kappa\leq 1$. Then there exists a constant $C>0$ such that*

(3.3) $$(1+t+|x|)^{1/2}(1+|t-|x||)^{(1-\kappa)/2}|u(t,x)|\leq C\int_{0}^{t}\frac{\|f(s)\|_{1,1}}{(1+s)^{\kappa/2}}ds.$$

Proof. See Hörmander [4]. In fact, in Corollary 6.2 of [4], it was shown that

$$\begin{aligned}&\left\{(1+|t^{2}-|x|^{2}|+(t^{2}+|x|^{2})^{1/2})^{1-\kappa}(1+(t^{2}+|x|^{2})^{1/2})^{\kappa}\right\}^{1/2}|u(t,x)|\\ &\qquad\leq C\sum_{|\alpha|\leq 1}\int_{0}^{t}\int_{R^{2}}\frac{|\Gamma^{\alpha}f(s,y)|}{\{1+(s^{2}+|y|^{2})^{1/2}\}^{\kappa/2}}dyds.\end{aligned}$$

Observing that $(1+t+|x|)(1+|t-|x||)\leq 2\sqrt{2}(1+|t^{2}-|x|^{2}|+(t^{2}+|x|^{2})^{1/2})$, the assertion follows immediately.

The next proposition is due to Li–Yu–Zhou [13].

Proposition 3. 2. *Suppose u satisfies* (3. 1), *then*

$$(3.4) \quad (\mathrm{i}) \quad \| \partial u(t) \|_2 \leq C\Big(\| \partial u(0) \|_2 + \int_0^t \| f(s) \|_2 \, ds \Big),$$

$$(3.5) \quad (\mathrm{ii}) \quad \| u(t) \|_2 \leq \| u(0) \|_2 + C_\delta t^{2\delta/(1+\delta)} \Big\{ \| \partial_t u(0) \|_{1+\delta} + \int_0^t \| f(s) \|_{1+\delta} \, ds \Big\}$$

for $0 < \delta < 1$, *where* C_δ *is a constant depending on* δ.

Proof. (i) is a standard energy estimate. So we prove here only (ii). First, let u be a solution of $\square u = 0$ with initial data $u(0) = u_0$ and $\partial_t u(0) = u_1$. Let $\mathscr{F}$ denote Fourier transform and $\hat{v}(\xi) = \mathscr{F}[v](\xi)$. Then it is well known that

$$\hat{u}(t, \xi) = \hat{u}_0(\xi) \cos |\xi| t + \frac{\hat{u}_1(\xi)}{|\xi|} \sin |\xi| t.$$

Fix any ρ satisfying $0 < \rho < 1$. We have

$$\begin{aligned} \left| \frac{\sin |\xi| t}{|\xi|} \hat{u}_1 \right| &= \left| \frac{\sin |\xi| t}{|\xi|} \right|^\rho \left| \frac{(\sin |\xi| t)^{1-\rho} \hat{u}_1}{|\xi|^{1-\rho}} \right| \\ &\leq \left| \frac{\sin |\xi| t}{|\xi|} \right|^\rho \left| \frac{\hat{u}_1}{|\xi|^{1-\rho}} \right| \leq C_\rho t^\rho \left| \frac{\hat{u}_1}{|\xi|^{1-\rho}} \right|. \end{aligned}$$

Let $\phi_a(x) = |x|^{-2/a}$ for $x \in \boldsymbol{R}^2$, then $\mathscr{F}[\phi_a](\xi) = C_a |\xi|^{-2+2/a}$ for any $a > 1$. (See Mizohata [11] for instance.) So it follows from Parseval's formula that

$$\left\| \frac{\hat{u}_1}{|\xi|^{1-\rho}} \right\|_2 = C_\rho \| \phi_{\frac{2}{(1+\rho)}} * u_1 \|_2.$$

Hardy–Littlewood inequality implies that

$$\| \phi_{\frac{2}{(1+\rho)}} * u_1 \|_2 \leq C_\rho \| u_1 \|_p,$$

where $1/p = 3/2 - (1+\rho)/2 = (2-\rho)/2$. (See Hörmander [3; Theorem 4. 5. 3].) Therefore we get

$$\| u(t, \cdot) \|_2 \leq \| u_0 \|_2 + C_\rho t^\rho \| u_1 \|_{1+\rho/(2-\rho)}.$$

Choose $\rho = 2\delta/(1+\delta)$ $(0 < \delta < 1)$, then we obtain

$$(3.6) \qquad \| u(t, \cdot) \|_2 \leq \| u_0 \|_2 + C_\delta t^{2\delta/(1+\delta)} \| u_1 \|_{1+\delta}.$$

Now let u be a solution of $\square u = f$ with initial data 0. From Duhamel's principle we can write

$$u(t, x) = \int_0^t U(t, x; s) ds,$$

where $U(t, x; s)$ is a solution of $\square U(t, x; s) = 0$ for $t \geq s$ with initial data $U(s, x; s) = 0$ and $(\partial_t U)(s, x; s) = f(s, x)$. (3. 6) leads to

$$\| U(t, \bullet ; s)\|_2 \leq C_\delta (t-s)^{2\delta/(1+\delta)} \| f(s, \bullet)\|_{1+\delta}.$$

Thus it follows that

$$\| u(t)\|_2 = \| \int_0^t U(t, \bullet ; s) ds \|_2 \leq \int_0^t \| U(t, \bullet ; s)\|_2 ds \tag{3. 7}$$
$$\leq C_\delta t^{2\delta/(1+\delta)} \int_0^t \| f(s, \bullet)\|_{1+\delta} ds.$$

Combining (3. 6) with (3. 7), we obtain the result for the general case.

Remark 3. Proposition 3. 2 (ii) does not hold if we choose $\delta = 0$. In fact, consider the Cauchy problem

$$\square u(t, x) = 0 \quad \text{in} \quad (0, \infty) \times \boldsymbol{R}^2,$$

with $u(0, x) = 0$, $\partial_t u(0, x) = g(x)$. *Suppose that* $g \in C_0^\infty(\boldsymbol{R}^2)$, $g \geq 0$ and $m = \int_{\boldsymbol{R}^2} g(x) dx > 0$. Then $\hat{g}(0) = m > 0$. So there exists some constant $A > 0$ such that $|\hat{g}(\xi)| \geq m/2$ for $|\xi| < A$. As $\hat{u}(t, \xi) = (\sin|\xi| t)\hat{g}(\xi)/|\xi|$, from Parseval's formula we get

$$\begin{aligned} \| u(t, \bullet)\|_2^2 &= C \| \hat{u}(t, \bullet)\|_2^2 \\ &= \int_{\boldsymbol{R}^2} \frac{\sin^2|\xi| t}{|\xi|^2} |\hat{g}(\xi)|^2 d\xi \geq \frac{Cm^2}{4} \int_{|\xi| < A} \frac{\sin^2|\xi| t}{|\xi|^2} d\xi \\ &= \frac{\pi C m^2}{2} \int_0^A \frac{\sin^2 rt}{r} dr = \frac{\pi C m^2}{2} \int_0^{At} \frac{\sin^2 \nu}{\nu} d\nu. \end{aligned}$$

Assume that there exists some constant $B > 0$ such that $\| u(t, \circ)\|_2 \leq B$ for $0 \leq t < \infty$. Then it follows that

$$\int_0^\infty \frac{\sin^2 \nu}{\nu} d\nu \leq \frac{2B^2}{\pi C m^2}.$$

But as

$$\int_0^\infty \frac{\sin^2\nu}{\nu} d\nu = \sum_{j=0}^\infty \int_{2\pi j}^{2\pi(j+1)} \frac{\sin^2\nu}{\nu} d\nu$$
$$\geq \int_0^{2\pi} \sin^2\nu \, d\nu \sum_{j=0}^\infty \frac{1}{2\pi(j+1)} = \infty,$$

this is a contradiction. So $\|u(t, \cdot)\|_2$ cannot be bounded.

Lemma 3. 3. *Let u be a smooth solution of*

$$\Box u(t, x) = \sum_{a=0}^2 C_a \partial_a f(t, x), \quad (t, x) \in \overline{R}_+ \times R^2$$

with initial data 0. *Then*

$$\|u(t)\|_2 \leq C_\delta t^{2\delta/(1+\delta)} \|f(0)\|_{1+\delta} + C \int_0^t \|f(s)\|_2 \, ds \tag{3. 8}$$

for $0 < \delta < 1$.

Proof. This is an analogue of the result of Lindblad [10]. (In fact his result concerns the case $n = 3$, but there is no difference in the proof.)

Let v be a solution of $\Box v = f$ with initial data 0, and let w be a solution of $\Box w = 0$ with initial data $w(0, x) = 0$, $\partial_t w(0, x) = f(0, x)$. Then u can be written as $u = \sum_{a=0}^2 C_a \partial_a v - C_0 w$. So we obtain the result using Proposition 3. 2.

Proposition 3. 4. *Let $u(t, x)$ be a solution of* (3. 1) *with initial data* 0. *Then it follows that*

$$\|u(t)\|_{2,1} \leq C_\delta t^{2\delta/(1+\delta)} \int_0^t \|f(s)\|_{1+\delta} \, ds + C \int_0^t (s+R) \|f(s)\|_2 \, ds \tag{3. 9}$$

for $0 < \delta < 1$, *where R is the same constant as in* (3. 2).

Proof. From Proposition 3. 2 (ii), we get

$$\|u(t)\|_2 \leq C_\delta t^{2\delta/(1+\delta)} \int_0^t \|f(s)\|_{1+\delta} \, ds.$$

From (2. 2) it follows that

$$\Box(\Gamma_0 u) = (\Gamma_0 + 2) f = \partial_t(tf) + \sum_{i=1}^2 \partial_i(x_i f) - f.$$

As $(\Gamma_0 u)(0) = (\partial_t \Gamma_0 u)(0) = 0$, noting that $|x| \leq t + R$ in supp f, Proposition 3. 2 and Lemma 3. 3 imply that

$$\begin{aligned}\|\Gamma_0 u(t)\|_2 &\le C\int_0^t \Big\{\|sf(s)\|_2+\sum_{i=1}^{2}\|(x_i f)(s)\|_2\Big\}ds\\ &\qquad\qquad + C_\delta t^{2\delta/(1+\delta)}\int_0^t \|f(s)\|_{1+\delta}\,ds\\ &\le C\int_0^t (s+R)\|f(s)\|_2\,ds + C_\delta t^{2\delta/(1+\delta)}\int_0^t \|f(s)\|_{1+\delta}\,ds.\end{aligned}$$

For $i=1, 2$, it follows that

$$\Box(\Omega_{0i}u) = \Omega_{0i}f = \partial_i(tf)+\partial_t(x_i f),$$

$$(\Omega_{0i}u)(0) = 0,\ (\partial_t\Omega_{0i}u)(0) = x_i f(0, x).$$

As in the proof of Lemma 3. 3, we obtain

$$\begin{aligned}\|\Omega_{0i}u(t)\|_2 &\le C\int_0^t \{\|sf(s)\|_2+\|(x_i f)(s)\|_2\}\,ds\\ &\le C\int_0^t (s+R)\|f(s)\|_2\,ds.\end{aligned}$$

Similarly, we get from Lemma 3. 3 that

$$\begin{aligned}\|\Omega_{12}u(t)\|_2 &\le C\int_0^t \{\|(x_2 f)(s)\|_2+\|(x_1 f)(s)\|_2\}\,ds\\ &\le C\int_0^t (s+R)\|f(s)\|_2\,ds.\end{aligned}$$

Finally, $\|\partial u(t)\|_2 \le C\int_0^t \|f(s)\|_2 ds$ from Proposition 3. 2 (i) and this completes the proof.

Lemma 3. 5. *Let $u\in C^\infty(\overline{R}_+\times R^2)$ be a solution of*

$$\Box u(t, x)-\sum_{a,b}\gamma_{ab}(t, x)\partial_a\partial_b u(t, x) = f(t, x).$$

Suppose that

$$\sum_{a,b}|\gamma_{ab}(t, x)| \le \frac{1}{2}$$

for all $(t, x)\in\overline{R}_+\times R^2$. Define

$$\|u(t)\|_E^2 = \int_{R^2}\{\beta_{00}|\partial_t u|^2+\sum_{i,j=1}^{2}\beta_{ij}\partial_i u\partial_j u\}\,dx,$$

where $\beta_{00}=1-\gamma_{00}$, $\beta_{ij}=\delta_{ij}+\gamma_{ij}$ for any $i, j=1, 2$. Then there exist some constants c, $C>0$ such that

(3. 10) $$\frac{1}{c}\|u(t)\|_E^2 \leq \|\partial u(t)\|_2^2 \leq c\|u(t)\|_E^2$$

and

(3. 11) $$\frac{d}{dt}\|u(t)\|_E \leq C\|\gamma'(t)\|_\infty \|u(t)\|_E + \|f(t)\|_2,$$

where $\|\gamma'(t)\|_\infty = \max_{a,b,c}\|\partial_c \gamma_{ab}(t,\cdot)\|_\infty$.

Proof. See Klainerman [6].

To conclude this section, we state two technical lemmata to be used in the proof of Theorem 1.

Lemma 3. 6. *Let*

(3. 12) $$\phi(t,x) = (1+t+|x|)^{-\alpha/2}(1+|t-|x||)^{-\alpha(1-\kappa)/2},\ 0<\kappa<1.$$

If $\alpha p(1-\kappa) > 2$, *then*

(3. 13) $$\|\phi(t,\cdot)\|_p \leq C_{\alpha,\kappa,p}(1+t)^{-\alpha/2+1/p}.$$

Proof.

$$\|\phi(t,\cdot)\|_p^p \leq (1+t)^{-\alpha p\kappa/2}\int_{R^2}\{(1+t+|x|)(1+|t-|x||)\}^{-\alpha p(1-\kappa)/2}dx.$$

Switching to the polar coordinate, we get

$$\begin{aligned}
&\int_{R^2}\{(1+t+|x|)(1+|t-|x||)\}^{-\alpha p(1-\kappa)/2}dx \\
&= 2\pi\int_0^\infty \{(1+t+r)(1+|t-r|)\}^{-\alpha p(1-\kappa)/2} r\,dr \\
&= 2\pi\Big\{\int_0^t \{(1+t)^2-r^2\}^{-\alpha p(1-\kappa)/2} r\,dr + \int_t^\infty \{(1+r)^2-t^2\}^{-\alpha p(1-\kappa)/2} r\,dr\Big\} \\
&\leq \frac{2\pi}{\alpha p(1-\kappa)-2}\Big(\Big[\{(1+t)^2-r^2\}^{1-\alpha p(1-\kappa)/2}\Big]_{r=0}^t \\
&\qquad\qquad + \Big[-\{(1+r)^2-t^2\}^{1-\alpha p(1-\kappa)/2}\Big]_{r=t}^\infty\Big) \\
&= \frac{2\pi}{\alpha p(1-\kappa)-2}\Big\{(1+2t)^{1-\alpha p(1-\kappa)/2} - (1+t)^{2-\alpha p(1-\kappa)} \\
&\qquad\qquad + (1+2t)^{1-\alpha p(1-\kappa)/2}\Big\} \\
&\leq C_{\alpha,\kappa,p}(1+t)^{-\alpha p(1-\kappa)/2+1}.
\end{aligned}$$

This completes the proof.

Lemma 3.7. *Suppose v, w be smooth functions in $\overline{R}_+ \times R^2$, and*

$$\operatorname{supp} w \subset \{|x| \leq t+R\}$$

with some constant $R > 0$. Then

$$\|(\partial_a v \circ w)(t, \circ)\|_2 \leq C\|v(t, \circ)\|_{\infty, 1}\|\partial w(t, \circ)\|_2 \tag{3.14}$$

for $0 \leq a \leq 2$.

Proof. First one can verify that

$$(1+|t-|x||)^2|(\partial v)(t, x)|^2 \leq 4|v(t, x)|_1^2$$

and

$$\left\|\frac{w(t, \circ)}{1+|t-|\circ||}\right\|_2 \leq C\|\partial w(t, \circ)\|_2.$$

Therefore we get

$$\begin{aligned}\|(\partial_a v \circ w)(t, \circ)\|_2 &\leq C\left\||v(t, \circ)|_1 \frac{w(t, \circ)}{1+|t-|\circ||}\right\|_2 \\ &\leq C\|v(t, \circ)\|_{\infty, 1}\left\|\frac{w(t, \circ)}{1+|t-|\circ||}\right\|_2 \\ &\leq C\|v(t, \circ)\|_{\infty, 1}\|\partial w(t, \circ)\|_2\end{aligned}$$

See Lindblad [10] and Godin [2] for details.

§ 4. Proof of Theorem 1

In this section we prove Theorem 1. First we make some *a priori* estimates. Let u be a solution of (1.1)–(1.2) for $0 \leq t < T$. Fix some integer $k \geq 4$. Define

$$M_1(t; u) = \sup_{0 \leq s < t} \sup_{y \in R^2} (1+s+|y|)^{1/2}(1+|s-|y||)^{1/4}|u(s, y)|_{k+2}, \tag{4.1}$$

$$M_2(t; u) = \sup_{0 \leq s < t} (1+s)^{-\lambda}\|u(s, \circ)\|_{2, 2k}, \tag{4.2}$$

$$M_3(t; u) = \sup_{0 \leq s < t} (1+s)^{-\mu}\|\partial u(s, \circ)\|_{2, 2k} \tag{4.3}$$

for $0 \leq t < T$ with $\lambda = 1/10$, $\mu = 1/30$, say. We show that there exist some constants $\varepsilon_0 > 0$ and $M > 0$ (which are independent of T), such that if $\varepsilon < \varepsilon_0$, then $\max_{i=1,2,3} M_i(t; u) \leq M\varepsilon$ holds for the solution $u(t, x)$ of (1. 1)–(1. 2) on $t \in [0, T)$. Combining this *a priori* estimate with the classical local existence theorem, we can prove the theorem.

Let

$$(4.4) \qquad \tau = \sup\{0 \leq t < T;\ \max_{i=1,2,3} M_i(s; u) \leq M\varepsilon \text{ for } 0 \leq s < t\}.$$

In the following, we are going to show that $\tau = T$ for any $\varepsilon \leq \varepsilon_0$, if M is sufficiently large and ε_0 is sufficiently small, and this proves the above *a priori* estimate.

The local existence theorem implies that $\tau > 0$, if M is chosen appropriately large Let ε be so small that $M\varepsilon < 1$.

§ 4. 1. L^∞–estimates

For $|\alpha| \leq k+2$, it follows that

$$(4.5) \qquad \square(\Gamma^\alpha u_i) = \sum_{|\beta| \leq k+2} C^\alpha_\beta(\Gamma^\beta G_i + \Gamma^\beta H_i), \quad i = 1, \cdots, N.$$

Lemma 2. 2 implies that

$$(4.6) \qquad \begin{aligned} \|\Gamma^\beta G_i\|_{1,1} &\leq C(1+s)^{-1}\|u(s)\|_{\infty,[(k+3)/2]+2} \\ &\quad \cdot \{\|u(s)\|_{2,k+4} + \|\partial u(s)\|_{2,k+4}\}^2 \\ &\leq C(1+s)^{2\lambda-3/2} M^3 \varepsilon^3, \quad 0 \leq s < \tau \end{aligned}$$

for $|\beta| \leq k+2$, because $[(k+3)/2]+2 \leq k+2$ and $k+4 \leq 2k$.

As $H = O(|u|^4 + |u'|^4 + |u''|^4)$, we get

$$\|\Gamma^\beta H_i\|_{1,1} \leq C\||u(s,\cdot)|^3_{[(k+3)/2]+2}\|_2(\|u(s,\cdot)\|_{2,k+4} + \|\partial u(s,\cdot)\|_{2,k+4}).$$

(4. 4) and Lemma 3. 6 imply that

$$(4.7) \qquad \begin{aligned} \||u(s,\cdot)|^3_{[(k+3)/2]+2}\|_2 &\leq CM^3\varepsilon^3\|(1+s+|\cdot|)^{-3/2}(1+|s-|\cdot||)^{-3/4}\|_2 \\ &\leq CM^3\varepsilon^3(1+s)^{-1} \end{aligned}$$

for $0 \leq s < \tau$. Therefore we get

$$\| \Gamma^{\beta} H_i \|_{1,1} \leq CM^4 \varepsilon^4 (1+s)^{\lambda-1} \quad \text{for } 0 \leq s < \tau. \tag{4.8}$$

Let $\tilde{u}_i^{\alpha}(t, x)$ be a solution of $\square \tilde{u}_i^{\alpha} = 0$ with initial data $\tilde{u}_i^{\alpha}(0) = (\Gamma^{\alpha} u_i)(0)$ and $(\partial_t \tilde{u}_i^{\alpha})(0) = (\partial_t \Gamma^{\alpha} u_i)(0)$. From the well-known decay estimate we can show that

$$(1+t+|x|)^{1/2}(1+|t-|x||)^{1/2} |\tilde{u}_i^{\alpha}(t, x)| \leq C_k \varepsilon, \tag{4.9}$$

where C is a positive constant which may depend on k, f and g.

From (4. 5), (4. 6), (4. 8), (4. 9) and Lemma 3. 1 with $\kappa = 1/2$, it follows that

$$\begin{aligned}
&(1+t+|x|)^{1/2}(1+|t-|x||)^{1/4} |\Gamma^{\alpha} u_i(t, x)| \\
&\leq C(\varepsilon + M^3 \varepsilon^3 \int_0^t (1+s)^{-1/4} \{(1+s)^{2\lambda - 3/2} + (1+s)^{\lambda-1} M\varepsilon\} \, ds \\
&\leq C(1+M^3 \varepsilon^2) \varepsilon
\end{aligned}$$

for $|\alpha| \leq k+2$, because $2\lambda - 7/4 < -1$ and $\lambda - 5/4 < -1$. So we obtain

$$M_1(\mathrm{t}\,; u) \leq C(1+M^3 \varepsilon^2) \varepsilon \qquad \text{for } 0 \leq t < \tau, \tag{4.10}$$

where C is a constant independent of M and ε.

§ 4. 2. L^2-estimates

For $|\alpha| \leq 2k-1$, we get as before

$$\square(\Gamma^{\alpha} u_i) = \sum_{|\beta| \leq 2k-1} C_{\beta}^{\alpha} \Gamma^{\beta} G_i + \sum_{|\beta| \leq 2k-1} C_{\beta}^{\alpha} \Gamma^{\beta} H_i, \quad i = 1, \cdots, N.$$

Write $H_i = I_i + J_i$, $i = 1, \cdots, N$, where $I_i(u, u', u'')$ is a homogeneous polynomial of degree four, and $J_i = O(|u|^5 + |u'|^5 + |u''|^5)$.

Let v_i^{α} be a solution of $\square v_i^{\alpha} = \sum_{|\beta| \leq 2k-1} C_{\beta}^{\alpha} \Gamma^{\beta}(G_i + J_i)$ with initial data 0, w_i^{α} be a solution of $\square w_i^{\alpha} = \sum_{|\beta| \leq 2k-1} C_{\beta}^{\alpha} \Gamma^{\beta} I_i$ with $(\partial_t^m w_i^{\alpha})(0) = (\partial_t^m \Gamma^{\alpha} u_i)(0)$ for $m = 0, 1$. Then $\Gamma^{\alpha} u_i = v_i^{\alpha} + w_i^{\alpha}$.

Lemma 2. 2 implies that

$$\begin{aligned}
&\| \Gamma^{\beta} G_i \|_2 \\
&\leq C(1+s)^{-1} \| u(s, \cdot) \|_{\infty, k+2}^2 \{ \| u(s, \cdot) \|_{2, 2k} + \| \partial u(s, \cdot) \|_{2, 2k} \} \\
&\leq C(1+s)^{\lambda-2} M^3 \varepsilon^3
\end{aligned} \tag{4.11}$$

and for $0 < \delta < 1$,

$$\| \Gamma^\beta G_i \|_{1+\delta} \leq (1+s)^{-1} \| |u(s,\cdot)|_{k+2}^2 \|_{2(1+\delta)/(1-\delta)} \{ \| u(s,\cdot) \|_{2,2k} + \| \partial u(s,\cdot) \|_{2,2k} \}$$

$$\leq C(1+s)^{\lambda-1} M\varepsilon \| |u(s,\cdot)|_{k+2}^2 \|_{2(1+\delta)/(1-\delta)} \tag{4.12}$$

for $|\beta| \leq 2k-1$. As $J_i = O(|u|^5 + |u'|^5 + |u''|^5)$, we have

$$\begin{aligned} &\| \Gamma^\beta J_i \|_2 \\ &\leq C \| u(s,\cdot) \|_{\infty,[(2k-1)/2]+2}^4 \{ \| u(s,\cdot) \|_{2,2k-1} + \| \partial u(s,\cdot) \|_{2,2k-1} \} \\ &\leq C(1+s)^{\lambda-2} M^5 \varepsilon^5 \leq C(1+s)^{\lambda-2} M^3 \varepsilon^3 \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \| \Gamma^\beta J_i \|_{1+\delta} &\leq C \| u(s,\cdot) \|_{\infty,[(2k-1)/2]+2}^2 \| |u(s,\cdot)|_{[(2k-1)/2]+2}^2 \|_{2(1+\delta)/(1-\delta)} \\ &\quad \cdot \{ \| u(s,\cdot) \|_{2,2k-1} + \| \partial u(s,\cdot) \|_{2,2k-1} \} \\ &\leq C(1+s)^{\lambda-1} M^3 \varepsilon^3 \| |u(s,\cdot)|_{k+2}^2 \|_{2(1+\delta)/(1-\delta)} \end{aligned} \tag{4.14}$$

for $|\beta| \leq 2k-1$, as $M\varepsilon \leq 1$. Because $2(1+\delta)/(1-\delta) > 2$, we can use Lemma 3.6 to get

$$\| |u(s,\cdot)|_{k+2}^2 \|_{2(1+\delta)/(1-\delta)} \leq C_\delta M^2 \varepsilon^2 (1+s)^{-1+(1-\delta)/2(1+\delta)}$$

for $0 \leq s < \tau$. Therefore Proposition 3.2 (ii) and Proposition 3.4 imply that

$$\begin{aligned} \| v_i^\alpha(t) \|_{2,1} &\leq C \int_0^t (1+s)^{\lambda-1} M^3 \varepsilon^3 ds \\ &\quad + C_\delta (1+t)^{2\delta/(1+\delta)} \int_0^t (1+s)^{\lambda-2+\frac{1-\delta}{2(1+\delta)}} M^3 \varepsilon^3 ds \\ &\leq C_\delta (1+t)^\lambda M^3 \varepsilon^3 \end{aligned} \tag{4.15}$$

for $|\alpha| \leq 2k-1$, provided that $2\delta/(1+\delta) \leq \lambda$ and $\lambda - 2 + \frac{1-\delta}{2(1+\delta)} < -1$. This holds if we choose $\delta = 1/20$, say.

Now we are going to estimate $\| w_i^\alpha(t) \|_{2,1}$. As $F(u, 0, 0) = O(|u|^5)$, it follows that $I_i(u, 0, 0) = 0$, $i = 1, \cdots, N$, so we can write

$$(4.16)\qquad I_i(u, u', u'') = \sum_{a,b} I_i^{ab}(u, u')\partial_a\partial_b u_i + \sum_{a,j} I_i^{a,j}(u, u')\partial_a u_j.$$

For $|\gamma| \leq 1$, we have

$$(4.17)\qquad \square(\Gamma^\gamma w_i^\alpha) = \sum_{|\beta|\leq 2k} C_\beta^{\alpha,\gamma}\Gamma^\beta I_i = \sum_{|\beta|=2k} C_\beta^{\alpha,\gamma}\sum_{a,b}\partial_a\{I_i^{ab}\bullet\Gamma^\beta\partial_b u_i\} + R_i^{\alpha,\gamma},$$

where

$$(4.18)\qquad R_i^{\alpha,\gamma} = \sum_{|\beta|\leq 2k} C_\beta^{\alpha,\gamma}\Gamma^\beta I_i - \sum_{|\beta|=2k} C_\beta^{\alpha,\gamma}\sum_{a,b} I_i^{ab}\bullet\Gamma^\beta(\partial_a\partial_b u_i) + \sum_{|\beta|=2k} C_\beta^{\alpha,\gamma}\sum_{a,b} I_i^{ab}[\Gamma^\beta, \partial_a]\partial_b u_i - \sum_{|\beta|=2k} C_\beta^{\alpha,\gamma}\sum_{a,b}(\partial_a I_i^{ab})\bullet\Gamma^\beta\partial_b u_i.$$

First it is easily seen that

$$(4.19)\qquad \| I_i^{ab}\bullet\Gamma^\beta\partial_b u_i\|_2 \leq C\| u(s,\bullet)\|_{\infty,1}^3 \|\partial u(s,\bullet)\|_{2,2k} \leq C(1+s)^{\mu-3/2}M^4\varepsilon^4 \quad \text{for } |\beta| = 2k.$$

$R_i^{\alpha,\gamma}$ is a linear combination of terms of the form

$$\Gamma^{\gamma_1}\partial^{\nu_1}u_{j_1}\Gamma^{\gamma_2}\partial^{\nu_2}u_{j_2}\Gamma^{\gamma_3}\partial^{\nu_3}u_{j_3}\Gamma^{\gamma_4}\partial^{\nu_4}u_{j_4}$$

with $j_l \in \{1, \cdots, N\}$, $0 \leq |\nu_1|, |\nu_2|, |\nu_3| \leq 2$, $1 \leq |\nu_4| \leq 2$, $|\gamma_1|, |\gamma_2| \leq k$, $\sum_{l=1}^4 |\gamma_l| \leq 2k$ and $|\gamma_l| + |\nu_l| \leq 2k+1$ for $l = 1, \cdots, 4$. Here ∂^{ν_l} denotes $\partial_0^{\nu_{l,0}}\partial_1^{\nu_{l,1}}\partial_2^{\nu_{l,2}}$ for $\nu_l = (\nu_{l,0}, \nu_{l,1}, \nu_{l,2})$.

When $|\gamma_4| \geq |\gamma_3|$, Hörder's inequality implies that

$$\|\prod_{l=1}^4 \Gamma^{\gamma_l}\partial^{\nu_l}u_{j_l}\|_{1+\delta} \leq \|\Gamma^{\gamma_1}\partial^{\nu_1}u_{j_1}\Gamma^{\gamma_2}\partial^{\nu_2}u_{j_2}\|_{2(1+\delta)/(1-\delta)}\bullet\|\Gamma^{\gamma_3}\partial^{\nu_3}u_{j_3}\bullet\Gamma^{\gamma_4}\partial^{\nu_4}u_{j_4}\|_2 \leq \|| u(t,\bullet)|_{k+2}^2\|_{2(1+\delta)/(1-\delta)}\| u(t,\bullet)\|_{\infty,k+2}\|\partial u(t,\bullet)\|_{2,2k}$$

for any $0 < \delta < 1$. When $|\gamma_4| < |\gamma_3|$ and $|\nu_3| \geq 1$, we can estimate the term in the same way as above.

When $|\gamma_4| < |\gamma_3|$ and $|\nu_3| = 0$, we obtain with the help of Lemma 3.7 and (2.8) that

$$\|\prod_{l=1}^4 \Gamma^{\gamma_l}\partial^{\nu_l}u_{j_l}\|_{1+\delta} \leq \|\Gamma^{\gamma_1}\partial^{\nu_1}u_{j_1}\Gamma^{\gamma_2}\partial^{\nu_2}u_{j_2}\|_{2(1+\delta)/(1-\delta)}\|\Gamma^{\gamma_3}u_{j_3}\bullet\Gamma^{\gamma_4}\partial^{\nu_4}u_{j_4}\|_2$$

$$\leq C \| | u(t, \cdot) |_{k+2}^{2} \|_{2(1+\delta)/(1-\delta)} \| u_{j_4}(t, \cdot) \|_{\infty, k+2} \| \partial(\Gamma^{\gamma_3} u_{j_3}) \|_2$$

$$\leq \| | u(t, \cdot) |_{k+2}^{2} \|_{2(1+\delta)/(1-\delta)} \| u(t, \cdot) \|_{\infty, k+2} \| \partial u(t, \cdot) \|_{2, 2k}.$$

Therefore

$$\| R_i^{\alpha, \gamma} \|_{1+\delta} \leq CM^2 \varepsilon^2 (1+t)^{\mu-1/2} \| | u(t, \cdot) |_{k+2}^{2} \|_{2(1+\delta)/(1-\delta)}.$$

As $2 \cdot 2(1+\delta)/(1-\delta) \cdot (1/2) > 2$ for any $0 < \delta < 1$, Lemma 3. 6 implies

$$\begin{aligned} &\| | u(t, \cdot) |_{k+2}^{2} \|_{2(1+\delta)/(1-\delta)} \\ &\quad \leq M^2 \varepsilon^2 \| (1+t+|\cdot|)^{-1} (1+|t-|\cdot||)^{-1/2} \|_{2(1+\delta)/(1-\delta)} \\ &\quad \leq C(1+t)^{-1+\frac{1-\delta}{2(1+\delta)}} M^2 \varepsilon^2. \end{aligned}$$

So, from Proposition 3. 2 and Lemma 3. 3, it follows that

$$\begin{aligned} \| w_i^{\alpha} \|_{2,1} \leq C \Big\{ (1+t)^{2\delta/(1+\delta)} \varepsilon + (1+t)^{2\delta/(1+\delta)} M^4 \varepsilon^4 \int_0^t (1+s)^{\mu - \frac{3}{2} + \frac{(1-\delta)}{2(1+\delta)}} ds & \\ + M^4 \varepsilon^4 \int_0^t (1+s)^{\mu-3/2} ds \Big\} & \qquad (4.20) \\ \leq C(1+t)^{\lambda} (1+M^4 \varepsilon^3) \varepsilon \quad \text{for } 0 \leq t < \tau, & \end{aligned}$$

provided that $\mu + \frac{1-\delta}{2(1+\delta)} - \frac{3}{2} < -1$ and $\lambda \geq 2\delta/(1+\delta)$. This holds if we choose $\delta = 1/20$.

Finally, combining above estimates, we conclude that

$$\| u(t, \cdot) \|_{2, 2k} \leq C(1+t)^{\lambda} (1+M^3 \varepsilon^2) \varepsilon \qquad (4.21)$$

for $0 \leq t < \tau$, that is,

$$M_2(t; u) \leq C(1+M^3 \varepsilon^2) \varepsilon \qquad \text{for } 0 \leq t < \tau. \qquad (4.22)$$

§ 4. 3. The Energy Estimates

Finally we make estimates for L^2–norms of the derivatives of the solution. For $|\alpha| \leq 2k$, we can write

$$\begin{aligned} &\Box(\Gamma^{\alpha} u_i) - \sum_{a,b=0}^{2} F_i^{ab}(u, u') \cdot \partial_a \partial_b (\Gamma^{\alpha} u_i) \\ &\quad = \sum_{|\beta| \leq 2k} C_{\beta}^{\alpha} \Gamma^{\beta} F_i - \sum_{a,b=0}^{2} F_i(u, u') \cdot \Gamma^{\alpha}(\partial_a \partial_b u_i) \qquad (4.23) \end{aligned}$$

$$+\sum_{a,b=0}^{2} F_i^{ab}(u, u')[\Gamma^\alpha, \partial_a\partial_b]u_i,$$

where $F_i^{ab}=\partial F_i/\partial u_{i,ab}$. Note that $C_\beta^\alpha=1$ if $\beta=\alpha$, and $C_\beta^\alpha=0$ if $|\beta|=2k$ and $\beta\neq\alpha$. Because $F=O(|u|^3+|u'|^3+|u''|^3)$ and $F(u, 0, 0)=O(|u|^5)$, L^2–norms of the right–hand side of (4. 23) is bounded by

$$\begin{aligned}&\|u(t,\cdot)\|_{\infty,k+2}^2(1+\|u(t,\cdot)\|_{\infty,k+2})\|\partial u(t,\cdot)\|_{2,2k}\\&\qquad+\|u(t,\cdot)\|_{\infty,k+2}^4\{\|u(t,\cdot)\|_{2,2k}+\|\partial u(t,\cdot)\|_{2,2k}\}\\&\leq C(1+t)^{\mu-1}M^3\varepsilon^3+C(1+t)^{\lambda-2}M^5\varepsilon^5,\end{aligned}$$

with the help of Lemma 3. 7.

As $\sup_{a,b,c}|\partial_c F_i^{ab}(u, u')|\leq|u(t,x)|_2^2\leq C(1+t)^{-1}M^2\varepsilon^2$ and $\sum_{a,b}|F_i^{ab}(u, u')|\leq C(1+t)^{-1}M^2\varepsilon^2$, we can apply Lemma 3. 5 to (4. 11) if ε is sufficiently small, and we obtain

$$\begin{aligned}\frac{d}{dt}\|\Gamma^\alpha u(t)\|_E&\leq C\|u(t,\cdot)\|_{\infty,2}^2\|\Gamma^\alpha u(t,\cdot)\|_E\\&\quad+C\|u(t,\cdot)\|_{\infty,k+2}^2(1+\|u(t,\cdot)\|_{\infty,k+2})\|\partial u(t,\cdot)\|_{2,2k}\\&\quad+C\|u(t,\cdot)\|_{\infty,k+2}^4\{\|u(t,\cdot)\|_{2,2k}+\|\partial u(t,\cdot)\|_{2,2k}\}\\&\leq C(1+t)^{\mu-1}M^3\varepsilon^3\end{aligned}\tag{4. 24}$$

for $|\alpha|\leq 2k$, because $\|u(t)\|_E\leq C\|\partial u(t)\|_{2,2k}$. This leads to

$$\|\Gamma^\alpha u(t)\|_E\leq C(1+M^3\varepsilon^2)(1+t)^\mu\varepsilon\quad\text{for } |\alpha|\leq 2k.$$

As $\frac{1}{c}\|\partial\Gamma^\alpha u(t)\|_2\leq\|\Gamma^\alpha u(t)\|_E$, we have

$$\|\partial(\Gamma^\alpha u)(t)\|_2\leq C(1+M^3\varepsilon^2)(1+t)^\mu\varepsilon\quad\text{for } |\alpha|\leq 2k.$$

In view of (2. 8), this means that

$$\|\partial u(t)\|_{2,2k}\leq C(1+M^3\varepsilon^2)(1+t)^\mu\varepsilon\quad\text{for } 0\leq t<\tau,\tag{4. 25}$$

i. e.,

$$M_3(t; u)\leq C(1+M^3\varepsilon^2)\varepsilon\quad\text{for } 0\leq t<\tau.\tag{4. 26}$$

§ 4. 4. Completion of the Proof of A Priori Estimates

We have proved in sections 4. 1–4. 3 that if $u(t, x)$ is a solution to (1. 1)–(1. 2) for $0 \leq t < T$, and

$$\tau = \sup\{0 \leq t < T;\ \max_{i=1,2,3} M_i(s; u) \leq M\varepsilon \text{ for } 0 \leq s < t\},$$

then

$$\max_{i=1,2,3} M_i(t; u) \leq C(1+M^3\varepsilon^2)\varepsilon \quad \text{for } 0 \leq t < \tau, \tag{4. 27}$$

where $C > 0$ is a constant independent of ε, τ, T and M. Choose M and ε_0 to satisfy $M \geq 2C$ and $M^3\varepsilon_0^2 \leq 1/2$, then from (4. 27) it follows that

$$\max_{i=1,2,3} M_i(t; u) \leq \frac{3}{2}C\varepsilon \leq \frac{3}{4}M\varepsilon,\ 0 \leq t < \tau$$

for any $\varepsilon \leq \varepsilon_0$. By usual continuation arguments, we conclude that $\tau = T$. This completes the proof.

§ 5. Some Remarks for the Single and Semilinear Equations

In this section we consider the Cauchy problem for semilinear wave equations of the type

$$\Box u = F(u, u') \quad \text{in } \overline{\boldsymbol{R}}_+ \times \boldsymbol{R}^2, \tag{5. 1}$$

$$u(0, x) = \varepsilon f(x),\ \partial_t u(0, x) = \varepsilon g(x), \quad x \in \boldsymbol{R}^2, \tag{5. 2}$$

where u and F are scalar–valued functions and $f, g \in C_0^\infty(\boldsymbol{R}^2)$.

Assume F satisfies the following :

(A6) $F = O(|u|^2 + |u'|^2)$ near $(u, u') = 0$,

(A7) $F(u, 0) = O(|u|^5)$,

(A8) F can be written as

$$F(u, u') = G_2(u, u') + G_3(u, u') + H(u, u'),$$

where G_2, the quadratic term of F, satisfies the null condition, G_3 is a cubic term of F, and

$$H(u, u') = O(|u|^4 + |u'|^4) \text{ near } (u, u') = 0,$$

(A9) In addition, G_3 satisfies the null condition.

Then F can be written as

$$F(u, u') = cQ(u, u) + P(u, u')Q(u, u) + H(u, u'),$$

where c is a constant, $Q(u, u) = (\partial_t u)^2 - \sum_{i=1}^2 (\partial_i u)^2$ as in section 2, P is a linear combination of u and $(\partial_a u)_{a=0,1,2}$, and $H(u, u') = O(|u|^4 + |u'|^4)$ satisfying $H(u, 0) = O(|u|^5)$.

Let u satisfies (5. 1)–(5. 2), and let $v = (1-\exp(-cu))/c$. (See [2].) Then

$$\begin{aligned}\square v &= (1-cv)\{\square u - cQ(u, u)\}\\ &= (1-cv)\{P(u, u')Q(u, u) + H(u, u')\}.\end{aligned}$$

Define

$$\begin{aligned}\tilde{F}(v, v') &= \tilde{F}(v, (\partial_a v)_{a=0,1,2})\\ &= \frac{1}{(1-cv)} P\left(-c^{-1}\log(1-cv), \left((1-cv)^{-1}\partial_a v\right)_{a=0,1,2}\right) Q(v, v)\\ &\quad + (1-cv) H\left(-c^{-1}\log(1-cv), \left((1-cv)^{-1}\partial_a v\right)_{a=0,1,2}\right).\end{aligned} \tag{5. 3}$$

Then v satisfies

$$\square v = \tilde{F}(v, v'), \tag{5. 4}$$

with initial data

$$v(0) = (1-\exp(-\varepsilon cf))/c, \ \partial_t v(0) = \varepsilon g \exp(-\varepsilon cf). \tag{5. 5}$$

One can verify that $\tilde{F}(v, v')$ satisfies (A1)–(A4). So Theorem 1 and Remark 2 imply that there exists a unique solution to (5. 4)–(5. 5) for $0 \leq t < +\infty$, if ε is sufficiently small. Therefore the Cauchy problem (5. 1)–(5. 2) also has a global solution.

Similarly, if we only assume (A6)–(A8) and not (A9), we can show that the reduced equation (5. 4) satisfies (A1)–(A3). So the result in Remark 1 implies that there exists a unique solution for $0 \leq t < \exp\{A\varepsilon^{-2}\}$ with some constant $A > 0$, if ε is sufficiently small.

Summing up, we have proved the following :

Theorem 2. *Assume* (A6)–(A8). *Then there exist some constants* $\varepsilon_0 > 0$ *and* $A > 0$, *such that for any* $\varepsilon \leq \varepsilon_0$, *there exists a unique solution* $u(t, x)$ *to* (5. 1)–(5. 2) *for* $0 \leq t < \exp\{A\varepsilon^{-2}\}$.

If we assume (A9) *in addition, then there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \leq \varepsilon_0$, *the Cauchy problem* (5. 1)–(5. 2) *has a global solution.*

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