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A Characterization for Fourier Hyperfunctions

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Abstract

The space of test functions for Fourier hyperfunctions is characterized by two conditions $\sup|\varphi(x)|\exp k|x|<\infty$ and $\sup|\hat{\varphi}(\xi)|\exp h|\xi|<\infty$ for some $h, k>0$. Combining this result and the new characterization of Schwartz space in [1] we can easily compare two important spaces $\mathcal{F}$ and $\mathcal{S}$ which are both invariant under Fourier transformations.

§0. Introduction

The purpose of this paper is to give new characterization of the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions.

In [6], K. W. Kim, S. Y. Chung and D. Kim introduce the real version of the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions as follows,

$$\mathcal{F}=\left\{\varphi\in C^\infty \mid \sup_{\alpha,x}\frac{|\partial^\alpha\varphi(x)|\exp k|x|}{h^{|\alpha|}\alpha!}<\infty \qquad \text{for some} \quad k,\ h>0\right\}.$$

They also show the equivalence of the above definition and Sato-Kawai's original definition in complex form.

Also, in [1] J. Chung, S. Y. Chung and D. Kim give new characterization of the Schwartz space $\mathcal{S}$, i.e., show that for $\varphi\in C^\infty$ the following are equivalent:

(1) $\varphi\in\mathcal{S}$;

(2) $\sup|x^\alpha\varphi(x)|<\infty$, $\sup|\partial^\beta\varphi(x)|<\infty$ for all multi-indices α and β;

(3) $\sup|x^\alpha\varphi(x)|<\infty$, $\sup|\xi^\beta\hat{\varphi}(\xi)|<\infty$ for all multi-indices α and β.

In a similar fashion as above we will give new characterization of the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions as the main theorem in this paper which says that for $\varphi\in C^\infty$ the following are equivalent:

(1) $\varphi\in\mathcal{F}$;

(2) $\sup|\varphi(x)|\exp k|x|<\infty$, $\sup|\hat{\varphi}(\xi)|\exp h|\xi|<\infty$ for some $h, k>0$.

Observing the above growth conditions we can easily see that the space $\mathcal{F}$ which is invariant under the Fourier transformation is much smaller than

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Schwartz space $\mathcal{S}$. Since an element in the strong dual $\mathcal{F}'$ of the space $\mathcal{F}$ is called a Fourier hyperfunction, the space $\mathcal{F}'$ of Fourier hyperfunctions which is also invariant under the Fourier transformation is much bigger than the space $\mathcal{S}'$ of tempered distributions.

Section 1 is devoted to providing the necessary definitions and preliminaries. We prove the main theorem in Section 2.

§ 1. Preliminaries

We use the multi-index notations; for $x=(x_1, \cdots, x_n)$, $\xi=(\xi_1, \cdots, \xi_n)\in \boldsymbol{R}^n$ and a multi-index $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)\in \boldsymbol{N}_0^n$, $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$, $|\alpha|=\alpha_1+\cdots+\alpha_n$ with $\partial_j=\partial/\partial x_j$, and $\boldsymbol{N}_0$ the set of non-negative integers.

For $f\in L^1(\boldsymbol{R}^n)$ the Fourier transform $\hat{f}$ is the bounded continuous function in $\boldsymbol{R}^n$ defined by

$$\hat{f}(\xi)=\int e^{-ix\cdot\xi}f(x)dx, \qquad \xi\in\boldsymbol{R}^n \tag{1.1}$$

Definition 1.1. We denote by $\mathcal{S}$ or $\mathcal{S}(\boldsymbol{R}^n)$ the Schwartz space of all $\varphi\in C^\infty(\boldsymbol{R}^n)$ such that

$$\sup_x|x^\alpha\partial^\beta\varphi(x)|<\infty \tag{1.2}$$

for all multi-indices α and β.

We need the following characterization to compare the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions with the above space.

Theorem 1.2 [1]. (i) *The Schwartz space $\mathcal{S}$ consists of all $\varphi\in C^\infty(\boldsymbol{R}^n)$ satisfying the conditions*

$$\sup_x|x^\alpha\varphi(x)|<\infty, \qquad \sup_x|\partial^\beta\varphi(x)|<\infty \tag{1.3}$$

for all multi-indices α and β.

(ii) *Also, the Schwartz space can be characterized by the following two conditions*

$$\sup|x^\alpha\varphi(x)|<\infty, \qquad \sup|\xi^\beta\hat{\varphi}(\xi)|<\infty \tag{1.4}$$

for all multi-indices α and β.

Now, we are going to introduce the original complex version and new real definition of test functions for the Fourier hyperfunctions as in [6], and state their equivalence.

Definition 1.3 [6]. A real valued function φ is in $\mathcal{F}$ if $\varphi \in C^\infty(\boldsymbol{R}^n)$ and if there are positive constants h and k such that

$$|\varphi|_{k,h} = \sup_{\alpha, x} \frac{|\partial^\alpha \varphi(x)|}{h^{|\alpha|}\alpha!} \exp k|x| < \infty .$$

Definition 1.4 [5]. A complex valued function $\varphi(z)$ is in $\mathcal{P}_*$ if $\varphi(z)$ is holomorphic in a tubular neighborhood $\boldsymbol{R}^n + i\{|y| \leqq r\}$, for some r, of $\boldsymbol{R}^n$ and if for some $k > 0$

$$\sup_{z \in \boldsymbol{R}_n + i\{|y| \leqq r\}} |\varphi(z)| \exp k|z| < \infty .$$

Theorem 1.5 [6]. *The space $\mathcal{F}$ is isomorphic to the space $\mathcal{P}_*$.*

Definition 1.6. We denote by $\mathcal{F}'$ the strong dual space of $\mathcal{F}$ and call its elements *Fourier hyperfunctions.*

Thus the global theory of the Fourier hyperfunctions is nothing but the duality theory for the space $\mathcal{F}$.

§ 2. Main Theorem

Now we shall give new characterization of the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions which is the main result in this paper.

First, we prove

Theorem 2.1. *The following conditions for $\varphi \in C^\infty$ are equivalent:*

(i) *There are positive constants k and h such that*

$$\sup_{\alpha, x} \frac{|\partial^\alpha \varphi(x)| \exp k|x|}{h^{|\alpha|}\alpha!} < \infty . \tag{2.1}$$

(ii) *There are positive constants C, k and h such that*

$$\sup_x |\varphi(x)| \exp k|x| < \infty , \tag{2.2}$$

$$\sup_x |\partial^\alpha \varphi(x)| \leqq C h^{|\alpha|}\alpha! . \tag{2.3}$$

(iii) *There are positive constants k and h such that*

$$\sup_x |\varphi(x)| \exp k|x| < \infty , \tag{2.4}$$

$$\sup_\xi |\hat{\varphi}(\xi)| \exp h|\xi| < \infty . \tag{2.5}$$

Proof. The implications (i) ⇒ (ii), (i) ⇒ (iii) are trivial. So it suffices to prove the implications (iii) ⇒ (ii) and (ii) ⇒ (i) in order.

(iii) ⇒ (ii): By the inequality (2.5) we have

$$
\begin{aligned}
|\partial^\alpha \varphi(x)| &\leqq \frac{1}{(2\pi)^n}\int |\xi^\alpha||\hat{\varphi}(\xi)|d\xi \\
&\leqq \frac{M}{(2\pi)^n}\int |\xi|^{|\alpha|}\exp(-h|\xi|)d\xi \\
&\leqq \frac{M}{(2\pi)^n}\sup_\xi \frac{|\xi|^{|\alpha|}}{\exp(h|\xi|/2)}\int \exp(-h|\xi|/2)d\xi \\
&\leqq CA^{|\alpha|}\alpha!
\end{aligned}
$$

for some positive constants M, A and C. Thus, we obtain the condition (2.3) which completes the proof of the implication (iii) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i): First, we can assume that φ is real valued. By integration by parts we obtain that

$$
\|x^\beta \partial^\alpha \varphi(x)\|_{L^2}^2 = \left|\int_{R^n} \partial^\alpha[x^{2\beta}\partial^\alpha \varphi(x)]\varphi(x)dx\right|.
$$

Note that the boundary terms tend to zero by Theorem 1.2. Therefore, applying the Leibniz formula we have, for some constant A,

$$
\begin{aligned}
&\|x^\beta \partial^\alpha \varphi(x)\|_{L^2}^2 \\
&\leqq \int_{R^n} \sum_{\substack{\gamma\leqq\alpha \\ \gamma\leqq 2\beta}} \binom{\alpha}{\gamma}\binom{2\beta}{\gamma}\gamma!\,|x^{2\beta-\gamma}\partial^{2\alpha-\gamma}\varphi(x)||\varphi(x)|dx \\
&\leqq n^{|\alpha|}\int \sum_{\substack{\gamma\leqq\alpha \\ \gamma\leqq 2\beta}} \binom{2\beta}{\gamma}\gamma!(2\beta-\gamma)!A^{|2\beta-\gamma|}|\varphi(x)|\exp k|x|\,|\partial^{2\alpha-\gamma}\varphi(x)|\exp(-k|x|/2)dx \\
&\leqq C_1 n^{|\alpha|}\sum_{\substack{\gamma\leqq\alpha \\ \gamma\leqq 2\beta}} \binom{2\beta}{\gamma}(2\beta)!A^{|2\beta-\gamma|}MCh^{|2\alpha-\gamma|}(2\alpha-\gamma)!
\end{aligned}
$$

where $C_1=\int_{R^n}\exp(-k|x|/2)$ and $M \geqq \sup_x |\varphi(x)|\exp k|x|$. Here we use the inequality

$$
x^\alpha \leqq \alpha!\exp|x|
$$

for any $\alpha\in N_0^n$. If we choose positive constants A, $h>1$ if necessary, and use the inequalities

$$
(\alpha!)^2 \leqq (2\alpha)! \leqq n^{2|\alpha|}(\alpha!)^2
$$

we have, for some C_2

$$
\begin{aligned}
\|x^\beta \partial^\alpha \varphi\|_{L^2}^2 &\leqq C_2 n^{|\alpha|+2|\beta|}(2\alpha)!(2\beta)!A^{2|\beta|}h^{2|\alpha|} \\
&\leqq C_2(nA)^{4|\beta|}(n\sqrt{n}h)^{2|\alpha|}(\alpha!)^2(\beta!)^2.
\end{aligned}
$$

Thus we obtain that for some positive constants C_0, C_1 and C_2 such that

$$
\frac{\|x^\beta \partial^\alpha \varphi(x)\|_{L^2}}{C_2^{|\beta|}\beta!} \leqq C_0 C_1^{|\alpha|}\alpha!.
$$

Therefore, summing up with respect to β we can choose a positive constant

k such that

$$\|\partial^{\alpha}\varphi(x)\exp k|x|\|_{L^2}\leqq C_0C_1^{|\alpha|}\alpha!.$$

By the Cauchy-Schwarz inequality there exists a positive constant C_3 such that

$$\left\|\partial^{\alpha}\varphi(x)\exp\frac{k}{2}|x|\right\|_{L^1}\leqq\|\partial^{\alpha}\varphi(x)\exp k|x|\|_{L^2}\left[\int\exp(-k|x|)dx\right]^{1/2}$$
$$\leqq C_3C_1^{|\alpha|}\alpha!.$$

Also, there exist positive constants k and C_1 such that

$$\|\partial^{\alpha}\varphi(x)\exp k\sqrt{1+|x|^2}\|_{L^1}\leqq C_0C_1^{|\alpha|}\alpha!.$$

Hence

$$|\partial^{\alpha}\varphi(x)\exp k\sqrt{1+|x|^2}|$$
$$=\left|\int_{-\infty}^{x_n}\cdots\int_{-\infty}^{x_1}\partial_1\cdots\partial_n(\partial^{\alpha}\varphi(x)\exp k\sqrt{1+|x|^2})dx\right|$$
$$=\left|\int_{-\infty}^{x_n}\cdots\int_{-\infty}^{x_1}\partial_n\cdots\partial_2[(\partial_1\partial^{\alpha}\varphi)\exp k\sqrt{1+|x|^2}\right.$$
$$\left.+\partial^{\alpha}\varphi(x)\cdot\partial_1(\exp k\sqrt{1+|x|^2})]dx\right|$$
$$\leqq\int_{-\infty}^{x_n}\cdots\int_{-\infty}^{x_1}\sum|\partial_{j_1}\cdots\partial_{j_r}(\partial^{\alpha}\varphi)||(\partial_{j_{r+1}}\cdots\partial_{j_n})\exp k\sqrt{1+|x|^2}|dx$$

where the summation is taken over all $r=0, 1, \cdots, n$ and $\{j_1, \cdots, j_n\}$ is a permutation of $\{1, \cdots, n\}$.

We can prove by induction

$$|(\partial_{j_1}\cdots\partial_{j_r})\exp k\sqrt{1+|x|^2}|\leqq P_r(k)\exp k\sqrt{1+|x|^2}$$

where $P_r(k)$ is a polynomial of k of r-th degree. Hence we derive that

$$|\partial^{\alpha}\varphi(x)\exp k\sqrt{1+|x|^2}|$$
$$\leqq\int C\sum|P_{n-r}(k)||(\partial_{j_1}\cdots\partial_{j_r})(\partial^{\alpha}\varphi)|\exp k\sqrt{1+|x|^2}dx$$
$$\leqq C\sum|P_{n-r}(k)|C_0C_1^{|\alpha|+r}(\alpha+\beta)!$$
$$\leqq C(k, n)C_1^{|\alpha|}\alpha!$$

where β is a multi-index with $|\beta|=r$. Therefore, using the relation

$$\exp k|x|\leqq\exp k\sqrt{1+|x|^2}\leqq e^k\exp k|x|$$

we obtain

$$\sup_x|\partial^{\alpha}\varphi(x)|\exp k|x|\leqq C(k, n)C_1^{|\alpha|}\alpha!$$

which completes the proof.

Now we can rephrase Theorem 2.1 as follows.

Theorem 2.2. *The space $\mathcal{F}$ of test functions for the Fourier hyperfunctions consists of all locally integrable functions such that for some h, $k>0$*

$$\sup_x |\varphi(x)| \exp k|x| < \infty ,$$

$$\sup_\xi |\hat{\varphi}(\xi)| \exp h|\xi| < \infty .$$

Remark. Combining Theorem 1.2 on the Schwartz space $\mathcal{S}$ and Theorem 2.2 on the space $\mathcal{F}$ we can easily compare the spaces $\mathcal{S}$ and $\mathcal{F}$ which are both invariant under the Fourier transformations as follows:

(i) The space $\mathcal{S}$ consists of all C^∞ functions φ such that φ itself and its Fourier transform $\hat{\varphi}$ are both rapidly decreasing.

(ii) The space $\mathcal{F}$ consists of all C^∞ functions φ such that φ itself and its Fourier transform $\hat{\varphi}$ are both exponentially decreasing.

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