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PBW Basis of Quantized Universal Enveloping Algebras

By

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§ 0. Introduction

In [4], Lusztig constructed PBW bases of U_q^+. Then, he introduced the canonical base of U_q^+ [6] in case of A, D, E type. His results can be reformulated in case of U_q^- as follows.

Let L be the sub $\boldsymbol{Z}[q]$-module of U_q^- generated by a PBW basis of U_q^-. This submodule is independent of the choice of the PBW basis. Let $\pi': L \to L/qL$ be the canonical projection. Then the image B of the PBW basis is a $\boldsymbol{Z}$-basis of L/qL and it is independent of the choice of the PBW bases. Let $-: U_q \to U_q$ be the $\boldsymbol{Q}$-algebra involution defined by $e_i \mapsto e_i$, $f_i \mapsto f_i$, $q^h \mapsto q^{-h}$, $q \mapsto q^{-1}$. Then π' induces a $\boldsymbol{Z}$-module isomorphism $\pi'': L \cap \bar{L} \to L/qL$, $\boldsymbol{B} = (\pi'')^{-1}(B)$ is a $\boldsymbol{Z}$-basis of $L \cap \bar{L}$ and $\boldsymbol{Z}[q]$-basis of L. Moreover each element of $\boldsymbol{B}$ is fixed by $-$. $\boldsymbol{B}$ is called the canonical base of U_q^-.

On the other hand, in [1] Kashiwara constructed the global crystal base of U_q^-. Let $(L(\infty), B(\infty))$ be the crystal base of U_q^- and let $U_{\boldsymbol{Q}}^-$ be the sub-$\boldsymbol{Q}[q, q^{-1}]$-algebra of U_q^- generated by $f_i^{(n)}$. Then $U_{\boldsymbol{Q}}^- \cap L(\infty) \cap L(\infty)^- \to L(\infty)/qL(\infty)$ is an isomorphism. Let G be the inverse of this isomorphism. Then $G(B(\infty))$ is a base of U_q^- and called the global crystal base of U_q^-.

In [7], Lusztig showed $\boldsymbol{B} = G(B(\infty))$ in the simply laced case.

In this paper, we show that the monomials of the root vectors form a base of U_q^- and they give a crystal base at $q=0$, when $\mathfrak{g}$ is an arbitrary finite dimensional semisimple Lie algebra.

In Section 1, we define the braid group action on the integrable U_q-module. Let M be an integrable U_q-module. Then we shall define the automorphism S_i of M as follows:

$$S_i v = \exp_{q_i^{-1}}(q_i^{-1} e_i t_i^{-1}) \exp_{q_i^{-1}}(-f_i) \exp_{q_i^{-1}}(q_i e_i t_i) q_i^{h_i(h_i+1)/2} v \qquad \text{for} \quad v \in M.$$

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Here $\exp_q(x)$ denotes the q-analogue of the exponential function $\sum_{k=0}^{\infty}(q^{k(k-1)/2}/[k]!)x^k$. The operator $q_i^{h_i(h_i+1)/2}$ sends u to $q_i^{m(m+1)/2}u$ if $t_iu=q_i^m u$. Here $q_i=q^{(\alpha_i,\alpha_i)}$ and $t_i=q^{(\alpha_i,\alpha_i)h_i}$. Since $q_i^{-1}e_it_i^{-1}$, f_i and $q_ie_it_i$ act on M in locally nilpotent way, S_i is well-defined. Moreover $\exp_q(x)\exp_{q^{-1}}(-x)=1$ implies that S_i is invertible. There exists a unique automorphism T_i of U_q such that $S_i(xv)=(T_ix)S_iv$ for $x\in U_q$ and $v\in M$. This automorphism T_i coincides with the automorphism T_i introduced by Lusztig [4] with a small modification. In Proposition 1.4.1 we shall show that $\{S_i\}$ satisfies the braid relation. In Section 2, we show Ker $e'_i=T_iU_q^-\cap U_q^-$. (e'_i is defined in 2.1.) This is the key of this paper. In Section 3, we shall give a relation of cystal base and the braid group action. Let P be an element of $T_i^{-1}U_q^-\cap U_q^-$. We assume that P belongs to $L(\infty)$ and $P \bmod qL(\infty)$ belongs to $B(\infty)$. Using the fact that Ker $e'_i=T_iU_q^-\cap U_q^-$, we show that T_iP belongs to $L(\infty)$ and $T_iP \bmod qL(\infty)$ belongs to $B(\infty)$. Thus $f_i^{(k)}T_iP$ belongs again to $L(\infty)$ and gives a crystal base at $q=0$. In Section 4, we introduce PBW basis $\{f^k;\, k=(k_1,\cdots,k_N)\in \boldsymbol{Z}_{\geq 0}^N\}$. Chosing a reduced expression $s_{i_1}\cdots s_{i_N}$ of the longest element of the Weyl group we define

$$f^k=f_{i_1}^{(k_1)}T_{i_1}(f_{i_2}^{(k_2)}T_{i_2}(\cdots f_{i_{N-1}}^{(k_{N-1})}T_{i_{N-1}}f_{i_N}^{(k_N)})\cdots).$$

By the consequence of Section 3 we show that f^k forms a base of $L(\infty)$ and $\{f^k \bmod qL(\infty)\}=B(\infty)$ when $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra (Main Theorem). This generalizes the result of Lusztig [7].

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§1. Braid Group Action on Integrable Modules

1.1. The operator Φ

We follow the notations in [1, 2, 3]. For example, $\mathfrak{g}$ is a symmetrizable Kac-Moody Lie algebra, $\{\alpha_i\}_{i\in I}$ is the set of simple roots, $\boldsymbol{P}$ is a weight lattice, U_q is the corresponding quantized universal enveloping algebra generated by e_i, f_i, $q^h(h\in \boldsymbol{P}^*)$, etc.

Let $U_q(sl_2)_i$ be the subalgebra of U_q generated by e_i, f_i, $t_i=q^{(\alpha_i,\alpha_i)h_i}$.

Introduce the $\boldsymbol{Q}(q)$-algebra anti-automorphism $*$ of U_q by

$$e_i^*=e_i,\qquad f_i^*=f_i,\qquad (q^h)^*=q^{-h}. \tag{1.1.1}$$

We define the $\boldsymbol{Q}(q)$-algebra homomorphism $\Phi: U_q\to \mathrm{End}(U_q)$ by

$$\Phi(x)(y)=\sum(S(x_{(2)}))^*y\,x_{(1)}^* \tag{1.1.2}$$

where

$$\Delta(x)=\sum x_{(1)}\otimes x_{(2)}.$$

Then we have

(1.1.3)
$$\Phi(e_i)(x)=t_i^{-1}[x, e_i]$$
$$\Phi(f_i)(x)=xf_i-f_it_ixt_i^{-1}$$
$$\Phi(t_i)(x)=t_ixt_i^{-1}$$

for $x\in U_q$.

Lemma 1.1.1. *For* $i\neq j$

(1.1.4)
$$\Phi(f_i^{(n)})(f_j)=\sum_{k=0}^{n}(-1)^{n-k}q_i^{(n-k)(-a_{ij}-n+1)}f_i^{(n-k)}f_jf_i^{(k)}.$$

Proof. Let A and B be the endomorphisms of U_q defined by $Ax=xf_i$ and $Bx=f_it_ixt_i^{-1}$. Then we have

$$\Phi(f_i)(x)=Ax-Bx$$

for $x\in U_q$. The operators A and B satisfy the commutation relation:

$$AB=q_i^2BA.$$

By the q-analogue of the binomial formula, we obtain

$$\Phi(f_i^n)=(A-B)^n$$
$$=\sum_{k=0}^{n}(-1)^{n-k}q_i^{-k(n-k)}\begin{bmatrix} n \\ k \end{bmatrix}_i A^kB^{n-k}.$$

$$\Phi(f_i^{(n)})(f_j)=\sum_{k=0}^{n}(-1)^{n-k}q_i^{(-n+1)(n-k)}f_i^{(n-k)}t_i^{n-k}f_jt_i^{-n+k}f_i^{(k)}$$
$$=\sum_{k=0}^{n}(-1)^{n-k}q_i^{(n-k)(-a_{ij}-n+1)}f_i^{(n-k)}f_jf_i^{(k)}.$$

Q. E. D.

For $n=1-a_{ij}$, the Serre relation implies

$$\Phi(f_i^{(1-a_{ij})})(f_j)=\sum_{k=0}^{-a_{ij}+1}(-1)^{-a_{ij}+1-k}f_i^{(-a_{ij}+1-k)}f_jf_i^{(k)}=0.$$

Along with $\Phi(e_i)f_j=0$, we conclude that f_j is $U_q(sl_2)_i$-finite and it is a highest weight vector. Here U_q is regarded as a $U_q(sl_2)_i$-module through Φ.

1.2. Definition of S_i

Let $V(l)\,(l\in \mathbf{Z}_{\geq 0})$ be the irreducible $U_q(sl_2)_i$-module of dimension $l+1$. Let us take a highest weight vector $u_0^{(l)}$ of $V(l)$. Then we have

(1.2.1)
$$e_iu_0^{(l)}=0,$$
$$t_iu_0^{(l)}=q_i^lu_0^{(l)}.$$

Let $u_k^{(l)}=f_i^{(k)}u_0^{(l)}$. Then we have

$$V(l)=\bigoplus_{k=0}^{l}\boldsymbol{Q}(q)u_k^{(l)}.$$

Next, we define the endomorphism S_i of the vector space $V(l)$ by

$$S_i v=\exp_{q_i^{-1}}(q_i^{-1}e_i t_i^{-1})\exp_{q_i^{-1}}(-f_i)\exp_{q_i^{-1}}(q_i e_i t_i)q_i^{h_i(h_i+1)/2}v \tag{1.2.2}$$

for $v\in V(l)$, where

$$\exp_q(X)=\sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}}{[k]!}X^k.$$

The operator $q_i^{h_i(h_i+1)/2}$ sends u to $q_i^{m(m+1)/2}u$ for a vector u with $t_i u=q_i^m u$. Since the action of e_i and f_i are nilpotent, this endomorphism is well defined.

For $x\in U_q$ and $n\geqq 0$, we set

$$x_i=\frac{x-x^{-1}}{q_i-q_i^{-1}}$$

$$\begin{Bmatrix} x \\ n \end{Bmatrix}_i=\frac{\prod_{k=1}^{n}(q_i^{1-k}x)_i}{[n]_i!}.$$

Hence we have

$$\begin{Bmatrix} q_i^m \\ n \end{Bmatrix}_i=\begin{cases} \begin{bmatrix} m \\ n \end{bmatrix}_i, & \text{for } m\geqq n\geqq 0 \\ 0, & \text{for } n>m\geqq 0 \\ (-1)^n\begin{bmatrix} n-1-m \\ n \end{bmatrix}_i, & \text{for } n\geqq 0>m \end{cases} \tag{1.2.3}$$

and

$$\begin{Bmatrix} x \\ n \end{Bmatrix}_i=(-1)^n\begin{Bmatrix} q_i^{n-1}x^{-1} \\ n \end{Bmatrix}_i. \tag{1.2.4}$$

Ordinary, $\begin{bmatrix} m \\ n \end{bmatrix}_i$ is defined for $m\geqq 0$, $n\geqq 0$. But we will extend $\begin{bmatrix} m \\ n \end{bmatrix}_i$ for $m\in\boldsymbol{Z}$, $n\geqq 0$. We set

$$\begin{bmatrix} m \\ n \end{bmatrix}_i=\begin{Bmatrix} q_i^m \\ n \end{Bmatrix}_i. \tag{1.2.5}$$

Then we have

$$\begin{bmatrix} m \\ n \end{bmatrix}_i=(-1)^n\begin{bmatrix} n-1-m \\ n \end{bmatrix}_i. \tag{1.2.6}$$

Proposition 1.2.1.

$$S_i u_k^{(l)}=(-1)^{l-k}q_i^{(l-k)(k+1)}u_{l-k}^{(l)} \tag{1.2.7}$$

for any $0\leqq k\leqq l$.

The first step is the next lemma.

Lemma 1.2.2. *For integers s, k, l with* $0 \leqq k, s \leqq l$, *we have*

$$\sum(-1)^b q^{(l-2k)(c-a)-c(b-c)+b-a(a-b+c)} \begin{bmatrix} l-k+c \\ c \end{bmatrix} \begin{bmatrix} l-s+a \\ b \end{bmatrix} \begin{bmatrix} s \\ a \end{bmatrix}$$
$$=(-1)^{l-k} q^{(l-k)(k+1)} \delta_{k,s}$$

where the sum ranges over non-negative integers a, b, c such that $s=l-k+a-b+c$.

Proof. We recall the next formula

$$\sum_{n=0}^{k} x^{-n} y^{k-n} \begin{Bmatrix} y \\ n \end{Bmatrix} \begin{Bmatrix} x \\ k-n \end{Bmatrix} = \begin{Bmatrix} xy \\ k \end{Bmatrix} \tag{1.2.8}$$

for $x, y \in U_q$. Then, (1.2.5) and (1.2.8) imply

$$\sum_{n=0}^{k} q^{-an+b(k-n)} \begin{bmatrix} b \\ n \end{bmatrix} \begin{bmatrix} a \\ k-n \end{bmatrix} = \begin{bmatrix} a+b \\ k \end{bmatrix}. \tag{1.2.9}$$

By (1.2.6) and (1.2.9), we have

$$\sum(-1)^c q^{-c(l-s+a)+(k-c)(-l+k-1)} \begin{bmatrix} l-k+c \\ c \end{bmatrix} \begin{bmatrix} l-s+a \\ k-c \end{bmatrix}$$
$$=\sum q^{-c(l-s+a)+(k-c)(-l+k-1)} \begin{bmatrix} -l+k-1 \\ c \end{bmatrix} \begin{bmatrix} l-s+a \\ k-c \end{bmatrix}$$
$$=\begin{bmatrix} k+a-s-1 \\ k \end{bmatrix}$$
$$=(-1)^k \begin{bmatrix} s-a \\ k \end{bmatrix}$$
$$=(-1)^k \begin{bmatrix} s-a \\ s-a-k \end{bmatrix}$$
$$=(-1)^{s-a} \begin{bmatrix} -k-1 \\ s-a-k \end{bmatrix}$$

and

$$\sum q^{s(s-a-k)+a(k+1)} \begin{bmatrix} -k-1 \\ s-a-k \end{bmatrix} \begin{bmatrix} s \\ a \end{bmatrix} = \begin{bmatrix} s-k-1 \\ s-k \end{bmatrix}$$
$$=\delta_{s,k}.$$

Therefore

$$\sum(-1)^{c+s-a} q^{-c(l-s+a)+(k-c)(-l+k-1)+s(s-a-k)+a(k+1)} \begin{bmatrix} l-k+c \\ c \end{bmatrix} \begin{bmatrix} l-s+a \\ b \end{bmatrix} \begin{bmatrix} s \\ a \end{bmatrix} = \delta_{s,k}.$$

Assume that s equals k. Since $s=l-k+a-b+c$, we have

$$-c(l-s+a)+(k-c)(-l+k-1)+s(s-a-k)+a(k+1)$$
$$=(l-2k)(c-a)-c(b-c)+b-a(a-b+c)-(k+1)(l-k),$$

which implies

$$\sum(-1)^b q^{(l-2k)(c-a)-c(b-c)+b-a(a-b+c)} \begin{bmatrix} l-k+c \\ c \end{bmatrix} \begin{bmatrix} l-s+a \\ b \end{bmatrix} \begin{bmatrix} s \\ a \end{bmatrix} = (-1)^{l-k} q^{(l-k)(k+1)} \delta_{s,k}.$$

We proved the lemma. *Q. E. D.*

We define the endomorphism s_i of $V(l)$ by

(1.2.10) $$s_i = \sum_{a,b,c} (-1)^b q_i^{c(c-b)-a(a-b+c)+b} e_i^{(a)} f_i^{(b)} e_i^{(c)} t_i^{c-a}.$$

Lemma 1.2.3.

(1.2.11) $$s_i u_k^{(l)} = (-1)^{l-k} q_i^{(l-k)(k+1)} u_{l-k}^{(l)}.$$

Proof. We get

(1.2.12)
$$s_i u_k^{(l)} = \sum(-1)^b q_i^{(l-2k)(c-a)+c(c-b)+b-a(a-b+c)} \begin{bmatrix} l-k+c \\ c \end{bmatrix}_i \begin{bmatrix} l-s+a \\ b \end{bmatrix}_i \begin{bmatrix} s \\ a \end{bmatrix}_i u_{l-s}^{(l)}.$$

Indeed

$$s_i u_k^{(l)} = \sum(-1)^b q_i^{c(c-b)-a(a-b+c)+b} e_i^{(a)} f_i^{(b)} e_i^{(c)} (q_i^{(l-2k)(c-a)}) u_k^{(l)}$$
$$= \sum(-1)^b q_i^{(l-2k)(c-a)+c(c-b)+b-a(a-b+c)}$$
$$\begin{bmatrix} l-k+c \\ c \end{bmatrix}_i \begin{bmatrix} k-c+b \\ b \end{bmatrix}_i \begin{bmatrix} l-k+c-b+a \\ a \end{bmatrix}_i u_{k-a+b-c}^{(l)}$$
$$= \sum(-1)^b q_i^{(l-2k)(c-a)+c(c-b)+b-a(a-b+c)} \begin{bmatrix} l-k+c \\ c \end{bmatrix}_i \begin{bmatrix} l-s+a \\ b \end{bmatrix}_i \begin{bmatrix} s \\ a \end{bmatrix}_i u_{l-s}^{(l)}$$

where $s=l-k+c-b+a$. By Lemma 1.2.2, we have

$$s_i u_k^{(l)} = (-1)^{l-k} q_i^{(l-k)(k+1)} u_{l-k}^{(l)}.$$

Q. E. D.

Proof of Proposition 1.2.1. It is enough to show that s_i equals S_i. First, we get

$$e_i^{(a)} f_i^{(b)} e_i^{(c)} t_i^{c-a} = q_i^{2ac-2ab+a(a-1)-c(c-1)} \frac{(e_i t_i^{-1})^a}{[a]_i!} f_i^{(b)} \frac{(e_i t_i)^c}{[c]_i!}.$$

Then, we have

$$s_\iota=\sum(-1)^b q_i^{c(c-b)-a(a-b+c)+b+2ac-2ab+a(a-1)-c(c-1)}\frac{(e_\iota t_\iota^{-1})^a}{[a]_\iota!}f_i^{(b)}\frac{(e_\iota t_\iota)^c}{[c]_i!}.$$

On the other hand, we have

$$\begin{aligned}&c(c-b)-a(a-b+c)+b+2ac-2ab+a(a-1)-c(c-1)\\&=-\frac{a(a-1)}{2}-a-\frac{b(b-1)}{2}-\frac{c(c-1)}{2}+c+\frac{(-a+b-c)(-a+b-c+1)}{2}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}s_\iota u_k^{(l)}&=\sum_{a,b,c}\frac{q_i^{-a(a-1)/2}}{[a]_\iota!}(q_i^{-1}e_\iota t_i^{-1})^a\frac{q_i^{-b(b-1)/2}}{[b]_\iota!}f_i^b\frac{q_i^{-c(c-1)/2}}{[c]_i!}(q_\iota e_\iota t_\iota)^c q_i^{(-a+b-c)(-a+b-c+1)/2}u_k^{(l)}\\&=S_\iota u_k^{(l)},\end{aligned}$$

since $k=s=l-k+a-b+c$. *Q. E. D.*

We define the endomorphism S'_ι of $V(l)$ by

(1.2.13) $$S'_\iota u_k^{(l)}=\exp_{q_i^{-1}}(-q_i^{-1}f_\iota t_\iota)\exp_{q_i^{-1}}(e_\iota)\exp_{q_i^{-1}}(-q_\iota f_\iota t_i^{-1})q_i^{h_\iota(h_\iota+1)/2}.$$

We can prove the next result similarly,

(1.2.14) $$S'_i u_k^{(l)}=(-1)^{l-k}q_i^{(l-k)(k+1)}u_{l-k}^{(l)}.$$

Therefore, $S_\iota=S'_\iota$, and (1.2.13) is another expression of S_ι.

Let M be an integrable U_q-module. M is a direct sum of irreducible $U_q(sl_2)_\iota$-modules. So, we regard S_ι as an endomorphism of M.

1.3. Definition of T_ι

Let $Int(\mathfrak{g}, P)$ be the category of integrable U_q-modules, for an object M of $Int(\mathfrak{g}, P)$ let $\Psi(M)$ be the underlying $\boldsymbol{Q}(q)$-vectorspace. Then Ψ is the functor from $Int(\mathfrak{g}, P)$ to the category of $\boldsymbol{Q}(q)$ vector spaces. Let R be the endomorphism ring. Then R contains S_ι as well as U_q.

We define the algebra automorphism T_ι of R by

(1.3.1) $$T_\iota x=\mathrm{Ad}S_\iota(x)$$

for $x\in U_q$.

Proposition 1.3.1. *We have*

(1.3.2) $$T_\iota(e_\iota)=-f_\iota t_\iota$$

(1.3.3) $$T_i(f_\iota)=-t_i^{-1}e_\iota$$

(1.3.4) $$T_\iota(t_j)=t_j t_i^{-a_{ij}}$$

(1.3.5) $$T_i(e_j)=\sum_{k=0}^{-a_{ij}}(-1)^{-a_{ij}-k}q_i^{a_{ij}+k}e_i^{(k)}e_je_i^{(-a_{ij}-k)} \quad for \quad i\neq j$$

(1.3.6) $$T_i(f_j)=\sum_{k=0}^{-a_{ij}}(-1)^{-a_{ij}-k}q_i^{-a_{ij}-k}f_i^{(-a_{ij}-k)}f_jf_i^{(k)} \quad for \quad i\neq j.$$

In particular U_q is stable by T_i.

Proof. Let $\{u_k^{(l)}\}$ be as in 1.2. Then we have

$$S_ie_iu_k^{(l)}=S_ie_iS_i^{-1}S_iu_k^{(l)}$$
$$=T_i(e_i)(-1)^{l-k}q_i^{(l-k)(k+1)}u_{l-k}^{(l)}$$

and

$$S_ie_iu_k^{(l)}=[l-k+1]_iS_iu_{k-1}^{(l)}$$
$$=[l-k+1]_i(-1)^{l-k+1}q_i^{k(l-k+1)}u_{l-k+1}^{(l)}.$$

Therefore $T_i(e_i)=-f_it_i$. Similarly, $T_i(f_i)=-t_i^{-1}e_i$ and $T_i(t_j)=t_jt_i^{-a_{ij}}$. We shall prove (1.3.6).

Lemma 1.3.2. *Let*

$$S_i''=\exp_{q_i^{-1}}(-f_i)\exp_{q_i^{-1}}(q_ie_it_i).$$

Then we have

$$\mathrm{Ad}(S_i'')(f_jt_j^{-1})=q_i^{-a_{ij}(a_{ij}-1)/2}\,\mathrm{Ad}(\exp_{q_i}(-q_i^{-1}e_it_i^{-1}))(\Phi(f_i^{(-a_{ij})})(f_j)t_j^{-1}).$$

Proof. Since $[q_ie_it_i,\ f_jt_j^{-1}]=0$, we get

$$\mathrm{Ad}(\exp_{q_i^{-1}}(q_ie_it_i))(f_jt_j^{-1})=f_jt_j^{-1}.$$

Therefore

$$\mathrm{Ad}(S_i'')(f_jt_j^{-1})=\mathrm{Ad}(\exp_{q_i^{-1}}(-f_i))(f_jt_j^{-1})$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{n-k}q_i^{-(n(n-1)/2)+k(n-1+a_{ij})}f_i^{(n-k)}f_jf_i^{(k)}t_j^{-1}$$
$$=\sum_{n=0}^{\infty}q_i^{-(n(n-1)/2)-n(-a_{ij}-n+1)}\Phi(f_i^{(n)})(f_j)t_j^{-1}$$
$$=\sum_{n=0}^{-a_{ij}}q_i^{(n(n-1)/2)+na_{ij}}\Phi(f_i^{(n)})(f_j)t_j^{-1}.$$

By (1.1.3) and Lemma 1.1.1, $\bigoplus_{k=0}^{-a_{ij}}\boldsymbol{Q}(q)\Phi(f_i^{(k)})(f_j)$ is a $(-a_{ij}+1)$-dimensional irreducible $U_q(sl_2)_i$-submodule of U_q, and f_j is a highest weight vector of weight $-a_{ij}$. Therefore

(1.3.7) $$\Phi(e_i^{(k)})\Phi(f_i^{(-a_{ij})})(f_j)=\Phi(f_i^{(-a_{ij}-k)})(f_j).$$

Then, we have

$$\mathrm{Ad}(\exp_{q_i}(-q_i^{-1}e_it_i^{-1}))(\Phi(f_i^{(-a_{ij})})(f_j)t_j^{-1})$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k q_i^{(n(n+1)/2)+k(n-1)}t_i^{-n}e_i^{(k)}(\Phi(f_i^{(-a_{ij})})(f_j))e_i^{(n-k)}t_j^{-1}$$
$$=\sum_{n=0}^{\infty}q_i^{(n+1)/2}\Phi(e_i^{(n)})\Phi(f_i^{(-a_{ij})})(f_j)t_j^{-1}$$
$$=\sum_{n=0}^{-a_{ij}}q_i^{n(n+1)/2}\Phi(f_i^{(-a_{ij}-n)})(f_j)t_j^{-1}$$
$$=\sum_{n=0}^{-a_{ij}}q_i^{(n(n-1)/2)+na_{ij}+(a_{ij}(a_{ij}-1)/2)}\Phi(f_i^{(n)})(f_j)t_j^{-1}.$$

Q. E. D.

Let

$$S_i'=\exp_{q_i^{-1}}(q_i^{-1}e_it_i^{-1})S_i''. \tag{1.3.8}$$

By Lemma 1.3.2, we have

$$\mathrm{Ad}(S_i')(f_jt_j^{-1})=q_i^{-a_{ij}(a_{ij}-1)/2}\Phi(f_i^{(-a_{ij})})(f_j)t_j^{-1}. \tag{1.3.9}$$

On the other hand, we get

$$f_jt_j^{-1}=q_i^{-a_{ij}(a_{ij}-1)/2}q_i^{h_i(h_i+1)/2}f_jt_j^{-1}t_i^{a_{ij}}q_i^{-h_i(h_i+1)/2}.$$

Therefore,

$$\mathrm{Ad}(S_i')(f_jt_j^{-1})=q_i^{-a_{ij}(a_{ij}-1)/2}\mathrm{Ad}(S_i)(f_jt_j^{-1}t_i^{a_{ij}})$$
$$=q_i^{-a_{ij}(a_{ij}-1)/2}\mathrm{Ad}(S_i)(f_j)t_j^{-1}.$$

This and (1.3.9) imply (1.3.6).

Introduce the $\boldsymbol{Q}$-algebra anti-automorphism ω by

$$\omega e_i=f_i$$
$$\omega f_i=e_i$$
$$\omega t_i=t_i^{-1}$$
$$\omega q=q^{-1}.$$

Applying ω to (1.3.6), we obtain (1.3.5). *Q. E. D.*

This proposition immediately imply the next corollary.

Corollary 1.3.3.

$$*T_i*=T_i^{-1}.$$

1.4. The braid group action

Proposition 1.4.1. $\{S_i\,;\, i\in I\}$ *satisfies the braid relations for the Weyl group*

W of $\mathfrak{g}$.

Proof. In case of $a_{ij}=a_{ji}=-1$, we have to prove

$$S_iS_jS_i=S_jS_iS_j.$$

This is equivalent to

$$S_j=(\mathrm{Ad}S_i)(\mathrm{Ad}S_j)(S_i)=T_iT_j(S_i).$$

By (1.3.2)-(1.3.6), we have

$$\begin{aligned} &T_iT_j(f_i)=f_j \\ (1.4.1)\qquad &T_iT_j(e_i)=e_j \\ &T_iT_j(t_i)=t_j. \end{aligned}$$

Therefore we get

$$\begin{aligned} &T_iT_j(\exp_{q_i^{-1}}(q_i^{-1}e_it_i^{-1})\exp_{q_i^{-1}}(-f_i)\exp_{q_i^{-1}}(q_ie_it_i)q_i^{h_i(h_i+1)/2}) \\ &=\exp_{q_i^{-1}}(q_i^{-1}T_iT_j(e_it_i^{-1}))\exp_{q_i^{-1}}(-T_iT_j(f_i))\exp_{q_i^{-1}}(q_iT_iT_j(e_it_i))q_i^{h_j(h_j+1)/2} \\ &=\exp_{q_j^{-1}}(q_j^{-1}e_jt_j^{-1})\exp_{q_j^{-1}}(-f_j)\exp_{q_j^{-1}}(q_je_jt_j)q_j^{h_j(h_j+1)/2} \\ &=S_j. \end{aligned}$$

The remaining cases are similarly proved by the corresponding identities to (1.4.1) due to Lusztig [4]. *Q. E. D.*

This proposition immediately implies the following result.

Corollary 1.4.2 ([5]). *$\{T_i\,;\ i\in I\}$ satisfies the braid relations.*

§2. $T_iU_q^-\cap U_q^-=$ Ker e_i'

2.1. Proof of $T_iU_q^-\cap U_q^-=$ Ker e_i'

Let U_q^- be the subalgebla over $\boldsymbol{Q}(q)$ of U_q generated by f_i.

Lemma 2.1.1 ([1]. *For any $P\in U_q^-$, there exist unique Q, $R\in U_q^-$ such that*

$$[e_i,\ P]=\frac{t_iQ-t_i^{-1}R}{q_i-q_i^{-1}}.$$

By this lemma, if we set $e_i''(P)=Q$ and $e_i'(P)=R$, then e_i' and e_i'' are endomorphism of U_q^-. Moreover, we get

$$e_i''f_j=q_i^{a_{ij}}f_je_i''+\delta_{ij}$$

and

$$e_i'f_j=q_i^{-a_{ij}}f_je_i'+\delta_{ij}.$$

Here f_j acts on U_q^- by the left multiplication.

The aim of this section is to prove the following result.

Proposition 2.1.2.

$$T_i(U_q^-)\cap U_q^-=\mathrm{Ker}\ e_i'. \tag{2.1.1}$$

The first step is the next lemma.

Lemma 2.1.3.

$$f_i(U_q^-)\cap T_i(U_q^-)=\{0\}. \tag{2.1.2}$$

Proof. Let $u\in f_i(U_q^-)\cap T_i(U_q^-)$. Then $T_i^{-1}u\in U_q^-$. On the other hand, choosing $x\in U_q^-$ such that $u=f_ix$, we have

$$T_i^{-1}u=-e_it_iT_i^{-1}x.$$

Since $e_iU_q\cap U_q^-=0$, we obtain $u=0$. *Q. E. D.*

Lemma 2.1.4. *For* $i\neq j$

$$f_jf_i^{(n)}=\sum_{k=0}^{n}q_i^{k(n-k-a_{ij})}f_i^{(k)}\Phi(f_i^{(n-k)})(f_j). \tag{2.1.3}$$

Proof. By the definition of Φ we have

$$\Phi(f_i)(\Phi(f_i^{(n)})(f_j))=\Phi(f_i^{(n)})(f_j)f_i-f_it_i\Phi(f_i^{(n)})(f_j)t_i^{-1}.$$

Therefore we have

$$\Phi(f_i^{(n)})(f_j)f_i=[n+1]_i\Phi(f_i^{(n+1)})(f_j)+q_i^{2n-a_{ij}}f_i\Phi(f_i^{(n)})(f_j). \tag{2.1.4}$$

We shall show this formula by induction on n,

$$\begin{aligned}
f_jf_i^{(n+1)}&=\frac{1}{[n+1]_i}\Big(\sum_{k=0}^{n}q_i^{k(n-k-a_{ij})}f_i^{(k)}\Phi(f_i^{(n-k)})(f_j)\Big)f_i\\
&=\frac{1}{[n+1]_i}\sum_{k=0}^{n}q_i^{k(n-k-a_{ij})}f_i^{(k)}([n-k+1]_i\Phi(f_i^{(n-k+1)})(f_j)\\
&\quad+q_i^{2(n-k)-a_{ij}}f_i\Phi(f_i^{(n-k)})(f_j))\\
&=\frac{1}{[n+1]_i}\sum_{k=0}^{n}q_i^{k(n-k-a_{ij})}[n-k+1]_if_i^{(k)}\Phi(f_i^{(n-k+1)})(f_j)\\
&\quad+\frac{1}{[n+1]_i}\sum_{k=1}^{n+1}q_i^{k(n-k-a_{ij})+n+1}[k]_if_i^{(k)}\Phi(f_i^{(n-k+1)})(f_j)\\
&=\sum_{k=0}^{n+1}q_i^{k(n+1-k-a_{ij})}f_i^{(k)}\Phi(f_i^{(n-k+1)})(f_j).
\end{aligned}$$

Q. E. D.

Corollary 2.1.5.

$$U_q^-=f_i(U_q^-)+M \tag{2.1.5}$$

where M is the $\boldsymbol{Q}(q)$-subalgebra of U_q^- generated by $\Phi(f_i^{(n)})(f_j)$ for $i\neq j$ and $n\geqq 0$.

Lemma 2.1.6. *For $i\neq j$*

$$T_i^{-1}\Phi(f_i^{(n)})(f_j)=(\Phi(f_i^{(-a_{ij}-n)})(f_j))^*. \tag{2.1.6}$$

Proof. We shall show this formula by induction on n,

$$\begin{aligned}
&T_i^{-1}\Phi(f_i^{(n+1)})(f_j)\\
&=\frac{1}{[n+1]_i}T_i(\Phi(f_i)\Phi(f_i^{(n)})(f_j))\\
&=\frac{1}{[n+1]_i}T_i(\Phi(f_i^{(n)})(f_j)f_i-q_i^{-2n-a_{ij}}f_i\Phi(f_i^{(n)})(f_j))\\
&=\frac{1}{[n+1]_i}((\Phi(f_i^{(-a_{ij}-n)})(f_j))^*(T_if_i)^*-q_i^{-2n-a_{ij}}(T_if_i)^*(\Phi(f_i^{(-a_{ij}-n)})(f_j))^*)\\
&=\frac{1}{[n+1]_i}(-t_i^{-1}e_i\Phi(f_i^{(-a_{ij}-n)})(f_j)+q_i^{-2n-a_{ij}}\Phi(f_i^{(-a_{ij}-n)})(f_j)t_i^{-1}e_i)^*\\
&=\frac{1}{[n+1]_i}(\Phi(e_i)\Phi(f_i^{(-a_{ij}-n)})(f_j))^*\\
&=(\Phi(f_i^{(-a_{ij}-n-1)})(f_j))^*.
\end{aligned}$$

Q. E. D.

Corollary 2.1.7. *For $i\neq j$,*

$$\Phi(f_i^{(n)})(f_j)\in T_i(U_q^-)\cap U_q^-. \tag{2.1.7}$$

Proof. U_q^- is stable under $*$. Lemma 2.1.6 immediately implies this corollary. Q. E. D.

Lemma 2.1.8. *For $i\neq j$*

$$e_i'\Phi(f_i^{(n)})(f_j)=q_i^{-2n-a_{ij}}\Phi(f_i^{(n)})(f_j)e_i'. \tag{2.1.8}$$

Proof. First we have

$$e_i'f_i^{(n)}=q_i^{-2n}f_i^{(n)}e_i'+q_i^{1-n}f_i^{(n-1)}.$$

Then, we have

$$\begin{aligned}
e_i'f_i^{(n-k)}f_jf_i^{(k)}=&q_i^{-2n-a_{ij}}f_i^{(n-k)}f_jf_i^{(k)}e_i'+q_i^{-2n+k+1-a_{ij}}f_i^{(n-k)}f_jf_i^{(k-1)}\\
&+q_i^{-n+k+1}f_i^{(n-k-1)}f_jf_i^{(k)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
e_i'\Phi(f_i^{(n)})(f_j)&=\sum_{k=0}^{n}(-1)^{n-k}q_i^{(n-k)(-a_{ij}-n+1)}e_i'f_i^{(n-k)}f_jf_i^{(k)}\\
&=q_i^{-2n-a_{ij}}\Phi(f_i^{(n)})(f_j)e_i'
\end{aligned}$$

$$+\sum_{k=0}^{n}(-1)^{n-k}q_i^{(n-k)(-a_{ij}-n+1)}$$

$$\cdot(q_i^{-2n+k+1-a_{ij}}f_i^{(n-k)}f_jf_i^{(k-1)}+q_i^{-n+k+1}f_i^{(n-k-1)}f_jf_i^{(k)}).$$

It is easy to show that the second term is 0. *Q. E. D.*

Lemma 2.1.9.

$$M=\mathrm{Ker}\ e_i'. \tag{2.1.9}$$

Proof. By Lemma 2.1.8, we have

$$M\subset\mathrm{Ker}\ e_i'. \tag{2.1.10}$$

We recall the result in [1],

$$U_q^-=\mathrm{Ker}\ e_i'\oplus f_i(U_q^-). \tag{2.1.11}$$

Then this lemma follows from Corollary 2.1.5. *Q. E. D.*

Proof of Proposition 2.1.2. By Lemmas 2.1.6 and 2.1.9, we have

$$\mathrm{Ker}\ e_i'\subset T_i(U_q^-)\cap U_q^-. \tag{2.1.12}$$

Then this proposition follows from Lemma 2.1.3, (2.1.10) and (2.1.12). *Q. E. D.*

§ 3. Crystals

3.1. Definition of crystal

Definition 3.1.1. *A crystal B is a set with*

(3.1.1) *maps* $wt: B\to P$, $\varepsilon_i: B\to \boldsymbol{Z}\cup\{-\infty\}$ *and* $\varphi_i: B\to\boldsymbol{Z}\cup\{-\infty\}$,

(3.1.2) $\tilde{e}_i: B\longrightarrow B\cup\{0\},\qquad \tilde{f}_i: B\longrightarrow B\cup\{0\}.$

They are subject to the following axioms:

(C1) $\varphi_i(b)=\varepsilon_i(b)+\langle h_i, wt(b)\rangle.$

(C2) *If* $b\in B$ *and* $\tilde{e}_ib\in B$ *then,*

$wt(\tilde{e}_ib)=wt(b)+\alpha_i$, $\varepsilon_i(\tilde{e}_ib)=\varepsilon_i(b)-1$ *and* $\varphi_i(\tilde{e}_ib)=\varphi_i(b)+1$.

(C2′) *If* $b\in B$ *and* $\tilde{f}_ib\in B$, *then*

$wt(\tilde{f}_ib)=wt(b)-\alpha_i$, $\varepsilon_i(\tilde{f}_ib)=\varepsilon_i(b)+1$ *and* $\varphi_i(\tilde{f}_ib)=\varphi_i(b)-1$.

(C3) *For* $b, b'\in B$ *and* $i\in I$, $b'=\tilde{e}_ib$ *if and only if* $b=\tilde{f}_ib'$.

(C4) *For* $b\in B$, *if* $\varphi_i(b)=-\infty$, *then* $\tilde{e}_ib=\tilde{f}_ib=0$.

For two crystals B_1 and B_2, a morphism ψ from B_1 to B_2 is a map $B_1 \to B_2 \cup \{0\}$ that satisfies the following conditions:

(3.1.3) *If* $b \in B_1$ *and* $\psi(b) \in B_2$, *then* $wt(\psi(b)) = wt(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, *and* $\varphi_i(\psi(b)) = \varphi_i(b)$,

(3.1.4) *For* $b \in B_1$, *we have* $\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b)$ *provided* $\psi(b)$ *and* $\psi(\tilde{e}_i b) \in B_2$,

(3.1.5) *For* $b \in B_1$, *we have* $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$ *provided* $\psi(b)$ *and* $\psi(\tilde{f}_i b) \in B_2$.

A morphism ψ: $B_1 \to B_2$ is called *strict*, if it commutes with all $\tilde{e}_i$ and $\tilde{f}_i$.

A morphism ψ: $B_1 \to B_2$ is called an *embedding*, if ψ induces an injective map $B_1 \cup \{0\} \to B_2 \cup \{0\}$.

For two crystals B_1 and B_2, we define its tensor product $B_1 \otimes B_2$ as follows:

$$B_1 \otimes B_2 = \{b_1 \otimes b_2 ;\ b_1 \in B_1 \text{ and } b_2 \in B_2\}$$

$$\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1),\ \varepsilon_i(b_2) - wt_i(b_1)\}$$

$$\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_1) + wt_i(b_2),\ \varphi_i(b_2)\}$$

$$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2).$$

Here $wt_i(b) = \langle h_i, wt(b) \rangle$. The action of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geqq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leqq \varepsilon_i(b_2). \end{cases}$$

Example 3.1.2. For $i \in I$, B_i is the crystal defined as follows

$$B_i = \{b_i(n) ;\ n \in \boldsymbol{Z}\}$$

$$wt(b_i(n)) = n$$

$$\varphi_i(b_i(n)) = n, \qquad \varepsilon_i(b_i(n)) = -n$$

$$\varphi_j(b_i(n)) = \varepsilon_j(b_i(n)) = -\infty \qquad \text{for } i \neq j.$$

We define the action of $\tilde{e}_i$ and $\tilde{f}_i$ by

$$\tilde{e}_i(b_i(n)) = b_i(n+1)$$

$$\tilde{f}_i(b_i(n)) = b_i(n-1)$$

$$\tilde{e}_j(b_i(n)) = \tilde{f}_j(b_i(n)) = 0 \qquad \text{for } i \neq j.$$

We write b_i for $b_i(0)$.

Example 3.1.3. For $\lambda\in P_+$, $B(\lambda)$ is the crystal associated with the crystal base of the simple module with highest weight λ. The unique element of $B(\lambda)$ of weight λ is denoted by u_λ.

Example 3.1.4. $B(\infty)$ is the crystal associated with the crystal base of U_q^-. We set $\varepsilon_i(b)=\max\{k\geqq 0;\ \tilde{e}_i^k b\neq 0\}$, $\varphi_i(b)=\varepsilon_i(b)+\langle h_i, wt(b)\rangle$. The unique element of $B(\infty)$ of weight 0 is denoted by u_∞.

3.2. Some results

Theorem 3.2. ([3]).

$$B(\infty)^*=B(\infty). \tag{3.2.1}$$

We define the operators $\tilde{e}_i^*$, $\tilde{f}_i^*$ of U_q^- by

$$\tilde{e}_i^*=*\tilde{e}_i*, \quad \text{and} \quad \tilde{f}_i^*=*\tilde{f}_i*. \tag{3.2.2}$$

Theorem 3.2.2 ([3]). 1. *For any i, there exists a unique strict embedding of crystals*

$$\Psi_i:\ B(\infty)\hookrightarrow B(\infty)\otimes B_i$$

that sends u_∞ to $u_\infty\otimes b_i$.

2. *If $\Psi_i(b)=b'\otimes\tilde{f}_i^n b_i (n\geqq 0)$, then $\varepsilon_i(b^*)=n$, $\varepsilon_i(b'^*)=0$ and*

$$\Psi_i(\tilde{e}_i^* b)=\begin{cases} b'\otimes\tilde{f}_i^{n-1}b_i & \text{if } n\geqq 0\\ 0 & \text{if } n=0\end{cases} \tag{3.2.3}$$

$$\Psi_i(\tilde{f}_i^* b)=b'\otimes\tilde{f}_i^{n+1}b_i \tag{3.2.4}$$

3. *$Im\ \Psi_i=\{b\otimes\tilde{f}_i^n b_i:\ b\in B(\infty),\ \varepsilon_i(b^*)=0,\ n\geqq 0\}$.*

3.3. Action on $L(\lambda)$

Lemma 3.3.1 ([3]). *For $b\in B(\infty)$*

$$f_i^{(a)}G(b)\equiv\begin{bmatrix}\varepsilon_i(b)+a\\ a\end{bmatrix}_i G(\tilde{f}_i^a b) \bmod f_i^{a+1}U_q^- \tag{3.3.1}$$

$$G(b)f_i^{(a)}\equiv\begin{bmatrix}\varepsilon_i(b^*)+a\\ a\end{bmatrix}_i G(\tilde{f}_i^{*a} b) \bmod U_q^- f_i^{a+1}. \tag{3.3.2}$$

Let $\lambda\in P_+$, and let $V(\lambda)$ be the irreducible U_q-module generated by the highest weight vector u_λ of highest weight λ. For $w\in W$, let us denote by $u_{w\lambda}$ the global base of weight $w\lambda$. Then we have

(3.3.3) $$u_{w\lambda}=u_{\lambda} \quad \text{if} \quad w=1.$$

If $w=s_iw'>w'$, then we have

(3.3.4) $$u_{w\lambda}=f_i^{(c)}u_{w'\lambda} \quad \text{where} \quad c=\langle h_i, w\lambda\rangle.$$

Lemma 3.3.2. *For* $b\in B(\infty)$

(3.3.5) $$G(b)u_{w\lambda}=G(\tilde{f}_i^{*c}b)u_{w'\lambda}.$$

Proof. Note that $f_iu_{w\lambda}=0$ and $f_i^{1+c}u_{w'\lambda}=0$. By Lemma 3.3.1,

$$\begin{aligned} G(b)u_{w\lambda}&=G(b)f_i^{(c)}u_{w'\lambda} \\ &=\left(\begin{bmatrix}\varepsilon_i(b^*)+c \\ c\end{bmatrix}_i G(\tilde{f}_i^{*c}b)+U_q^-f_i^{c+1}\right)u_{w'\lambda} \\ &=\begin{bmatrix}\varepsilon_i(b^*)+c \\ c\end{bmatrix}_i G(\tilde{f}_i^{*c}b)u_{w'\lambda}. \end{aligned}$$

If $\varepsilon_i(b^*)=0$, it is obvious. If $\varepsilon_i(b^*)\neq 0$, then we have

(3.3.6) $$G(b)u_{w\lambda}\in U_q^-f_iu_{w\lambda}=0$$

and

$$G(\tilde{f}_i^{*c}b)u_{w'\lambda}\in U_q^-f_i^{c+1}u_{w'\lambda}=0$$

since $G(b)\in U_q^-f_i^{\varepsilon_i(b^*)}$ and $\varepsilon_i((\tilde{f}_i^{*c}b)^*)=\varepsilon_i(b^*)+c$. *Q. E. D.*

Corollary 3.3.3. *Let* $\lambda\in P_+$, $P\in L(\infty)$ *and* $b\in B(\infty)$ *such that* $b\equiv P \bmod qL(\infty)$. *Then we have*

(3.3.7) $$Pu_{s_i\lambda}\equiv G(\tilde{f}_i^{*\langle h_i,\lambda\rangle}b)u_{\lambda} \mod qL(\lambda).$$

Proposition 3.3.4.

(3.3.8) $$T_i(P)u_{\lambda}\in L(\lambda)$$

and

(3.3.9) $$T_i(P)u_{\lambda}\equiv\begin{cases} \tilde{e}_i^mG(\tilde{f}_i^{*\langle h_i,\lambda\rangle}b)u_{\lambda} & \text{if} \quad \varphi(\tilde{f}_i^{*\langle h_i,\lambda\rangle}b)+\langle h_i,\lambda\rangle=0 \\ 0 & \text{if} \quad \varphi_i(\tilde{f}_i^{*\langle h_i,\lambda\rangle}b)+\langle h_i,\lambda\rangle\neq 0 \end{cases}$$

where $m=\varepsilon_i(\tilde{f}_i^{*\langle h_i,\lambda\rangle}b)$.

Proof. We recall (1.2.4)

$$S_iu_k^{(l)}=(-1)^{l-k}q_i^{(l-k)(k+1)}u_{l-k}^{(l)}.$$

So, $L(\lambda)$ and $L(\lambda)/qL(\lambda)$ are stable by S_i, and for $b\in B(\lambda)$

$$S_i b = \begin{cases} \tilde{e}_i^{\varepsilon_i(b)} b & \text{if } \varphi_i(b)=0 \\ 0 & \text{if } \varphi_i(b)\neq 0 . \end{cases} \tag{3.3.10}$$

In particular,

$$S_i P u_{s_i\lambda} \equiv \begin{cases} \tilde{e}_i^m G(\tilde{f}_i^{*\langle h_i, \lambda\rangle} b) u_\lambda & \text{if } \varphi_i(\tilde{f}_i^{*\langle h_i, \lambda\rangle} b)+\langle h_i, \lambda\rangle=0 \\ 0 & \text{if } \varphi_i(\tilde{f}_i^{*\langle h_i, \lambda\rangle} b)+\langle h_i, \lambda\rangle\neq 0 . \end{cases} \tag{3.3.11}$$

On the other hand, by (3.3.10)

$$\begin{aligned} S_i P u_{s_i\lambda} &= T_i(P) S_i u_{s_i\lambda} \\ &= T_i(P) u_\lambda . \end{aligned}$$

Therefore, we have (3.3.8) and (3.3.9). *Q. E. D.*

3.4. Action on $L(\infty)$

Let $b\in B(\infty)$ and $\lambda\in P_+$.

Lemma 3.4.1. *$G(b)u_\lambda\neq 0$ if and only if $\varepsilon_i(b^*)\leqq\langle h_i, \lambda\rangle$ for any i.*

Proof.

$$\{P\in U_q^- ;\ Pu_\lambda=0\} = \sum_i U_q^- f_i^{1+\langle h_i, \lambda\rangle}$$

and

$$U_q^- f_i^a = \bigoplus_{\varepsilon_i(b^*)\geqq a} \boldsymbol{Q}(q) G(b) .$$

So, we have

$$\{P\in U_q^- ;\ Pu_\lambda=0\} = \sum_i \bigoplus_{\varepsilon_i(b^*)>\langle h_i, \lambda\rangle} \boldsymbol{Q}(q) G(b) .$$

Q. E. D.

Lemma 3.4.2. ([2]).

$$\varepsilon_j(\tilde{f}_i^a b)\leqq\varepsilon_j(b) \quad \textit{for} \quad i\neq j . \tag{3.4.1}$$

Now, let us assume that $\langle h_i, \lambda\rangle$ is sufficiently large for any i.

Lemma 3.4.3. *If $\varepsilon_i(b^*)=0$, then $G(\tilde{f}_i^{*\langle h_i, \lambda\rangle} b)u_\lambda\neq 0$.*

Proof. By Lemma 3.4.1, $G(\tilde{f}_i^{*\langle h_i, \lambda\rangle} b)u_\lambda \neq 0$ if and only if $\varepsilon_j^*(\tilde{f}_i^{*\langle h_i, \lambda\rangle} b)\leqq \langle h_j, \lambda\rangle$ for any j, where we set $\varepsilon_i^*(b)=\varepsilon_i(b^*)$ for $b\in B(\infty)$. If $i=j$, then

$$\varepsilon_j^*(\tilde{f}_i^{*\langle h_i, \lambda\rangle} b)=\langle h_i, \lambda\rangle+\varepsilon_i^*(b)=\langle h_i, \lambda\rangle .$$

If $i\neq j$, then

$$\varepsilon_j^*(\tilde{f}_i^{*\langle h_i, \lambda\rangle} b)\leqq\varepsilon_j^*(b)\leqq\lambda_j$$

since $\lambda\gg 0$. *Q. E. D.*

Let b be an element of $B(\infty)$ and let us assume that $P=G(b)$ belongs to $T_i^{-1}U_q^-\cap U_q^-$.

Lemma 3.4.4.

(3.4.2) $$G(\tilde{f}_i^{*\langle h_i, \lambda\rangle}b)u_\lambda \neq 0.$$

Proof. Since $T_i^{-1}=*T_i*$ and U_q^- is stable under $*$

$$T_i^{-1}(U_q^-)\cap U_q^-=(T_i(U_q^-)\cap U_q^-)^* =(\mathrm{Ker}\ e_i')^*.$$

Then we have $e_i'P^*=0$, and therefore $\tilde{e}_iP^*=0$. This implies

(3.4.3) $$\varepsilon_i^*(b)=0.$$

By Lemma 3.4.3, we have (3.4.2). *Q. E. D.*

Lemma 3.4.5.

(3.4.4) $$T_i(P)u_\lambda \neq 0.$$

Proof. By Proposition 3.3.4, it is enough to show that $\varphi_i(\tilde{f}_i^{*\langle h_i, \lambda\rangle}b)+\langle h_i, \lambda\rangle =0$. We have

(3.4.5) $$\varphi_i(\tilde{f}_i^{*\langle h_i, \lambda\rangle}b)+\langle h_i, \lambda\rangle=\varepsilon_i(\tilde{f}_i^{*\langle h_i, \lambda\rangle}b)+\langle h_i, wt(b)-\lambda\rangle.$$

Since $\varepsilon_i^*(b)=0$, $\Psi_i(b)=b\otimes b_i$ and $\Psi_i(\tilde{f}_i^{*\langle h_i, \lambda\rangle}b)=b\otimes\tilde{f}_i^{\langle h_i, \lambda\rangle}b_i$,

$$\begin{aligned}\varepsilon_i(\tilde{f}_i^{*\langle h_i, \lambda\rangle}b)&=\varepsilon_i(b\otimes\tilde{f}_i^{\langle h_i, \lambda\rangle}b_i)\\&=\max\{\varepsilon_i(b),\ \varepsilon_i(\tilde{f}_i^{*\langle h_i, \lambda\rangle}b_i)-\langle h_i, wt(b)\rangle\}\\&=\max\{\varepsilon_i(b),\ \langle h_i, \lambda\rangle-\langle h_i, wt(b)\rangle\}.\end{aligned}$$

Therefore we have

(3.4.6) $$\varepsilon_i(\tilde{f}_i^{*\langle h_i, \lambda\rangle}b)=\langle h_i, \lambda-wt(b)\rangle.$$

Since $\langle h_i, \lambda\rangle$ is sufficently large, it equals $\langle h_i, \lambda-wt(b)\rangle$. Therefore $\varphi_i(\tilde{f}_i^{*\langle h_i, \lambda\rangle}b)+\langle h_i, \lambda\rangle=0$. *Q. E. D.*

Lemma 3.4.6.

(3.4.7) $$\varphi_i(b)+\varepsilon_i^*(b)\geqq 0 \quad \textit{for any} \quad b\in B(\infty).$$

Proof. By Theorem 3.2.2

$$\Psi_i(b)=b'\otimes\tilde{f}_i^n b_i$$

where $n=\varepsilon_i^*(b)=-\varphi_i(\tilde{f}_i^n b_i)$. We have

$$\begin{aligned}\varphi_i(b)&=\varphi_i(b'\otimes\tilde{f}_i^n b_i)\\&=\max\{\varphi_i(b')+\langle h_i, wt(\tilde{f}_i^n b_i)\rangle,\ \varphi_i(\tilde{f}_i^n b_i)\}.\end{aligned}$$

Therefore we have

$$\varphi_\iota(b)+\varepsilon_\iota^*(b)=\varphi_\iota(b)-\varphi_\iota(\tilde{f}_\iota^n b_\iota)\geqq 0. \qquad Q.E.D.$$

Proposition 3.4.7.

$$T_\iota P\in L(\infty). \tag{3.4.8}$$

$$T_\iota P\equiv \tilde{f}_\iota^{*\varphi_\iota(b)}\tilde{e}_\iota^{\varepsilon_\iota(b)}b \mod qL(\infty). \tag{3.4.9}$$

Proof. By Lemma 3.4.6, we have

$$\varphi_\iota(b)=\varphi_\iota(b)+\varepsilon_\iota^*(b)\geqq 0. \tag{3.4.10}$$

By (3.4.3), (3.4.6) and (3.4.10), we have

$$\begin{aligned}\Psi_\iota(\tilde{e}_\iota^m\tilde{f}_\iota^{*\langle h_\iota,\lambda\rangle}b)&=\tilde{e}_\iota^{\langle h_\iota,\lambda-wt(b)\rangle}(b\otimes\tilde{f}_\iota^{\langle h_\iota,\lambda\rangle}b_\iota)\\&=\tilde{e}_\iota^{\varepsilon_\iota(b)}b\otimes\tilde{f}_\iota^{\varphi_\iota(b)}b_\iota.\end{aligned}$$

On the other hand, by Theorem 3.2.2 we have

$$\begin{aligned}\Psi_\iota(\tilde{f}_\iota^{*\varphi_\iota(b)}\tilde{e}_\iota^{\varepsilon_\iota(b)}b)&=\tilde{f}_\iota^{*\varphi_\iota(b)}(\tilde{e}_\iota^{\varepsilon_\iota(b)}b\otimes b_\iota)\\&=\tilde{e}_\iota^{\varepsilon_\iota(b)}b\otimes\tilde{f}_\iota^{\varphi_\iota(b)}b_\iota.\end{aligned}$$

Since Ψ_ι is an embedding, we have

$$\tilde{e}_\iota^m\tilde{f}_\iota^{*\langle h_\iota,\lambda\rangle}b=\tilde{f}_\iota^{*\varphi_\iota(b)}\tilde{e}_\iota^{\varepsilon_\iota(b)}b. \tag{3.4.11}$$

Therefore we obtain

$$T_\iota(P)u_\lambda\equiv G(\tilde{f}_\iota^{*\varphi_\iota(b)}\tilde{e}_\iota^{\varepsilon_\iota(b)}b)u_\lambda. \tag{3.4.12}$$

For $\lambda\gg 0$, Proposition 3.3.4 and (3.4.12) imply this proposition. Q.E.D.

Corollary 3.4.8. *Let us define the map* $\Lambda_\iota:\{b\in B(\infty);\ \varepsilon_\iota^*(b)=0\}\to\{b\in B(\infty);\ \varepsilon_\iota(b)=0\}$ *by* $b\mapsto\tilde{f}_\iota^{*\varphi_\iota(b)}\tilde{e}_\iota^{\varepsilon_\iota(b)}b$. *Then* Λ_ι *is bijective and* $\Lambda_\iota^{-1}(b)=\tilde{f}_\iota^{\varphi_\iota^*(b)}\tilde{e}_\iota^{*\varepsilon_\iota^*(b)}b$.

Proof. Let $b\in\{b\in B(\infty);\ \varepsilon_\iota^*(b)=0\}$, $b'=\tilde{f}_\iota^{*\varphi_\iota(b)}\tilde{e}_\iota^{\varepsilon_\iota(b)}b$. By Proposition 3.4.7 $\varepsilon_\iota(b')=0$. We have

$$\varepsilon_\iota^*(b')=\varphi_\iota(b). \tag{3.4.13}$$

Indeed we have $\Psi_\iota(b')=\tilde{e}_\iota^{\varepsilon_\iota(b)}b\otimes\tilde{f}_\iota^{\varphi_\iota(b)}b_\iota$. Theorem 3.2.2 implies (3.4.13). We get

$$\begin{aligned}\varphi_\iota^*(b')&=\varepsilon_\iota^*(b')+\langle h_\iota,\ wt(b')\rangle\\&=\varphi_\iota(b)+\langle h_\iota,\ (\varepsilon_\iota(b)-\varphi_\iota(b))\alpha_\iota+wt(b)\rangle\\&=2\varepsilon_\iota(b)-\varphi_\iota(b)+\langle h_\iota,\ wt(b)\rangle\\&=\varepsilon_\iota(b).\end{aligned}$$

By Theorem 3.2.2 and (3.4.13), we have

$$\Psi_i(\tilde{e}_i^{*\varepsilon_i^*(b')}b')=\tilde{e}_i^{\varepsilon_i(b)}b\otimes b_i$$
$$=\Psi_i(\tilde{e}_i^{\varepsilon_i(b)}b).$$

Therefore we obtain

$$\tilde{e}_i^{*\varepsilon_i^*(b')}b'=\tilde{e}_i^{\varepsilon_i(b)}b.$$

Hence, we have

$$\tilde{f}_i^{\varphi_i^*(b')}\tilde{e}_i^{*\varepsilon_i^*(b')}b'=\tilde{f}_i^{\varepsilon_i(b)}\tilde{e}_i^{\varepsilon_i(b)}b=b.$$

Let $b\in\{b\in B(\infty);\ \varepsilon_i(b)=0\}$ and $b'=\tilde{f}_i^{\varphi_i^*(b)}\tilde{e}_i^{*\varepsilon_i^*(b)}b$. We have the following formulas similarly

$$\varepsilon_i^*(b')=0 \tag{3.4.14}$$

$$\tilde{f}_i^{*\varphi_i(b)}\tilde{e}_i^{\varepsilon_i(b)}b'=b. \tag{3.4.15}$$

The corollary is proved. *Q.E.D.*

§ 4. Main Theorem

4.1. Proof of Main Theorem

In this section, we assume that $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra.

Proposition 4.1.1 [4]. (1) *Let $w\in W$ and let $s_{i_1}\cdots s_{i_k}$ be a reduced expression of w. Then the automorphism $T_w=T_{i_1}\cdots T_{i_k}$ of U_q is independent of the choice of the reduced expression of w.*

(2) *If $w\alpha_i\in R^+$, then $T_w f_i\in U_q^-$.*

Fix a reduced expression $s_{i_1}\cdots s_{i_N}$ of the longest element of W. This gives us an ordering of the set of all positive roots R^+

$$\beta_1=\alpha_{i_1},\qquad \beta_2=s_{i_1}\alpha_{i_2},\ \cdots,\qquad \beta_N=s_{i_1}\cdots s_{i_{N-1}}\alpha_{i_N}. \tag{4.1.1}$$

We define

$$f_{\beta_m}=T_{i_1}\cdots T_{i_{m-1}}(f_{i_m}) \tag{4.1.2}$$

and

$$f^k=f_{\beta_1}^{(k_1)}f_{\beta_2}^{(k_2)}\cdots f_{\beta_N}^{(k_N)}\qquad \text{where}\quad k=(k_1,\cdots,k_N)\in \mathbf{Z}_{\geqq 0}^N. \tag{4.1.3}$$

Theorem 4.1.2 (Main Theorem).

(i) $$f^k\in L(\infty)\qquad \textit{for any}\quad k=(k_1,\cdots,k_N)\in\mathbf{Z}_{\geqq 0}^N.$$

(ii) $$\{f^k \bmod qL(\infty);\ k\in\mathbf{Z}_{\geqq 0}^N\}=B(\infty).$$

Proof. Let $P\in L(\infty)$ such that $T_iP\in U_q^-$. Then we have $e_i'T_iP=0$.

Therefore

$$f_i^{(n)}T_\iota P=\tilde{f}_i^n T_\iota P. \tag{4.1.4}$$

Hence we obtain

$$f_i^{(n)}T_\iota P\in L(\infty) \tag{4.1.5}$$

and

$$f_i^{(n)}T_\iota P\equiv\tilde{f}_i^n\tilde{f}_i^{*\varphi_\iota(b)}\tilde{e}_i^{\varepsilon_\iota(b)}b \mod qL(\infty) \quad \text{if } P\equiv b \mod qL(\infty). \tag{4.1.6}$$

By (4.1.5), we have $f^k\in L(\infty)$ immediately.

By Proposition 3.4.7, we have $f^k \mod qL(\infty)\in B(\infty)$ for any k. So, there exist the cannonical map π: $\{f^k\}\to B(\infty)$ by $f^k\mapsto f^k \mod qL(\infty)$. We write b^k for $f^k \mod qL(\infty)$.

The first step is the next lemma.

Lemma 4.1.3. *π is injective.*

Proof. Let $b_{(1)}=T_{\iota_1}f_{i_2}^{(k_2)}\cdots T_{\iota_{N-1}}f_{i_N}^{(k_N)} \mod qL(\infty)\in B(\infty)$. Then we have

$$b^k=\tilde{f}_{i_1}^{k_1}b_{(1)}$$

and

$$T_{\iota_1}f_{i_2}^{(k_2)}\cdots T_{\iota_{N-1}}f_{i_N}^{(k_N)}\in T_{\iota_1}(U_q^-)\cap U_q^-=\text{Ker } e'_{\iota-1}.$$

Therefore we have $\tilde{e}_{\iota_1}b_{(1)}=0$. Hence we obtain $k_1=\varepsilon_{\iota_1}(b^k)$. This implies

$$b_{(1)}=\tilde{e}_{i_1}^{\varepsilon_{i_1}(b^k)}b^k.$$

By Corollary 3.4.8, we have

$$f_{i_2}^{(k_2)}T_{\iota_3}\cdots T_{\iota_{N-1}}f_{i_N}^{(k_N)}\equiv\Lambda_{i_1}^{-1}(b_{(1)}) \mod qL(\infty).$$

Let $b_{(2)}=\tilde{e}_{i_2}^{\varepsilon_{\iota_1}(\Lambda_{i_1}^{-1}(b_{(1)}))}\Lambda_{i_1}^{-1}(b_1)$. Then similarly we have

$$\varepsilon_{\iota_2}(\Lambda_{i_1}^{-1}(b_2))=k_2,$$

$$b_{(2)}\equiv T_{\iota_2}f_{i_3}^{(k_3)}\cdots T_{\iota_{N-1}}f_{i_N}^{(k_N)},$$

and

$$f_{i_3}^{(k_3)}\cdots T_{\iota_{N-1}}f_{i_N}^{(k_N)}\equiv\Lambda_{i_2}^{-1}(b_{(2)}) \mod qL(\infty).$$

Repeating this, $k=(k_1,\cdots,k_N)$ is uniquly determined by b^k.

Now we define a map $b^k\mapsto f^{\rho(b^k)}$. It is trivial that this map is π^{-1}. Therefore π is injective. *Q. E. D.*

Let $Q_-=\sum \boldsymbol{Z}_{\leq 0}\alpha_\iota$.

$$t_i f_{\beta_m}t_i^{-1}=q_i^{-\langle h_\iota,\beta_m\rangle}f_{\beta_m}. \tag{4.1.7}$$

For $\xi\in Q_-$, we set

$$B_\xi=\{b^k\,;\ k\in\boldsymbol{Z}_{\geq 0}^N,\ wt(b^k)=\xi\}.$$

Hence we have $B_\xi \hookrightarrow B(\infty)_\xi$. By (4.1.7), we obtain

$$\#B_\xi=\#\{(c_1,\cdots,c_N)\in\boldsymbol{Z}_{\geq 0}^N\,;\ \xi=-\sum c_i\beta_i\}.$$

On the other hand, the PBW theorem for finite-dimensional semisimple Lie algebra implies

$$\begin{aligned}\#B(\infty)_\xi&=\dim_{\boldsymbol{Q}(q)}(U_q^-)_\xi\\&=\dim_{\boldsymbol{C}}(U^-)_\xi\\&=\#\{(c_1,\cdots,c_N)\in\boldsymbol{Z}_{\geq 0}^N\,;\ \xi=-\sum c_i\beta_i\}.\end{aligned}$$

Therefore

$$B_\xi=B(\infty)_\xi.$$

Hence we obtain (ii). *Q. E. D.*

4.2. Examples

Example 4.2.1.

$$\mathfrak{g}=A_2,\qquad I=\{1,2\},\qquad a_{ij}=a_{ji}=-1,$$

$$\beta_1=\alpha_1,\qquad \beta_2=\alpha_1+\alpha_2,\qquad \beta_3=\alpha_2\,.$$

In this case, $\Psi_{121}:\ B(\infty)\hookrightarrow u_\infty\otimes B_1\otimes B_2\otimes B_1$. We shall calculate $\Psi_{121}(f^{(k_1k_2k_3)})$. First we have

$$\Psi_1(f_1^{(k_3)})=u_\infty\otimes\tilde{f}_1^{k_3}b_1\,,$$

$$T_2f_1^{(k_3)}\equiv\tilde{f}_2^{*\varphi_2(b)}\tilde{e}_2^{\varepsilon_2(b)}b\qquad\text{where}\quad f_1^{(k_3)}\equiv b \mod qL(\infty)$$

and

$$\varphi_2(b)=k_3,\qquad \varepsilon_2(b)=0\,.$$

Therefore we have

$$\begin{aligned}T_2f_1^{(k_3)}&\equiv\tilde{f}_2^{*k_3}b\\&\xrightarrow{\Psi_2}b\otimes\tilde{f}_2^{k_3}b_2\\&\xrightarrow{\Psi_1}u_\infty\otimes\tilde{f}_1^{k_3}b_1\otimes\tilde{f}_2^{k_3}b_2\,.\end{aligned}$$

Since $\varphi_2(u_\infty\otimes\tilde{f}_1^{k_3}b_1)=k_3$ and $\varepsilon_2(\tilde{f}_1^{k_3}b_2)=k_3$, we have

$$\begin{aligned}f_2^{(k_2)}T_2f_1^{(k_3)}&\longmapsto\tilde{f}_2^{k_2}(u_\infty\otimes\tilde{f}_1^{k_3}b_1\otimes\tilde{f}_2^{k_3}b_2)\\&=u_\infty\otimes\tilde{f}_1^{k_3}b_1\otimes\tilde{f}_2^{k_2+k_3}b_2\,.\end{aligned}$$

$$T_1f_2^{(k_2)}T_2f_1^{(k_3)}\equiv\tilde{f}_1^{*\varphi_1(b')}\tilde{e}_1^{\varepsilon_1(b')}b,\qquad\text{where}\quad b'\equiv f_2^{(k_2)}T_2f_1^{(k_3)}\,.$$

Since $\varphi_1(b')=\varphi_1(u_\infty\otimes\tilde{f}_1^{k_3}b_1\otimes\tilde{f}_2^{k_2+k_3}b_2)=k_2$ and $\varepsilon_1(b')=k_3$ we have

$$T_1 b' \longmapsto \tilde{e}_1^{\varepsilon_1(b')} b' \otimes \tilde{f}_1^{k_2} b_1$$
$$\longmapsto \tilde{e}_1^{k_3}(u_\infty \otimes \tilde{f}_1^{k_3} b_1 \otimes \tilde{f}_2^{k_2+k_3} b_2) \otimes \tilde{f}_1^{k_2} b_1$$
$$= u_\infty \otimes b_1 \otimes \tilde{f}_2^{k_2+k_3} b_2 \otimes \tilde{f}_1^{k_1} b_1 .$$

Therefore we obtain

$$f^{(k_1, k_2, k_3)} = f_{\beta_1}^{(k_1)} f_{\beta_2}^{(k_2)} f_{\beta_3}^{(k_3)}$$
$$= f_1^{(k_1)} T_1 f_2^{(k_2)} T_2 f_1^{(k_3)}$$
$$\equiv f_1^{(k_1)} T_1 b'$$
$$\longmapsto \tilde{f}_1^{k_1}(u_\infty \otimes b_1 \otimes \tilde{f}_2^{k_2+k_3} b_2 \otimes \tilde{f}_1^{k_2} b_1)$$
$$= \begin{cases} u_\infty \otimes \tilde{f}_1^{k_1} b_1 \otimes \tilde{f}_2^{k_2+k_3} b_2 \otimes \tilde{f}_1^{k_2} b_1 & (k_1 < k_3) \\ u_\infty \otimes \tilde{f}_1^{k_3} b_1 \otimes \tilde{f}_2^{k_2+k_3} b_2 \otimes \tilde{f}_1^{k_1+k_2-k_3} b_1 & (k_1 \geqq k_3) . \end{cases}$$

By [3], we know

$$B(\infty) = \{u_\infty \otimes \tilde{f}_1^{k_2} b_1 \otimes \tilde{f}_2^{k_2+k_3} b_2 \otimes \tilde{f}_1^{k_1} b_1 \,;\; 0 \leqq k_1,\ 0 \leqq k_2,\ 0 \leqq k_3\} .$$

We shall calculate $(u_\infty \otimes \tilde{f}_1^{k_2} b_1 \otimes \tilde{f}_2^{k_2+k_3} b_2 \otimes \tilde{f}_1^{k_1} b_1)^*$. First we have

$$(\tilde{f}_1^{k_1} \tilde{f}_2^{k_2+k_3} \tilde{f}_1^{k_2} u_\infty)^* \longmapsto u_\infty \otimes \tilde{f}_1^{k_2} b_1 \otimes \tilde{f}_2^{k_2+k_3} b_2 \otimes \tilde{f}_1^{k_1} b_1 .$$

And we have

$$\tilde{f}_1^{k_1} \tilde{f}_2^{k_2+k_3} \tilde{f}_1^{k_2} u_\infty \longmapsto \tilde{f}_1^{k_1} \tilde{f}_2^{k_2+k_3}(u_\infty \otimes \tilde{f}_1^{k_2} b_1)$$
$$\longmapsto \tilde{f}_1^{k_1}(u_\infty \otimes \tilde{f}_2^{k_2+k_3} b_2 \otimes \tilde{f}_1^{k_2} b_1)$$
$$\longmapsto \begin{cases} u_\infty \otimes \tilde{f}_1^{k_1} b_1 \otimes \tilde{f}_2^{k_2+k_3} b_2 \otimes \tilde{f}_1^{k_2} b_1 & (k_1 \leqq k_3) \\ u_\infty \otimes \tilde{f}_1^{k_3} b_1 \otimes \tilde{f}_2^{k_2+k_3} b_2 \otimes \tilde{f}_1^{k_1+k_2-k_3} b_1 & (k_1 \geqq k_3) . \end{cases}$$

Therefore we have

$$f^{(k_1, k_2, k_3)} \equiv \tilde{f}_1^{k_1} \tilde{f}_2^{k_2+k_3} \tilde{f}_1^{k_2} u_\infty$$
$$\longmapsto (u_\infty \otimes \tilde{f}_1^{k_2} b_1 \otimes \tilde{f}_2^{k_2+k_3} b_2 \otimes \tilde{f}_1^{k_1} b_1)^* .$$

Example 4.2.2.

$$\mathfrak{g} = B_2, \qquad I = \{1, 2\}, \qquad a_{ij} = -2, \qquad a_{ji} = -1,$$
$$\beta_1 = \alpha_1, \qquad \beta_2 = 2\alpha_1 + \alpha_2, \qquad \beta_3 = \alpha_1 + \alpha_2, \qquad \beta_4 = \alpha_2 .$$

We can calculate similarly,

$$f^{(k_1,k_2,k_3,k_4)} \longmapsto \begin{cases} u_\infty\otimes\tilde{f}_2^{k_2}b_2\otimes\tilde{f}_1^{k_1+2k_2}b_1\otimes\tilde{f}_2^{k_3+k_4}b_2\otimes\tilde{f}_1^{k_3}b_1 (k_2\leqq k_4,\ k_1\leqq -2k_2+k_3+2k_4) \\ u_\infty\otimes\tilde{f}_2^{k_2}b_2\otimes\tilde{f}_1^{k_3+2k_4}b_1\otimes\tilde{f}_2^{k_3+k_4}b_2\otimes\tilde{f}_1^{k_1+2k_2-2k_4}b_1 (k_2\leqq k_4,\ k_1\geqq -2k_2+k_3+2k_4) \\ u_\infty\otimes\tilde{f}_2^{k_4}b_2\otimes\tilde{f}_1^{k_1+2k_4}b_1\otimes\tilde{f}_2^{k_2+k_3}b_2\otimes\tilde{f}_1^{2k_2+k_3-2k_4}b_1 (k_2\geqq k_4,\ k_1\leqq k_3) \\ u_\infty\otimes\tilde{f}_2^{k_4}b_2\otimes\tilde{f}_1^{k_3+2k_4}b_1\otimes\tilde{f}_2^{k_2+k_3}b_2\otimes\tilde{f}_1^{k_1+2k_2-2k_4}b_1 (k_2\geqq k_4,\ k_1\geqq k_3). \end{cases}$$

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