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The Singularities of Type A_k of Holonomic Systems

By

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§ 1. Introduction

Let us consider the microdifferential holonomic system $\mathcal{M}$ whose characteristic variety is in a generic position (see [2], Ch. I, § 6) of the point $(0, dx_0) \in P^*X$, $X = C^{n+1} \ni (x_0, x_1, \cdots, x_n)$. Suppose that the conditions in the Theorem 5.9 in [3] are satisfied, there is a basis $u_1, \cdots, u_m$ of $\mathcal{M}$ such that we have the following equations:

$$\begin{cases} x_0 u = (A_0(x) + A_1 \xi_0^{-1}) u \\ \xi_j \xi_0^{-1} u = B_j(x) u, \quad j = 1, \cdots, n \end{cases}$$

where $A_0(x)$, $B_j(x)$, $j = 1, \cdots, n$ are analytic matrices of order m and A_1 is a constant matrix. Note that these matrices have to verify the integrability conditions

$$[A_0(x), B_i(x)] = [B_i(x), B_j(x)] = 0,$$

$$-\frac{\partial A_0(x)}{\partial x_i} = [A_1, B_i(x)] + B_i(x),$$

$$\frac{\partial B_i(x)}{\partial x_j} = \frac{\partial B_j(x)}{\partial x_i}, \qquad i, j = 1, \cdots, n$$

By the last conditions we establish a generating matrix $H(x)$ of the system M such that $\mathrm{trac} H(x) = 0$,

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$$\frac{\partial H(x)}{\partial x_i}=B_i(x), \quad i=1, \cdots, n$$

$$H(0)=H_0,$$

where H_0 is the solution of $[A_1, H_0]+H_0=-A_0(0)$. In addition, the other conditions translate to the following

$$\left[[A_1, H(x)]+H(x), \frac{\partial H}{\partial x_i}\right]=0, \quad i=1, \cdots, n \tag{1}$$

$$\left[\frac{\partial H}{\partial x_i}, \frac{\partial H}{\partial x_j}\right]=0, \quad i, j=1, \cdots, n.$$

The matrix $H(x)$ satisfying the conditions (1) is called a D-matrix.

Thus, the investigation of singularities of holonomic systems reduces to that of D-matrices (see [1]).

In this article we give the explicite form of D-matrices of type A_k.

§2. Results

Let $H(x)$, $x=(x_1, \cdots, x_n)$ be a D-matrix of order $n+1$.

Definition. We say that $H(x)$ is of type A_{n+1} if $H(x)$ has the following properties:

1) $H(0)=0$,

2) $\frac{\partial H}{\partial x_1}(0)=X$,

$\frac{\partial^2 H}{\partial x_1^2}(0)=Y^n$, $Y^n=\underbrace{Y \cdots Y}_{n}$

where

$$X=\underbrace{\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}}_{n+1} ; \quad Y=\underbrace{\begin{bmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}}_{n+1}$$

Theorem. *There is a unique D-matrix $H(x)$ of type A_{n+1}. It follows that the matrix A_1 has the form*

$$A_1 = \operatorname{diag}\left(\frac{1}{n+2}, \cdots, \frac{n+1}{n+2}\right). \tag{2}$$

The proof of this theorem relies on the following lemma:

Lemma. There is unique matrix $A(x)$, $x=(x_1, \cdots, x_n)$ of order $n+1$ satisfying the following conditions

a) $A(0)=X, \quad \frac{\partial A}{\partial x_1}(0)=Y^n,$

b) $\frac{\partial [A]^k}{\partial x_j}=\frac{\partial [A]^j}{\partial x_k}, \quad k, j=1, \cdots, n$

where $[A]^k \equiv \underbrace{A \cdots A}_{k}$, $k=1, \cdots, n$,

c) If $A(x)=(a_{ij}(x))_{i,j=1,\dots,n+1}$ then the functions $a_{ij}(x)$ are quasihomogeneous with the weights

$$\sigma(a_{ij}(x))=\frac{i+1-j}{n+2}, \quad i, j=1, \cdots, n+1$$

where σ is a weight function

$$\sigma(x_k)\equiv\frac{n+2-k}{n+2}, \quad k=1, \cdots, n.$$

§3. The Proofs

We shall prove first the theorem by the lemma.
We set

$$B_k=[A]^k, \quad k=1, \cdots, n.$$

Then we have

$$[B_i, B_j]=0, \quad i, j=1, \cdots, n.$$

and

$$\frac{\partial B_k(x)}{\partial x_j}=\frac{\partial B_j(x)}{\partial x_k}, \quad j,\ k=1,\ \cdots,\ n.$$

Therefore, there is a unique matrix $H(x)$ such that

$$H(0)=0,$$

$$\frac{\partial H(x)}{\partial x_i}=B_i(x), \quad i=1,\ \cdots,\ n.$$

(See [1], for example).

On the other hand, by the properties of quasihomogeneous functions we have the equality

$$[A_1,\ H(x)]+H(x)=\frac{1}{n+2}\sum_{k=1}^{n}(n+2-k)\cdot x_k\cdot\frac{\partial H}{\partial x_k},$$

where A_1 is given by (2).

Thus, it is easy to check that the conditions (1) are satisfied and we have the theorem.

Proof of the lemma. We denote by $m=(x_1,\ \cdots,\ x_n)$ the maximal ideal of $\boldsymbol{C}\{x\}$. We will construct the matrices $A_p(x)$, $p=0,\ 1,\ \cdots$ such that

$$A_p(x)=A_{p+1}(x)\ \mathrm{mod}\ (m^{p+1}),\ p=0,\ 1,\ \cdots$$

and they have to respect the conditions

$$\frac{\partial[A_p(x)]^k}{\partial x_j}=\frac{\partial[A_p(x)]^j}{\partial x_k}\ \mathrm{mod}\ (m^p), \quad j,k=1,\cdots,n. \tag{b}_p$$

Starting with $A_0(x)=X$, we define

$$A_1(x)=X+\sum_{i=1}^{n}x_{n+1-i}\cdot Y^i.$$

It is easy to show that the matrix $A_1(x)$ satisfies the condition

$$\frac{\partial[A_1(x)]^k}{\partial x_j}=\frac{\partial[A_1(x)]^j}{\partial x_k} \quad \mathrm{mod}\,(m), \quad j, k=1, \cdots, n. \tag{(b)$_1$}$$

By the derivation, the left hand side of $(b)_1$ is equal to

$$\frac{\partial[A_1(x)]^k}{\partial x_j}=[A_1(x)]^{k-1}\frac{\partial A_1(x)}{\partial x_j}+[A_1(x)]^{k-2}\cdot\frac{\partial A_1(x)}{\partial x_j}\cdot A_1(x)$$
$$+\cdots+\frac{\partial A_1(x)}{\partial x_j}[A_1(x)]^{k-1}.$$

Remarking that

$$\frac{\partial A_1(x)}{\partial x_j}=Y^{n+1-j}$$

$$A_1(x) \text{ mod } (m)=X$$

it follows that

$$\frac{\partial[A_1(x)]^k}{\partial x_j} \text{ mod } (m)=X^{k-1}Y^{n+1-j}+X^{k-2}Y^{n+1-j}X+\cdots+Y^{n+1-j}X^{k-1}.$$

By a similar argument, we obtain

$$\frac{\partial[A_1(x)]^j}{\partial x_k} \text{ mod } (m)=X^{j-1}Y^{n+1-k}+X^{j-2}Y^{n+1-k}X+\cdots+Y^{n+1-k}X^{j-1}.$$

Let us remark that the matrices X, Y verify the following equalities

$$Y^l=X^{n-l-1}Y^n+X^{n-l-2}Y^nX+\cdots+Y^nX^{n-l-1}, \quad l=1, \cdots, n.$$

Setting $l=n-k+1$ and $l=n-j+1$, respectively, we are thus reduced to showing that

$$\begin{aligned}
&X^{k-1}(X^{j-1}Y^n+X^{j-2}Y^nX+\cdots+Y^nX^{j-1})\\
&+X^{k-2}(\qquad\text{--------------}\qquad)\,X\\
&\vdots\\
&+\qquad(\qquad\text{--------------}\qquad)\,X^{k-1}
\end{aligned}$$

$$
\begin{aligned}
&= X^{j-1}(X^{k-1}Y^n + X^{k-2}Y^nX + \cdots + Y^nX^{k-1}) \\
&+ X^{j-2}(\qquad \text{--------------} \qquad) X \\
&\vdots \\
&+ \quad (\qquad \text{--------------} \qquad) X^{j-1}
\end{aligned}
$$

It is easy to see that the rows of the left hand side are equal to the columns of the right side. We hence have the desired equalities and with it the conditions $(\mathrm{b})_1$.

Furthermore, we can show that the matrix $A_1(x)$ is uniquely determined by the conditions (a), $(\mathrm{b})_1$.

Suppose now we have already constructed $A_p(x)$, $p \geqslant 1$ satisfying $(\mathrm{b})_p$ and $(\mathrm{c})_p$, where the conditions $(\mathrm{c})_p$ mean that if $A_p(x) = (a_{ij}^p(x))_{i,j=1,\cdots,n+1}$ then

$$\sigma(a_{ij}^p(x)) = (i-j+1)/(n+2).$$

By the conditions $(\mathrm{c})_p$, we have

$$\frac{\partial A_p(x)}{\partial x_1} = Y^n$$

and therefore we obtain the decompositions

$$[A_p(x)]^k = x_1 \cdot P_p^k(x) + Q_p^k(x), \qquad k=1, \cdots, n$$

where $P_p^k(x)$, Q_p^k, $k=1, \cdots, n$ are matrices not depending on x_1. Thus, we can write

$$
\begin{aligned}
P_p^k(x) &= \frac{\partial [A_p(x)]^k}{\partial x_1} \\
&= [A_p(x)]^{k-1}\frac{\partial A_p(x)}{\partial x_1} + \cdots + Y\frac{\partial A_p(x)}{\partial x_1}[A_p(x)]^{k-1} \\
&= [A_p(x)]^{k-1}Y^n + \cdots + Y^n[A_p(x)]^{k-1}.
\end{aligned}
$$

On the other hand, derivating both sides of $(\mathrm{b})_p$ by x_1 we have

$$\frac{\partial}{\partial x_1}\frac{\partial [A_p(x)]^k}{\partial x_j} = \frac{\partial}{\partial x_1}\frac{\partial [A_p(x)]^j}{\partial x_k} \mod (m^p), \qquad j, k=1, \cdots, n$$

which implies

$$\frac{\partial}{\partial x_j}\frac{\partial[A_p(x)]^k}{\partial x_1}=\frac{\partial}{\partial x_k}\frac{\partial[A_p(x)]^j}{\partial x_1} \bmod (m^p), \qquad j,\ k=1,\ \cdots,\ n$$

that is

$$\frac{\partial}{\partial x_j}P_p^k(x)=\frac{\partial}{\partial x_k}P_p^j(x) \bmod (m^p), \qquad j,\ k=1,\ \cdots,\ n$$

From these equalities we can construct the matrix $A_{p+1}(x)$ such that

$$\frac{\partial}{\partial x_k}A_{p+1}(x)=P_p^k(x) \bmod (m^{p+1}), \qquad k=1,\ \cdots,\ n.$$

By induction on p, we can easily show that

$$A_p(x)=A_{p+1}(x) \bmod (m^{p+1}).$$

Next, we shall prove that the matrix $A_{p+1}(x)$ satisfies $(\mathrm{b})_{p+1}$:

$$\frac{\partial[A_{p+1}(x)]^k}{\partial x_j}=\frac{\partial[A_{p+1}(x)]^j}{\partial x_k} \bmod (m^{p+1}), \qquad j,\ k=1,\ \cdots,\ n$$

Note that

$$\frac{\partial[A_{p+1}(x)]^k}{\partial x_j}=[A_{p+1}(x)]^{k-1}\frac{\partial A_{p+1}(x)}{\partial x_j} + \cdots + \frac{\partial A_{p+1}(x)}{\partial x_j}[A_{p+1}(x)]^{k-1}.$$

It follows by the above arguments that:

$$\begin{aligned}&\frac{\partial[A_{p+1}(x)]^k}{\partial x_j} \bmod (m^{p+1})\\ &=[A_p(x)]^{k-1}P_p^j(x)+[A_p(x)]^{k-2}P_p^j(x)A_p(x)+\cdots+P_p^j(x)[A_p(x)]^{k-1}\\ &=[A_p(x)]^{k-1}([A_p(x)]^{j-1}Y^n+\cdots+Y^n[A_p(x)]^{j-1})\end{aligned}$$

$$
\begin{aligned}
&+[A_p(x)]^{k-2}(\qquad \text{-------------} \qquad) A_p(x)\\
&\vdots\\
&+ \qquad (\qquad \text{-------------} \qquad) [A_p(x)]^{k-1}.
\end{aligned}
$$

Similarly we prove that

$$
\begin{aligned}
&\frac{\partial[A_{p+1}(x)]^j}{\partial x_k} \bmod (m^{p+1})\\
&\quad = [A_p(x)]^{j-1}([A_p(x)]^{k-1}Y^n + \cdots + Y^n[A_p(x)]^{k-1})\\
&\quad + [A_p(x)]^{j-2}(\qquad \text{-------------} \qquad) A_p(x)\\
&\quad \vdots\\
&\quad + \qquad (\qquad \text{-------------} \qquad) [A_p(x)]^{j-1}.
\end{aligned}
$$

Then the conditions $(b)_{p+1}$ follow by the same remarks as in the proof of $(b)_1$. And we obtain the conditions $(c)_{p+1}$ by a simple calculation of the quasihomogeneous functions. Now we observe that $A_p(x) = A_n(x)$, $p \geqslant n$ by the condition $(c)_p$.

Finally we define $A(x) = A_n(x)$, then the matrix $A(x)$ satisfies the conditions of the lemma, so that the proof of lemma is complete.

§4. The Explicit Form of the Matrix $A(x)$

In this section we shall find the matrix $A(x)$ in the form

$$A(x) = X + \sum_{j=1}^{n} w_j(x)\, Y^j,$$

where $(w_j(x))$, $j=1, \cdots, n$ are quasihomogeneous functions with the weights

$$\sigma(w_j(x)) = j+1, \qquad j=1, \cdots, n.$$

First, we denote by

$$B_k(x) \equiv \frac{\partial A(x)}{\partial x_k} \left(= \frac{\partial [A(x)]^k}{\partial x_1}\right), \qquad k=1, \cdots, n.$$

Suppose that

$$B_k(x) = \sum_{j=1}^{n} a_j^k(x)\, Y^{n-k+j}, \qquad k=1, \cdots, n$$

where $a_1^k(x) = 1$.

We will calculate a_j^k as the functions of $w_1, \cdots, w_n$.

For $k=1$, it follows from the conditions (c) of the lemma that

$$B_1(x)=\frac{\partial A(x)}{\partial x_1}=Y^n.$$

For convenience we use the notations

$$E_{n-j+1}=\begin{pmatrix} & 0 & \\ \underbrace{0 \quad 1}_{j} & & \end{pmatrix}, \qquad j=1, \cdots, n$$

$$Y_j^* \equiv Y^j - E_j, \qquad j=1, \cdots, n.$$

We will show by induction on k the following expressions

$$(*)_k \quad \begin{cases} Y^n[A(x)]^k = \sum_{j=1}^{k} a_j^k(x) E_{n-k+j-1} + b_k(x) E_n, \\ \text{where} \qquad b_k(x) = \sum_{j=1}^{k-1} a_j^{k-1}(x)\, w_{k-j} \end{cases}$$

$$(**)_{k+1} \quad \begin{cases} a_j^{k+1}(x) = a_j^k(x) + \sum_{i=1}^{j-2} a_i^k(x)\, w_{j-i-1}, \qquad j=1, \cdots, k \\ a_{k+1}^{k+1}(x) = b_k(x) + \sum_{i=1}^{k-1} a_i^k(x)\, w_{k-i}. \end{cases}$$

To prove $(*)_k \Rightarrow (**)_{k+1}$ we compute

$$A(x)\cdot B_k(x) = \sum_{j=1}^{k} a_j^k(x)\, Y_{n-k+j-1}^* + \sum_{j=2}^{k} \Big(\sum_{i=1}^{j-1} a_i^k(x) w_{j-i}\Big) Y^{n-k+j}.$$

And then we have

$$B_{k+1}(x) = \frac{\partial [A(x)]^{k+1}}{\partial x_1} = \frac{\partial A(x)}{\partial x_1}\, [A(x)]^k + A(x)\, \frac{\partial [A(x)]^k}{\partial x_1}$$

$$= Y^n[A(x)]^k + A(x)\cdot B_k(x)$$

$$\overset{(*)_k}{=}\sum_{j=1}^{k} a_j^k(x)\, Y^{n-k+j-1} + b_k(x)E_n + \sum_{j=2}^{k}(\sum_{i=1}^{j-1} a_i^k w_{j-i})\, Y^{n-k+j}$$

$$= \sum_{j=1}^{n} (a_j^k(x) + \sum_{i=1}^{j-2} a_i^k(x)\; w_{j-i-1})\, Y^{n-k+j-1} + (b_k(x) + \sum_{i=1}^{k-1} a_i^k(x)\; w_{k-i})\, Y^n.$$

Thus we obtain $(**)_{k+1}$. Let us now prove the implication $(**)_{k+1} \Rightarrow (*)_{k+1}$.

By the straightforward computations it follows that

$$Y^n[A(x)]^{k+1} = (Y^n[A(x)]^k) \cdot A(x)$$

$$= (\sum_{j=1}^{k} a_j^k(x)\; E_{n-k+j-1} + b_k(x)E_n) \cdot (X + \sum_{j=1}^{n} \omega_j Y^j)$$

$$= \sum_{j=1}^{k} (a_j^k(x) + \sum_{i=1}^{j-2} a_i^k(x)\; w_{j-i-1})E_{n-k+j-2}$$

$$+ (b_k(x) + \sum_{i=1}^{k-1} a_i^k(x)\; w_{k-i})E_{n-1} + (\sum_{j=1}^{k} a_j^k(x)\; w_{k-j+1})E_n$$

$$\overset{(**)^{k+1}}{=} \sum_{j=1}^{k+1} a_j^{k+1}\; E_{n-k+j-2} + b_{k+1}(x)E_n,$$

where we put $b_{k+1}(x) = \sum_{j=1}^{k} a_j^k(x)\; w_{k-j+1}$. This proves $(*)_{k+1}$.

We also observe that $a_1^k(x) = 1$ and $a_2^k(x) = 0$ for all $k = 1, \cdots, n$.

Next, by the recurrence formulae $(**)_k$ we will calculate explicitly a_j^k as the functions of $w_1, \cdots, w_n$.

Proposition. The function a_j^k is given by the formula

$$a_j^k \sum_{\alpha} R_j^k\,(\alpha) \cdot w^\alpha, \qquad 3 \leqslant j \leqslant k$$

where

$$w^\alpha = w_1^{\alpha_1} \cdots w_n^{\alpha_n}$$

$$R_j^k(\alpha) = \frac{k-j+2}{\alpha!} \prod_{s=0}^{r-2} (k-s)$$

and

$$\begin{cases}\alpha_1+\cdots+\alpha_n=r,\\ 2\alpha_1+\cdots+(n+1)\alpha_n=j-1.\end{cases}$$

Note here that

$$\prod_{s=0}^{r-2}(k-s)=\begin{cases}1 & \text{if } r=1\\ k & \text{if } r=2.\end{cases}$$

Proof. Let us prove by the induction on k. For $j=1$ we have $a_1^k=1$, so that $a_1^1=1$. For $j\leqslant k$, it is enough to show that for $\alpha=(\alpha_1, \cdots, \alpha_n)$ satisfying

$$\alpha_1+\cdots+\alpha_n=r$$

$$2\alpha_1+\cdots+(n+1)\alpha_n=j-1$$

we have

$$R_j^{k+1}(\alpha)=R_j^k(\alpha)+\sum_{i=1}^n R_{j-i-1}^k(\alpha^*(i)), \qquad j\leqslant k$$

and

$$R_{k+1}^{k+1}(\alpha)=\sum_{i=1}^{k-1} R_{k-i}^{k-i}(\alpha^*(i)) \;+\; R_{k-i}^k\,(\alpha^*(i)))$$

where we put

$$\alpha^*(i)=(\alpha_1, \cdots, \alpha_i^{-1}, \cdots, \alpha_n), \qquad i=1, \cdots, n$$

$$R_j^k(\alpha^*(i))=0, \text{ if } \alpha_i=0.$$

By the induction hypothesis we have

$$R_j^k(\alpha)=\frac{1}{\alpha!}(k-j+2)\prod_{s=0}^{r-2}(k-s)$$

$$R_{j-i-1}^k\,(\alpha^*(i))=\frac{\alpha_i}{\alpha!}(k-j+i+3)\prod_{s=0}^{r-3}(k-s), \qquad i=1, \cdots, n$$

$$R^{k-1}_{k-i}\ (\alpha^*(i))=\frac{\alpha_i}{\alpha!}\ (i+1)\prod_{s=0}^{r-3}\ (k-1-s), \qquad i=1, \cdots, n$$

Then, by a direct computation, one obtains

$$R^{k+1}_j\ (\alpha)=\frac{k-j+3}{\alpha!}\prod_{s=0}^{r-2}\ (k+1-s), \qquad j\leqslant k+1.$$

This proves the proposition.

Remark. By the definition we have

$$\frac{\partial w_m}{\partial x_{n+j-m}}=a^{n+j-m}_j, \quad j\leqslant m$$

By remarking that a^{n+j-m}_j depends only $w_1, \cdots, w_{j-1}$ (the consequence of the proposition). Then by the induction the functions $w_m(x)$, $m=1, \cdots, n$ are uniquely determined with initial value $w_m(0)=0$.

We make a conjecture.

Conjecture. The function $w_k(x)$ is given by

$$w_k(x)=\Sigma Q_k(\alpha)x^\alpha$$

where

$$x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$$

$$Q_k(\alpha)=\frac{1}{\alpha!}\prod_{j=1}^{r-1}\ (j(n+2)-k)$$

with

$$\begin{cases}\alpha_1+\cdots+\alpha_n=r\\ 2\alpha_n+3\alpha_{n-1}+\cdots+(n+1)\alpha_1=k+1\end{cases}$$

(Note that for $r=1$ we put $Q_k(\alpha)=1$).

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