Publ. RIMS, Kyoto Univ.
30 (1994), 249-259

Operators Characterized by Certain Cauchy-Schwarz Type Inequalities

By

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Abstract

A Hilbert space operator T satisfying
either (**) $|(T\xi, \eta)|^2 \leqq (|T|\xi, \xi)(|T|\eta, \eta)$ for all $\xi, \eta \in \mathscr{H}$,
or (*) $|(T\xi, \xi)| \leqq (|T|\xi, \xi)$ for all $\xi \in \mathscr{H}$
is studied. The condition (**) defines a slightly larger class than the hyponormality, and for compact operators (**) is equivalent to the normality. The condition (*) is characterized by using an operator whose numerical radius is less than 1, and among other things we show that (*) and the normality are equivalent for matrices. Moreover, we show that (*) and the normality are equivalent for trace class operators in Appendix.

§ 0. Introduction

The purpose of the present paper is to study operators T satisfying either

(**) $$|(T\xi, \eta)|^2 \leqq (|T|\xi, \xi)(|T|\eta, \eta) \quad \text{for all } \xi, \eta \in \mathscr{H},$$

or

(*) $$|(T\xi, \xi)| \leqq (|T|\xi, \xi) \quad \text{for all } \xi \in \mathscr{H}.$$

Here, T is an operator on a Hilbert space $\mathscr{H}$ with absolute value $|T| = (T^*T)^{1/2}$.

It is obvious that (**) implies (*). Based on the Cauchy-Schwarz inequality, one can show that (**) is equivalent to the operator inequality $|T^*| \leqq |T|$ (Theorem 1.1). In particular, if T is normal (i.e., $TT^* = T^*T$), then (**) holds. In the operator theory, several extensions of the notion of the normality are known (see, for example, [8]). One of the most important and most widely studied classes among them is the hyponormality (i.e., $TT^* \leqq T^*T$) (see, for example, [5]). Since the square root function $t^{1/2}$ $(t \geqq 0)$ preserves the (natural) order among positive operators ([7]), a hyponormal operator T actually satisfies $|T^*| \leqq |T|$ (and hence (**)). Therefore, we are looking at a slightly (and strictly ⋯ see the end of § 1) larger class than the hyponormality.

Communicated by H. Araki, April 22, 1993.
1991 Mathematics Subject Classifications: 47B20.
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In § 1, we will identify the class of operators satisfying (**). We will also show that for compact operators the validity of (**) is equivalent to the normality based on the following result due to T. Ando ([1]): A compact hyponormal operator is automatically normal (see also [4] and [9]).

In § 2, we will consider the condition (*). Firstly we will characterize (*) by making use of an operator X whose numerical radius $w(X)$ satisfies $w(X)\leqq 1$ (Theorem 2.1). Secondly we will also show that for (finite) matrices ($\dim(\mathcal{H})<\infty$) the condition (**) actually implies the normality (Theorem 2.3). Consequently, (*), (**), and the normality are all equivalent for matrices.

A beautiful characterization of an operator X with $w(X)\leqq 1$ was obtained by T. Ando ([2]). Based on this characterization and Theorem 2.1, in § 3 we will show that the class of operators satisfying (*) is strictly larger than the class of operators satisfying (**) (when $\dim(\mathcal{H})=\infty$).

Finally, in Appendix, we will extend to the result obtained in § 2 to trace class operators. Based on T. Ando's factorization of a numerical contraction X (i.e., $w(X)\leqq 1$) ([2]), we will show that a numerical contraction and its adjoint have the same invariant vectors, and that for trace class operators the validity of (*) is equivalent to the normality.

The results in Appendix were suggested by the referee, and the author would like to thank the referee for the suggestion.

§ 1. Inequality (**)

In this section, we consider the following inequality for an operator $T\in\mathcal{B}(\mathcal{H})$:

$$(**)\qquad |(T\xi,\eta)|^2\leqq(|T|\xi,\xi)(|T|\eta,\eta)\qquad\text{for all }\xi,\eta\in\mathcal{H}.$$

Theorem 1.1. *For an operator $T\in\mathcal{B}(\mathcal{H})$, (**) holds for all $\xi, \eta\in\mathcal{H}$ if and only if $|T^*|\leqq|T|$.*

Proof. Let $T=U|T|$ be the polar decomposition of T. Then, since $|T^*|=U|T|U^*$,

$$\begin{aligned}|(T\xi,\eta)|^2&=|(U|T|^{1/2}|T|^{1/2}\xi,\eta)|^2\\&=|(|T|^{1/2}\xi,|T|^{1/2}U^*\eta)|^2\\&\leqq\||T|^{1/2}\xi\|^2\cdot\||T|^{1/2}U^*\eta\|^2\\&=(|T|\xi,\xi)(U|T|U^*\eta,\eta)\\&=(|T|\xi,\xi)(|T^*|\eta,\eta)\end{aligned}$$

for all $\xi, \eta\in\mathcal{H}$. Therefore, if $|T^*|\leqq|T|$, then we get (**).

Conversely when (**) is valid, by replacing ξ, η by $U^*\xi, \xi$, we get

$$\begin{aligned}|(|T^*|\xi,\xi)|^2&=|(U|T|U^*\xi,\xi)|^2\\&=|(TU^*\xi,\xi)|^2\\&\leqq(|T|U^*\xi,U^*\xi)(|T|\xi,\xi)\\&=(U|T|U^*\xi,\xi)(|T|\xi,\xi)\\&=(|T^*|\xi,\xi)(|T|\xi,\xi).\end{aligned}$$

Hence, we conclude $|T^*|\leqq|T|$. q. e. d.

From the above theorem, we easily see that the normality of T implies (**). But, in general, the inequality (**) does not imply that T is normal (for example, an isometry). However, when T is compact, we obtain:

Theorem 1.2. *Let $T\in\mathcal{B}(\mathcal{H})$ be compact. Then* (**) *holds for all $\xi,\eta\in\mathcal{H}$ if and only if T is normal.*

To prove Theorem 1.2, we need the following fact due to T. Ando ([1] ··· see also [4] and [9]):

Proposition 1.3. *A compact hyponormal operator in $\mathcal{B}(\mathcal{H})$ is normal.*

Proof of Theorem 1.2. Let $T=U|T|$ be the polar decomposition of a compact operator T. We must show that $|T^*|\leqq|T|$ implies the normality of T. By setting $S=U|T|^{1/2}$, we observe that

$$\begin{aligned}SS^*&=U|T|U^*=|T^*|\\&\leqq|T|=|T|^{1/2}U^*U|T|^{1/2}=S^*S\,,\end{aligned}$$

i. e., S is hyponormal. Since T is compact, S is also compact. Thus S is actually normal by Proposition 1.3. On the other hand, since $S=U|T|^{1/2}$ is the polar decomposition of S, the normality of S implies $UU^*=U^*U$ and $U|T|^{1/2}=|T|^{1/2}U$. Thus, $U|T|=|T|U$, and hence T is normal. q. e. d

The function $t^{1/2}$ $(t\geqq 0)$ is operator monotone ([7]). Therefore, the hyponormality (i. e., $TT^*\leqq T^*T$) implies $|T^*|\leqq|T|$. But $|T^*|\leqq|T|$ does not necessarily imply the hyponormality of T. For example, consider the 2×2-matrices

$$r=\begin{bmatrix}1&0\\0&0\end{bmatrix}\quad\text{and}\quad s=\begin{bmatrix}2&1\\1&1\end{bmatrix}.$$

Note that $r\leqq s$ and $r^2\not\leqq s^2$. We set

$$T=\begin{pmatrix} 0 & & & & \\ r & 0 & & & \\ & s & 0 & & \\ & & s & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

Then we compute

$$TT^*=\begin{pmatrix} 0 & & & & \\ & r^2 & & & \\ & & s^2 & & \\ & & & s^2 & \\ & & & & \ddots \end{pmatrix} \quad \text{and} \quad T^*T=\begin{pmatrix} r^2 & & & & \\ & s^2 & & & \\ & & s^2 & & \\ & & & s^2 & \\ & & & & \ddots \end{pmatrix}.$$

Therefore $|T^*|\leqq|T|$, but T is not hyponormal (because of $r^2\nleqq s^2$).

§ 2. Inequality (*)

In this section, we consider the following inequality for an operator $T\in\mathcal{B}(\mathcal{H})$:

$$(*) \qquad |(T\xi, \xi)|\leqq(|T|\xi, \xi) \qquad \text{for all } \xi\in\mathcal{H}.$$

Obviously the inequality (**) implies (*), and hence the normality of T implies (*).

For an operator $T\in\mathcal{B}(\mathcal{H})$, $\sup\{|(T\xi, \xi)| : \xi\in\mathcal{H}, \|\xi\|=1\}$ is called the *numerical radius* of T and denoted by $w(T)$. Then the following inequality is standard ([6]):

$$(1/2)\|T\|\leqq w(T)\leqq\|T\|.$$

Theorem 2.1. *For an operator $T\in\mathcal{B}(\mathcal{H})$, (*) holds for all $\xi\in\mathcal{H}$ if and only if*

$$U|T|^{1/2}=|T|^{1/2}X$$

for some $X\in\mathcal{B}(\mathcal{H})$ with $w(X)\leqq 1$, where $T=U|T|$ is the polar decomposition of T.

Related results were obtained in [3].

Proof. Suppose that (*) holds for all $\xi\in\mathcal{H}$. For each positive integer $n\in\boldsymbol{N}$, we define $X_n\in\mathcal{B}(\mathcal{H})$ by

$$X_n=\{|T|+(1/n)I\}^{-1/2}U\{|T|+(1/n)I\}^{1/2}.$$

Then, for all $\xi\in\mathcal{H}$,

$$
\begin{aligned}
(X_n\xi,\ \xi) &= (U\{|T|+(1/n)I\}^{1/2}\xi,\ \{|T|+(1/n)I\}^{-1/2}\xi) \\
&= (U\{|T|+(1/n)I\}\{|T|+(1/n)I\}^{-1/2}\xi,\ \{|T|+(1/n)I\}^{-1/2}\xi) \\
&= (T\{|T|+(1/n)I\}^{-1/2}\xi,\ \{|T|+(1/n)I\}^{-1/2}\xi) \\
&\quad + (1/n)(U\{|T|+(1/n)I\}^{-1/2}\xi,\ \{|T|+(1/n)I\}^{-1/2}\xi).
\end{aligned}
$$

Thus, by (*) $(X_n\xi,\ \xi)$ is majorized by

$$
\begin{aligned}
&(|T|\{|T|+(1/n)I\}^{-1/2}\xi,\ \{|T|+(1/n)I\}^{-1/2}\xi) \\
&\quad + (1/n)\|\{|T|+(1/n)I\}^{-1/2}\xi\|^2 \\
&= (\{|T|+(1/n)I\}\{|T|+(1/n)I\}^{-1/2}\xi,\ \{|T|+(1/n)I\}^{-1/2}\xi) \\
&= (\xi,\ \xi).
\end{aligned}
$$

Therefore, we get

$$w(X_n)\leqq 1 \quad \text{and} \quad \|X_n\|\leqq 2\,.$$

Thus, by the Alaoglu theorem, we can construct a subnet $\{X_j\}_{j\in J}$ converging weakly to some $X\in\mathcal{B}(\mathcal{H})$ with $\|X\|\leqq 2$ from the sequence $\{X_n\}_{n\in\boldsymbol{N}}$. Then, we have $w(X)\leqq 1$ since

$$(X\xi,\ \xi)=\lim_j (X_j\xi,\ \xi)\leqq(\xi,\ \xi)\,.$$

Now, from the definition of $\{X_j\}_{j\in J}$, we get

$$U\{|T|+(1/F(j))I\}^{1/2}=\{|T|+(1/F(j))I\}^{1/2}X_{F(j)} \tag{1}$$

for some mapping $F\colon J\to\boldsymbol{N}$ (in fact, $X_j=X_{F(j)}$). Hence, we conclude

$$U|T|^{1/2}=|T|^{1/2}X$$

by taking weak limits of both sides of (1) (see [7]).

Conversely, assume that $U|T|^{1/2}=|T|^{1/2}X$ for some $X\in\mathcal{B}(\mathcal{H})$ with $w(X)\leqq 1$. Then, for all $\xi\in\mathcal{H}$,

$$
\begin{aligned}
|(T\xi,\ \xi)| &= |(U|T|^{1/2}|T|^{1/2}\xi,\ \xi)| \\
&= |(|T|^{1/2}X|T|^{1/2}\xi,\ \xi)| \\
&= |(X|T|^{1/2}\xi,\ |T|^{1/2}\xi)| \\
&\leqq (|T|^{1/2}\xi,\ |T|^{1/2}\xi) \\
&= (|T|\xi,\ \xi).
\end{aligned}
$$

q. e. d.

From the above theorem, we easily obtain:

Corollary 2.2. *When $T\in\mathcal{B}(\mathcal{H})$ satisfies (*) for all $\xi\in\mathcal{H}$, we have*

$$|T^*| \leqq 4|T|$$

and

$$UU^* \leqq U^*U \, .$$

For matrices (i.e., $\dim(\mathcal{H})<\infty$), we obtain the following characterization:

Theorem 2.3. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space. Then for $T\in \mathcal{B}(\mathcal{H})$, $(*)$ holds for all $\xi\in\mathcal{H}$ if and only if T is normal.*

Proof. We may assume that $\mathcal{H}=\boldsymbol{C}^n$. Then $\mathcal{B}(\mathcal{H})=M_n(\boldsymbol{C})$ ($:=$ {complex $n\times n$-matrix}). Thanks to the obvious unitary invariance, we may and do assume that T is of the form

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ & & a_{nn} \end{bmatrix}$$

(i.e., T is an upper triangular matrix).

We will show that

$$a_{ij}=0 \qquad \text{if } i<j$$

by induction on the size n of a matrix.

For $n=1$ this result is trivial. Let

$$T=\begin{bmatrix} a & \beta^* \\ 0 & B \end{bmatrix} \quad \text{and} \quad |T|=\begin{bmatrix} x & \zeta^* \\ \zeta & Z \end{bmatrix}.$$

Here, $a=a_{11}$, x is a non-negative number, β and ζ are (column) vectors in $\boldsymbol{C}^n$, B is an upper triangular matrix in $M_n(\boldsymbol{C})$, and Z is a positive matrix in $M_n(\boldsymbol{C})$. Then, since $T^*T=|T|^2$, we have

$$|a|^2=x^2+\zeta^*\zeta$$

by comparing the $1-1$ components. Therefore $|a|\geqq x$. On the other hand, with $\xi={}^t(1, 0)$ in $\boldsymbol{C}^{n+1}$, we have $|a|\leqq x$ by the assumption $(*)$. Hence

$$x=|a| \quad \text{and} \quad \zeta=0\, .$$

Furthermore, since $T^*T=|T|^2$, we have

(2) $$\bar{a}\beta^*=0$$

by comparing the $1-2$ components.

We will show $\beta=0$ by the contradiction argument. Then, the result follows from the induction hypothesis.

Assume $\beta\neq 0$, and hence $x=|a|=0$ from (2). We choose and fix a (column) vector ξ' $(=\beta)$ in $\boldsymbol{C}^n$ such that

$$k := \beta^*\xi'(=\beta^*\beta)>0\,.$$

Let $\xi={}^t(p,\ {}^t\xi')$ in $\boldsymbol{C}^{n+1}$ ($p>0$). Since $a=0$ and $\zeta=0$, straight forward computations show

$$(T\xi,\ \xi)=kp+(B\xi',\ \xi')$$

and

$$(|T|\xi,\ \xi)=(Z\xi',\ \xi').$$

Therefore (*) does not hold for p sufficiently large, a contradiction. q. e. d.

§ 3. Relation of (*) and (**)

From Theorem 1.2 and 2.3, for a finite-dimensional Hilbert space $\mathcal{H}$, (*) is equivalent to (**) (and to the normality of T). Recall that (**) implies (*). But, in general, (*) does not imply (**) (i.e., $|T^*|\leqq|T|$ by Theorem 1.1).

We will consider an operator T of the form

$$T=\begin{bmatrix} 0 & & & \\ \alpha_1 & 0 & & \\ & \alpha_2 & 0 & \\ & & \ddots & \ddots \end{bmatrix}$$

to explain this phenomenon. Here, α_n's are positive numbers to be fixed later. We note that

$$|T^*|=\begin{bmatrix} 0 & & & \\ & \alpha_1 & & \\ & & \alpha_2 & \\ & & & \ddots \end{bmatrix} \quad\text{and}\quad |T|=\begin{bmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \ddots \end{bmatrix}.$$

Therefore, if the sequence $\{\alpha_n\}$ is strictly decreasing, then $|T^*|\leqq|T|$ (i.e., (**)) does not hold. On the other hand, Corollary 2.3 indicates that, if $\{\alpha_n\}$ decreases too rapidly, then (*) does not hold either. Thus, we are forced to choose a slowly decreasing sequence $\{\alpha_n\}$ so that T does not satisfy (**) but (*).

We set

$$e_n^{-1}=3\cdot 2^n-4 \qquad (n\geqq 1).$$

By using this sequence $\{e_n\}$ (of positive numbers converging to 0), we set

$$\alpha_1=1 \quad\text{and}\quad \alpha_{n+1}=\alpha_n(1+e_n)^{-1} \quad (n\geqq 1).$$

Then $\{\alpha_n\}$ is obviously decreasing, and it remains to show that (*) holds. For this purpose, we need the following result due to T. Ando ([2]): $w(Y)\leqq 1$ if and only if $Y=(I+A)^{1/2}B(I-A)^{1/2}$ with $-I\leqq A\leqq I$ and $\|B\|\leqq 1$ (we actually need just the easier half).

We define the sequence $\{a_n\}\subset[-1,\ 1]$ (in fact, $a_n\in[-1,\ 0)$) by

$$a_n^{-1}=-3\cdot 2^{n-1}+2 \qquad (n\geqq 1),$$

and we set

$$X=\begin{bmatrix} \sqrt{1+a_1} & & \\ & \sqrt{1+a_2} & \\ & & \ddots \end{bmatrix} U \begin{bmatrix} \sqrt{1-a_1} & & \\ & \sqrt{1-a_2} & \\ & & \ddots \end{bmatrix}$$

with

$$U=\begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \end{bmatrix}.$$

In fact, the above U is the partial isometry appearing in the polar decomposition of T. Then, by T. Ando's result, we have $w(X)\leqq 1$. It is straightforward to see

$$X=\begin{bmatrix} 0 & & & \\ \sqrt{1+e_1} & 0 & & \\ & \sqrt{1+e_2} & 0 & \\ & & \ddots & \ddots \end{bmatrix},$$

thanks to

$$1+e_n=(1+a_{n+1})(1-a_n).$$

It is also easy to see

$$U|T|^{1/2}=|T|^{1/2}X.$$

Therefore, we see that T satisfies (*) by Theorem 2.1.

Since $\sum_{n=1}^{\infty} e_n$ is convergent, so is

$$\alpha_1\cdot\alpha_n^{-1}=\prod_{i=1}^{n-1}(1+e_i).$$

Therefore, $\lim_n \alpha_n \gneqq 0$, and the above T is not compact.

The author does not know whether the condition (*) and the normality are different for compact operators (in fact, we can confine ourselves to the case T is compact quasi-nilpotent according to the way used in Theorem 2.3), and this problem seems to deserve further investigation.

Appendix

Theorem 2.3 is extended to trace class operators, that is, for them the condition (*) is equivalent to the normality. We show this by the a method different from that of Theorem 2.3.

Lemma. *Let $X\in\mathcal{B}(\mathcal{H})$ be a numerical contraction, i.e., $w(X)\leqq 1$. Then $X\xi=\xi$ implies $X^*\xi=\xi$.*

Proof. By T. Ando's factorization of a numerical contraction ([2]), there exist a self-adjoint contraction $A\in\mathcal{B}(\mathcal{H})$ and a contraction $B\in\mathcal{B}(\mathcal{H})$ such that

$$X=(I+A)^{1/2}B(I-A)^{1/2}$$

and B is isometric on the range of $I-A$. Since

$$\begin{aligned}(\xi,\xi)=(X\xi,\xi)&=((I+A)^{1/2}B(I-A)^{1/2}\xi,\xi)\\&\leqq\|B(I-A)^{1/2}\xi\|\cdot\|(I+A)^{1/2}\xi\|\\&\leqq(1/2)\{\|(I-A)^{1/2}\xi\|^2+\|(I+A)^{1/2}\xi\|^2\}\\&=(\xi,\xi),\end{aligned}$$

we have

(3) $$B(I-A)^{1/2}\xi=c(I+A)^{1/2}\xi$$

for some scalar c and

(4) $$((I-A)\xi,\xi)=((I+A)\xi,\xi).$$

From (4), we have

$$(A\xi,\xi)=0.$$

Since B is isometric on the range of $(I-A)$, we have from (3)

$$\begin{aligned}(I-A)\xi&=(I-A)^{1/2}B^*B(I-A)^{1/2}\xi\\&=c(I-A)^{1/2}B^*(I+A)^{1/2}\xi\\&=cX^*\xi\end{aligned}$$

by the polarization identity. Therefore

$$\begin{aligned}(\xi,\xi)=((I-A)\xi,\xi)&=c(X^*\xi,\xi)\\&=c(\xi,\xi)\end{aligned}$$

and hence $c=1$ and $X^*\xi=\xi-A\xi$. But since

$$\begin{aligned}\xi=X\xi&=(I+A)^{1/2}B(I-A)^{1/2}\xi\\&=(I+A)\xi,\end{aligned}$$

we obtain $X^*\xi=\xi$. q.e.d.

Theorem. *Let $T\in\mathcal{B}(\mathcal{H})$ be of trace class. Then $(*)$ holds for all $\xi\in\mathcal{H}$ if and only if T is normal.*

Proof. By Theorem 2.1, there exists a numerical contraction $X\in\mathcal{B}(\mathcal{H})$ such that

$$U|T|^{1/2}=|T|^{1/2}X,$$

where $T=U|T|$ is the polar decomposition of T. The space $\mathcal{C}_2(\mathcal{H})$ of Hilbert-Schmidt class operators becomes a Hilbert space with the inner product $\langle K, L\rangle = Tr(L^*K)$ for $K, L\in\mathcal{C}_2(\mathcal{H})$. We define the operator Φ on $\mathcal{C}_2(\mathcal{H})$ by $\Phi(K)=U^*KX$. Then $\Phi(|T|^{1/2})=U^*|T|^{1/2}X=|T|^{1/2}$.

Now, by T. Ando's factorization, we are led to the representation

$$X=(I+A)^{1/2}B(I-A)^{1/2}$$

with a self-adjoint contraction $A\in\mathcal{B}(\mathcal{H})$ and a contraction $B\in\mathcal{B}(\mathcal{H})$. Then

$$\begin{aligned}\Phi &= L_{U^*}\circ R_X\\ &= L_{U^*}\circ R_{(I-A)^{1/2}}\circ R_B\circ R_{(I+A)^{1/2}}\\ &= R_{(I-A)^{1/2}}\circ L_{U^*}\circ R_B\circ R_{(I+A)^{1/2}}\\ &= (I-R_A)^{1/2}\circ(L_{U^*}\circ R_B)\circ(I+R_A)^{1/2}\,.\end{aligned}$$

Here, L_D and R_D are the left- and right-multiplication operator on $\mathcal{C}_2(\mathcal{H})$ induced by $D\in\mathcal{B}(\mathcal{H})$ respectively. Again by T. Ando's result, we get $w(\Phi)\leqq 1$. By virtue of the above lemma, we have $\Phi^*(|T|^{1/2})=U|T|^{1/2}X^*=|T|^{1/2}$ and hence

$$U^*|T|^{1/2}=|T|^{1/2}X^*\,.$$

Therefore, we get $\mathrm{Re}\,(U)|T|^{1/2}=|T|^{1/2}\,\mathrm{Re}\,(X)$ and $\mathrm{Im}\,(U)|T|^{1/2}=|T|^{1/2}\,\mathrm{Im}\,(X)$.

Let $\{E_{\mathrm{Re}(U)}(S): S$ is a Borel subset of $\boldsymbol{R}\}$ and $\{E_{\mathrm{Re}(X)}(S): S$ is a Borel subset of $\boldsymbol{R}\}$ be the spectral projections of $\mathrm{Re}\,(U)$ and $\mathrm{Re}\,(X)$ respectively. Then, since $\mathrm{Re}\,(U)|T|^{1/2}=|T|^{1/2}\,\mathrm{Re}\,(X)$, we have $E_{\mathrm{Re}(U)}(S)|T|^{1/2}=|T|^{1/2}E_{\mathrm{Re}(X)}(S)$. This implies

$$E_{\mathrm{Re}(U)}(S)|T|E_{\mathrm{Re}(U)}(S)\leqq|T|\,.$$

But $E_{\mathrm{Re}(U)}(S)|T|E_{\mathrm{Re}(U)}(S)\leqq|T|$ is possible only when $E_{\mathrm{Re}(U)}(S)$ commutes with $|T|$. Therefore, we are led to the commutativity of $\mathrm{Re}\,(U)$ and $|T|$. In a similar fashion, $\mathrm{Im}\,(U)$ commutes with $|T|$. Hence, we obtain

$$U|T|=|T|U\,,$$

i.e., T is quasi-normal. Since T is of trace class, T is normal by Proposition 1.3. q.e d.

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