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On Non–real Eigenvalues of Schrödinger Operators in a Weighted Hilbert Space

Dedicated to Professor Teruo Ikebe on his sixtieth birthday

By

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§ 1. Introduction

In this paper we are concerned with the Schrödinger operator

$$L = -\sum_{j,k=1}^{n}\left(\frac{\partial}{\partial x_j} + ib_j(x)\right) a_{jk}(x)\left(\frac{\partial}{\partial x_k} + ib_k(x)\right) + V(x), \tag{1.1}$$

where $i = \sqrt{-1}$ and $b_j(x)$, $V(x)$ are real valued functions. For simplicity, we consider only the case $a_{jk}(x) \equiv \delta_{jk}$ in the present section. Our aim is to study the existence or the non-existence of eigenfunctions $u(x)$ of L such that

$$\begin{aligned} &Lu = zu, \\ &u \in H_{2,loc}, \quad (1+|x|^2)^{-s/2}u(x) \in L^2(\boldsymbol{R}^n) \end{aligned} \tag{1.2}$$

for a non–real number z and $s \geq 0$. Under our conditions stated in § 2 the non-existence of non-trivial solutions of $u(x)$ satisfying (1.2) is equivalent to the following Proposition (P);

(P) *the set* $\{(L-\bar{z})u \mid u \in C_0^\infty(\boldsymbol{R}^n)\}$ *is dense in the weighted space*

$$L_s^2 = \{f \mid (1+|x|^2)^{s/2} f \in L^2(\boldsymbol{R}^n)\}$$

(see Remark 2.2 in § 2), where $\bar{z}$ is the complex conjugate of z. It is well–known that the Proposition (P) with $s=0$ and $z = \pm\sqrt{-1}$ is equivalent

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to the essential self–adjointness of the symmetric operator L restricted to $C_0^\infty(\boldsymbol{R}^n)$, which is the case under our condition so that the equation (1.2) with $s \leq 0$ has no non–trivial solutions. But the problem with $s>0$ is not trivial.

Ikebe-Saitō [2] proves the Proposition (P) in order to study the limiting absorption method for Schrödinger operators with short range potentials. The second writer [5] treats exploding potentials, i.e., those satisfying the Stummel condition and

$$(1.3) \qquad V(x) \to -\infty \quad \text{as} \quad r=|x| \to \infty,$$

$$V(x)=O(r^\alpha) \quad \text{as} \quad r \to \infty,$$

for some $\alpha<2$ and shows (P) and the limiting absorption principle. But he has encountered a difficulty to prove (P) for $\alpha=2$ in (1.3)

We note here that the non–existence of eigenfunctions of (1.2) is also shown by Simon [4], Theorem 2.2 for a class of potentials, roughly speaking, bounded from below.

Our interest in the present paper is to investigate the case that $V(x)$ satisfies

$$V(x) \geq -C|x|^2, \ |x| \geq R$$

for some positive C and R. We shall show the non-existence of eigenfunctions of (1.2) under the condition

$$(1.4) \qquad |\text{Im } z| > 2s\sqrt{C},$$

where Im z is the imaginary part of z. This condition (1.4) is optimal in the sense stated in §4.

Our assertion will be stated in §2 and will be proved in §3. The optimality of the condition (1.4) will be shown in §4 by giving some examples. We shall consider these problems not only for $[a_{jk}(x)] \equiv I$ (identity matrix) but also for positive symmetric matrices $[a_{jk}(x)]$ for $x \in \boldsymbol{R}^n$.

§2. Assumptions and Results

The following condition (A) on L of (1.1) is assumed throughout this paper:

(A–1) The $n \times n$ matrices $A(x)=[a_{jk}(x)]$ are real symmetric and positive definite for $x \in \boldsymbol{R}^n$ with C^2 components.

(A–2) Let $a^+(x)$ denote the greatest eigenvalue of $A(x)$ and put

$$a^*(r)=\max_{|x|=r} a^+(x).$$

Then

$$\rho(r) \equiv \int_0^r \frac{ds}{\sqrt{a^*(s)}} \to \infty, \qquad r \to \infty.$$

(If $a_{jk}(x) \equiv \delta_{jk}$, $a^+(x)=a^*(r) \equiv 1$ and $\rho(r)=r$.)

(A–3) Each component $b_j(x)$ of the vector potential is a real valued C^1 function.

(A–4) The potential $V(x)$ is real valued and can be decomposed into $V(x)=V_1(x)+V_2(x)$ as follows;
i) $V_1(x)$ satisffes

$$V_1(x) \geq -C_1\ \rho(|x|)^2 - C_2 \qquad (x \in \boldsymbol{R}^n)$$

for some positive constants C_1 and C_2, and belongs to the Stummel class $Q_{\alpha,loc}$ for a positive number $\alpha<1$ i.e.,

$$M[V_1](x) \equiv \left[\int_{|x-y| \leq 1} \frac{|V_1(y)|^2}{|x-y|^{n-4+\alpha}}\, dy \right]^{1/2}$$

is a locally bounded function of $x \in \boldsymbol{R}^n$.
ii) $V_2(x)$ also belongs to $Q_{\alpha,loc}$ and satisfies the following conditions (A–5) and (A–6).

(A–5) Let $a^-(x)$ be the least eigenvalue of $A(x)$ and let $V^-(x)=\max(-V(x), 0)$. Then, there exists a positive constant C_3 such that

$$M[V_2^-](x) \leq C_3\ a^-(x) \qquad (x \in \boldsymbol{R}^n).$$

(A–6) $M[V_2^-](x)=o(\rho(r)^2)$, $r=|x| \to \infty$.

For a real number s and the function $\rho(r)$ above, $L^2_{s,\rho}$ denotes the weighted Hilbert space consisting of all functions $f(x)$ satisfying

$$\|f\|_{s,\rho}=\left(\int_{\boldsymbol{R}^n} [1+\rho(r)^2]^s\ |f(x)|^2\, dx \right)^{1/2} < \infty$$

with the inner product

$$(f, g)_{s,\rho} = \int_{R^n} [1 + \rho(r)^2]^s f(x) \overline{g(x)}\, dx.$$

When $\rho(r) = r$, $L^2_{s,\rho}$ is written simply as L^2_s. $L^2_{0,\rho} = L^2_0$ is the usual Hilbert space $L^2(R^n)$.

$H_2 = H_2(R^n)$ is the Sobolev space of all functions $u \in L^2(R^n)$ such that $\Delta u \in L^2(R^n)$. $H_{2,loc}$ is the class of all locally H_2 functions.

Theorem 2.1. *Assume the condition (A) and let $z \in C$ and $s \geq 0$ satisfy*

(2.1) $$|\mathrm{Im}\ z| > 2s\sqrt{C_1}$$

Then, there exist no non–trivial solutions $u \in L^2_{-s,\rho} \cap H_{2,loc}$ satisfying

(2.2) $$Lu = zu.$$

This theorem will be proved in the next section.

Remark 2.2. Under the condition (A), the set

$$(L - \bar{z})C_0^\infty = \{(L - \bar{z})f \mid f \in C_0^\infty(R^n)\}$$

is dense in $L^2_{s,\rho}$ for non–real z, if and only if (2.2) has no non–trivial solutions in $L^2_{-s,\rho} \cap H_{2,loc}$. This fact can be obtained by the same argument as in Ikebe–Saitō [2], Lemma 1.10.

Remark 2.3 If $V_1(x)$ satisfies

$$V_1(x) = o(\rho(r)^2), \qquad r \to \infty,$$

then the constant C_1 in (A.4) can be taken arbitrarily small, and the assumption (2.1) of Theorem 2.1 is valid for every non–real z.

§3. Proof of Theorem 2.1

We start with the following Lemma.

Lemma 3.1. *Let $f(x)$ and $g(x)$ be real valued C^1-functions on $\boldsymbol{R}^n$ and $u(x)$ be an $H_{2,loc}$ function. Assume that $f(x)$ has compact support. Then we have*

$$\int_{\boldsymbol{R}^n} f^2g^2(Lu)\bar{u}\,dx=\int_{\boldsymbol{R}^n}\{f^2g^2[\langle ADu,\overline{Du}\rangle+(V_1+V_2)|u|^2]$$

$$+2fg[f\langle A\nabla g,Du\rangle+g\langle A\nabla f,Du\rangle]\bar{u}\}dx,$$

where

$$\langle\vec{a}\ ,\vec{b}\rangle=\sum_{j=1}^{n}a_jb_j,\qquad \langle A\vec{a}\ ,\vec{b}\rangle=\sum_{j,k=1}^{n}a_{jk}(x)a_kb_j$$

$$\text{for } \vec{a}\ =(a_1,a_2,\cdots,a_n),\qquad \vec{b}=(b_1,b_2,\cdots,b_n),$$

$$\nabla\ =\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\cdots,\frac{\partial}{\partial x_n}\right)$$

$$D\ =(D_1,D_2,\cdots,D_n),\qquad D_j=\frac{\partial}{\partial x_j}+ib_j(x)$$

$$\overline{Du}=(\overline{D_1u},\overline{D_2u},\cdots,\overline{D_nu}).$$

The above Lemma can be proved directly by integration by parts.

Lemma 3.2. *Assume the condition* (A), *and let* $z\in\boldsymbol{C}$, $s\geq 0$ and $u\in L^2_{-s,\rho}\cap H_{2,loc}$ *be a solution of* (2.2). *Then, for any* $C_1'>C_1$ *there exists a positive constant* m_0 *such that*

$$\int_{\boldsymbol{R}^n}(m_0+\rho(r)^2)^{-s-1}\langle ADu,\overline{Du}\rangle\,dx\leq C_1'\int_{\boldsymbol{R}^n}(m_0+\rho(r)^2)^{-s}|u|^2dx.$$

Proof. Take a C^∞ function φ on $\boldsymbol{R}$ such that

(3.1) $\qquad \varphi(t)=1\ (t\leq 1),\ \varphi(t)=0\ (t\geq 2)\quad$ and $\quad 0\leq\varphi(t)\leq 1\ (1\leq t\leq 2)$

and put

(3.2) $\qquad f_R(x)=\varphi(\rho(|x|)-R+1).$

Making use of Lemma 3.1 with $f=f_R$ and

$$g=(m+\rho(r)^2)^{-(s+1)/2} \qquad (m>0)$$

for the integral

$$\int_{R^n} f_R^2 g^2 (Lu)\bar{u}\,dx = z\int_{R^n} f_R^2 g^2 |u|^2 dx$$

and taking the real part, we have

$$\begin{aligned}
(3.3)\qquad & \int_{R^n}(m+\rho(r)^2)^{-s-1} f_R^2 \langle ADu, \overline{Du}\rangle\, dx \\
&= -\int_{R^n}\{f_R^2 g^2 (V_1+V_2-\mathrm{Re}\, z)\,|u|^2 \\
&\quad + 2f_R^2 g\,\mathrm{Re}\,[\langle A\nabla g, Du\rangle \bar{u}] + 2f_R g^2 \mathrm{Re}\,[\langle A\nabla f_R, Du\rangle \bar{u}]\}dx \\
&\le \int_{R^n} f_R^2 g^2(-V_1+|\mathrm{Re}\, z|)\,|u|^2 dx + \int_{R^n} f_R^2 g^2 V_2^- \,|u|^2 dx \\
&\quad + \int_{R^n} f_R^2 g^2 |u|^2 dx + \int_{R^n} f_R^2 \,|\langle A\nabla g, Du\rangle|\,^2 dx \\
&\quad + \eta\int_{R^n} f_R^2 g^2 |\langle A\nabla f_R, Du\rangle^2 dx + \frac{1}{\eta}\int_{R^n} g^2|u|^2 dx \\
&\equiv I_1+I_2+I_3+I_4+I_5+I_6
\end{aligned}$$

for any $\eta>0$. We shall estimate $I_1 \sim I_6$ as follows.

The condition (A.4) gives

$$\begin{aligned}
(3.4)\qquad I_1 &= \int_{R^n} f_R^2 g^2(-V_1+|\mathrm{Re}\ z|)\,|u|^2 dx \\
&\le \int_{R^n}(m+\rho(r)^2)^{-s-1}(C_1\rho(r)^2+C_2+|\mathrm{Re}\ z|)\,|u|^2 dx \\
&\le \left(C_1+\frac{C_2'}{m}\right)\int_{R^n}(m+\rho(r)^2)^{-s}|u|^2 dx,
\end{aligned}$$

where we put $C_2' = C_2 + |\mathrm{Re}\, z|$. Ikebe–Kato [1], Lemma 2 and (A.4) yield

$$(3.5)\qquad \begin{aligned} I_2 &= \int_{\boldsymbol{R}^n} f_R^2 g^2\, V_2^-(x)\,|u|^2 dx \\ &\leq C_0\, \eta^{\alpha/2} \int_{\boldsymbol{R}^n} M[V_2^-]\, \{f_R^2 g^2 |Du|^2 + |\nabla(f_R g)|^2\, |u|^2 + \eta^{-2} f_R^2 g^2 |u|^2\}\, dx \end{aligned}$$

for any $0<\eta<1$ and some positive constant C_0 independent of η and R (see also Jörgens [3], § 3). The above inequality (3.5), the condition (A–5), and

$$M[V_2^-](x) \leq \eta^2 \rho(r)^2 + C(\eta)$$

for any $\eta>0$ and some constant $C(\eta)>0$, which is a consequence of (A–6), imply

$$(3.6)\qquad \begin{aligned} I_2 \leq C_0 \eta^{\alpha/2} \int_{\boldsymbol{R}^n} &\{C_3 f_R^2 g^2 \langle ADu, \overline{Du}\rangle + 2C_3 f_R^2 \langle A\nabla g, \nabla g\rangle\, |u|^2 \\ &+ 2C_3 g^2 \langle A\nabla f_R, \nabla f_R\rangle\, |u|^2 + \rho(r)^2 f_R^2 g^2 |u|^2 + \frac{C(\eta)}{\eta^2} f_R^2 g^2 |u|^2\}\, dx \end{aligned}$$

for any $0<\eta<1$. In view of the definition (3.2) of f_R we have

$$(3.7)\qquad \begin{aligned} \langle A\nabla f_R, \nabla f_R\rangle &= \langle A\hat{x}, \hat{x}\rangle\, \rho'(r)^2 \varphi'(\rho(r) - R + 1)^2 \\ &= \langle A\hat{x}, \hat{x}\rangle\, a^*(r)^{-1} \varphi'(\rho(r) - R + 1)^2 \\ &\leq \varphi'(\rho(r) - R + 1)^2 \leq C_4 \\ &\qquad (\hat{x} = x/|x|) \end{aligned}$$

for a positive constant C_4, and similarly

$$(3.8)\qquad \begin{aligned} \langle A\nabla g, \nabla g\rangle &= \langle A\hat{x}, \hat{x}\rangle\, a^*(r)^{-1} (s+1)^2 \rho(r)^2 (m + \rho(r)^2)^{-s-3} \\ &\leq (s+1)^2 (m + \rho(r)^2)^{-s-2} \end{aligned}$$

$$\leq \frac{(s+1)^2}{m}(m+\rho(r)^2)^{-s-1}$$

$$\leq \frac{(s+1)^2}{m^2}(m+\rho(r)^2)^{-s}$$

for any $m>0$. Gathering (3.6), (3.7) and (3.8) gives

$$I_2 \leq C_0\ \eta^{\alpha/2}\int_{\mathbf{R}^n}(m+\rho(r)^2)^{-s}\left[C_3 f_R^2(m+\rho(r)^2)^{-1}\langle ADu, \overline{Du}\rangle + 2C_3\left(\frac{s+1}{m}\right)^2|u|^2 + \frac{2C_3C_4}{m}|u|^2 + |u|^2 + \frac{C(\eta)}{m\eta^2}|u|^2\right]dx \tag{3.9}$$

for any positive m and $\eta<1$. For I_3 and I_6 we have

$$\begin{aligned} I_3+I_6 &= \int_{\mathbf{R}^n}\{f_R^2 g^2|u|^2 + \frac{g^2}{\eta}|u|^2\}\,dx \\ &\leq \frac{1}{m}\left(1+\frac{1}{\eta}\right)\int_{\mathbf{R}^n}(m+\rho(r)^2)^{-s}|u|^2\,dx. \end{aligned} \tag{3.10}$$

In view of Schwarz' inequality

$$\langle A\vec{\xi}, \vec{\eta}\rangle^2 \leq \langle A\vec{\xi}, \vec{\xi}\rangle\langle A\vec{\eta}, \vec{\eta}\rangle \qquad (\vec{\xi}, \vec{\eta} \in \mathbf{C}^n)$$

the estimates (3.7) and (3.8) give

$$\begin{aligned} I_4+I_5 &= \int_{\mathbf{R}^n}\{f_R^2|\langle A\nabla g, Du\rangle|^2 + \eta f_R^2 g^2|\langle A\nabla f_R, Du\rangle|^2\}\,dx \\ &\leq \int_{\mathbf{R}^n} f_R^2\langle ADu, \overline{Du}\rangle[\langle A\nabla g, \nabla g\rangle + \eta g^2\langle A\nabla f_R, \nabla f_R\rangle]\,dx \\ &\leq \left[\frac{(s+1)^2}{m} + \eta C_4\right]\int_{\mathbf{R}^n} f_R^2(m+\rho(r)^2)^{-s-1}\langle ADu, \overline{Du}\rangle\,dx \end{aligned} \tag{3.11}$$

for $\eta, m>0$. Now, we obtain from (3.3), (3.4), (3.9), (3.10), and (3.11) that

$$\left[1-C_0C_3\eta^{\alpha/2}-\frac{(s+1)^2}{m}-C_4\eta\right]\times\int_{\boldsymbol{R}^n} f_R^2(m+\rho(r)^2)^{-s-1}\langle ADu,\overline{Du}\rangle\,dx$$

$$\leq\left\{C_1+\frac{C_2'}{m}+C_0\eta^{\alpha/2}\left[2C_3\,(\frac{s+1}{m})^2+\frac{2C_3C_4}{m}+1+\frac{C(\eta)}{m\eta^2}\right]\right.$$

$$\left.+\frac{1}{m}\left(1+\frac{1}{\eta}\right)\right\}\int_{\boldsymbol{R}^n}(m+\rho(r)^2)^{-s}|u|^2\,dx,$$

which shows that for any given $C_1'>C_1$ one can find a sufficiently small $\eta>0$ and a sufficiently large m_0 so that

$$\int_{\boldsymbol{R}^n} f_R^2(m_0+\rho(r)^2)^{-s-1}\langle ADu,\overline{Du}\rangle^2\,dx \tag{3.12}$$

$$\leq C_1'\int_{\boldsymbol{R}^n}(m_0+\rho(r)^2)^{-s}|u|^2\,dx.$$

The condition (A.2) implies

$$f_R(x)=\varphi(\rho(r)-R+1)\ \to\ 1 \qquad (R\ \to\ \infty)$$

which, $u\in L^2_{-s,\rho}$ and (3.12) give Lemma 3.2 by means of Fatou's lemma. Q.E.D.

Proof of Theorem 2.1. Let $u\in L^2_{-s,\rho}\cap H_{2,loc}$ be a solution of $Lu=zu$ such that $|\mathrm{Im}\ z|>2s\sqrt{C_1}$. We shall show $u=0$ below. In view of (2.1) there exists a positive number $C_1'>C_1$ such that

$$|\mathrm{Im}\ z|>2s\sqrt{C_1'}. \tag{3.13}$$

Choose a sufficiently large number m_0 as in Lemma 3.2. Lemma 3.1 is used again with

$$f=f_R=\varphi\left(\frac{\sqrt{m_0+\rho(r)^2}}{R}\right),\qquad g=(m_0+\rho(r)^2)^{-s/2}$$

(φ is the function in the proof of Lemma 3.2) and gives, by taking the imaginary part of the integral,

$$(3.14)\qquad (\mathrm{Im}\ z)\int_{\boldsymbol{R}^n} f_R^2 g^2 |u|^2\,dx = \mathrm{Im}\int f_R^2 g^2 (Lu)\,\bar{u}\,dx$$

$$= 2\int_{\boldsymbol{R}^n} f_R g\,\{f_R\,\mathrm{Im}\,[\langle A\nabla g, Du\rangle \bar{u}] + g\,\mathrm{Im}\,\langle A\nabla f_R, Du\rangle \bar{u}]\}\,dx.$$

We have

$$(3.15)\qquad \langle A\nabla f_R, \nabla f_R\rangle$$

$$= \langle A\hat{x}, \hat{x}\rangle\,\rho'(r)^2 \rho(r)^2 (m_0+\rho(r)^2)^{-1}\,\varphi'(\sqrt{m_0+\rho(r)^2}/R)^2/R^2$$

$$\le R^{-2}\varphi'(\sqrt{m_0+\rho(r)^2}/R)^2$$

and

$$(3.16)\qquad \langle A\nabla g, \nabla g\rangle = \langle A\hat{x}, \hat{x}\rangle\, s^2 \rho'(r)^2 \rho(r)^2 (m_0+\rho(r)^2)^{-s-2}$$

$$\le s^2 (m_0+\rho(r)^2)^{-s-1}.$$

It follows from (3.14), (3.15), (3.16) and Schwarz' inequality that

$$(3.17)\qquad |\mathrm{Im}\ z|\int_{\boldsymbol{R}^n} f_R^2 (m_0+\rho(r)^2)^{-s} |u|^2\,dx$$

$$\le 2\left[\int_{\boldsymbol{R}^n} (m_0+\rho(r)^2)^{-s-1}\langle ADu, \overline{Du}\rangle\,dx\right]^{1/2}$$

$$\times\left[s\left(\int_{\boldsymbol{R}^n} (m_0+\rho(r)^2)^{-s} |u|^2\,dx\right)^{1/2}\right.$$

$$\left.+\left(\int_{\boldsymbol{R}^n} (m_0+\rho(r)^2)^{-s+1} R^{-2}\,\varphi'(\sqrt{m_0+\rho(r)^2}/R)^2 |u|^2\,dx\right)^{1/2}\right].$$

Since the support of $\varphi'(t)$ is included in the closed interval $[1,2]$ and $|\varphi'(t)|^2 \le C_4$, the last integral in (3.17) is estimated by

$$4C_4\int_{R<\sqrt{m_0+\rho^2}<2R} (m_0+\rho(r))^{-s} |u|^2\,dx,$$

which converges to 0 as $R \to \infty$ by means of (A–2) and $u\in L^2_{-s,\rho}$. Thus, letting R tend to infinity in (3.17), we have

$$|\mathrm{Im}\ z|\int_{\boldsymbol{R}^n}(m_0+\rho(r)^2)^{-s}|u|^2\,dx$$

$$\leq 2s\left[\int_{\boldsymbol{R}^n}(m_0+\rho(r)^2)^{-s-1}\langle ADu,\overline{Du}\rangle\,dx\right]^{1/2}$$

$$\times\left[\int_{\boldsymbol{R}^n}(m_0+\rho(r)^2)^{-s}|u|^2\,dx\right]^{1/2}$$

and, by Lemma 3.2

$$\left(|\mathrm{Im}\ z|-2s\sqrt{C_1'}\right)\int_{\boldsymbol{R}^n}(m_0+\rho(r)^2)^{-s}|u|^2\,dx\leq 0,$$

which and (3.13) show $u=0$. Q.E.D.

Remark 3.3. In the above proof the condition

(3.18) $V_1\in Q_{\alpha,loc}$

is used only to assure $V_1|u|^2\in L^1_{loc}$ for $u\in H_{2,loc}$. Therefore, we can adopt in (A–4)

$$V_1|u|^2\in L^1_{loc}\qquad\text{for any}\quad u\in H_{2,loc}$$

instead of (3.18).

§ 4. Counterexamples

In this section we show examples of L with a non–trivial solution $u(x)\in L^2_{-s,\rho}\cap C^\infty$ of

$$Lu=zu$$

for $|\mathrm{Im}\ z|<2s\sqrt{C_1}$.

The following example in $\boldsymbol{R}^1$ is simple.

Example 4.1. Let $L=-\dfrac{d^2}{dx^2}-x^2$ and $u(x)=\exp(-ix^2/2)$.

Then $u(x)$ satisfies

$$Lu = -u'' - x^2 u = iu,$$

$$u \in L^2_{-s} \cap C^\infty \text{ for any } s > \frac{1}{2}.$$

This is an example which gives $C_1 = 1$, $\mathrm{Im}\ z = 1$ and

$$|\mathrm{Im}\ z| - 2s\sqrt{C_1} = 1 - 2s < 0$$

for any $s > 1/2$.

The following example is a generalization of the above.

Example 4.2. Let $t < \frac{n}{4}$, $\delta \neq 0$ and put

$$u(x) = \frac{\exp(-i\delta r^2/2)}{(1+r^2)^t} \qquad (x \in \boldsymbol{R}^n,\ r = |x|)$$

Then, $u(x)$ satisfies

(4.1) $$u \in L^2_{-s} \cap C^\infty$$

for any real s such that

(4.2) $$s > \frac{n-4t}{2}$$

and

$$Lu = \left[-\sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} + ix_j g(r) \right)^2 + V_1(x) + V_2(x) \right] u = i\delta(n-4t)u,$$

where

(4.3) $$g(r) = 4\delta t r^{-n}(1+r^2)^{2t} \int_0^r \frac{s^{n-1}}{(1+s^2)^{2t+1}} ds,$$

$$V_1(x) = -\delta^2 r^2 + 2\delta g r^2 - g^2 r^2,$$

$$V_2(x)=\frac{f''}{f}+\frac{(n-1)f'}{rf}$$

where

$$f(r)=(1+r^2)^{-t}.$$

The identity (4.3) can be shown by a straightforward calculation. We should remark here that we choose $g(r)$ as a solution of

$$rg'+(n-4t)g+\frac{4t}{1+r^2}g=\frac{4\delta t}{1+r^2}.$$

One can check immediately that $V_2(x)\in C^\infty(\boldsymbol{R}^n)$,

$$V_2(x)=O(r^{-2}) \text{ as } r \to \infty$$

and $g(|x|)\in C^1(\boldsymbol{R}^n)$ satisfy

$$g(r)=0(r^{-\gamma}) \text{ as } r \to \infty$$

for some $\gamma>0$, which yields that for any positive number $C_1<\delta^2$ there exists a positive number C_2 such that

$$V_1(x)\geq -C_1r^2-C_2.$$

Thus, it turns out that the assumptions of Theorem 2.1 are satisfied with $z=\delta i(n-4t)$ except (2.1). In fact, we have

$$|\text{Im } z|-2s\sqrt{C_1}=|\delta|(n-4t)-2s\sqrt{C_1}<0$$

in view of (4.2) and by taking C_1 sufficiently close to δ^2.

The above example implies that for any real number $s>0$, $C_1>0$ and $\varepsilon\neq 0$ such that

$$|\varepsilon|<2s\sqrt{C_1}$$

there exist L and $u\neq 0$ satisfying the condition (A), (4.1) and $Lu=i\varepsilon u$.

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