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Generalised Mean Averaging Interpolation by Discrete Cubic Splines

By

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Abstract

The aim of this work is to introduce for a discrete function, certain discrete integrals which may reduce in particular to usual Riemann Stieltjes integrals. We name them as Discrete Stieltjes integrals. The existence and convergence of a discrete cubic interpolatory spline whose discrete Stieltjes integrals between consecutive meshpoints match with the corresponding integrals of a given periodic discrete function, are studied.

KEY WORDS: Discrete Stieltjes Integrals, forward differences, central differences, Discrete Splines.

§1. Introduction

Discrete integrals play a significant role in the theory of interpolation and approximation of functions defined on discrete subsets of the real line. Schumaker [8] and Lyche [4] have studied extensively the properties of discrete integrals. Here we introduce certain discrete integrals which we prefer to call Discrete Stieltjes (DS-) integrals, as they reduce in particular to the usual Riemann-Stieltjes integrals.

Schoenberg [7] and de Boor [1] have considered area matching interpolatory condition for even-degree splines. Considering Lebesgue integrals with respect to a non-negative measure, Sharma and Tzimbalario [9] have studied quadratic spline interpolants satisfying a fairly general mean-averaging condition. Similar interpolation problems for cubic splines and discrete cubic splines have been investigated in Dikshit [2] and Dikshit and Powar [3] respectively. Discrete splines are piecewise polynomials which satisfy smoothness requirements at knots in terms of differences. Our aim

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in this paper is to study the existence and convergence properties of a discrete cubic spline whose discrete Stieltjes integrals between consecutive meshpoints match with the corresponding integrals of a given discrete function. For terms and notations we refer to [11].

§2. Discrete Cubic Interpolatory Spline

Given a real number $h>0$, let f be a bounded function and α be a non decreasing function defined over a discrete interval $[a,b]_h$. The Discrete Stieltjes integral of f with respect to α over $[a,b]_h$ is defined as:

$$\int_a^b f(x)\,d_h\,\alpha(x)=\sum_{i=0}^{N-1} f(a+ih)\cdot[\alpha(a+(i+1)h)-\alpha(a+ih)], \tag{2.1}$$

where it is assumed that $b-a=Nh$, N being a positive integer. The definition (2.1) remains valid if α is monotonic non-increasing, or in fact, if α is a function of bounded variation.

Let $P=\{x_i\}_{i=0}^n$ with $0=x_0<x_1<\cdots<x_n=1$, be a uniform sequence of points in $[0,1]_h$ such that $x_i-x_{i-1}=p$, $i=1,2,\cdots,n$. A discrete cubic spline with knots in P is a piecewise cubic polynomial over $[0,1]$ which satisfies the conditions:

$$D_h^{\{j\}}s_i(x_i)=D_h^{\{j\}}s_{i+1}(x_i) \qquad j=0,1 \text{ and } 2, \quad i=1,2,\cdots n-1, \tag{2.2}$$

where s_i is the restriction of s in $[x_{i-1},x_i]$ and $D_h^{\{j\}}g$ is the j^{th} central difference of a function g. The space of discrete cubic splines with knots in P is denoted by $S(4,P,h)$. Consider a non decreasing function α defined over $[0,1]_h$ such that

$$\alpha(x+p)-\alpha(x)=K; \tag{2.3}$$

K being a constant.

We shall investigate the following:

Problem 2.1. *Given a 1-periodic discrete function f over $[0,1]_h$, does there exist a unique 1-periodic discrete cubic spline s in $S(4,P,h)$ satisfying the interpolatory condition*

$$\int_{x_{i-1}}^{x_i} [f(x)-s(x)]d_h\alpha(x)=0, \qquad i=1,2,\cdots,n? \tag{2.4}$$

A discrete cubic spline s can be represented in terms of its second central differences at meshpoints, as follows:

$$6\ p\ s(x) = M_{i-1}\ (x_i-x)^{\{3\}}+M_i(x-x_{i-1})^{\{3\}}+6c_i\ (x_i-x)+6d_i(x-x_{i-1})$$
$$x_{i-1}\le x\le x_i, \qquad i=1,2,\cdots,n; \tag{2.5}$$

where $M_i=D_h^{\{2\}}s(x_i)$. Also, c_i and d_i are arbitrary constants, which in view of conditions (2.2), are given by following relations

$$d_i=c_{i+1}$$
$$p^2M_i=d_{i-1}-2d_i+d_{i+1}. \tag{2.6}$$

For convenience, we set

$$\int_{x_{i-1}}^{x_i} f(x)d_h\alpha(x)=F_i, \quad \int_{x_{i-1}}^{x_i} (x_i-x)^{\{j\}}\ d_h\alpha(x)=A(j),$$

and

$$\int_{x_{i-1}}^{x_i} (x-x_{i-1})^{\{j\}}\ d_h\alpha(x)=B(j). \qquad j=1,2,\cdots,n.$$

In view of (2.3) we find that

$$A(j) = \int_{x_{r-1}}^{x_r} (x_r-x)^{\{j\}}\ d_h\alpha(x)$$

$$B(j) = \int_{x_{r-1}}^{x_r} (x-x_{r-1})^{\{j\}}\ d_h\alpha(x) \qquad r=1,2,\cdots,n;$$
$$j=1,2,\cdots$$

and

$$\int_{x_{i-1}}^{x_i} d_h\alpha=K=(1/p)[A(1)+B(1)], \qquad \text{for each } i.$$

Thus, from interpolatory condition (2.4) we obtain the following

$$6p\ F_i = M_{i-1}A(3) + M_i\ B(3) + 6\ d_{i-1}\ A(1) + 6\ d_i\ B(1).$$

Eliminating d_i's in (2.6) and the above equation we get

$$\begin{aligned} &B(3)\ M_{i+1} + [A(3) - 2\ B(3) + 6\ p^2\ B(1)]M_i \\ &+ [-2\ A(3) + B(3) + 6p^2\ A(1)]M_{i-1} + A(3)\ M_{i-2} \\ &= 6p(F_{i+1} - 2F_i + F_{i-1}), \qquad i = 1,2,\cdots,n, \end{aligned} \tag{2.7}$$

where $M_n = M_0$, $M_{n+1} = M_1$, $F_n = F_0$ and $F_{n+1} = F_1$. Now in view of the properties of Discrete Stieltjes integrals, it is easy to see that when $p > 2h$,

$$\begin{aligned} &A(3) \geq 0,\ B(3) \geq 0 \\ &p^2 A(1) \geq A(3) \text{ and } p^2\ B(1) \geq B(3). \end{aligned}$$

Therefore the coefficients of M_{i+1}, M_i, M_{i-1} and M_{i-2} are all non-negative. Also, the excess of coefficient of M_{i-1} over the sum of coefficients of M_{i+1}, M_i and M_{i-2} is

$$\begin{aligned} &2[-2A(3) + B(3) + 3p^2(A(1) - B(1))] \\ &= 2\int_0^p [p^{\{3\}} + 3x^{\{3\}} - 6px^2 + 3ph^2]d_h\alpha. \end{aligned} \tag{2.8}$$

Now if non-decreasing function α is such that it remains constant after $x = .466p$ in each mesh interval then the expression (2.8) is positive. The coefficient matrix of the system of equations (2.7) is then diagonally dominant and the system admits a unique solution.

Again, considering the excess of coefficient of M_i over the sum of coefficients of M_{i+1}, M_{i-1}, M_{i-2} we observe that if the function α is such that it remains constant upto $x = .533p$ in each subinterval $[0,p]_h$, then the coefficient matrix of the system of equations (2.7) is invertible and the system is uniquely solved.

We have thus proved the following:

Theorem 2.1. *Given a* 1*-periodic function* f *and a non-decreasing function* α *defined over* $[0,1]_h$ *such that* (2.3) *holds, there exists a unique* 1*-periodic discrete cubic spline* $s \in S(4,P,h)$ *with* $p > 2h$, *satisfying* (2.4) *provided* α *is a function such that it remains constant either in* $[.466p,p]_h$ *or in* $[0,.533p]_h$ *for* *each subinterval* $[0,p]_h$ *of the mesh* P.

§3. Convergence

Now we aim to establish the convergence properties of the discrete cubic spline interpolant of Theorem 2.1. Let $e=s-f$ denote the error function. We estimate the error-bounds in terms of '*discrete norm*' and '*discrete modulus of smoothness*' denoted by $\|f\|$ and $w(f,t)$ respectively (cf. [11]).

We shall prove the following:

Theorem 3.1. *If* f,α, *and* $s\in S(4,P,h)$ *be as in Theorem* 2.1, *then*

$$\| e_i^{\{2\}} \| \leq K_1 w(f^{(2)},p) \tag{3.1}$$

and

$$\| e^{\{2\}} \| \leq (K_1+1) w(f^{(2)},p), \tag{3.2}$$

where K_1 *is a constant.*

Proof of the theorem. Replacing M_i *in* (2.7) *by* $e_i^{\{2\}}+f_i^{\{2\}}$, we have

$$\begin{aligned}&B(3)\ e_{i+1}^{\{2\}}+[A(3)-2B(3)+6p^2\ B(1)]e_i^{\{2\}}\\&+[-2A(3)+B(3)+6p^2\ A(1)]e_{i-1}^{\{2\}}+A(3)e_{i-2}^{\{2\}}=6p(F_{i+1}-2F_i+F_{i-1})\\&-B(3)f_{i+1}^{\{2\}}-[A(3)-2B(3)+6p^2\ B(1)]f_i^{\{2\}}\\&-[-2A(3)+B(3)+6p^2A(1)]f_{i-1}^{\{2\}}-A(3)f_{i-2}^{\{2\}}\equiv R(\text{say}).\end{aligned}$$

Expanding $f(x)$ in each subinterval by Discrete Taylor formula we get

$$F_i=f_{i-1}K+f_{i-1}^{(1)}\ B(1)+\theta_i f^{(2)}(z_i)\ \bar{B}(2)$$

where $0\leq\theta_i\leq 1$, $z_i\in(x_{i-1},x_i)_h$; $(x-x_{i-1})^{(2)}=(x-x_{i-1})(x-x_{i-1}-h)$ and

$$\bar{B}(2)=\int_{x_{i-1}}^{x_i}(x-x_{i-1})^{(2)}d_h\alpha(x).$$

We observe that

$$f_{i-2}-2f_{i-1}+f_i=p^2f^{(2)}(y_i);$$

where $y_i\in(x_{i-2},x_i)_h$; and

$$| f_{i-2}^{(1)}-2f_{i-1}^{(1)}+f_i^{(1)} | \leq 2p\ w(f^{(2)},p).$$

Therefore,

$$|R| \leq 2[A(3)+B(3)+6p\bar{B}(2)+6p^2B(1)+3p^3K]\ w(f^{(2)},p).$$

If $|e_j^{\{2\}}| \geq |e_i^{\{2\}}|$, $i=1,2,\cdots,n$; then from (2.7) we have

$$2[A(3)-2B(3)+3p^2(B(1)-A(1))]\,|e_j^{\{2\}}| \leq |R|.$$

This directly leads to (3.1). It is easy to see from (2.5) that in $[x_{i-1},x_i]$,

$$ps^{\{2\}}(x)=M_{i-1}(x_i-x)+M_i(x-x_{i-1}).$$

Therefore,

$$\begin{aligned} p\ e^{\{2\}}(x) &= [e_{i-1}^{\{2\}}+f^{(2)}(x)-f_{i-1}^{\{2\}}](x-x_{i-1}) \\ &\quad +[e_i^{\{2\}}+f^{(2)}(x)-f_i^{\{2\}}](x_i-x). \end{aligned}$$

A little calculation then leads to (3.2). This completes the proof of Theorem 3.1.

Remarks.

1. In the case when $\alpha(x)=x$ and $h\to 0$, the mean averaging condition (2.4) reduces to the area matching condition considered in [10].

2. When α is a step function, for suitable choices of function α, the interpolatory condition (2.4) reduces to different conditions of interpolation at one or more interior points in each mesh interval (cf. Meir and Sharma [6]). When α has a single jump at one end point in each mesh interval then the discrete cubic spline of Theorem 2.1 reduces to that considered in Lyche [5]. For an other appropriate choice of function α the interpolatory condition (2.4) reduces to the average-interpolation condition considered in [11].

3. The estimates (3.1) and (3.2) in Theorem 3.1 are sharp, i.e., as a functions of n, they decrease to zero, when $n\to\infty$ like $\beta\cdot n^{-1}$ where β is a constant.

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