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# Oriented $Z_4$ Actions without Stationary Points

By

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## § 0. Introduction

Let $\boldsymbol{Z}_{2^k}$ denote the cyclic group of order $2^k$ ($k\geqq 2$). In [8], we have studied the theory $W_*(\boldsymbol{Z}_{2^k}; Af)$ of almost free $\boldsymbol{Z}_{2^k}$ actions on closed Wall manifolds, i.e., an element $g\in\boldsymbol{Z}_{2^k}$ has no fixed point on the manifolds unless $g$ is 1 or the unique element of order two. When $k=2$, such objects are the stationary point free ("proper") $\boldsymbol{Z}_4$ actions and the above theory is denoted by $W_*(\boldsymbol{Z}_4; p)$.

On the other hand let $\Omega_*(\boldsymbol{Z}_4; p)$ be the theory of oriented (orientation-preserving), stationary point free $\boldsymbol{Z}_4$ actions, which has been studied in [16]. Letting $\Omega_*$ be the oriented cobordism ring, then

**Theorem** (R. E. Stong). *For the map $\sigma: \Omega_*(\boldsymbol{Z}_4; p)\to\Omega_*$ which forgets actions on manifolds, the image* $\mathrm{Im}(\sigma)$ *is precisely the ideal of classes $\alpha\in\Omega_*$ having even Euler characteristic.*

This was proved in [16] for arbitrary stationary point free $\boldsymbol{Z}_{2^k}$ actions, but the proof is reduced to the case $k=2$.

In connection with this result, we treat here the restriction map $r; \Omega_*(\boldsymbol{Z}_4; p)\to\Omega_*(\boldsymbol{Z}_2; All)$ induced by $\boldsymbol{Z}_2\subset\boldsymbol{Z}_4$ where $\Omega_*(\boldsymbol{Z}_2; All)$ is the theory of all oriented involutions.

In section 1, we first state some basic facts on $\Omega_*(\boldsymbol{Z}_4; p)$, and summarize the theories $W_*(\boldsymbol{Z}_2; -)$ in [13] which are important to further arguments.

We show in section 2 that the image of $r$ lies in a homology $H_\beta(d)$ which is obtained from two differentials $\beta$ and $d$ on the relative theory $W_*(\boldsymbol{Z}_2; rel)$. The kernel $\mathcal{E}$ of the induced map $r_*: \Omega_*(\boldsymbol{Z}_4; p)\to H_\beta(d)$ consists of the images of two types of extensions from $\Omega_*(\boldsymbol{Z}_2; All)$ and $\mathcal{T}_2$, the torsion part of order 2 in $\Omega_*(\boldsymbol{Z}_4; p)$ studied in [8]. Hence an embedding $r_*: \Omega_*(\boldsymbol{Z}_4; p)/\mathcal{E}\hookrightarrow H_\beta(d)$ is obtained (Theorem 2.4). In conclusion of this section, we calculated the homology $H_\beta(d)$ (Proposition 2.7).

From these, in section 3 we obtain a necessary and sufficient condition for

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an element in $\Omega_*(\boldsymbol{Z}_2; All)$ to belong to the image of $r$ (Theorem 3.1). Using this, we give some examples which belong to $\mathrm{Im}(r)$ (Example 3.3) and a necessary condition for an element in $\Omega_*(\boldsymbol{Z}_2; All)$ to come from the theory $\Omega_*(\boldsymbol{Z}_4; All)$ of all oriented $\boldsymbol{Z}_4$ actions (Proposition 3.4). We return to the embedding $r_*$ in section 2, and show an example which doesn't belong to $\mathcal{E}$ (Example 3.5). Next we consider an $\Omega_*$ algebra $\mathcal{R}_*$ generated by the standard involutions on the complex projective spaces $CP(n)$ and determine the ideal $\mathcal{J}_*=\{y\in\mathcal{R}_*|y\in \mathrm{Im}(r)\}$ by using the above example (Theorem 3.7). Some torsion elements $y$ in $\mathcal{J}_*$ come from those $x$ of order 4 in $\Omega_*(\boldsymbol{Z}_4; p)$: that is, those which don't belong to $\mathcal{E}$. We show that such elements $x$ also have order 4 in $\Omega_*(\boldsymbol{Z}_4; All)$ (Theorem 3.9 and Example 3.10). Finally we give examples such that they don't belong to $\mathcal{E}$ and their restriction don't belong to $\mathcal{J}_*$ (Proposition 3.12).

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## § 1. Preliminaries

As an oriented analogue of the unoriented bordism theory $\mathfrak{N}_*(\boldsymbol{Z}_2; All)$ of all involutions on closed manifolds in Conner and Floyd [6, Sec. 28], the theory $\Omega_*(\boldsymbol{Z}_2; All)$ of all oriented involutions on closed oriented manifolds has been introduced and studied by Rosenzweig [14], Conner [5], Stong [17], and Kosniowski and Ossa [13]. The basic notations of this theory are found there, so we omit these here. Next we summarize the theory $\Omega_*(\boldsymbol{Z}_4; p)$ of oriented, stationary point free $\boldsymbol{Z}_4$ actions, which has been studied in Conner and Floyd [6, (45.5)] and Stong [16]. On the other hand Rowlett [15] contains some results on this theory as a special case of even-order group actions. Detailed results have been obtained for the corresponding theory $\mathfrak{N}_*(\boldsymbol{Z}_4; p)$ in the unoriented category by Beem [2]. In theories $\Omega_*(G; -)$ we denote by $[M, t]$ the bordism class of an oriented $G$ action $t$ on a (closed) oriented manifold $M$ in general (here $G=\boldsymbol{Z}_2$ or $\boldsymbol{Z}_4$ with generator $t$).

**Definition 1.1.** Let $e$ and $s$ be the maps defined by $e([M, A])=[\boldsymbol{Z}_4\times_{\boldsymbol{Z}_2}M, i\times\mathrm{id}]$ and $s([M, A])=[S^1\times_{\boldsymbol{Z}_2}M, i\times\mathrm{id}]$ for each $[M, A]\in\Omega_*(\boldsymbol{Z}_2; All)$ where $S^1$ is the unit circle with $\boldsymbol{Z}_4$ action $i=\sqrt{-1}$. On the other hand, let $d:\Omega_*(\boldsymbol{Z}_2; All)\to\Omega_{*+1}(\boldsymbol{Z}_2; All)$ be the map given by $d([M, A])=[S^1\times_{\boldsymbol{Z}_2}M, -1\times\mathrm{id}]$. Then the relation $r\circ s=d$ holds for the restriction map $r$ in Introduction.

We list some basic properties of the theory $\Omega_*(\boldsymbol{Z}_4; p)$:

(1.2) The composition $e\circ r$ (resp. $r\circ e$) is the multiplication by 2 in $\Omega_*(\boldsymbol{Z}_4; p)$ (resp. $\Omega_*(\boldsymbol{Z}_2; All)$) (cf. [15, Prop. 4.2]), hence
(1.3) $e:\Omega_*(\boldsymbol{Z}_2; All)\otimes R_2\cong\Omega_*(\boldsymbol{Z}_4; p)\otimes R_2$ with the inverse map $e^{-1}=(1/2)r$ where $R_2$ is the subring of $\boldsymbol{Q}$ generated by $\boldsymbol{Z}$ and 1/2.
(1.4) If $x$ is torsion free in $\Omega_*(\boldsymbol{Z}_4; p)$, so is $r(x)$ in $\Omega_*(\boldsymbol{Z}_2; All)$. Equivalently,

if $y$ is a torsion free in $\Omega_*(\boldsymbol{Z}_2; All)$, so is $e(y)$ in $\Omega_*(\boldsymbol{Z}_4; p)$.
(1.5) A torsion element in $\Omega_*(\boldsymbol{Z}_4; p)$ is of order 2 or 4.

By (1.3), $\Omega_*(\boldsymbol{Z}_4; p)\otimes R_2$ is freely generated (over $\Omega_*\otimes R_2$) by the class $\{[\boldsymbol{Z}_4, i], e(C_{2I+2})|I\}$ where $C_{2I+2}=C_{2m_1+2}\times\cdots\times C_{2m_p+2}$ is the monomial on $C_{2m+2}$ defined at (1.8) for each $I=(m_1, \cdots, m_p)$ with $m_1\geqq\cdots\geqq m_p\geqq 0$ (cf. [5, p. 101]). On the other hand, the above (1.5) is obtained from (1.2) and the following:
(1.6) Any torsion element in $\Omega_*(\boldsymbol{Z}_2; All)$ is of order 2 (cf. [14, Theorem 3.4]).

**Example.**

(1.7) For $m\geqq 1$, define an oriented $\boldsymbol{Z}_4$ action $T$ on the complex projective space $\boldsymbol{C}P(2m+2)$ by

$$T([z_0:z_1:z_2:\cdots:z_{2m+1}:z_{2m+2}])=[\bar{z}_0:-\bar{z}_2:\bar{z}_1:\cdots:-\bar{z}_{2m+2}:\bar{z}_{2m+1}].$$

We note that the only stationary point of $T$ is $*=[1:0:\cdots:0]$. Then $T\times\cdots\times T$ acts on $\boldsymbol{C}P(2)^{m+1}$ with one stationary point $(*, \cdots, *)$, and the action at this point is the same as the action at the point $*$ of $T$ on $\boldsymbol{C}P(2m+2)$. By excising neighborhoods of these points of $\boldsymbol{C}P(2m+2)$ and $(\boldsymbol{C}P(2))^{m+1}$ (suitably oriented), and fitting together along the resulting boundaries, we get an orientable manifold $V^{2m+2}$ with the stationary point free $\boldsymbol{Z}_4$ action $T$ (cf. [6, p. 142]).
(1.8) Let $C^n=[\boldsymbol{C}P(n), I_n]$ be an element in $\Omega_*(\boldsymbol{Z}_4; All)$ defined by

$$I_n([z_0:z_1:\cdots:z_n])=[iz_0:z_1:\cdots:z_n] \qquad (n\geqq 1),$$

and put $C_n=[\boldsymbol{C}P(n), A_n]$ in $\Omega_*(\boldsymbol{Z}_2; All)$ where $A_n=I_n^2$.

We see that $C_n$ doesn't come from a stationary point free $\boldsymbol{Z}_4$ action. If $n$ is even, this follows from Theorem in Introduction. See Theorem 3.7 in general.

Next we view the bordism theories $W_*(G; -)$ as an equivariant analogue of the Wall cobordism ring $W_*$ in Wall [19]. Our objects are Wall manifolds of type $(G, 1)$ in the sense of Komiya [11] and Stong [17]. An oriented $G$ action $(M, t)$ falls into this category. Suppose that $M$ admits an orientation-reversing involution $R$ which commutes with $t$. Then
(1.9) $S^1\times_R M=S^1\times M/-1\times R$ with $G$ action $\mathrm{id}\times t$ has the induced Wall structure of type $(G, 1)$ as $\beta([S^1\times_R M, \mathrm{id}\times t])=[M, t]$ where $\beta: W_*(G; -)\to\Omega_{*-1}(G; -)$ is the Bockstein homomorphism.

This induces a universal coefficient sequence:

$$\text{(1.10)}\quad 0\longrightarrow\Omega_*(G; -)\otimes\boldsymbol{Z}_2\xrightarrow{\ i\ }W_*(G; -)\xrightarrow{\ \beta\ }\mathrm{Tor}\,(\Omega_{*-1}(G; -), \boldsymbol{Z}_2)\longrightarrow 0$$

(*cf.* [17, Prop. 6.1]).

Now we summarize the theory $W_*(\boldsymbol{Z}_2; -)$ which is denoted by $\mathcal{O}_*^{(2)}(-)$ in [13].

(1.11) As regards (1.10) $\beta(W_*(\boldsymbol{Z}_2; All))=\mathrm{Tor}\,\Omega_*(\boldsymbol{Z}_2; All)$, the torsion part of $\Omega_*(\boldsymbol{Z}_2; All)$ and the above $i$ induces embedding $i: \mathrm{Tor}\,\Omega_*(\boldsymbol{Z}_2; All) \hookrightarrow W_*(\boldsymbol{Z}_2; All)$ by (1.6).

(1.12) There is a splitting $W_*(\boldsymbol{Z}_2; All)=\mathrm{Im}(i_*)\oplus Q_*^{(2)}$ as $W_*$ modules in the usual long exact sequence $(i_*, j_*, \partial)$ for the triple $(All, Free, \emptyset)$, where $\mathrm{Im}(i_*)$ is the image of the free involutions and freely generated by $[\boldsymbol{Z}_2, -1]$ as a $W_*/E_*$ module. Here $E_*$ is the ideal of $x\in W_*$ having even Euler characteristic, so $W_*/E_*=\boldsymbol{Z}_2[w_4]$, $w_4=[\boldsymbol{C}P(2)]$ as a $\boldsymbol{Z}_2$ polynomial ring. On the other hand, $Q_*^{(2)}$ is the kernel of a map $q: W_*(\boldsymbol{Z}_2; All)\to \mathrm{Im}(i_*)$ with $q\circ \bar{i}=\mathrm{id}$ for the inclusion $\bar{i}: \mathrm{Im}(i_*)\hookrightarrow W_*(\boldsymbol{Z}_2; All)$. The definition of $q$ is as follows: $q(y)=\bar{\chi}(y)[\boldsymbol{C}P(2)]^n\cdot[\boldsymbol{Z}_2, -1]$ if $\dim y=4n$ and $q(y)=0$ otherwise (Here $\bar{\chi}([M, A])=\chi([M/A])$, the Euler characteristic modulo 2 of the orbit space $M/A$.) (cf. [13, Theorem 3.2, Cor. 6.4, Cor. 7.5 and Sec. 8]).

Denote the theory $W_*(\boldsymbol{Z}_2; All, Free)$ by $W_*(\boldsymbol{Z}_2; rel)$. From the above,

(1.13) there is an embedding $j_*: Q_*^{(2)}\hookrightarrow W_*(\boldsymbol{Z}_2; rel)$ (cf. [13, Sec. 9]), and

(1.14) the images $d(W_*(\boldsymbol{Z}_2; All))$ and $\beta(W_*(\boldsymbol{Z}_2; All))=\mathrm{Tor}\,\Omega_*(\boldsymbol{Z}_2; All)$ are contained in $Q_*^{(2)}$ by definition and [13, Lemma 8.2].

(1.15) $W_*(\boldsymbol{Z}_2; rel)$ is the free $W_*$ module generated by the class $\{\xi_\omega | \omega\in\Gamma\}$, $\xi_\omega=\xi_{n_1}\times\cdots\times\xi_{n_{2p}}$ where $\Gamma$ consists of all sequences of integers $\omega=(n_1, \cdots n_{2p})$ of even length with $n_1\geqq\cdots\geqq n_{2p}\geqq 0$. Here $\{\xi_n | n\geqq 0\}$ is the class such that each $\xi_{2n}$ is the canonical line bundle over the real projective space $\boldsymbol{R}P(2n)$, and $\xi_{2n+1}=d(\xi_{2n})$ by the map $d$ as Definition 1.1 (cf. [12, Lemma 3.4.3]). From this $d(\xi_{2n+1})=0$ and $d$ acts on $\xi_\omega$ by the derivation (cf. [9, Lemma 1], [1, Theorem 3] and [18, Prop. 3.3]). In this way the properties on $d$ are inherited from the corresponding unoriented theory $\mathfrak{N}_*(\boldsymbol{Z}_2; rel)$ via the embedding $W_*(\boldsymbol{Z}_2; rel)\hookrightarrow \mathfrak{N}_*(\boldsymbol{Z}_2; rel)$. On the other hand $\beta(\xi_{2n})=0$ and $\beta(\xi_{2n+1})=\xi_{2n}$ (cf. (1.9)), and $\beta$ also acts on $\xi_\omega$ by the derivation (cf. [13, Theorem 4.2]). The map $\beta$ commutes with $d$ in $W_*(\boldsymbol{Z}_2; rel)$.

According to the above derivations, let $H_d$ or $H_\beta$ be the homology of the complex $W_*(\boldsymbol{Z}_2; rel)$ with differential $d$ or $\beta$, respectively. Then

(1.16) $H_d\cong W_*[\xi_{2m}^2 | m\geqq 0]$ as a free $W_*$ algebra (cf. [1, Lemma 7]), and

(1.17) $H_\beta\cong C_*[\xi_{2m+1}^2 | m\geqq 0]$ as a free $C_*$ algebra where $C_*$ is the $\boldsymbol{Z}_2$ polynomial ring on the class $\{[\boldsymbol{C}P(2n)] | n\geqq 1\}$.

Denote by $B_*$ the module of $W_*$ indecomposables in $W_*(\boldsymbol{Z}_2; rel)$, then $W_*(\boldsymbol{Z}_2; rel)\cong W_*\otimes_{\boldsymbol{Z}_2}B_*$ as graded differential algebras. Thus $H_\beta\cong C_*\otimes_{\boldsymbol{Z}_2} H_*(B_*, \beta)$ by the Künneth formula since $H_*(W_*, \beta)\cong(\Omega_*/\mathrm{Tor}\,\Omega_*)\otimes\boldsymbol{Z}_2\cong C_*$ (cf. [19, Lemma 13]). Then (1.17) is obtained from $H_*(B_*, \beta)\cong\boldsymbol{Z}_2[\xi_{2m+1}^2 | m\geqq 0]$ (cf. [13, Lemma 5.2]).

(1.18) For the class $\{C_n\}$ in (1.8),

(i) $j_*(C_{2m+1})=\xi_{2m}^2+\xi_0^{4m+2}$, and

(ii) $j_*(C_{2m+2})=\xi_{2m+1}^2+\xi_0^{4m+4}+\delta_{m+1}$

in $W_*(\boldsymbol{Z}_2; rel)$ where $\delta_{m+1}$ is the part of $W_*$ decomposables (cf. [13, Lemma 5.1]).

## §2. On the Homology $H_\beta(d)$

In this section we study the map $r: \Omega_*(\boldsymbol{Z}_4; p) \to \Omega_*(\boldsymbol{Z}_2; All)$ induced by the restriction $\boldsymbol{Z}_2 \subset \boldsymbol{Z}_4$. We have studied the theory $W_*(\boldsymbol{Z}_4; p)$ of Wall manifolds with stationary point free $\boldsymbol{Z}_4$ actions, and obtained the torsion part $\mathcal{T}_2$ of order 2 in $\Omega_*(\boldsymbol{Z}_4; p)$ as the image of the Bockstein homomorphism $\beta: W_*(\boldsymbol{Z}_4; p) \to \Omega_{*-1}(\boldsymbol{Z}_4; p)$ in (1.10) (cf. [8, Theorems 1.19 and 2.3]). Let $r_W: W_*(\boldsymbol{Z}_4; p) \to W_*(\boldsymbol{Z}_2; All)$ be the restriction as mentioned above. Then

**Lemma 2.1.**
(1) $r_W(W_*(\boldsymbol{Z}_4; p)) = d(W_*(\boldsymbol{Z}_2; All))$ *hence*
(2) $r(\mathcal{T}_2) = d(\mathrm{Tor}\, \Omega_*(\boldsymbol{Z}_2; All))$.

*Proof.* As a $W_*$ module, $W_*(\boldsymbol{Z}_4; p)$ is generated by the following (i) and (ii):

(i) the parts $\mathrm{Im}(t)$ where $t=e$ and $s$, the maps from $W_*(\boldsymbol{Z}_2; All)$ as Definition 1.1,

(ii) $V(\varepsilon, 2)$ ($\varepsilon=0$ and 1) and $V(q, 2K)$ for each $q \geqq 2$ and $2K=(2k_1, \cdots, 2k_n)$ with $k_1 \geqq \cdots \geqq k_n \geqq 0$.

In the above $V(\varepsilon, 2)$ is defined by $j_*(V(\varepsilon, 2)) = t(\xi_0^2)$ for the map $j_*: W_*(\boldsymbol{Z}_4; p) \to W_*(\boldsymbol{Z}_4; p, Free)$ in [8, Prop. 1.11 (i)] where if $\varepsilon=0$ or 1, then $t=e$ or $s$, the map from $W_*(\boldsymbol{Z}_2; rel)$, respectively. Further, let $\eta_{2K} \to \boldsymbol{C}P^{2K} = \boldsymbol{C}P(2k_1) \times \cdots \times \boldsymbol{C}P(2k_n)$ be the product of the canonical complex line bundles $\eta_{2k_j} \to \boldsymbol{C}P(2k_j)$ and let $S(\eta_{2K})$ or $D(\eta_{2K})$ be the associated sphere or disk bundle of $\eta_{2K}$, respectively. Then

(ii-1) $V(2p+1, 2K) = D^{2p+2} \times S(\eta_{2K}) \cup -(S^{2p+1} \times D(\eta_{2K}))$ with an oriented, stationary point free $\boldsymbol{Z}_4$ action $T_V = -1 \times i \cup -1 \times i$, and

(ii-2) $V(2p, 2K) = S^1 \times_R V(2p-1, 2K)$ with action $\mathrm{id} \times T_V$ in (1.9), where $R$ is the reflection in the first coordinate of $D^{2p}$ (See [8, Def. 1.17 and Theorem 1.19]. $V(q, 2K)$ is denoted by $V_{(2)}(q, 2K)$ there.).

It is easy to show that $r(V(2p+1, 2K))$ vanishes in $\Omega_*(\boldsymbol{Z}_2; All)$ by definition, so does $r_W(V(2p, 2K))$ in $W_*(\boldsymbol{Z}_2; All)$ naturally. On the other hand $j_*(r_W V(\varepsilon, 2)) = r_W(t(\xi_0^2)) = 0$ in $W_*(\boldsymbol{Z}_2; rel)$ since $r_W \circ e = 2 \times \mathrm{id}$ from (1.2) and $r_W \circ s = d$ from Definition 1.1. Note that $r_W(V(\varepsilon, 2)) \in Q_*^{(2)}$ since $\dim V(\varepsilon, 2) = 2$ or 3. These imply that $r_W(V(\varepsilon, 2)) = 0$ in $W_*(\boldsymbol{Z}_2; All)$ by (1.12) and (1.13). Further, $r_W(\mathrm{Im}(e)) = 2W_*(\boldsymbol{Z}_2; All) = \{0\}$. Therefore, $r_W(W_*(\boldsymbol{Z}_4; p)) = r_W(\mathrm{Im}(s)) = d(W_*(\boldsymbol{Z}_4; All))$ and the result (1) follows. Multiply both sides of (1) by $\beta$, then (2) is obtained by (1.11). *q.e.d.*

From this lemma we see that $\mathrm{Im}(r) \subset d(W_*(\boldsymbol{Z}_2; All))$ in particular. Since $\beta(r(x)) = 0$ for each $x \in \Omega_*(\boldsymbol{Z}_4; p)$, the image $r(x)$ belongs to $\overline{H}_\beta(d)$, the homology of the complex $(d(W_*(\boldsymbol{Z}_2; All)), \beta)$. Hence we have natural maps $\bar{r}_*: \Omega_*(\boldsymbol{Z}_4; p)\to \bar{H}_\beta(d)$ and $r_*=\bar{j}_*\circ \bar{r}_*: \Omega_*(\boldsymbol{Z}_4; p)\to H_\beta(d)$, the homology of the complex $(d(W_*(\boldsymbol{Z}_2; rel)), \beta)$, through the map $j_*: W_*(\boldsymbol{Z}_2; All)\to W_*(\boldsymbol{Z}_2; rel)$. The latter homology is comparatively easy to handle.

**Lemma 2.2.** $\mathrm{Ker}(\bar{j}_*: \bar{H}_\beta(d)\to H_\beta(d))=d(F)$ *where* $F=\Omega_*(\boldsymbol{Z}_2; All)/\mathrm{Tor}$ *is the torsion free part of* $\Omega_*(\boldsymbol{Z}_2; All)$.

*Proof.* If $y\in \mathrm{Ker}(\bar{j}_*)$, then $j_*(y)=\beta(d\xi)=d(\beta\xi)$ for some $\xi\in W_*(\boldsymbol{Z}_2; rel)$. Therefore $\partial(\beta\xi)\in \mathrm{Ker}(d: W_*(\boldsymbol{Z}_2; Free)\to W_{*+1}(\boldsymbol{Z}_2; Free))$ in the exact sequence in (1.12). As a free $W_*$ module, $W_*(\boldsymbol{Z}_2; Free)$ is generated by the class $\{X(2n), Y_{2n+1} \mid n\geqq 0\}$ where $Y_{2n+1}=[S^{2n+1}, -1]$ or $d(X(2n))$ in [8, Prop. 1.4]. Since $d([S^{2n+1}, -1])=0$ by [9, Lemma 1], we have $\partial(\beta\xi)=\sum_{n\geqq 0} M_{2n+1}[S^{2n+1}, -1]$ for some $M_{2n+1}\in W_*$. Take $\bar{y}\in W_*(\boldsymbol{Z}_2; All)$ such that $j_*(\bar{y})=\beta(\xi)-\sum_{n\geqq 0} M_{2n+1}\xi_0^{2n+2}$, then $y=d(\bar{y})$ in $W_*(\boldsymbol{Z}_2; All)$ since $y$, $d(\bar{y})\in Q_*^{(2)}$ and $j_*(y)=j_*(d(\bar{y}))=d(\beta\xi)$ (cf. (1.14)). Further $j_*(\beta(\bar{y}))=-\sum_{n\geqq 0}\beta(M_{2n+1})\xi_0^{2n+2}$ and $\beta(M_{2n+1})=0$ in $W_*$ since $\partial\circ j_*=0$. These imply that $\beta(\bar{y})\in Q_*^{(2)}$ vanishes in $W_*(\boldsymbol{Z}_2; All)$ hence in $\mathrm{Tor}\,\Omega_*(\boldsymbol{Z}_2; All)$ by (1.11). Thus $\bar{y}\in\Omega_*(\boldsymbol{Z}_2; All)$. If $\bar{y}$ is a torsion element, then $\bar{y}=\beta(z)$ for some $z\in W_*(\boldsymbol{Z}_2; All)$. So $y=d(\bar{y})$ vanishes in $\bar{H}_\beta(d)$ by definition. Therefore we may consider that $\bar{y}\in F$ and this proves $\mathrm{Ker}(\bar{j}_*)\subset d(F)$. Conversely, take any $y=d(\bar{y})\in d(F)$. The part $F$ is generated by the following (i) and (ii):

(i) monomials on $C_{2m+2}$ for $m\geqq 0$ (cf. (1.8)),

(ii) $[\boldsymbol{Z}_2, -1]$ and $r_{4m}$ $(m>1)$ which satisfies $2r_{4m}=W_{4m}[\boldsymbol{Z}_2, -1]$ for a suitable generator $W_{4m}\in\Omega_{4m}$ of the polynomial algebra $\Omega_*/\mathrm{Tor}\,\Omega_*$. Note that an element as $W_{4m}r_{4n}-W_{4n}r_{4m}$ $(m>n>1)$ is a torsion by definition, so it is excluded (cf. [13, Introduction and Theorem 10.1]).

In the first case, we see that $A_{2m+2}$ in (1.8) is the reduction of the $S^1$ action on $\boldsymbol{C}P(2m+2)$ by $\boldsymbol{Z}_2\subset S^1$, so $d([\boldsymbol{C}P(2m+2), A_{2m+2}])=[S^1\times \boldsymbol{C}P(2m+2), \mathrm{id}\times A_{2m+2}]=0$ in $\Omega_*(\boldsymbol{Z}_2; All)$ (cf. [1, Theorem 5]). Further, for each monomial $C_{2I+2}=C_{2m_1+2}\times\cdots\times C_{2m_p+2}$, we see that $j_*(d(C_{2I+2}))$ vanishes in $W_*(\boldsymbol{Z}_2; rel)$ by the derivation of $d$ and the above. Therefore $d(C_{2I+2})\in Q_*^{(2)}$ vanishes in $W_*(\boldsymbol{Z}_2; All)$ hence in $\mathrm{Tor}\,\Omega_*(\boldsymbol{Z}_2; All)$. When $C_\emptyset=[\mathrm{pt}, \mathrm{id}]$, $d(C_\emptyset)=[S^1, \mathrm{id}]=0$. In the second case, if $\bar{y}=[\boldsymbol{Z}_2, -1]$, then $d(\bar{y})=[S^1, -1]=0$. Finally we note that $j_*(r_{4m})=\beta(\xi)$ for some $\xi\in W_*(\boldsymbol{Z}_2; rel)$ since $j_*(r_{4m})\in\mathrm{Tor}\,\Omega_*(\boldsymbol{Z}_2; rel)=\beta(W_*(\boldsymbol{Z}_2; rel))$ by the relation in (ii) and [14, Theorem 3.4]. Such a $\xi$ is shown in [13, Sec. 12 and 13] concretely. Hence $d(r_{4m})$ vanishes in $H_\beta(d)$, and this completes the proof. *q. e. d.*

**Lemma 2.3.** $\mathrm{Ker}(r_*)=e(F)\oplus(\mathcal{T}_2+s(F))$.

*Proof.* We first show that $\mathrm{Ker}(\bar{r}_*)=e(F)\oplus\mathcal{T}_2$. Note that $e(F)\oplus\mathcal{T}_2\subset\mathrm{Ker}(\bar{r}_*)$ by (1.2), (1.11) and Lemma 2.1 (2). Conversely, suppose that $\bar{r}_*(x)=0$ in $\bar{H}_\beta(d)$

for $x\in\Omega_*(\boldsymbol{Z}_4;p)$, i.e., $r(x)=\beta(dy)=d(\beta y)$ for some $y\in W_*(\boldsymbol{Z}_2;All)$. Therefore $r(x)=d(\beta y)+2z$ for some $\Omega_*(\boldsymbol{Z}_2;All)$ by (1.10). Then $t=x-s(\beta y)-e(z)\in\mathcal{T}_2$ by Definition 1.1 and (1.2), and $x=e(z)+(t+s(\beta y))\in e(F)\oplus\mathcal{T}_2$. Thus we have $\mathrm{Ker}(\bar{r}_*)=e(F)\oplus\mathcal{T}_2$. The result follows immediately from Lemma 2.2. *q.e.d.*

**Theorem 2.4.** *There is an embedding* $r_*:\Omega_*(\boldsymbol{Z}_4;p)/\mathcal{E}\hookrightarrow H_\beta(d)$ *where* $\mathcal{E}=e(F)\oplus(\mathcal{T}_2+s(F))$.

In the above, we see that only $C_*$ in (1.17) acts nontrivially on both sides and $r_*$ is a $C_*$ module homomorphism.

*Remark* 2.5. The part $s(F)$ in $\mathcal{E}$ consists of torsion elements, since $[S^1,i]$ is of order 4 in $\Omega_*(\boldsymbol{Z}_4;Free)$ (cf. [10, Lemma 2.13 (i)]). For part (i) in the proof of Lemma 2.2, note that $s(C_{2I+2})\in\mathcal{T}_2$ since its restriction $d(C_{2I+2})=0$ in $\Omega_*(\boldsymbol{Z}_2;All)$ (cf. (1.2)). We see that it never vanishes in $\Omega_*(\boldsymbol{Z}_4;p)$. In fact, by (1.18 (ii)) $j_*(C_{2I+2})=\xi_0^{4\|I\|}+d(\lambda)$ (mod $W_*$ decomposables) for some $\lambda$ where $\|I\|=m_1+\cdots+m_p+p$. Hence in $W_*(\boldsymbol{Z}_4;p,Free)$, $j_*(s(C_{2I+2}))=s(j_*(C_{2I+2}))=s(\xi_0^{4\|I\|})$ (mod $W_*$ decomposables) which doesn't vanish there (cf. [8, Lemma 1.9 (ii) and (iii)]). Therefore $s(C_{2I+2})\neq 0$ in $\Omega_*(\boldsymbol{Z}_4;p)$. For part (ii), we see that $[S^1,i]\neq 0$ and belongs to $\mathcal{T}_2$ (cf. [8, Cor. 1.15]). On the other hand, $s(\sum_{m>1}M_{4m}r_{4m})$ may be of order 4 for some $M_{4m}\in\Omega_*/\mathrm{Tor}\,\Omega_*$.

Now we calculate the homology $H_\beta(d)$.

**Definition 2.6.** For each sequence $(I;J)=(m_1,\cdots,m_p;n_1,\cdots,n_q)$ of non-negative integers with $m_1\geqq\cdots\geqq m_p\geqq n_1\geqq\cdots\geqq n_q\geqq 0$, put $\xi_{(I;J)}=\xi^2_{2m_1+1}\cdots\xi^2_{2m_p+1}\xi^2_{2n_1}\cdots\xi^2_{2n_q}$ in $W_*(\boldsymbol{Z}_2;All)$. When $p\geqq 1$, each $\xi_{(I;J)}=d(\xi_{2m_1+1}\xi_{2m_1}\xi_{(I_0;J)})\in d(W_*(\boldsymbol{Z}_2;rel))$, $I_0=(m_2,\cdots,m_p)$, and is a class in the homology $H_\beta(d)$. Since $\xi_{(m;n)}=\xi_{(n;m)}$ in $H_\beta(d)$, the above condition for $(I;J)$ does not lose the generality.

**Proposition 2.7.** $H_\beta(d)\cong C_*\{\{\xi_{(I;J)}\,|\,I\neq\emptyset\}\}$ *as a free* $C_*$ *modules.*

In the same way as (1.17), it is sufficient to prove that

**Lemma 2.8.** $H_*(d(B_*),\beta)\cong\boldsymbol{Z}_2\{\{\xi_{(I;J)}\,|\,I\neq\emptyset\}\}$ *as a* $\boldsymbol{Z}_2$ *vector spaces.*

*Proof.* For each $x\in B_*$, we examine the form of $d(x)$ in $H_*(d(B_*),\beta)$. For any sequence $N=(n_1,\cdots,n_p)$ of integers with $n_1>\cdots>n_p\geqq 0$, define $B_N$ to be the $\boldsymbol{Z}_2$ vector subspace of $B_*$ generated by the monomials $\xi=\xi^{a_1}_{2n_1}\xi^{b_1}_{2n_1+1}\cdots\xi^{a_p}_{2n_p}\xi^{b_p}_{2n_p+1}$ such that $(a_i,b_i)\neq(0,0)$ for each $i$. Then $B_*=\sum_N B_N$ as $\boldsymbol{Z}_2$ vector spaces and we may suppose that $x\in B_N$ for some $N$ since $d$ and $\beta$ leave $B_N$ invariant by (1.15). Further, note that $d$ and $\beta$ preserve the length $2k=\sum a_i+\sum b_i$ of $\xi$. Hence we suppose that $x$ is a sum of monomials of the same length $2k$ (in $B_N$) and use induction on the length of $x$. For convenience, we represent $x$ by using the variables $\xi_{2n}$ and $\xi_{2n+1}$ (here $n=n_1$) for example. Then $x$ may have the following form (i) or (ii), i.e., in (i) the length $a_1+b_1$ is even and in (ii) that is odd, since $d$ and $\beta$ never change the length $a_1+b_1$ in particular:

(i) $x=\xi_{2n}\xi_{2n+1}(\sum P\cdot p)+\xi_{2n}^2(\sum Q\cdot q)+\xi_{2n+1}^2(\sum R\cdot r)$ or

(ii) $x=\xi_{2n}(\sum S\cdot s)+\xi_{2n+1}(\sum T\cdot t)$

where $P\cdot p, \cdots, T\cdot t$ are the monomials on $\{\xi_{2n_i}, \xi_{2n_i+1} \mid 1\leqq i\leqq p\}$, each of which is divided into the part $P, \cdots$, or $T$ on the squares $\{\xi_{2n_i}^2, \xi_{2n_i+1}^2\}$ and the remaining one $p, \cdots$, or $t$ which never has both $\xi_{2n}$ and $\xi_{2n+1}$. Note that $d$ and $\beta$ act trivially on the parts $P, \cdots$, and $T$. For saving the trouble, we admit $x$ is non homogeneous on the total dimension in $B_N$. When $k=1$, $x\in B_N$ where $N=(n_1)$ or $N=(n_1, n_2)$. The former is of type (i), while the latter is of type (ii). If $d(x)$ is a class in $H_*(d(B_*), \beta)$, then $d(x)=\varepsilon\xi_{2n_1+1}^2=\varepsilon\xi_{(n_1;\emptyset)}$ ($\varepsilon\in Z_2$) for the case (i) and $\varepsilon\beta d(\xi_{2n_1+1}\xi_{2n_2})$ which vanishes in this homology for the case (ii). Suppose that for any $x_0\in B_{N_0}$ in $B_*$ with the length $\leqq 2(k-1)$, $d(x_0)$ is a sum of monomials $\xi_{(I;J)}$ with $I\neq\emptyset$ in our homology. Let $x\in B_N$ be an element with the length $2k$ for some $N=(n_1, \cdots, n_p)$. We first consider the case (i). Unlike (ii), note that $p$, $q$ and $r$ have even length, so $d$ commutes with $\beta$ on them (cf. (1.15)). Now by (i),

(2.8.1) $$d(x)=\xi_{2n+1}^2(\sum P\cdot p)+\xi_{2n}\xi_{2n+1}(\sum P\cdot d(p))+\xi_{2n}^2(\sum Q\cdot d(q))+\xi_{2n+1}^2(\sum R\cdot d(r)).$$

The condition $0=\beta d(x)$ yields that

(2.8.2) $$\sum P\cdot\beta(p)+\sum R\cdot\beta d(r)=\xi_{2n}^2\eta, \quad \text{and}$$

(2.8.3) $$\sum P\cdot d(p)+\sum Q\cdot\beta d(q)=\xi_{2n+1}^2\eta$$

for some $\eta\in B_*$ by comparing the coefficient of $\xi_{2n}^2$ with that of $\xi_{2n+1}^2$ in $\beta d(x)$. Moreover note that $\beta(\eta)=d(\eta)=0$ by multiplying (2.8.2) or (2.8.3) by $\beta$ or $d$, respectively. Then

(2.8.4) $$\sum P\cdot p=\sum R\cdot d(r)+\xi_{2n}\xi_{2n+1}\eta+\lambda+\beta(\bar{\lambda})$$

by (2.8.2) and the structure of $H_*(B_*, \beta)$ in (1.17), where $\bar{\lambda}\in B_*$ and $\lambda$ is a sum of monomials $\xi_{(I;\emptyset)}$. Note that

(2.8.5) $$d\beta(\bar{\lambda})=\sum Q\cdot\beta d(q)$$

by (2.8.3). Substituting (2.8.3) and (2.8.4) into (2.8.1), we obtain that

(2.8.6) $$d(x)=\xi_{2n+1}^2\gamma+\beta d(\xi_{2n}\xi_{2n+1}(\sum Q\cdot q))=\xi_{2n+1}^2\gamma$$

in $H_*(d(B_*), \beta)$, where $\gamma=\lambda+\beta\bar{\lambda}+\sum Q\cdot\beta(q)$. Since $d(\gamma)=0$ by (2.8.5), $\gamma=\gamma_1+d(\gamma_2)$ where $\gamma_2\in B_*$ and $\gamma_1$ is a sum of monomials $\xi_{(\emptyset;J)}$ by (1.16). Since $\beta(\gamma)=0$, we have $\beta d(\gamma_1)=0$, i.e., $d(\gamma_1)$ is a class in $H_*(d(B_*), \beta)$ and the length of $d(\gamma_1)=2(k-1)$ by the definition of $\gamma$ in (2.8.6). Therefore $d(x)$ is the desired form by induction. For the case (ii), if $d(x)$ is a class in $H_\beta(d)$, then $d(x)=\beta d(\xi_{2n+1}(\sum S\cdot s))$ by the same way as $k=1$. Next we prove the linear independence of the class $\{\xi_{(I;J)}|I\neq\emptyset\}$. Suppose that

$$\sum\varepsilon_{(I,\emptyset)}\xi_{(I;\emptyset)}+\sum_{l(J)\geq 1}\varepsilon_{(I,J)}\xi_{(I;J)}=\beta(d\eta) \tag{2.8.7}$$

for some $\eta\in W_*(\boldsymbol{Z}_2; rel)$ where $\varepsilon_{(I,\emptyset)}$, $\varepsilon_{(I,J)}\in\{0, 1\}$ and $l(J)=q$ for $J=(n_1, \cdots, n_q)$. If $l(J)\geqq 1$, then $\xi_{(I;J)}=\beta(\bar{\xi}_{(I;J)})$ where $\bar{\xi}_{(I;J)}=\xi_{(I;\emptyset)}\xi_{2n_1+1}\xi_{2n_1}\xi_{(\emptyset;J_0)}$, $J_0=(n_2, \cdots, n_q)$ for $(I; J)=(m_1, \cdots, m_p; n_1, \cdots, n_q)$. Hence $\sum\varepsilon_{(I,\emptyset)}\xi_{(I;\emptyset)}=0$ in $H_*(B_*, \beta)$ and $\varepsilon_{(I,\emptyset)}=0$ for any $(I, \emptyset)$ by (1.17). Next we represent (2.8.7) as

$$\sum_{l(J)=1}\varepsilon_{(I,J)}\xi_{(I;J)}+\sum_{l(J)\geq 2}\varepsilon_{(I,J)}\xi_{(I;J)}=\beta(d\eta). \tag{2.8.8}$$

The left side has the form $\beta(x)$ where

$$x=\sum_{l(J)=1}\varepsilon_{(I,J)}\bar{\bar{\xi}}_{(I;J)}+\sum_{l(J)\geq 2}\varepsilon_{(I,J)}\bar{\bar{\xi}}_{(I;J)}. \tag{2.8.9}$$

Therefore $x$ has the form:

$$x=d(\eta)+\sum\varepsilon_{(I_0,\emptyset)}\xi_{(I_0;\emptyset)}+\beta(\bar{\eta}) \tag{2.8.10}$$

for some $\bar{\eta}\in W_*(\boldsymbol{Z}_2; rel)$ by using $H_*(B_*, \beta)$ again. Multiply this by $d$, then

$$\sum_{l(J)=1}\varepsilon_{(I,J)}\xi_{(I';J')}+\sum_{l(J)\geq 2}\varepsilon_{(I,J)}\xi_{(I';J')}=\beta(d\bar{\eta}) \tag{2.8.11}$$

by (2.8.9) and (2.8.10), where $(I'; J')=(m_1, \cdots, m_p, n_1; n_2, \cdots, n_q)$ for the above $(I; J)$. Since $(I'; J')=(I'; \emptyset)$ if $l(J)=1$, we have $\varepsilon_{(I,J)}=0$ for any $(I, J)$ with $l(J)=1$ in (2.8.11) in the same way as (2.8.7). Hence the result follows by induction on $l(J)$, and this completes the proof of the lemma. *q.e.d.*

## § 3. The Restriction from $Z_4$ Actions

We first consider a condition for an element in $\Omega_*(\boldsymbol{Z}_2; All)$ to come from the theory $\Omega_*(\boldsymbol{Z}_4; p)$ by the restriction $r$.

**Theorem 3.1.** *Let $y$ be an element in $\Omega_*(\boldsymbol{Z}_2; All)$ which lies in* $Q_*^{(2)}$ (cf. (1.12)). *In order that $y\in\mathrm{Im}(r)$, a necessary and sufficient condition is that $j_*(y)=\sum_{I\neq\emptyset}C_{(I,J)}\xi_{(I;J)}+\beta d(\lambda)$ $(C_{(I,J)}\in C_*)$ in $W_*(\boldsymbol{Z}_2; rel)$, i.e., $j_*(y)$ is a class in $H_\beta(d)$.*

*Proof.* Suppose that $y$ has a fixed point data $j_*(y)$ as above. Put $\xi_{(I;J)}=d(\bar{\xi}_{(I;J)})$ as Definition 2.6 and $\eta=\sum C_{(I,J)}\bar{\xi}_{(I;J)}+\beta(\lambda)$. Then we have $\bar{y}\in W_*(\boldsymbol{Z}_2; All)$ such that $j_*(\bar{y})=\eta-\sum_{n\geq 0}M_{2n+1}\xi_0^{2n+2}$ for some $M_{2n+1}\in W_*$. This implies that $y=d(\bar{y})$ since $y$, $d(\bar{y})\in Q_*^{(2)}$ and $j_*(y)=j_*(d(\bar{y}))$. If $j_*(y)=\beta d(\lambda)$, then $\bar{y}\in\Omega_*(\boldsymbol{Z}_2; All)$. Therefore $y=r(s(\bar{y}))\in\mathrm{Im}(r)$ in this case (See the first half of the proof of Lemma 2.2.). Next we suppose that $j_*(y)\neq 0$ in $H_\beta(d)$, i.e., it has terms $\xi_{(I;J)}$ with coefficients in $C^*$. In this case, $\dim y$ is even, i.e., $\dim y\equiv 0$ or $2 \pmod 4$. Consider the above $s(\bar{y})\in W_*(\boldsymbol{Z}_4; p)$ again. Then $j_*(\beta s(\bar{y}))=s(j_*(\beta\bar{y}))=s(\beta\eta-\sum_{n\geqq 0}\beta(M_{2n+1})\xi_0^{2n+2})$ for the map $j_*: W_*(\boldsymbol{Z}_4; p)\to W_*(\boldsymbol{Z}_4; p, Free)$ in the exact sequence in [8, Prop. 1.11 (i)]. We note that $\beta(\bar{\xi}_{(I;J)})=\xi_{(I_0;J_0)}$ where $(I_0; J_0)=(m_2, \cdots, m_p; m_1, n_1, \cdots, n_q)$ for $(I; J)=(m_1, \cdots, m_p; n_1, \cdots, n_q)$. Therefore if $p\geqq 2$, $\xi_{(I_0;J_0)}\in\mathrm{Im}(d)=\mathrm{Ker}(s)$ in $W_*(\boldsymbol{Z}_2; rel)$ (cf. [8, Lemma 1.9 (iii)]). So

$$j_*(\beta s(\bar{y}))=s(\sum_{J_0} C_{J_0}\xi_{(\emptyset;J_0)}-\sum_{n\geqq 0}\beta(M_{2n+1})\xi_0^{2n+2}) \tag{3.1.1}$$

in $W_*(\boldsymbol{Z}_4; p, Free)$ where $C_{J_0}=C_{(I,J)}$ with $p=1$. Put

$$\bar{x}=s(\bar{y})-(\sum_{J_0} C_{J_0}V(2, 2J_0)-\sum_{n\geqq 0}\beta(M_{2n+1})V(2, \underline{0})) \tag{3.1.2}$$

in $W_*(\boldsymbol{Z}_4; p)$, where in general $V(2, 2K)$ is defined at (ii-2) in the proof of Lemma 2.1 and $\underline{0}=(0, \cdots, 0)$ ($(n+1)$ times of 0), i.e., $\eta_{\underline{0}}=\boldsymbol{C}^{n+1}\to\{\mathrm{pt}\}$. Note that $\beta(V(2, 2K))=V(1, 2K)$ by (1.9). Then $j_*(\beta\bar{x})=0$ in $W_*(\boldsymbol{Z}_4; p, Free)$ since $j_*(V(1, 2K))=Q(1, 2K)=s(\eta_{2K})$ in [8, Prop. 1.8 (i)] and $\eta_{2K}=\xi_{(\emptyset;K)}$ in $\mathfrak{N}_*(\boldsymbol{Z}_2; rel)$ hence in $W_*(\boldsymbol{Z}_2; rel)$ (cf. [3, p. 446]). Therefore $\beta(\bar{x})\in\mathcal{P}=\mathrm{Ker}(j_*)$ in the above exact sequence. Recall that $\dim\bar{x}=\dim y\equiv 0$ or $2 \pmod 4$. Hence in $\Omega_*(\boldsymbol{Z}_4; p)$, $\beta(\bar{x})=2\alpha$ if $\dim\bar{x}\equiv 0 \pmod 4$ and $\varepsilon[\boldsymbol{C}P(2)]^n[S^1, i]+2\alpha$ ($\varepsilon\in\{0, 1\}$) if $\dim\bar{x}\equiv 2 \pmod 4$ by the structure of $\mathcal{P}$ and (1.10). We see that $\alpha$ is of order 4 if $2\alpha$ does not vanish. Such an element may belong to $s(F)$ in $\mathcal{E}$ or $\Omega_*(\boldsymbol{Z}_4; p)/\mathcal{E}$ (cf. Theorem 2.4 and Remark 2.5). If $\alpha\in s(F)$, then $\dim\alpha\equiv 1 \pmod 4$ and if $\alpha\in\Omega_*(\boldsymbol{Z}_4; p)/\mathcal{E}$, then $\dim\alpha\equiv 0$ or $2 \pmod 4$ since $r_*(\alpha)\neq 0$ in $H_\beta(d)$ as $j_*(y)$ in this case. Thus, if $\dim\bar{x}\equiv 0 \pmod 4$, then $\beta(\bar{x})=2\alpha=0$ and the element $\bar{x}$ (denoted by $x_1$) belongs to $\Omega_*(\boldsymbol{Z}_4; p)$. If $\dim\bar{x}\equiv 2 \pmod 4$, we may consider the case $\beta(\bar{x})=\varepsilon[\boldsymbol{C}P(2)]^n[S^1, i]+2\alpha$ with $\alpha=s(\sum_{m>1}M_{4m}r_{4m})$ for suitable $M_{4m}\in\Omega_*/\mathrm{Tor}\,\Omega_*$. Note that $2\alpha=M'[S^1, i]$ for some $M'$ by the definition of $r_{4m}$ (See (ii) in the proof of Lemma 2.2.). Therefore $\beta(\bar{x})=M[S^1, i]$ where $M=\varepsilon[\boldsymbol{C}P(2)]^n+M'$. Now we put

$$x_2=\bar{x}-M\cdot V(0, 2) \tag{3.1.3}$$

where $V(0, 2)\in W_2(\boldsymbol{Z}_4; p)$ is an element such that $\beta(V(0, 2))=[S^1, i]$ (cf. [8, Def. 1.17 and Lemma 2.5]). Then $\beta(x_2)=0$ and $x_2\in\Omega_*(\boldsymbol{Z}_4; p)$. Consider now the restriction $r(x_k)$ in $\Omega_*(\boldsymbol{Z}_2; All)$ for $k=1$ or 2. It is shown in the proof of Lemma 2.1 that $r_W(V(2, 2K))$ and $r_W(V(0, 2))$ vanish in $W_*(\boldsymbol{Z}_2; All)$. Therefore $r_W(x_k)=d(\bar{y})=y\in W_*(\boldsymbol{Z}_2; All)$ by (3.1.2) and (3.1.3), and $r(x_k)-y=2z$ for some $z\in\Omega_*(\boldsymbol{Z}_2; All)$ by (1.10). Hence we have $y=r(x_k-e(z))\in\mathrm{Im}(r)$. The converse follows from Theorem 2.4 and Proposition 2.7. This completes the proof. *q.e.d.*

*Remark* 3.2.

(i) In the above theorem, $y\notin Q_*^{(2)}$ occurs only if $\dim y=4n$ by (1.12). In this case, $y+[CP(2)]^n[\boldsymbol{Z}_2, -1]\in Q_*^{(2)}$ and belongs to $\mathrm{Im}(r)$ if $j_*(y)$ is a class in $H_\beta(d)$ as above. We see whether $y\in Q_*^{(2)}$ or not by using the formula in [4, Chap. III, Theorem 4.3] for example.

(ii) The map $\bar{r}_*: \Omega_*(\boldsymbol{Z}_4; p)\to\bar{H}_\beta(d)$ is epic from the above theorem, hence $\bar{r}_*: \Omega_*(\boldsymbol{Z}_4; p)/(e(F)\oplus\mathcal{T}_2)\cong\bar{H}_\beta(d)$ (cf. Lemma 2.3).

(iii) The image of the embedding $r_*$ in Theorem 2.4 is properly contained in $H_\beta(d)$. For example, $\xi=\xi_1^2+\xi_0^4$ is the only fixed point data which includes a class $\xi_1^2$ in $H_\beta(d)$. In fact $j_*(C_2)=\xi$. But $\xi$ is not a class in this homology.

**Example 3.3.** Take any $y\in\mathrm{Im}(r)$ in $\Omega_*(\boldsymbol{Z}_2; All)$. Then

(i) if $z\in\Omega_*(\boldsymbol{Z}_2; All)$ has a fixed point data

$$j_*(z)=\sum_{(I,J)\neq(\emptyset,\emptyset)} M_{(I,J)}\xi_{(I;J)}+\beta d(\lambda)$$

in $W_*(\boldsymbol{Z}_2; rel)$ where $M_{(I;J)}\in\Omega_*$, then $z\cdot y\in\mathrm{Im}(r)$. In particular,

(ii) $z^2\cdot y\in\mathrm{Im}(r)$ for any $z\in\Omega_*(\boldsymbol{Z}_2; All)$ since $j_*(z^2)$ has the above form by [19, Prop. 3].

In (i), note that $j_*(z\cdot y)$ has the form in Theorem 3.1. Since $y=d(\bar{y})$ for some $\bar{y}\in W_*(\boldsymbol{Z}_2; All)$ by Lemma 2.1 (1), we have $z\cdot y=z\cdot d(\bar{y})\in Q_*^{(2)}$ by the formula in [4] as mentioned above. Thus the result (i) follows. In (ii), if $\dim z$ is even and $z=[M, A]$, then $M\times M$ admits an oriented $\boldsymbol{Z}_4$ action $I$ with $I^2=A$ defined by $I(a, b)=(A(b), a)$ for $(a, b)\in M\times M$. Consider $[M\times M, I]\cdot x\in\Omega_*(\boldsymbol{Z}_4; p)$ naturally for $x\in\Omega_*(\boldsymbol{Z}_4; p)$ with $r(x)=y$, then it restricts to $z^2\cdot y$.

Relating to the above example, let $\Omega_*(\boldsymbol{Z}_4; All)$ be the theory of all oriented $\boldsymbol{Z}_4$ actions. Then

**Proposition 3.4.** *For the restriction $r_0: \Omega_*(\boldsymbol{Z}_4; All)\to\Omega_*(\boldsymbol{Z}_2; All)$, the fixed point data $j_*(z)$ of each $z\in\mathrm{Im}(r_0)$ has the form of Example* 3.3 (i).

*Proof*. Let $z=r_0(x)$ for some $x\in\Omega_*(\boldsymbol{Z}_4; All)$ and put $j_*(z)=\eta$. Choose any $x_0\in\Omega_*(\boldsymbol{Z}_4; p)$ such that $j_*(r(x_0))=\eta_0\neq0$ in $W_*(\boldsymbol{Z}_2; rel)$ (Such an $x_0$ is given in the next example 3.5.). Then $d(\eta_0)=0$ since $\eta_0$ is a class in $H_\beta(d)$. Moreover $r_*(x\cdot x_0)=\eta\eta_0\in H_\beta(d)$ since $x\cdot x_0\in\Omega_*(\boldsymbol{Z}_4; p)$. From these $0=d(\eta\eta_0)=d(\eta)\eta_0$ and $d(\eta)=0$ in $W_*(\boldsymbol{Z}_2; rel)$, i.e., $\eta$ is a class in the homology $H_d$. Hence $\eta=\sum M_{(\emptyset,J)}\xi_{(\emptyset;J)}+d(\bar{\eta})$ for some $\bar{\eta}$ (cf. (1.16)). Further $\beta(\eta)=0$ implies that $\sum\beta(M_{(\emptyset,J)})\xi_{(\emptyset;J)}\in\mathrm{Im}(d)$. Thus $\beta(M_{(\emptyset,J)})=0$, i.e., $M_{(\emptyset,J)}\in\Omega_*$ and $d(\bar{\eta})$ is a class in $H_\beta(d)$. Hence $\eta$ has the desired form by Proposition 2.7. *q.e.d.*

**Example 3.5.** For $m\geqq1$, let $V^{2m+2}\in\Omega_*(\boldsymbol{Z}_4; p)$ be the element in Example (1.7). It restricts to $C_{2m+2}\pm(C_2)^{m+1}$ by definition, which is torsion free in $\Omega_*(\boldsymbol{Z}_2; All)$ (cf. the part $P_*$ in [13, Introduction]), and so is $V^{2m+2}$ in $\Omega_*(\boldsymbol{Z}_4; p)$. Further

(3.5.1) $$r_*(V^{2m+2})=j_*(C_{2m+2})+j_*((C_2)^{m+1})$$
$$=(\xi^2_{2m+1}+\xi_0^{4m+4}+\delta_{m+1})+(\xi_1^2+\xi_0^4)^{m+1}$$
$$=\xi^2_{2m+1}+\eta+\delta_{m+1}$$

where $\eta$ is the sum of monomials $(\xi_1^2)^a(\xi_0^2)^b$ and $\delta_{m+1}$ is the $W_*$ decomposable part in (1.18 (ii)). Hence $r_*(V^{2m+2})\neq 0$ in $H_\beta(d)$, and $V^{2m+2}\notin e(F)$ by Lemma 2.3. Note that the relation $2V^{2m+2}=e(C_{2m+2}\pm(C_2)^{m+1})$ holds by (1.2)

*Remark* 3.6. The part $\delta_{m+1}\in H_\beta(d)$ in particular, i.e.,

$$\delta_{m+1}=\sum_{I\neq\emptyset} C_{(I,J)}\xi_{(I;J)}+\beta d(\lambda)$$

in $W_*(\boldsymbol{Z}_2; rel)$ formally where $C_{(I,J)}\in C_*$ with $\dim C_{(I,J)}>0$.

Let $\mathcal{R}_*$ be an $\Omega_*$ algebra generated by the class $\{C_n \mid n\geqq 2\}$ in $\Omega_*(\boldsymbol{Z}_2; All)$. We examine $\mathcal{I}_*=\{y\in\mathcal{R}_* \mid y\in\operatorname{Im}(r)\}$. Note that it is an ideal in $\mathcal{R}_*$ since any element in $\mathcal{R}_*$ comes from $\Omega_*(\boldsymbol{Z}_4; All)$ (cf. (1.8)). By (1.2), $2\mathcal{R}_*\subset\mathcal{I}_*$ and so it is sufficient to study an ideal $\mathcal{I}_*\otimes\boldsymbol{Z}_2$ in $\mathcal{R}_*\otimes\boldsymbol{Z}_2$.

**Theorem 3.7.** *$\mathcal{R}_*\otimes\boldsymbol{Z}_2$ is a free $\Omega_*\otimes\boldsymbol{Z}_2$ polynomial algebra on the class $\{C_n\}$, and $\mathcal{I}_*\otimes\boldsymbol{Z}_2$ is an ideal generated by the class $\{C_{2m+2}-(C_2)^{m+1} \mid m\geqq 1\}$.*

*Proof*. For each pair $I=(m_1, \cdots, m_p)$, $J=(n_1, \cdots, n_q)$ of sequences of integers with $m_1\geqq\cdots\geqq m_p\geqq 0$ and $n_1\geqq\cdots\geqq n_q\geqq 1$, the fixed point data of the monomial $C_{2I+2}C_{2J+1}$ has the following form by (1.18):

(3.7.1) $$j_*(C_{2I+2}C_{2J+1})=(\xi^2_{2m_1+1}+\xi_0^{4m_1+4}+\delta_{m_1+1})\cdots(\xi^2_{2m_p+1}+\xi_0^{4m_p+4}+\delta_{m_p+1})$$
$$\times(\xi^2_{2n_1}+\xi_0^{4n_1+2})\cdots(\xi^2_{2n_q}+\xi_0^{4n_q+2})$$
$$=\xi_{(I;J)}+\eta+\lambda+\delta$$

in $W_*(\boldsymbol{Z}_2; rel)$ where $\eta$ is the sum of monomials except $\xi_{(I;J)}$ which contain some $\xi^2_{2m_j+1}$ (and so do some $(\xi_0^2)^a$), $\lambda$; the sum of monomials $(\xi_0^2)^b\xi_{(\emptyset;J_0)}$ with $b>0$, $J_0\subseteqq J$ and $\delta$ is the $W_*$ decomposable part. Thus the elements $\{C_{2I+2}C_{2J+1}\}$ correspond to those $\{\xi_{(I;J)}\}$ which are linearly independent (over $W_*$) in $W_*(\boldsymbol{Z}_2; rel)$. Hence $\{C_{2I+2}C_{2J+1}\}$ is an $\Omega_*\otimes\boldsymbol{Z}_2$ base for $\mathcal{R}_*\otimes\boldsymbol{Z}_2$ by the embedding $\Omega_*\otimes\boldsymbol{Z}_2\hookrightarrow W_*$ in (1.10). Next we suppose that in $\mathcal{R}_*\otimes\boldsymbol{Z}_2$ an element $y=\sum_{(I,J)}M_{(I,J)}C_{2I+2}C_{2J+1}$ $(M_{(I,J)}\in\Omega_*\otimes\boldsymbol{Z}_2)$ belongs to $\operatorname{Im}(r)=\mathcal{I}_*\otimes\boldsymbol{Z}_2$. Here we consider the homology $H_d$ (cf. (1.16)). Then $i_*(y)$ is a class in $H_d$ and vanishes there by Lemma 2.1 (1). More precisely, let an integer $t$ with $t\geqq 0$ be fixed and put $S_t=\{(I, J) \mid \text{the total dimension of } \xi_{(I;J)}=t\}$. We then have

(3.7.2) $$0=j_*(y)=M_{(\emptyset;\emptyset)}\cdot 1+\sum_{t>0}\Big(\sum_{(I,J)\in S_t}M_{(I,J)}(\xi_{(I;J)}+\eta+\lambda+\delta)\Big)$$

in $H_d$ by (3.7.1) where $1=\xi_{(\emptyset;\emptyset)}$. Hence $M_{(\emptyset;\emptyset)}=0$. Further, if $t>0$, then $\xi_{(I;J)}$ with $I\neq\emptyset$, $\eta$ and $\delta$ belong to $\mathrm{Im}(d)$ by definition and the fact that each $\delta_{m_i+1}\in H_\beta(d)$ (cf. Remark 3.6). Therefore

$$0=j_*(y)=\sum_{t>0}\{\sum_{(\emptyset,J')\in S_t}M_{(\emptyset,J')}(\xi_{(\emptyset;J')}+\lambda')+\sum_{(I,J)\in S_t^0}M_{(I,J)}(\xi_0^{4\|I\|}\xi_{(\emptyset;J)}+\bar{\lambda})\} \tag{3.7.3}$$

in $H_d$ where $S_t^0=\{(I,J)\in S_t\,|\,I\neq\emptyset\}$, $\|I\|=m_1+\cdots+m_p+p$ and $\bar{\lambda}$ is the remaining part of $\lambda$. For each $t>0$, consider the part $\{\cdots\}$. Then it vanishes in $H_d$ by definition. First we have $M_{(\emptyset,J')}=0$ in $W_*$ hence in $\Omega_*\otimes\boldsymbol{Z}_2$ for each $J'$. Further, put $S(u,J)=\{I\,|\,(I,J)\in S_t^0$ with $\|I\|=u\}$ for each positive integer $u$ and the sequence $J$. Then the coefficient of $\xi_0^{4u}\xi_{(\emptyset;J)}+\bar{\lambda}$, $\sum_{I\in S(u,J)}M_{(I,J)}=0$ for any $(u,J)$ since the length of each monomial in $\bar{\lambda}$ is greater than that $\xi_0^{4u}\xi_{(\emptyset;J)}$. We write $y=\sum_t y_t$ where $y_t=\sum_{(I,J)\in S_t}M_{(I,J)}C_{2I+2}C_{2J+1}$. Then $y_0=0$ as mentioned above and for each $t>0$,

$$y_t=\sum_{(u,J)}(\sum_{I\in S(u,J)}M_{(I,J)}(C_{2I+2}-C_{2I_0+2})C_{2J+1}) \tag{3.7.4}$$

for suitable $I_0\in S(u,J)$. Since $C_{2I+2}-C_{2I_0+2}\in\mathcal{V}_*$, the ideal in $\mathcal{R}_*\otimes\boldsymbol{Z}_2$ generated by the class $\{C_{2m+2}-(C_2)^{m+1}\,|\,m\geqq 1\}$, we see that $y\in\mathcal{V}_*$ and $\mathcal{I}_*\otimes\boldsymbol{Z}_2\subset\mathcal{V}_*$. On the other hand, consider the element $V^{2m+2}$ in Example 3.5. Then $r(V^{2m+2})=C_{2m+2}-(C_2)^{m+1}$ in $\mathcal{R}_*\otimes\boldsymbol{Z}_2$. In general an element $V^{2m+2}C^{2K+2}C^{2L+1}$ in $\Omega_*(\boldsymbol{Z}_4;p)$ restricts to $(C_{2m+2}-(C_2)^{m+1})C_{2K+2}C_{2L+1}$ in $\mathcal{V}_*$ where $C^{2K-2}$ or $C^{2L+1}$ is a monomial on the class $\{C^{2k_j+2}\}$ or $\{C^{2l_j+1}\}$, respectively (cf. (1.8)). Hence $\mathcal{V}_*\subset\mathrm{Im}(r)=\mathcal{I}_*\otimes\boldsymbol{Z}_2$. This completes the proof. *q.e.d.*

**Corollary 3.8.** *For a class $\{(I,J)\}$ with $J\neq\emptyset$, let us consider a torsion element $y=\sum_{(I,J)}M_{(I,J)}C_{2I+2}C_{2J+1}$ in $\mathcal{R}_*$. Then $y$ comes from a (torsion) element in $\Omega_*(\boldsymbol{Z}_4;p)$ if and only if it is a sum of the polynomials* (3.7.4) *in $\mathcal{R}_*$. In this case, any counter-image $x$ of $y$ is of order* 4 *if and only if some $M_{(I,J)}$ in* (3.7.4) *is a torsion free element such that $i(M_{(I,J)})\neq 0$ where $i:\Omega_*\to C_*$ is the projection* (*cf.* (1.17)).

*Proof.* Note that $C_{2J+1}\in\mathrm{Tor}\,\Omega_*(\boldsymbol{Z}_2;All)$ since there is an orientation-reversing conjugation on each $CP(2n_j+1)$. Therefore, the above theorem applies to this case in $\mathcal{R}_*$ (without tensoring $\boldsymbol{Z}_2$) by (1.6) and the first half follows. By (1.4), any counter-image $x$ of $y$ is a torsion element in $\Omega_*(\boldsymbol{Z}_4;p)$. If such $x$ is of order 2, then $r_*(x)=j_*(y)=0$ in $H_\beta(d)$ by Lemma 2.3 and so is $j_*(y_t)$ for each $t$ by the definition of $y_t$. Therefore $i(M_{(I,J)})=0$ in $C_*$ for any $M_{(I,J)}$ in (3.7.4) since the terms $\{(C_{2I+2}-C_{2I_0+2})C_{2J+1}\}$ of $y_t$ correspond to those $\{\xi_{(I;J)}-\xi_{(I_0;J)}\}$ which are linearly independent (over $C_*$) in $H_\beta(d)$ by (3.7.1) and Proposition 2.7. We see that the counter-image $r^{-1}(y)$ consists of torsions of order 2 (or order 4) if $j_*(y)=0$ (or $\neq 0$) in $H_\beta(d)$, respectively by Lemma 2.3

and (1.5). Hence the second half follows. *q. e. d.*

Further, any element $x$ in the above corollary is also of order 4 in $\Omega_*(\boldsymbol{Z}_4; All)$ (cf. [15, Sec. 4]). More generally,

**Theorem 3.9.** *If $x$ is a torsion of order 4 in $\Omega_*(\boldsymbol{Z}_4; p)$ such that $r_*(x)\neq 0$ in $H_\beta(d)$, then it is also of order 4 in $\Omega_*(\boldsymbol{Z}_4; All)$.*

*Proof.* Let $r_*(x)=\sum_{I\neq\emptyset} C_{(I,J)}\xi_{(I;J)}$ with $C_{(I,J)}\neq 0$ in $C_*$ by assumption. Consider $x\cdot x$ in $\Omega_*(\boldsymbol{Z}_4; All)\times\Omega_*(\boldsymbol{Z}_4; p)\subset\Omega_*(\boldsymbol{Z}_4; p)$. If $x$ is of order 2 in $\Omega_*(\boldsymbol{Z}_4; All)$, so is $x\cdot x$ in $\Omega_*(\boldsymbol{Z}_4; p)$ and $r_*(x\cdot x)=\sum_{I\neq\emptyset} C^2_{(I,J)}\xi_{(I,I;J,J)}=0$ in $H_\beta(d)$ by Theorem 2.4. This implies that $C_{(I,J)}=0$ for any $(I, J)$ since the elements $\{\xi_{(I,I;J,J)}\}$ are linearly independent over $C_*$ by the remark in Definition 2.6. This is contrary to the assumption and the theorem follows. *q. e. d.*

**Example 3.10.** An element $V^{2m+2}C^{2K+2}C^{2L+1}$ has order 4 in $\Omega_*(\boldsymbol{Z}_4; All)$ where $m\geqq 1$, $K=(k_1, \cdots, k_p)$ and $L=(l_1, \cdots, l_q)$ with $k_1\geqq\cdots\geqq k_p\geqq 0$, $l_1\geqq\cdots\geqq l_q\geqq 1$ and $q\geqq 1$.

We obtain similar examples from $y$ in the second half of Corollary 3.8 in general.

Finally we consider the torsion free part $\mathcal{R}_F$ in $\mathcal{R}_*$, i.e., $\mathcal{R}_F=(\Omega_*/\mathrm{Tor}\,\Omega_*)[C_{2n+2}\,|\,n\geqq 0]$ as a polynomial algebra over $\Omega_*/\mathrm{Tor}\,\Omega_*$ (cf. [13, Introduction]). Then $\mathcal{R}_F\otimes\boldsymbol{Z}_2=C_*[C_{2n+2}\,|\,n\geqq 0]$ which is isomorphic to $H_*(P_*^{(2)}, \beta)$ where $P_*^{(2)}=W_*[C_{2n+2}\,|\,n\geqq 0]$ in the same way as (1.17). Using this, we describe the complementary part $\mathcal{A}=\Omega_*(\boldsymbol{Z}_2; All)\otimes\boldsymbol{Z}_2/\mathcal{R}_F\otimes\boldsymbol{Z}_2$ as an additive group. The map $j_*|_{P_*^{(2)}}: P_*^{(2)}\to W_*(\boldsymbol{Z}_2; rel)$ provides an isomorphism $j_*: P_*^{(2)}\cong j_*(P_*^{(2)})$ by (3.7.1) when $J=\emptyset$. In [13, Sec. 5], $j_*(P_*^{(2)})$ is denoted by $P_*^{(2)}(rel)$. Then we have

$$(3.11)\qquad j_*: \mathcal{R}_F\otimes\boldsymbol{Z}_2=H_*(P_*^{(2)}, \beta)\cong H_*(P_*^{(2)}(rel), \beta)\cong H_\beta$$

through the isomorphism $i_*: H_*(P_*^{(2)}(rel), \beta)\cong H_\beta$ by [13, Theorem 5.3] (cf. (1.17)). Let $j'_*: \Omega_*(\boldsymbol{Z}_2; All)\otimes\boldsymbol{Z}_2\to H_\beta$ be the natural map, then $j'_*|_{\mathcal{R}_F\otimes\boldsymbol{Z}_2}=j_*$ and $\mathcal{A}=\mathrm{Ker}\,(j'_*)$ by (3.11). Any torsion element belongs to $\mathcal{A}$ while the torsion free element $r_{4m}$ also belongs to $\mathcal{A}$ (cf. Proof of Lemma 2.2).

By Theorem 3.7 and the above, each element in $(\mathcal{R}_F\cap\mathcal{I}_*)\otimes\boldsymbol{Z}_2$ is a sum of terms $C_{2I+2}-C_{2I_0+2}$ (here $\|I\|=\|I_0\|$) with coefficients in $C_*$. Thus, for example an element $C_{2m+2}+\sum_I M_I C_{2I+2}$ ($M_I\in C_*$ and $\dim M_I>0$) in $\mathcal{R}_F\otimes\boldsymbol{Z}_2$ doesn't belong to $\mathcal{I}_*\otimes\boldsymbol{Z}_2$, i.e., it doesn't come from $\Omega_*(\boldsymbol{Z}_4; p)$.

**Proposition 3.12.** *For each $m\geqq 1$, there is a torsion free element*

$$y_{2m+2}=C_{2m+2}+\sum_I M_I C_{2I+2}+\alpha$$

*in* $\Omega_*(\boldsymbol{Z}_2; All)$ *such that it comes from* $\Omega_*(\boldsymbol{Z}_4; p)/\mathcal{E}$ *and does not belong to* $\mathcal{J}_*$ *where* $\sum_I M_I C_{2I+2}$ *is a decomposable element as mentioned above and* $\alpha\in\mathcal{A}$.

*Proof.* For each $m\geqq 1$, put $P_{2m+1+\varepsilon}=\boldsymbol{RP}(\xi_0^{2m+1}\times\xi_\varepsilon)$, the projective space bundle associated to $\xi_0^{2m+1}\times\xi_\varepsilon$ with an involution $R_{2m+1+\varepsilon}$ induced by the reflection $\mathrm{id}\times -1$ on $\xi_0^{2m+1}\times\xi_\varepsilon$ ($\varepsilon\in\{0, 1\}$). Note that $\beta(P_{2m+2})=P_{2m+1}$. Put $y_{2m+2}=d(P_{2m+2}\times P_{2m+1})$ in $W_{4m+4}(\boldsymbol{Z}_2; All)$. Then its fixed point data is as follows:

$$(3.12.1)\qquad j_*(y_{2m+2})=d((\xi_{2m+1}+\xi_1\xi_0^{2m})(\xi_{2m}+\xi_0^{2m+1}))$$
$$=\xi_{2m+1}^2+\xi_1^2\xi_0^{4m}$$

in $W_*(\boldsymbol{Z}_2; rel)$ by (1.15). Since $j_*(\beta(y_{2m+2}))=0$ in $W_*(\boldsymbol{Z}_2; rel)$, $\beta(y_{2m+2})\in Q_*^{(2)}$ vanishes in $W_*(\boldsymbol{Z}_2; All)$ hence in $\mathrm{Tor}\,\Omega_*(\boldsymbol{Z}_2; All)$. Therefore $y_{2m+2}\in\Omega_{4m+4}(\boldsymbol{Z}_2; All)$ and also belongs to $Q_*^{(2)}$ by definition (cf. (1.14)). Hence it comes from some $x_{2m+2}\in\Omega_{4m+4}(\boldsymbol{Z}_4; p)$ by (3.12.1) and Theorem 3.1. On the other hand,

$$(3.12.2)\qquad j_*(C_{2m+2})=\xi_{2m+1}^2+\sum_K M_K\xi_{(K;\emptyset)}$$

in $H_\beta$ for some $M_K\in C_*$ with $\dim M_K>0$ by (1.18 (ii)) and (1.17). From this, for each sequence $K=(k_1, \cdots, k_p)$ with $k_1\geqq\cdots\geqq k_p\geqq 0$,

$$(3.12.3)\qquad j_*(C_{2K+2})=\xi_{(K;\emptyset)}+\sum_L M_L\xi_{(L;\emptyset)}$$

in $H_\beta$ for some $M_L\in C_*$ with $\dim M_L>0$ by the product of $j_*(C_{2k_j+2})$. Let $\|K\|=k_1+\cdots+k_p+p$. Then $\|L\|<\|K\|$ for each $L$ in the above. Let $p_0=\max\{\|K\|\}$ for the class $\{K\}$ in (3.12.2) and let $\{K_0\}$ be the subset of $\{K\}$ with $\|K_0\|=p_0$. Then

$$(3.12.4)\qquad j_*(C_{2m+2}-\sum_{K_0} M_{K_0}C_{2K_0+1})=\xi_{2m+1}^2+\sum_S M_S\xi_{(S;\emptyset)}$$

in $H_\beta$ for some $M_S\in C_*$ with $\dim M_S>0$ by (3.12.2) and (3.12.3) when $K=K_0$. Then $\|S\|<p_0$ for each $S$. By easy induction on $\|\cdot\|$, we obtain $\bar{y}_{2m+2}=C_{2m+2}+\sum_I M_I C_{2I+2}$ such that $j_*(\bar{y}_{2m+2})=\xi_{2m+1}^2$ (in $H_\beta$) for some $I$ with $1\leqq\|I\|\leqq p_0$ and $M_I\in C_*$ with $\dim M_I>0$. Put $\alpha=y_{2m+2}-\bar{y}_{2m+2}$, then $j'_*(\alpha)=0$ in $H_\beta$, i.e., $\alpha\in\mathcal{A}$ by construction and (3.12.1). Since $r_*(x_{2m+2})=j_*(y_{2m+2})\neq 0$ in $H_\beta(d)$ by (3.12.1), we see that $x_{2m+2}\notin\mathcal{E}$ by Theorem 2.4. Since $j'_*(y_{2m+2})=j_*(\bar{y}_{2m+2})\neq 0$ in $H_\beta$ by (3.11), $y_{2m+2}$ is a torsion free element and so is $x_{2m+2}$ in $\Omega_*(\boldsymbol{Z}_4; p)$. Assume that $y_{2m+2}\in\mathcal{J}_*\otimes\boldsymbol{Z}_2$, then it is a sum of terms $(C_{2I+2}-C_{2I_0+2})C_{2J+1}$ with coefficients in $C_*$ by Theorem 3.7. If $J\neq\emptyset$, then such terms belong to $\mathcal{A}$. So $\bar{y}_{2m+2}=T \pmod{\mathcal{A}}$ where $T$ is a sum of terms $C_{2I+2}-C_{2I_0+2}$ with coefficients in $C_*$ by the definition of $\alpha$. Hence in $\mathcal{R}_F\otimes\boldsymbol{Z}_2$, $\bar{y}_{2m+2}=T$ by (3.11), i.e., $\bar{y}_{2m+2}\in\mathcal{J}_*\otimes\boldsymbol{Z}_2$. This is a contradiction. Hence $y_{2m+2}\notin\mathcal{J}_*\otimes\boldsymbol{Z}_2$ and so $y_{2m+2}\notin\mathcal{J}_*$.

*q.e.d.*

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