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Existence of the Greek Letter Elements in the Stable Homotopy Groups of $E(n)_*$-Localized Spheres

By

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Abstract

Let $E_2^{*,*}(X)$ denote the E_2-term of the Adams-Novikov spectral sequence converging to the homotopy groups of X. We can define the Greek letter elements in $E_2^{*,*}(L_nS^0)$ in the same way as those in $E_2^{*,*}(S^0)$. We show that these of $E_2^{*,*}(L_nS^0)$ are permanent cycles, though we know little for $E_2^{*,*}(S^0)$.

§ 1. Introduction

Let BP denote the Brown-Peterson spectrum whose homotopy groups are $\boldsymbol{Z}_{(p)}[v_1, v_2, \cdots]$ over the Hazewinkel generators v_n's. By the insight of Toda's construction of $V(3)$ [T], Z. Yosimura and the first named author constructed a BP version of Toda's construction [SY]. In [SY], it is studied the L_n-local spectrum YJ whose BP_*-homology is $v_n^{-1}BP_*/W(J)$ for an invariant regular sequence of length only n. Here n is a positive integer and L_n is the Bousfield localization with respect to the spectrum $v_n^{-1}BP$, which is the same one with respect to the Johnson-Wilson spectrum $E(n)$. Here $\pi_*(E(n)) = \boldsymbol{Z}_{(p)}[v_1, \cdots, v_n, v_n^{-1}]$.

In this paper, we show that [SY] is applied for the case that the length of the sequence J is less than n, and obtain

Theorem 1.1. *Let n and m be integers such that $0 \leq m \leq n$ and $n^2 + n < 2p$. Then there exists a spectrum YJ such that*

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$$E(n)_*(YJ) \cong E(n)_*/(J_m)$$

as $E(n)_*$-module.

F. Adams introduced the first Greek letter elements in the stable homotopy groups of spheres, and then the second and the third Greek letter elements were defined by H. Toda, L. Smith, S. Oka and so on. In their paper [MRW], H. Miller, D. Ravenel and S. Wilson defined the general Greek letter elements in the E_2-term of the Adams-Novikov spectral sequence based on BP converging to the stable homotopy groups of spheres. For the Greek letter elements given earlier in the stable homotopy, the corresponding elements in the E_2-term agree with the newly defined Greek letter elements in the E_2-term. Toda and et al's Greek letter elements are based on the spectrum XJ such that $BP_*(XJ)=BP_*/(J)$.

In most cases, it is still unknown whether or not those Greek letter elements in the E_2-term survive to E_∞, though we know that some of each n-th Greek letter elements are permanent cycles by a result of [HS].

Here we show that the Greek letter elements are all permanent cycles after replacing the sphere spectrum by the L_n-localized spheres under some condition on n. We define the Greek letter elements $g(J_m)$ in the E_2-term for an invariant regular sequence J_m with $m \leq n$ by the image of a composition of connecting homomorphisms associated to J_m (see (5.1)). We denote the corresponding homotopy element by $\alpha(J_m)$ (see (5.4)). As Theorem 1.1, we have an analogous spectrum YJ to XJ if we replace BP by $E(n)$. So in the same way as the construction of Toda and et al's Greek letter elements, we obtain

Theorem 1.2. *Let n and m be non-negative integers with $m \leq n$ and $n^2+n \leq 2p$. Then we have the homotopy element $\alpha(J_m)$ in $\pi_*(L_nS^0)$.*

We prove Theorem 1.1 in §4 by a similar fashion to that of [SY]. In other words, we construct a BP-based Adams tower from the E_1-term, which is shown to give an Adams-type spectral sequence converging to the homotopy groups of the $E(n)$-localized spectrum by M. Hopkins and D. Ravenel, recently. Unhappily, $BP_*(L_nX) \neq v_n^{-1}BP_*(X)$ for general X, and we cannot apply the theorem of [SY] without any change.

This paper is organized as follows: In § 2, we study the ring structure of a telescope of a ring spectrum related to BP. We adjust the results of [SY] to fit to our case in §3 and prove Theorem 1.1 and Theorem 1.2 in §4 and §5, respectively.

§2. *BP*-Hopf Module Spectra $v_n^{-1}BPJ$

Let E be a ring spectrum and A an E-module spectrum with structure map $\phi: E\wedge A\to A$. We call a spectrum A *E-quasi associative ring spectrum* ([JY]), if there is a multiplication $\mu: A\wedge A\to A$ satisfying the following three yconditions:

$$\mu(\phi\wedge 1)=\phi(1\wedge\mu): E\wedge A\wedge A\to A,$$

$$\mu(\phi\wedge 1)(T\wedge 1)=\mu(1\wedge\phi): A\wedge E\wedge A\to A, \text{ and}$$

$$\mu(1\wedge\phi)(1\wedge T)=\phi T(\mu\wedge 1): A\wedge A\wedge E\to A.$$

We fix a prime number p and consider the Brown-Peterson spectrum BP. BP is a ring spectrum, whose homotopy and BP_*-homology groups are

$$BP_*=\pi_*(BP)=\boldsymbol{Z}_{(p)}[v_1, v_2, \cdots], \text{ and}$$

$$BP_*(BP)=\pi_*(BP\wedge BP)=BP_*[t_1, t_2, \cdots],$$

where the degrees of the generators are $\deg v_n=\deg t_n=2p^n-2$, which we denote q_n. We denote the unit map and the multiplication of BP by $i: S^0\to BP$ and $\mu: BP\wedge BP\to BP$, respectively. We also denote $\overline{BP}$ for the cofiber of the unit map i.

Throughout this paper we fix a positive integer

$$n.$$

Let A be a BP-module spectrum with structure map $\nu: BP\wedge A\to A$. Then we have a self map $\dot{v}_n: A\to\Sigma^{-q_n}A$ as in the above theorem induced from the element $v_n\in BP_*$ whose degree is $q_n=2p^n-2$. Suppose that any composition of $\dot{v}_n$ is not trivial. We then have the homotopy colimit (telescope) $TA=\underrightarrow{\lim}_i\,\Sigma^{-iq_n}A$, and $\pi_*(TA)=v_n^{-1}\pi_*(A)$. We consider the following condition:

(2.1) *A consists of cells e_α such that* $\deg e_\alpha\equiv r \bmod 2p-2$ *with* $0\le r\le 2p-4$.

We have the next lemma whose proof is due to T. Ohkawa.

Lemma 2.2. *Let A be a BP-module spectrum and B a spectrum. Suppose that $\pi_r(B)=0$ unless $r\equiv 0 \bmod 2p-2$, and A satisfies the condition (2.1). Then we have an isomorphism*

$$[TA,B]\cong \varprojlim [A,B].$$

Proof. Consider the iq_n-skeleton $A^{(i)}$ of A. Then we obtain a map $(v_n)^{(i)}\colon A^{(i)}\to\Sigma^{-q_n}A^{(i+1)}$ induced from the map $\dot{v}_n\colon A\to\Sigma^{-q_n}A$. In other words, it satisfies that $\dot{v}_n\iota_i=\iota_{i+1}(v_n)^{(i)}$ for the inclusion $\iota_j\colon A^{(j)}\to A$. Let $T'A$ denote the homotopy colimit of the maps $(v_n)^{(i)}$. Then we have a map $\iota\colon T'A\to TA$ given by the inclusions ι_j, which induces an isomorphism $\iota_*\colon \pi_*(T'A)\to\pi_*(TA)$ at each dimension. Thus the map ι is a homotopy equivalence.

We have Milnor's exact sequence

$$0\to\varprojlim{}^1\left[\Sigma^{1-iq_n}A^{(i)},B\right]\to[T'A,B]\xrightarrow{i'^*}\varprojlim\left[\Sigma^{-iq_n}A^{(i)},B\right]\to 0.$$

We notice that the group $\left[\Sigma^{1-iq_n}A^{(i)},B\right]$ is a subquotient of a direct sum of homotopy groups $\pi_r(B)$ with deg $r\not\equiv 0 \bmod 2p-2$ by the assumption on A. Thus we have $\left[\Sigma^{1-iq_n}A^{(i)},B\right]=0$ by the assumption on B and we see that the $\varprojlim^1$ group is null. Therefore we have the commutative diagram

$$\begin{array}{ccc} [T'A,B] & \xrightarrow{i'^*} & \varprojlim\left[\Sigma^{-iq_n}A^{(i)},B\right] \\ \uparrow{\scriptstyle \iota^*} & & \uparrow{\scriptstyle \varprojlim \iota_i^*} \\ [TA,B] & \xrightarrow{i^*} & \varprojlim\left[\Sigma^{-iq_n}A,B\right], \end{array}$$

in which i'^* and ι^* are isomorphisms. So i^* is a monomorphism. On the other hand, it is an epimorphism by the Milnor's exact sequence on TA. q.e.d.

Let A be a BP-module spectrum with structure map $v\colon BP\wedge A\to A$. We call A *BP-Hopf module spectrum* if it is provided with BP-module map $\eta\colon A\to BP\wedge A$ such that $v\eta=1$ and $(1\wedge\eta)\eta=(1\wedge i\wedge 1)\eta$ for the identity map 1 (*cf.* [SY]).

It is known (*cf.* [R2]) that the pair (BP_*, $BP_*(BP)$) is a Hopf algebroid with structure maps: the right and the left units η_R, η_L: $BP_* \to BP_*(BP)$, the coproduct Δ: $BP_*(BP) \to BP_*(BP) \otimes_{BP_*} BP_*(BP)$ and the counit ε: $BP_*(BP) \to BP_*$, which are induced from $i \wedge 1$, $1 \wedge i$: $BP \to BP \wedge BP$, $1 \wedge i \wedge 1$: $BP \wedge BP \to BP \wedge BP \wedge BP$ and μ: $BP \wedge BP \to BP$, respectively. Let J: $a_0, a_1, \cdots$ be a sequence of BP_* of finite length, and denote J_m for a subsequence $a_0, a_1, \cdots, a_{m-1}$ of length m. (J_m) denotes the ideal of BP_* generated by each entry a_k of the sequence J_m, and especially, we put $(J_0)=0$. We call the sequence J_m *invariant* if $\eta_R(a_k) \equiv a_k \bmod (J_k)$ for $0 \le k < m$, and *regular* if the multiplication by a_k: $BP_*/(J_k) \to BP_*/(J_k)$ is a monomorphism for $0 \le k < m$.

Let J: $a_0, a_1, \cdots, a_{n-1}$ be an invariant regular sequence of BP_* of length n. Then there exists an BP-module spectrum BPJ with structure ν_J: $BP \wedge BPJ \to BPJ$, whose homotopy groups are given by $\pi_*(BPJ) = BP_*/(J)$ ([B]). For subsequences J_m of J, these BPJ_m's fit into the cofiber sequence

$$(2.3) \qquad \Sigma^{\deg a_m} BPJ_m \xrightarrow{\dot{a}_m} BPJ_m \xrightarrow{j_m} BPJ_{m+1} \xrightarrow{k_m} \Sigma^{\deg a_m + 1} BPJ_m,$$

where $\dot{a}_m$ denotes the composition $\nu_{J_m}(a_m \wedge 1)$ for $a_m \in \pi_*(BP)$. Furthermore, it is a BP-quasi associative ring spectrum and also is a BP-Hopf module spectrum if $n < 2p-2$, by [SY; Proposition 1.2].

By [BP] (*cf.* [P]), BP consists of cells of degree 0 mod $2p-2$. So using the cofiber sequences (2.3) inductively, we see that BPJ_m satisfies the condition (2.1) if $m \le 2p-4$. Put $A(m) = TBPJ_m$ for $m \le n$. Then we have the following from Lemma 2.2, by an easy argument on limit and colimit.

Theorem 2.4. *Let m be a positive integer with $m \le \min\{p-2, n\}$, and J_m is a subsequence of an invariant regular sequence J of length n. Then we have a BP-quasi associative ring spectrum $A(m)$ whose homotopy groups are $v_n^{-1}BP_*/(J_m)$, which is also a BP-Hopf module spectrum.*

Corollary 2.5. *For each integer n, $v_n^{-1}BP$ is a ring spectrum.*

§3. Geometric Resolution and k-Adams Tower

First recall [SY] some facts and notations on BP-Hopf module spectra. Define the structure maps by $\nu = \mu \wedge 1$ and $\eta = 1 \wedge i \wedge 1$, and we have a BP-Hopf module spectrum $BP \wedge X$ for any spectrum X. We call a BP-Hopf module spectrum A an *extended* BP-Hopf module spectrum if A is homotopic to $BP\wedge B$ for a BP-module spectrum B as BP-Hopf module spectrum. Note that an extended BP-Hopf module spectrum A gives rise to an extended $BP_*(BP)$-comodule $A_*(X)=\pi_*(A\wedge X)$ for any spectrum X. Consider a sequence

$$\cdots \to A_0 \xrightarrow{k_0} A_1 \to \cdots \to A_j \xrightarrow{k_j} A_{j+1} \to \cdots.$$

We call the sequence a *complex* if the composition k_jk_{j-1} is null for each j. Furthermore suppose that it is a sequence of E-module spectra for a ring spectrum E. We call it *split* if there exist E-module maps s_j: $A_{j+1} \to A_j$ such that $s_js_{j+1}=0$ and $k_js_j+s_{j+1}k_{j+1}=1$.

Now recall [SY] the definition of geometric resolution and k-Adams tower. Let E and M be BP-Hopf module spectra. We call a complex $G=\{G_j, d_j\colon G_j \to G_{j+1}\}_{j\geq 0}$ an *E-geometric resolution* over M if it satisfies the following three conditions:

(i) There exists a BP-Hopf module map $\delta\colon M\to E\wedge G_0$ with $(1\wedge d_0)\delta=0$.

(ii) The complex

$$*\to M\xrightarrow{\delta} E\wedge G_0\xrightarrow{1\wedge d_0} E\wedge G_1\to\cdots\to E\wedge G_k\xrightarrow{1\wedge d_k} E\wedge G_{k+1}\to\cdots$$

splits as a sequence of BP-module spectra.

(iii) The entry $E\wedge G_k$ of the complex is an extended BP-Hopf module spectrum for each $k\geq 0$.

Then we have

(3.1) [SY; Theorem 3.3] *There exists a BP-geometric resolution* $G=\{G_k=\overline{BP}^k\wedge E, d_k\colon G_k\to G_{k+1}\}$ *over a BP-Hopf module spectrum E.*

Consider a set $X(k)=\{X_j, a_j, b_j, c_j\}_{1\leq j\leq k}$ of spectra X_j and maps $a_j\colon X_j\to\Sigma X_{j-1}$, $b_j\colon X_j\to W_{j+1}$ and $c_j\colon W_j\to X_j$, and an E-geometric resolution $G=\{G_j, d_j\}_{j\geq 0}$. Then we call $X(k)$ *k–Adams tower* over G if it satisfies the following properties:

(i) $X_{j-1}\xrightarrow{b_{j-1}} W_j \xrightarrow{c_j} X_j \xrightarrow{a_j} \Sigma X_{j-1}$ is a cofiber sequence, and

(ii) $d_{j-1}=b_{j-1}c_{j-1}$ and $d_mb_{m-1}=0$ for each j with $1\leq j\leq m$,

where $X_0=W_0$, $b_0=d_0$, $c_0=1$ and $1\leq m\leq\infty$. We note that k-Adams tower is called k-factorized system in [SY]. If there exists a spectrum YJ such that $BP_*(YJ)=v_n^{-1}BP_*/(J)$, then the so-called BP-Adams tower for YJ is an ∞-Adams tower in our sense. So this naming seems reasonable.

We call BPJ-module spectrum M *quasi-associative* if $\varphi_M(v_J\wedge 1)=v_M(1\wedge\varphi_M)$: $BP\wedge BPJ\wedge M\to M$ and $v_M(1\wedge\varphi_M)(T\wedge 1)=\varphi_M(1\wedge v_M)$: $BPJ\wedge{}_{BP}\wedge M\to M$, where φ_M denotes the BPJ-module structure map and $v_M=\varphi_M(j\wedge 1)$ for the map $j: BP\to BPJ$.

Let F be a quasi-associative BPJ-module spectrum and $G=\{G_k, d_k\}$ a BPJ-geometric resolution over M. As we remarked in [SY], the Kronecker product κ: $[\Sigma^t G_k, F]\to \mathrm{Hom}^t_{BPJ_*}(BPJ_*(G_k), F_*)$ is an isomorphism for each $k\geq 0$ if $BPJ_*(G_k)$ is BPJ_*-free.

(3.2) [SY; Theorem 4.6] *Suppose that the length of J is less than $p-1$. Let $G=\{G_k, d_k\}_{k\geq 0}$ be a BP-geometric resolution over M such that M and G_k for $k\geq 0$, are quasi-associative BPJ-module spectra with $\pi_*(G_k)$ BPJ_*-free and $\pi_*(G_k)=0$ unless $*\equiv 0$ mod $2p-2$. If $\mathrm{Ext}^{m+2,m+t}_{BP_*(BP)}(M_*, M_*)=0$ for all $m\geq 1$ and $t\in\Lambda_J$, then G admits an ∞-Adams tower $X(\infty)$. Moreover, its ∞-Adams tower is uniquely given if $\mathrm{Ext}^{m+1,m+t}_{BP_*(BP)}(M_*, M_*)=0$ for all $m\geq 1$ and $t\in\Lambda_J$.*

Here,

$$\Lambda_J=\left\{\textstyle\sum_j \varepsilon_j d_j \mid d_j=\deg a_j+1 \text{ and } \varepsilon_j\in\{0,1\}\right\}$$

for an invariant regular sequence $J: a_0, \cdots, a_{n-1}$.

By virtue of Lemma 2.2, the Kronecker product κ stays an isomorphism even if we replace G_k by $TG_k=\varinjlim_{v_n} G_k$, under the assumption that F satisfies (2.1). By taking the telescope T, we obtain a BP-geometric resolution $TG=\{TG_k=\overline{BP}^k\wedge TE, d_k: TG_k\to TG_{k+1}\}$ over a BP-Hopf module spectrum TE from (3.1). Now put $E=BPJ_m$, and we see that it satisfies the condition of (3.2), and we rewrite (3.1–2) to fit to our case:

Theorem 3.3 *Assume that $0\leq m\leq n<p-1$. If $\mathrm{Ext}^{k+2,k+t}_{BP_*(BP)}(BP_*, v_n^{-1}BP_*/(J_m))=0$ for all $k>0$ and $t\in\Lambda_{J_m}$, then we have an ∞-Adams tower $X(\infty)$ over the BP-geometric resolution $TG=\{TG_k, d_k\}$ obtained from* (3.2) *over $v_n^{-1}BPJ_m$. Furthermore, it is uniquely given if $\mathrm{Ext}^{k+1,k+t}_{BP_*(BP)}(BP_*, v_n^{-1}BP_*/(J_m))=0$ for all $k>0$ and $t\in\Lambda_{J_m}$.*

Here the uniqueness follows from

Theorem 3.4. *Consider a map $f=\{f_n\}:G\to H$ of BP-geometric resolutions over $v_n^{-1}BPJ$. If $\mathrm{Ext}_{BP_*(BP)}^{k+1,k+d+t}(BP_*, v_n^{-1}BP_*/(J_m))=0$ for all $k>0$ and $t\in\Lambda_J$, then we have a map $\bar{f}=\{\bar{f}_n, f_n\}:X(\infty)\to Y(\infty)$ of ∞-Adams towers.*

Proof. Recall [SY] that

(3.5) [SY; Proposition 4.3] *Let $G=\{G_k, d_k\}_{k\geq 0}$ and $G'=\{G'_k, d'_k\}_{k\geq 0}$ be E-geometric resolutions over M and N, respectively, and $X(m)=\{X_j\}_{1\leq j\leq m}$ and $X'(m)=\{X'_j\}_{1\leq j\leq m}$ be their m-Adams towers. Given a map $g:G\to G'$ of complexes, there exists a map $f(m):X(m)\to X'(m)$ of m-Adams towers if $[\Sigma G_k, G'_{k+2}]=0$ and the sequence $[\Sigma X_{k-1}, G'_k]\to[\Sigma X_{k-1}, G'_{k+1}]\to[\Sigma X_{k-1}, G'_{k+2}]$ are exact for all k with $1\leq k<m$.*

Then the theorem follows immediately from (3.5), since we see by Lemma 2.2 that the hypothesis of the theorem satisfy the conditions of (3.5). q.e.d.

§4. Spectrum whose $v_n^{-1}BP_*$-Homology is $v_n^{-1}BP_*/(J_m)$

For $m\leq n$, we have the change of rings theorem

$$\mathrm{Ext}_{BP_*(BP)}^{s,t}(BP_*, v_n^{-1}BP_*/(J_m))=\mathrm{Ext}_{E(n)_*(E(n))}^{s,t}(E(n)_*, E(n)_*/(J_m)),$$

for the Johnson-Wilson spectrum $E(n)$ which has the coefficient ring $\mathbb{Z}_{(p)}[v_1,\cdots,v_n,v_n^{-1}]\subset v_n^{-1}BP_*$. The spectrum $E(n)$ is defined to be a spectrum that represents the homology theory $E(n)_*(X)=E(n)_*\otimes_{BP_*}BP_*(X)$. In [R1, (10.11)], Ravenel computed to show

$$\mathrm{Ext}_{E(n)_*(E(n))}^{s,*}(E(n)_*, E(n)_*/I_m)=0$$

for $s>n^2+n-m$ by using the chromatic spectral sequence. Now use the Filtration Theorem [JY; Theorem 1.16], and we have the following

Lemma 4.1. *If $0\leq m\leq n<p-1$, then*

$$\mathrm{Ext}_{BP_*(BP)}^{s,t}(BP_*, v_n^{-1}BP_*/(J_m))=0$$

for any integers s with $s>n^2+n-m$ and t.

Lemma 4.2. *Let $X(\infty)=(X_j, a_j, b_j, c_j)$ be an ∞-Adams tower over* *the geometric resolution* $TG=\{TG_k, d_k\}$ *over* $v_n^{-1}BPJ_*$ *in Theorem* 3.2. *Then*

$$v_n^{-1}BP_*(\varinjlim X_i) \cong v_n^{-1}BP_*/(J)$$

as BP_**-module. See* [R2; Definition 2.1.10] *for a definition of* $\varinjlim X_i$.

Proof. Applying BP_*-homology to the tower $X(\infty)$, we obtain an exact couple, which gives an Adams-type spectral sequence. By the definition of geometric resolution, we see that the E_2-term is given by $E_2^{s,*}=0$ for $s>0$ and $E_2^{0,*}=v_n^{-1}BP_*/(J)$. Therefore this spectral sequence collapses and converges to $BP_*(\varinjlim X_i)$. q.e.d.

Now applying Lemmas 4.1–2 to Theorem 3.3, we get

Theorem 4.3. *Let* J_m *be an invariant regular sequence of length* m. *If* $n^2+n<2p$, *then there exists a spectrum* YJ_m *such that*

$$v_n^{-1}BP_*(YJ_m) \cong v_n^{-1}BP_*/(J_m)$$

as BP_**-module.*

§5. The Greek Letter Elements

Let $E(n)$ be the Johnson-Wilson spectrum with $E(n)_*=\boldsymbol{Z}_{(p)}[v_1, \cdots, v_n, v_n^{-1}]$ and it gives rise to a homology theory $E(n)_*(X)=E(n)_*\otimes_{BP_*}BP_*(X)$ for any spectrum X. In the same way as [MRW], we define the Greek letter elements in the E_2-term of the Adams-Novikov spectral sequence computing $\pi_*(L_nS^0)$. Let J_m: $a_0, \cdots, a_{m-1}$ be an invariant regular sequence of $E(n)_*$ with $m\leq n$. Then a_m gives a cycle of $\mathrm{Ext}^{0,*}_{E(n)_*(E(n))}(E(n)_*, E(n)_*/(J_m))$ for J_m. We use here an abbreviation: $H^{s,t}M=\mathrm{Ext}^{s,t}_{E(n)_*(E(n))}(E(n)_*, M)$ for an $E(n)_*(E(n))$-comodule M. Using this abbreviation, we have $a_m\in H^{0,*}E(n)_*/(J_m)$. For each k, the short exact sequence $0 \to E(n)_*/(J_k) \xrightarrow{a_k} E(n)_*/(J_k) \to E(n)_*/(J_{k+1}) \to 0$ gives the associated connecting homomorphism δ_k: $H^{s,*}E(n)_*/(J_{k+1}) \to H^{s+1,*}E(n)_*/(J_k)$. Now define an m-th Greek letter element $g(J_m)$ associating to the sequence J_m by

$$g(J_m)=\delta_0\cdots\delta_{m-1}(a_m)\in H^{m,*}E(n)_*. \tag{5.1}$$

We have shown that the existence of spectra YJ_m such that

$E(n)_*(YJ_m)=E(n)_*/(J_m)$ by Theorem 4.3, since $E(n)_*(YJ_m)=E(n)_*\otimes_{BP_*} BP_*(YJ_m)$.

Lemma 5.2. *Let m be an integer with $0\leq m\leq n$ and $n^2+n<2p$. Then there exists a self map $\alpha_m: YJ_m \to YJ_m$ whose cofiber is YJ_{m+1}, and so $E(n)_*(\alpha_m)=a_m$.*

Proof. By Theorem 3.4 and Lemma 4.1, the map $\dot{a}_m: TBPJ_m \to TBPJ_m$ induces the map between the Adams towers, and hence we have a map $\alpha_m: YJ_m \to YJ_m$ such that $BP_*(\alpha_m)=a_m$. q.e.d.

The condition on m and n indicates that the Adams-Novikov spectral sequence collapses, and so the cycle a_m in $H^{0,*}E(n)_*/(J_m)$ survives to YJ_{m*}. By Lemma 5.2, we have cofiber sequences

$$YJ_k \xrightarrow{\alpha_k} YJ_k \xrightarrow{i_k} YJ_{k+1} \xrightarrow{j_k} \Sigma YJ_k$$

for each $k<n$ with $E(n)_*(YJ_k)=E(n)_*/(J_k)$ and $E(n)_*(\alpha_k)=a_k$, which induces the exact sequence

$$0 \to E(n)_*/(J_k) \xrightarrow{a_k} E(n)_*/(J_k) \xrightarrow{i_{k*}} E(n)_*/(J_{k+1}) \to 0.$$

Now recall [JMWZ] the Geometric Boundary Theorem. Let E denotes a ring spectrum with $E_*(E)$ is flat over E_*. Then the E-Adams spectral sequence has the E_2-term

$$E_2^{*,*}(X)=\mathrm{Ext}^{*,*}_{E_*(E)}(E_*, E_*(X)).$$

The class $\bar{x}\in E_2^{t,*}(X)$ is said to converge to $x\in\pi_*(X)$ provided that

(i) $\bar{x}$ is a permanent cycle representing the class $\{\bar{x}\}\in E_\infty^{t,*}(X)$;

(ii) $x\in F^t\pi_*(X)$ for the Adams filtration F^t; and

(iii) The homomorphism $F^t\pi_*(X)/F^{t+1}\pi_*(X) \to E_\infty^{t,*}(X)$ sends the coset $x+F^{t+1}\pi_*(X)$ to $\{\bar{x}\}$.

(5.3) [R2; Theorem 2.3.4] *Let E be a ring spectrum with unit such that E_* is commutative and $E_*(E)$ is flat over E_*. Let $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} \Sigma W$ be a* *cofiber sequence of finite spectra with* $E_*(h)=0$. *If* $\bar{x}\in E_2^{t,*}(Y)$ *converges to* $x\in\pi_*(Y)$, *then* $\delta(\bar{x})$ *converges to* $h_*(x)\in\pi_*(W)$.

Here $\delta: E_2^{*,*}(Y) \to E_2^{*,*}(W)$ is the connecting homomorphism of Ext groups. By [H; Lemma 6.2], we see that this theorem is applied to our case. For an invariant regular sequence J_m, we define a homotopy element $\alpha(J_m)$ by

$$\alpha(J_m)=j_0j_1\cdots j_{m-2}\,a_{m-1}\,i_{m-2}\cdots i_1i_0\in\pi_*(L_nS^0). \tag{5.4}$$

Then by (5.3), we see that $g(J_m)$ converges to $\alpha(J_m)\in\pi_*(L_nS^0)$. In fact, $YJ_0=L_nS^0$ since the spectrum $\varprojlim X_i$ of ∞-Adams tower for $v_n^{-1}BP$ is homotopic to L_nS^0, which is shown by Hopkins and Ravenel.

Consider the localization map $\eta_n: S^0 \to L_nS^0$. Then comparing the E_2-terms of the Adams-Novikov spectral sequences leads us to the statement that the map $\eta_{n*}: \pi_*(S^0) \to \pi_*(L_nS^0)$ sends the Greek letter elements to the Greek letter elements. For example, $\eta_{n*}(\alpha_k)=\alpha(p, v_1^k)$ and $\eta_{n*}(\beta_k)=\alpha(p, v_1, v_2^k)$. Furthermore, $\pi_*(L_nS^0)$ contains more such elements than $\pi_*(S^0)$ does. For example, it contains $\alpha(p, v_1, v_2, v_3, v_4^k)$ that may be called δ_k.

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