Publ. RIMS, Kyoto Univ.
30 (1994), 261-279

Fibrewise Hopf Construction and Hoo Formula for Pairings

By

Nobuyuki ODA*

Abstract

Let Γ be a fibrewise co-Hopf space over a topological space B. A Γ_B-suspension space $\Gamma_B X$ is a generalization of a fibrewise suspension space $\Sigma_B X$ for any fibrewise pointed space X over B. Making use of Γ_B-suspension space, we define Γ_B-Hopf construction and prove a Γ_B-suspension formula which generalizes the suspension formula of C. S. Hoo. The dual formula is also proved.

Introduction

Let $\mathbf{Top}_B^B$ be the category of fibrewise pointed topological spaces over a base space B and fibrewise pointed continuous maps over B (cf. James [4, 5]). We write simply $f: X \to Y$ for morphisms in $\mathbf{Top}_B^B$. Throughout this paper a *space* X means a fibrewise pointed topological space over B and a *map* $f: X \to Y$ means a fibrewise pointed continuous map over B between fibrewise pointed spaces X and Y over B.

Let Γ be a co-Hopf space in $\mathbf{Top}_B^B$. Then we define a Γ_B-suspension space $\Gamma_B X$ by $\Gamma \wedge_B X$ for any space X in $\mathbf{Top}_B^B$ and Γ_B-suspension map $\Gamma_B \alpha: \Gamma_B A \to \Gamma_B Z$ by $\Gamma_B \alpha = 1_\Gamma \wedge_B \alpha: \Gamma_B A = \Gamma \wedge_B A \to \Gamma \wedge_B Z = \Gamma_B Z$ for any map $\alpha: A \to Z$ in $\mathbf{Top}_B^B$ (cf. [7]). This defines a Γ_B-suspension map between fibrewise pointed homotopy sets:

$$\Gamma_B: [A, Z]_B^B \longrightarrow [\Gamma_B A, \Gamma_B Z]_B^B.$$

Let $\mu: X \times_B Y \to Z$ be a pairing in $\mathbf{Top}_B^B$. Then for any maps $\alpha: A \to X$ and $\beta: A \to Y$ in $\mathbf{Top}_B^B$, we can define a map $\alpha \dot{+} \beta: A \to Z$ in $\mathbf{Top}_B^B$. On the other hand, $[\Gamma_B A, \Gamma_B Z]_B^B$ has a binary operation (denoted by $\dot{+}$) induced by the fibrewise co-Hopf structure of $\Gamma_B A$.

A co-Hopf space Γ in $\mathbf{Top}_B^B$ is called a co-looplike space in $\mathbf{Top}_B^B$ if $[\Gamma, Z]_B^B$ is naturally an algebraic loop for any space Z in $\mathbf{Top}_B^B$. The following theorem shows that the pairing $\dot{+}$ and the binary operation $\dot{+}$ are closely related.

Communicated by K. Saito, June 1, 1993.
1991 Mathematics Subject Classifications: 55P30, 55P35, 55P40.
* Department of Applied Mathematics, Faculty of Science, Fukuoka University, 8-19-1 Nanakuma, Jonanku, Fukuoka, 814-01, Japan.

Theorem 3.1. *Let Γ be a co-looplike space in* $\mathbf{Top}_B^B$. *Let* $\mu: X\times_B Y\to Z$ *be a pairing in* $\mathbf{Top}_B^B$ *with axes* $f: X\to Z$ *and* $g: Y\to Z$. *Then the following relation holds in* $[\Gamma_B A, \Gamma_B Z]_B^B$ *for any maps* $\alpha: A\to X$ *and* $\beta: A\to Y$ *in* $\mathbf{Top}_B^B$:

$$\Gamma_B(\alpha\dotplus_B\beta)=J_{\Gamma,B}(\mu)\circ\Gamma_B(\alpha\wedge_B\beta)\circ\Gamma_B(q\circ\Delta_{A,B})\dotplus_B\{\Gamma_B(f\circ\alpha)\dotplus_B\Gamma_B(g\circ\beta)\}.$$

The Γ_B-Hopf construction $J_{\Gamma,B}(\mu)$ in the above theorem is defined in Definition 2.7 and $\Delta_{A,B}: A\to A\times_B A$ is a fibrewise diagonal map over B. The Γ_B-Hopf construction $J_{\Gamma,B}(\mu)$ is a generalization of the Hopf construction $J(\mu)$ of Hoo [2, 3]. If $B=\{*\}$ and $\Gamma=S^1$, then the Γ_B-Hopf construction coincides with the usual Hopf construction (e. g. on p. 502 of Whitehead [8]). The above formula is a generalization of Theorem 1 of Hoo [3].

In § 1 we review some properties of pairings, copairings, Γ_B-suspension spaces and Γ_B-loop spaces ([6, 7]). In § 2 we define Γ_B-Hopf construction. In § 3 we prove Γ_B-suspension formula and related results. In §§ 4 and 5 we prove the dual results of §§ 2 and 3 respectively.

We use the following symbols in $\mathbf{Top}_B^B$. An isomorphism in $\mathbf{Top}_B^B$ is denoted by $\cong_B$. Let X and Y be spaces in $\mathbf{Top}_B^B$. The space $X\vee_B Y$ is the *fibrewise wedge sum* over B which is a subspace of the *fibrewise product* $X\times_B Y$ over B. We have a natural inclusion map $j_B: X\vee_B Y\subset X\times_B Y$. The *fibrewise smash product* over B is defined by $X\wedge_B Y=(X\times_B Y)/_B(X\vee_B Y)$. The *fibrewise pointed mapping-space* over B is denoted by $\mathrm{map}_B^B(X, Y)$ (cf. § 9 of [5]).

We denote by $\Delta_{X,B}: X\to X\times_B X$ the *fibrewise diagonal map* over B and $\nabla_{X,B}: X\vee_B X\to X$ the *fibrewise folding map* over B. We denote by $*_B: X\to Y$ the *fibrewise constant map* over B.

A *fibrewise pointed homotopy relation* over B is denoted by $\simeq_B$ and the set of the fibrewise pointed homotopy classes over B is denoted by $[X, Y]_B^B$.

Let $\Sigma=B\times S^1$ in $\mathbf{Top}_B^B$. Then $\Sigma_B X=\Sigma\wedge_B X$ is the *fibrewise reduced suspension space* of X and $\Omega_B X=\Sigma_B^* X=\mathrm{map}_B^B(\Sigma, X)$ is the *fibrewise loop space* of X. (We remark that James [5] uses the symbol $\Sigma_B^B X$ for the fibrewise reduced suspension space and $\Omega_B^B X$ for the fibrewise loop space of a space X in $\mathbf{Top}_B^B$. We use our abbreviated symbols for simplicity.)

We assume that all the spaces are fibrewise pointed non-degenerated spaces with closed section (cf. § 22 of [5]). Moreover we assume that the co-Hopf space Γ is fibrewise locally compact and fibrewise regular (cf. [5]).

§ 1. Fibrewise Induced Pairings

We call a map $\mu: X\times_B Y\to Z$ in $\mathbf{Top}_B^B$ a *pairing* with the *axes* $f: X\to Z$ and $g: Y\to Z$ if it satisfies the condition that

$$\mu|X\vee_B Y\simeq_B\nabla_{Z,B}\circ(f\vee_B g): X\vee_B Y\longrightarrow Z.$$

Such a map $\mu: X\times_B Y\to Z$ is also called a *fibrewise pairing over* B or a *pairing* *in* $\mathbf{Top}_B^B$.

If we are given a pairing $\mu: X\times_B Y\to Z$ in $\mathbf{Top}_B^B$, we define a map $\alpha\dot{+}_B\beta: A\to Z$ in $\mathbf{Top}_B^B$ for any maps $\alpha: A\to X$ and $\beta: A\to Y$ in $\mathbf{Top}_B^B$ by

$$\alpha\dot{+}_B\beta=\mu\circ(\alpha\times_B\beta)\circ\Delta_{A,B}.$$

The relations $\alpha\dot{+}_B*_B=f_*(\alpha)$ and $*_B\dot{+}_B\beta=g_*(\beta)$ holds in $[A, Z]_B^B$. We have an equality of maps in $\mathbf{Top}_B^B$

$$(\alpha\dot{+}_B\beta)\circ\delta=(\alpha\circ\delta)\dot{+}_B(\beta\circ\delta)$$

for any maps $\alpha: A\to X$, $\beta: A\to Y$ and $\delta: D\to A$ in $\mathbf{Top}_B^B$.

We call a map $\theta: A\to X\vee_B Y$ in $\mathbf{Top}_B^B$ a *copairing* with the *coaxes* $h: A\to X$ and $r: A\to Y$ if it satisfies the condition that

$$j\circ\theta\simeq_B(h\times_B r)\circ\Delta_{A,B}: A\longrightarrow X\times_B Y$$

for the inclusion map $j: X\vee_B Y\to X\times_B Y$. Such a map $\theta: A\to X\vee_B Y$ in $\mathbf{Top}_B^B$ is also called a *fibrewise copairing over* B or a *copairing in* $\mathbf{Top}_B^B$.

If we are given a copairing $\theta: A\to X\vee_B Y$ in $\mathbf{Top}_B^B$, we define a map $\alpha\dot{+}_B\beta: A\to Z$ in $\mathbf{Top}_B^B$ for any maps $\alpha: X\to Z$ and $\beta: Y\to Z$ in $\mathbf{Top}_B^B$ by

$$\alpha\dot{+}\beta=\nabla_{Z,B}\circ(\alpha\vee_B\beta)\circ\theta.$$

The relations $\alpha\dot{+}_B*_B=h^*(\alpha)$ and $*_B\dot{+}_B\beta=r^*(\beta)$ hold in $[A, Z]_B^B$. We have an equality of maps in $\mathbf{Top}_B^B$

$$\zeta\circ(\alpha\dot{+}_B\beta)=(\zeta\circ\alpha)\dot{+}_B(\zeta\circ\beta)$$

for any maps $\alpha: X\to Z$, $\beta: Y\to Z$ and $\zeta: Z\to W$ in $\mathbf{Top}_B^B$.

A space X in $\mathbf{Top}_B^B$ is said to be a *Hopf space in* $\mathbf{Top}_B^B$ or a *fibrewise Hopf space over* B if there is a pairing $\mu: X\times_B X\to X$ in $\mathbf{Top}_B^B$ with axes $f\simeq_B g\simeq_B 1_X$. Such a pairing μ is called a *fibrewise multiplication* of X. If X is a Hopf space in $\mathbf{Top}_B^B$, then $[A, X]_B^B$ has a natural binary operation $\dot{+}_B$ defined above for any space A in $\mathbf{Top}_B^B$.

A space A in $\mathbf{Top}_B^B$ is said to be a *co-Hopf space in* $\mathbf{Top}_B^B$ or a *fibrewise co-Hopf space over* B if there is a copairing $\theta: A\to A\vee_B A$ in $\mathbf{Top}_B^B$ with coaxes $h\simeq_B r\simeq_B 1_A$. Such a copairing θ is called a *fibrewise co-multiplication* of A. If A is a co-Hopf space in $\mathbf{Top}_B^B$, then $[A, X]_B^B$ has a natural binary operation $\dot{+}_B$ defined above for any space X in $\mathbf{Top}_B^B$.

Let Γ be a **co-Hopf space** in $\mathbf{Top}_B^B$ with a fibrewise co-multiplication $\gamma: \Gamma\to\Gamma\vee_B\Gamma$. We write $\Gamma\wedge_B X=\Gamma_B X$ (the Γ_B-*suspension space* of X) for any space X in $\mathbf{Top}_B^B$. A map $\alpha: X\to Y$ in $\mathbf{Top}_B^B$ induces a Γ_B-*suspension map* $\Gamma_B\alpha: \Gamma_B X\to\Gamma_B Y$ by $\Gamma_B\alpha=1_\Gamma\wedge_B\alpha: \Gamma\wedge_B X\to\Gamma\wedge_B Y$. We see $\Gamma_B\alpha\circ\Gamma_B\beta=\Gamma_B(\alpha\circ\beta)$ for any maps $\alpha: Y\to Z$ and $\beta: X\to Y$ in $\mathbf{Top}_B^B$.

We define a fibrewise pointed map $\gamma_X=\gamma\wedge_B 1_X: \Gamma_B X\to\Gamma_B X\vee_B\Gamma_B X$ by

$$\gamma_X: \Gamma_B X = \Gamma \wedge_B X \longrightarrow (\Gamma \vee_B \Gamma) \wedge_B X \cong_B (\Gamma \wedge_B X) \vee_B (\Gamma \wedge_B X) = \Gamma_B X \vee_B \Gamma_B X$$

(cf. (6.1) of [5]). Then $\Gamma_B X$ is a co-Hopf space in $\mathbf{Top}_B^B$ with a fibrewise comultiplication γ_X for any space X in $\mathbf{Top}_B^B$. If α, $\beta: \Gamma_B X \to Y$ and $\delta: W \to X$ are maps in $\mathbf{Top}_B^B$, then as maps in $\mathbf{Top}_B^B$ we have

$$(\alpha \dotplus_B \beta) \circ \Gamma_B \delta = (\alpha \circ \Gamma_B \delta) \dotplus_B (\beta \circ \Gamma_B \delta).$$

There are following isomorphisms as co-Hopf spaces in $\mathbf{Top}_B^B$ (cf. (6.2) and (6.1) of [5], (3.80) of [4]);

$$\Gamma_B(X \wedge_B Y) \cong_B (\Gamma_B X) \wedge_B Y \cong_B X \wedge_B \Gamma_B Y$$

and

$$\Gamma_B(X \vee_B Y) \cong_B (\Gamma_B X) \vee_B (\Gamma_B Y).$$

Dually, we define $\Gamma_B^* X = \mathrm{map}_B^B(\Gamma, X)$ (the *Γ_B-loop space* of X) for any space X in $\mathbf{Top}_B^B$. A map $\alpha: X \to Y$ in $\mathbf{Top}_B^B$ induces a *Γ_B-loop map* $\Gamma_B^* \alpha: \Gamma_B^* X \to \Gamma_B^* Y$. We see $\Gamma_B^* \alpha \circ \Gamma_B^* \beta = \Gamma_B^*(\alpha \circ \beta)$ for any maps $\alpha: Y \to Z$ and $\beta: X \to Y$ in $\mathbf{Top}_B^B$. We define a fibrewise pointed map $\gamma_X^* = \mathrm{map}_B^B(\gamma, 1_X): \Gamma_B^* X \times_B \Gamma_B^* X \to \Gamma_B^* X$ by

$$\gamma_X^*: \mathrm{map}_B^B(\Gamma, X) \times_B \mathrm{map}_B^B(\Gamma, X) \cong_B \mathrm{map}_B^B(\Gamma \vee_B \Gamma, X) \longrightarrow \mathrm{map}_B^B(\Gamma, X)$$

(cf. (9.19) of [5]). Then $\Gamma_B^* X$ is a Hopf space in $\mathbf{Top}_B^B$ with fibrewise multiplication γ_X^* for any space X in $\mathbf{Top}_B^B$.

If α, $\beta: X \to \Gamma_B^* Y$ and $\zeta: Y \to Z$ are maps in $\mathbf{Top}_B^B$, then as maps in $\mathbf{Top}_B^B$ we have

$$(\Gamma_B^* \zeta) \circ (\alpha \dotplus_B \beta) = (\Gamma_B^* \zeta \circ \alpha) \dotplus_B (\Gamma_B^* \zeta \circ \beta).$$

There is the following isomorphism as Hopf spaces in $\mathbf{Top}_B^B$ (cf. (9.21) of [5]);

$$\Gamma_B^*(X \times_B Y) \cong_B \Gamma_B^* X \times_B \Gamma_B^* Y.$$

Proposition 1.1. *Let Γ be a co-Hopf space in $\mathbf{Top}_B^B$. Let X be a fibrewise locally compact and fibrewise regular space and Y any space in $\mathbf{Top}_B^B$. Then the adjoint map*

$$\tau: [\Gamma_B X, Y]_B^B \cong [X, \Gamma_B^* Y]_B^B$$

is an isomorphism of sets which satisfies

$$\tau(\alpha \dotplus_B \beta) = \tau(\alpha) \dotplus_B \tau(\beta)$$

for any elements α, $\beta \in [\Gamma_B X, Y]_B^B$. Moreover it satisfies the following relations.

(i) *$\tau(\alpha \circ \Gamma_B \beta) = \tau(\alpha) \circ \beta$ for any elements $\alpha \in [\Gamma_B Y, Z]_B^B$ and $\beta \in [X, Y]_B^B$.*

(ii) *$\tau(\zeta \circ \alpha) = \Gamma_B^* \zeta \circ \tau(\alpha)$ for any elements $\alpha \in [\Gamma_B Y, Z]_B^B$ and $\zeta \in [Z, W]_B^B$.*

Proof. By (9.20), (9.25) and (6.2) of [5], we see that $\tau: [\Gamma_B X, Y]_B^B \to [X, \Gamma_B^* Y]_B^B$ is a bijection of homotopy sets. The statement that τ is a homomorphism is proved by a similar argument as in the case of the usual adjoint map $\tau: [\Sigma X, Y] \cong [X, \Omega Y]$.

(i) and (ii) are direct consequences of definitions. *q.e.d.*

Let S be a set with a binary operation $+$. We call S an *algebraic loop* if S has two-sided identity (denoted by 0) and for any elements a, b of S, the equations

$$x+a=b \quad \text{and} \quad a+y=b$$

have a unique pair of solutions x, $y \in S$. A map $\phi: S \to L$ between two algebraic loops is called a *homomorphism* if $\phi(a+b)=\phi(a)+\phi(b)$ holds for any a, $b \in S$. If $\phi: S \to L$ is a homomorphism, we have $\phi(0)=0$. A sequence $S \xrightarrow{\phi} L \xrightarrow{\varphi} R$ of algebraic loops and homomorphisms is said to be *exact* if $\operatorname{Im}\phi = \operatorname{Ker}\varphi$.

A Hopf space X in $\mathbf{Top}_B^B$ is called a *looplike space in* $\mathbf{Top}_B^B$ or a *fibrewise looplike space over* B if the homotopy set $[A, X]_B^B$ is an algebraic loop with the binary operation $\dotplus_B$ for any space A in $\mathbf{Top}_B^B$.

A co-Hopf space A in $\mathbf{Top}_B^B$ is called a *co-looplike space in* $\mathbf{Top}_B^B$ or a *fibrewise co-looplike space over* B if the homotopy set $[A, X]_B^B$ is an algebraic loop with the binary operation $\dotplus_B$ for any space X in $\mathbf{Top}_B^B$.

§ 2. Fibrewise Γ-Hopf Construction

If X and Y are fibrewise non-degenerate spaces over B, then the cofiber of $j: X\vee_B Y \to X\times_B Y$ is fibrewise pointed homotopy equivalent to $X\wedge_B Y$ (cf. (22.5) of [5]). Consider a fibrewise pointed cofibration sequence

$$X\vee_B Y \xrightarrow{j} X\times_B Y \xrightarrow{q} X\wedge_B Y \xrightarrow{\delta} \Sigma_B(X\vee_B Y) \xrightarrow{\Sigma_B j} \Sigma_B(X\times_B Y) \longrightarrow \cdots$$

(cf. § 21 of [5]). Then we have a fibrewise pointed cofibration sequence

$$\Gamma_B(X\vee_B Y) \xrightarrow{\Gamma_B j} \Gamma_B(X\times_B Y) \xrightarrow{\Gamma_B q} \Gamma_B(X\wedge_B Y) \xrightarrow{\Gamma_B \delta} \Sigma_B\Gamma_B(X\vee_B Y) \xrightarrow{\Sigma_B\Gamma_B j} \cdots$$

for any co-Hopf space (Γ, γ) in $\mathbf{Top}_B^B$ (cf. (20.19) and (21.2) of [5]).

Let $p_1: X\times_B Y\to X$ and $p_2: X\times_B Y\to Y$ be the fibrewise projections and $i_1: X\to X\vee_B Y$ and $i_2: Y\to X\vee_B Y$ the inclusion maps. We define a fibrewise pointed map $\rho=\rho(\Gamma, X, Y): \Gamma_B(X\times_B Y)\to\Gamma_B(X\vee_B Y)$ by

$$\rho=\nabla_{Z,B}\circ\{\Gamma_B(i_1\circ p_1)\vee_B\Gamma_B(i_2\circ p_2)\}\circ\bar{\gamma}=\Gamma_B(i_1\circ p_1)\dotplus_B\Gamma_B(i_2\circ p_2),$$

where $Z=\Gamma_B(X\vee_B Y)$ and $\bar{\gamma}: \Gamma_B(X\times_B Y)\to \Gamma_B(X\times_B Y)\vee_B\Gamma_B(X\times_B Y)$ is the fibrewise co-Hopf structure of $\Gamma_B(X\times_B Y)$ induced by the fibrewise co-Hopf structure $\gamma: \Gamma\to\Gamma\vee_B\Gamma$ of Γ.

Proposition 2.1. *Let Γ be a co-Hopf space in $\mathbf{Top}_B^B$. Let $j: X\vee_B Y\to X\times_B Y$ be the inclusion map. Then the map $\rho: \Gamma_B(X\times_B Y)\to\Gamma_B(X\vee_B Y)$ defined above satisfies a relation*

$$\rho\circ\Gamma_B j\simeq_B 1_{\Gamma_B(X\vee_B Y)}.$$

Proof. We remark that $\Gamma_B(X\vee_B Y)\cong_B\Gamma_B X\vee_B\Gamma_B Y$ (cf. (6.1) of [5]). We have

$$\begin{aligned}\rho\circ\Gamma_B j\,|\,\Gamma_B X\times_B\{*_B\}&=\rho\circ\Gamma_B j\circ\Gamma_B i_1\\&=\{\Gamma_B(i_1\circ p_1)\dot{+}_B\Gamma_B(i_2\circ p_2)\}\circ\Gamma_B j\circ\Gamma_B i_1\\&=\Gamma_B(i_1\circ p_1\circ j\circ i_1)\dot{+}_B\Gamma_B(i_2\circ p_2\circ j\circ i_1)\\&=\Gamma_B i_1\dot{+}_B *_B\simeq_B\Gamma_B i_1=1_{\Gamma_B(X\vee_B Y)}\,|\,\Gamma_B X\times_B\{*_B\}.\end{aligned}$$

Similarly, we have $\rho\circ\Gamma_B j\,|\,\{*_B\}\times_B\Gamma_B Y\simeq_B 1_{\Gamma_B(X\vee_B Y)}\,|\,\{*_B\}\times_B\Gamma_B Y$.

Then the result follows. *q.e.d.*

We first prove naturality of the fibrewise pointed map ρ. Consider another fibrewise cofibration sequence

$$X'\vee_B Y'\xrightarrow{j'}X'\times_B Y'\xrightarrow{q'}X'\wedge_B Y'.$$

Then we have another fibrewise pointed map

$$\rho'=\Gamma_B(i_1'\circ p_1')\dot{+}_B\Gamma_B(i_2'\circ p_2'):\ \Gamma_B(X'\times_B Y')\longrightarrow\Gamma_B(X'\vee_B Y').$$

(We use the symbols i_1', p_1', ρ' etc. for the maps corresponding to i_1, p_1, ρ etc. respectively.)

Proposition 2.2. *Let Γ be a co-Hopf space in* $\mathbf{Top}_B^B$. *The following diagram is strictly commutative in* $\mathbf{Top}_B^B$ *for any maps α: $X'\to X$ and β: $Y'\to Y$ in* $\mathbf{Top}_B^B$.

$$\begin{array}{ccc}\Gamma_B(X'\times_B Y') & \xrightarrow{\rho'} & \Gamma_B(X'\vee_B Y')\\ {\scriptstyle\Gamma_B(\alpha\times_B\beta)}\downarrow & & \downarrow{\scriptstyle\Gamma_B(\alpha\vee_B\beta)}\\ \Gamma_B(X\times_B Y) & \xrightarrow{\rho} & \Gamma_B(X\vee_B Y)\end{array}$$

Proof. We have

$$\begin{aligned}\rho\circ\Gamma_B(\alpha\times_B\beta)&=\{\Gamma_B(i_1\circ p_1)\dot{+}_B\Gamma_B(i_2\circ p_2)\}\circ\Gamma_B(\alpha\times_B\beta)\\&=\Gamma_B\{i_1\circ p_1\circ(\alpha\times_B\beta)\}\dot{+}_B\Gamma_B\{i_2\circ p_2\circ(\alpha\times_B\beta)\}\\&=\Gamma_B\{(\alpha\vee_B\beta)\circ i_1'\circ p_1'\}\dot{+}_B\Gamma_B\{(\alpha\vee_B\beta)\circ i_2'\circ p_2'\}\\&=\Gamma_B(\alpha\vee_B\beta)\circ\{\Gamma_B(i_1'\circ p_1')\dot{+}_B\Gamma_B(i_2'\circ p_2')\}\\&=\Gamma_B(\alpha\vee_B\beta)\circ\rho'.\end{aligned}$$

q.e.d.

Lemma 2.3. *Let Γ be a co-looplike space in* $\mathbf{Top}_B^B$. *Then*

$$(\Gamma_B q)^*:\ [\Gamma_B(X\wedge_B Y),\ Z]_B^B\longrightarrow[\Gamma_B(X\times_B Y),\ Z]_B^B$$

is a monomorphism for any space Z in $\mathbf{Top}_B^B$.

Proof. Consider a fibrewise pointed cofibration sequence

$$\Gamma_B(X\vee_B Y)\xrightarrow{\Gamma_B j}\Gamma_B(X\times_B Y)\xrightarrow{\Gamma_B q}\Gamma_B(X\wedge_B Y)\xrightarrow{\Gamma_B \partial}\Sigma_B\Gamma_B(X\vee_B Y)\longrightarrow\cdots.$$

There is a fibrewise pointed map $\rho:\Gamma_B(X\times_B Y)\to\Gamma_B(X\vee_B Y)$ such that $\rho\circ\Gamma_B j\simeq_B 1_{\Gamma_B(X\vee_B Y)}$ by Proposition 2.1. Then by the long homotopy exact sequence of algebraic loops and homomorphisms, we have the result.

Theorem 2.4. *Let Γ be a co-looplike space in* $\mathbf{Top}_B^B$. *Then there exists a unique element $v=v(\Gamma, X, Y)$ of $[\Gamma_B(X\wedge_B Y), \Gamma_B(X\times_B Y)]_B^B$ such that*

$$(v\circ\Gamma_B q)\dotplus_B(\Gamma_B j\circ\rho)=1_{\Gamma_B(X\times_B Y)}$$

in $[\Gamma_B(X\times_B Y), \Gamma_B(X\times_B Y)]_B^B$.

Proof. Consider an exact sequene of algebraic loops and homomorphisms:

$$0\longrightarrow[\Gamma_B(X\wedge_B Y), \Gamma_B(X\times_B Y)]_B^B\xrightarrow{(\Gamma_B q)^*}[\Gamma_B(X\times_B Y), \Gamma_B(X\times_B Y)]_B^B$$
$$\xrightarrow{(\Gamma_B j)^*}[\Gamma_B(X\vee_B Y), \Gamma_B(X\times_B Y)]_B^B.$$

Since $[\Gamma_B(X\times_B Y), \Gamma_B(X\times_B Y)]_B^B$ is an algebraic loop, its elements $1_{\Gamma_B(X\times_B Y)}$ and $\Gamma_B j\circ\rho$ determine a unique element $t\in[\Gamma_B(X\times_B Y), \Gamma_B(X\times_B Y)]_B^B$ such that

$$t\dotplus_B(\Gamma_B j\circ\rho)=1_{\Gamma_B(X\times_B Y)}.$$

Then we see $(\Gamma_B j)^*\{t\dotplus_B(\Gamma_B j\circ\rho)\}=(\Gamma_B j)^*\{1_{\Gamma_B(X\times_B Y)}\}$. We note that

$$(\Gamma_B j)^*\{t\dotplus_B(\Gamma_B j\circ\rho)\}=\{t\dotplus_B(\Gamma_B j\circ\rho)\}\circ\Gamma_B j$$
$$=(t\circ\Gamma_B j)\dotplus_B(\Gamma_B j\circ\rho\circ\Gamma_B j)\simeq_B(t\circ\Gamma_B j)\dotplus_B\Gamma_B j$$

by Proposition 2.1. It follows that $(t\circ\Gamma_B j)\dotplus_B\Gamma_B j=\Gamma_B j$ as fibrewise homotopy classes and hence $t\circ\Gamma_B j=(\Gamma_B j)^*(t)=0$. Hence there exists an element v of $[\Gamma_B(X\wedge_B Y), \Gamma_B(X\times_B Y)]_B^B$ such that $v\circ\Gamma_B q=(\Gamma_B q)^*(v)=t$ or

$$\{v\circ(\Gamma_B q)\}\dotplus_B\{(\Gamma_B j)\circ\rho\}=1_{\Gamma_B(X\times_B Y)}.$$

The element v is unique by Lemma 2.3. *q.e.d.*

We now prove that the fibrewise pointed map v defined by Theorem 2.4 is natural. Consider another fibrewise pointed cofibration sequence

$$X'\vee_B Y'\xrightarrow{j'}X'\times_B Y'\xrightarrow{q'}X'\wedge_B Y'.$$

Then there exists an element $v'\in[\Gamma_B(X'\wedge_B Y'), \Gamma_B(X'\times_B Y')]_B^B$ such that

$$(v'\circ\Gamma_B q')\dot{+}_B(\Gamma_B j'\circ\rho')=1_{\Gamma_B(X'\times_B Y')}$$

in $[\Gamma_B(X'\times_B Y'), \Gamma_B(X'\times_B Y')]_B^B$ by Theorem 2.4.

Theorem 2.5. *Let Γ be a co-looplike space in* $\mathbf{Top}_B^B$. *The following diagram is homotopy commutative in* $\mathbf{Top}_B^B$ *for any maps $\alpha: X'\to X$ and $\beta: Y'\to Y$ in* $\mathbf{Top}_B^B$.

$$\begin{array}{ccc} \Gamma_B(X'\wedge_B Y') & \xrightarrow{v'} & \Gamma_B(X'\times_B Y') \\ {\scriptstyle \Gamma_B(\alpha\wedge_B\beta)}\downarrow & & \downarrow{\scriptstyle \Gamma_B(\alpha\times_B\beta)} \\ \Gamma_B(X\wedge_B Y) & \xrightarrow{v} & \Gamma_B(X\times_B Y) \end{array}$$

Proof. We have a monomorphism

$$(\Gamma_B q')^*: [\Gamma_B(X'\wedge_B Y'), \Gamma_B(X\times_B Y)]_B^B \longrightarrow [\Gamma_B(X'\times_B Y'), \Gamma_B(X\times_B Y)]_B^B$$

by Lemma 2.3. So we show that

$$(\Gamma_B q')^*\{\Gamma_B(\alpha\times_B\beta)\circ v'\}=(\Gamma_B q')^*\{v\circ\Gamma_B(\alpha\wedge_B\beta)\}.$$

Now, we have

$$\begin{aligned}
&(\Gamma_B q')^*\{\Gamma_B(\alpha\times_B\beta)\circ v'\}\dot{+}_B\{\Gamma_B(\alpha\times_B\beta)\circ\Gamma_B j'\circ\rho'\}\\
&=\{\Gamma_B(\alpha\times_B\beta)\circ v'\circ\Gamma_B q'\}\dot{+}_B\{\Gamma_B(\alpha\times_B\beta)\circ\Gamma_B j'\circ\rho'\}\\
&=\Gamma_B(\alpha\times_B\beta)\circ\{(v'\circ\Gamma_B q')\dot{+}_B(\Gamma_B j'\circ\rho')\}\\
&=\Gamma_B(\alpha\times_B\beta)\circ 1_{\Gamma_B(X'\times_B Y')} \text{ by 2.4}\\
&=1_{\Gamma_B(X\times_B Y)}\circ\Gamma_B(\alpha\times_B\beta)\\
&=\{(v\circ\Gamma_B q)\dot{+}_B(\Gamma_B j\circ\rho)\}\circ\Gamma_B(\alpha\times_B\beta) \text{ by 2.4}\\
&=\{v\circ\Gamma_B q\circ\Gamma_B(\alpha\times_B\beta)\}\dot{+}_B\{\Gamma_B j\circ\rho\circ\Gamma_B(\alpha\times_B\beta)\}\\
&=\{v\circ\Gamma_B(\alpha\wedge_B\beta)\circ\Gamma_B q'\}\dot{+}_B\{\Gamma_B j\circ\Gamma_B(\alpha\vee_B\beta)\circ\rho'\} \text{ by 2.2}\\
&=(\Gamma_B q')^*\{v\circ\Gamma_B(\alpha\wedge_B\beta)\}\dot{+}_B\{\Gamma_B(\alpha\times_B\beta)\circ\Gamma_B j'\circ\rho'\}.
\end{aligned}$$

Since the above equality holds in an algebraic loop, we have

$$(\Gamma_B q')^*\{\Gamma_B(\alpha\times_B\beta)\circ v'\}=(\Gamma_B q')^*\{v\circ\Gamma_B(\alpha\wedge_B\beta)\}$$

in $[\Gamma_B(X'\times_B Y'), \Gamma_B(X\times_B Y)]_B^B$. *q.e.d.*

Consider the following diagram

$$\Gamma_B(X\vee_B Y)\underset{\rho}{\overset{\Gamma_B j}{\rightleftarrows}}\Gamma_B(X\times_B Y)\underset{v}{\overset{\Gamma_B q}{\rightleftarrows}}\Gamma_B(X\wedge_B Y).$$

We have already shown that $\rho\circ\Gamma_B j\simeq_B 1_{\Gamma_B(X\vee_B Y)}$ (Proposition 2.1) and

$(v\circ\Gamma_B q)\dot{+}_B(\Gamma_B j\circ\rho)\simeq_B 1_{\Gamma_B(X\times_B Y)}$ (Theorem 2.4).

Proposition 2.6. *Let Γ be a co-looplike space in* $\mathbf{Top}_B^B$. *Then the following relations hold.*

(i) $\Gamma_B q\circ v\simeq_B 1_{\Gamma_B(X\wedge_B Y)}$.

(ii) $\rho\circ v\simeq_B *_B$.

Proof. (i) By Theorem 2.4, we have

$$\Gamma_B q\circ\{(v\circ\Gamma_B q)\dot{+}_B(\Gamma_B j\circ\rho)\}\simeq_B\Gamma_B q\circ 1_{\Gamma_B(X\times_B Y)}.$$

It follows then that $(\Gamma_B q\circ v\circ\Gamma_B q)\dot{+}_B(\Gamma_B q\circ\Gamma_B j\circ\rho)\simeq_B\Gamma_B q$ and hence

$$(\Gamma_B q\circ v)\circ\Gamma_B q\simeq_B(\Gamma_B q\circ v\circ\Gamma_B q)\dot{+}_B *_B\simeq_B\Gamma_B q=1_{\Gamma_B(X\wedge_B Y)}\circ\Gamma_B q.$$

Since $(\Gamma_B q)^*$: $[\Gamma_B(X\wedge_B Y),\ \Gamma_B(X\wedge_B Y)]_B^B\to[\Gamma_B(X\times_B Y),\ \Gamma_B(X\wedge_B Y)]_B^B$ is a monomorphism by Lemma 2.3, we have $\Gamma_B q\circ v\simeq_B 1_{\Gamma_B(X\wedge_B Y)}$.

(ii) By Proposition 2.1 and Theorem 2.4, we see

$$\begin{aligned}(\rho\circ v\circ\Gamma_B q)\dot{+}_B\rho&\simeq_B(\rho\circ v\circ\Gamma_B q)\dot{+}_B(\rho\circ\Gamma_B j\circ\rho)\\&=\rho\circ\{(v\circ\Gamma_B q)\dot{+}_B(\Gamma_B j\circ\rho)\}\simeq_B\rho\circ 1_{\Gamma_B(X\times_B Y)}=\rho.\end{aligned}$$

It follows that $(\rho\circ v)\circ\Gamma_B q\simeq_B *_B\circ\Gamma_B q$ and hence $\rho\circ v\simeq_B *_B$. *q.e.d.*

Remark. The existence of ρ (Proposition 2.1) and v (Theorem 2.4) corresponds to the following Γ_B-decomposition (cf. [7]):

$$\Gamma_B(X\times_B Y)\simeq_B\Gamma_B(X\wedge_B Y)\vee_B\Gamma_B(X\vee_B Y).$$

Definition 2.7. Let Γ be a co-looplike space in $\mathbf{Top}_B^B$. Let μ: $X\times_B Y\to Z$ be a pairing in $\mathbf{Top}_B^B$. We define the *Γ_B-Hopf construction*

$$J_{\Gamma,B}(\mu)\in[\Gamma_B(X\wedge_B Y),\ \Gamma_B Z]_B^B$$

by $J_{\Gamma,B}(\mu)=(\Gamma_B\mu)\circ v$ for the element $v\in[\Gamma_B(X\wedge_B Y),\ \Gamma_B(X\times_B Y)]_B^B$ obtained in Theorem 2.4.

Remark. If $B=\{*\}$ and $\Gamma=S^1$, then the Γ_B-Hopf construction coincides with the ordinary Hopf construction (cf. §4 of Chapter XI of Whitehead [8]).

Proposition 2.8. *Let μ: $X\times_B Y\to Z$ be a pairing in* $\mathbf{Top}_B^B$. *Then the following formulas hold.*

(i) $J_{\Gamma,B}(\zeta\circ\mu)=\Gamma_B\zeta\circ J_{\Gamma,B}(\mu)$ *for any map* ζ: $Z\to Z'$ *in* $\mathbf{Top}_B^B$.

(ii) $J_{\Gamma,B}\{\mu\circ(\alpha\times_B\beta)\}=\Gamma_B\mu\circ J_{\Gamma,B}(\alpha\times_B\beta)=J_{\Gamma,B}(\mu)\circ\Gamma_B(\alpha\wedge_B\beta)$ *for any maps* α: $X'\to X$ *and* β: $Y'\to Y$ *in* $\mathbf{Top}_B^B$.

Proof. (i) $J_{\Gamma,B}(\zeta\circ\mu)=\Gamma_B(\zeta\circ\mu)\circ v=\Gamma_B\zeta\circ\Gamma_B\mu\circ v=(\Gamma_B\zeta)\circ J_{\Gamma,B}(\mu)$.

(ii) The first equality is a result of (i). We prove the second one.

$$\begin{aligned} J_{\Gamma,B}\{\mu\circ(\alpha\times_B\beta)\} &= \Gamma_B\{\mu\circ(\alpha\times_B\beta)\}\circ v' \\ &= \Gamma_B\mu\circ\Gamma_B(\alpha\times_B\beta)\circ v' \\ &= \Gamma_B\mu\circ v\circ\Gamma_B(\alpha\wedge_B\beta) \text{ by 2.5} \\ &= J_{\Gamma,B}(\mu)\circ\Gamma_B(\alpha\wedge_B\beta). \end{aligned}$$ *q. e. d.*

§ 3. Fibrewise Γ-Suspension Formula

Now we generalize the suspension formula of Hoo [3] to the case of Γ_B-*suspension space* $\Gamma_B X$ for any space X in $\mathbf{Top}_B^B$. Consider a diagram

$$\Gamma_B(A\vee_B A) \underset{\rho}{\overset{\Gamma_B j}{\rightleftarrows}} \Gamma_B(A\times_B A) \underset{v}{\overset{\Gamma_B q}{\rightleftarrows}} \Gamma_B(A\wedge_B A).$$

In this situation we prove the following theorem.

Theorem 3.1. *Let Γ be a co-looplike space in $\mathbf{Top}_B^B$. Let $\mu: X\times_B Y\to Z$ be a pairing in $\mathbf{Top}_B^B$ with axes $f: X\to Z$ and $g: Y\to Z$. Then the following relation holds in $[\Gamma_B A, \Gamma_B Z]_B^B$ for any maps $\alpha: A\to X$ and $\beta: A\to Y$ in $\mathbf{Top}_B^B$:*

$$\Gamma_B(\alpha\dotplus_B\beta)=J_{\Gamma,B}(\mu)\circ\Gamma_B(\alpha\wedge_B\beta)\circ\Gamma_B(q\circ\Delta_{A,B})\dotplus_B\{\Gamma_B(f\circ\alpha)\dotplus_B\Gamma_B(g\circ\beta)\}.$$

$$(\textit{Remark.}\quad J_{\Gamma,B}(\mu)\circ\Gamma_B(\alpha\wedge_B\beta)\circ\Gamma_B(q\circ\Delta_{A,B})=J_{\Gamma,B}\{\mu\circ(\alpha\times_B\beta)\}\circ\Gamma_B(q\circ\Delta_{A,B})$$

$$=J_{\Gamma,B}(\mu)\circ\Gamma_B\{(\alpha\wedge_B\beta)\circ q\circ\Delta_{A,B}\}=J_{\Gamma,B}(\mu)\circ\Gamma_B\{q\circ(\alpha\times_B\beta)\circ\Delta_{A,B}\}.)$$

Proof. Since $1_{\Gamma_B(A\times_B A)}=(v\circ\Gamma_B q)\dotplus_B(\Gamma_B j\circ\rho)$ by Theorem 2.4, the map $\mu\circ(\alpha\times_B\beta): A\times_B A\to X\times_B Y\to Z$ induces

$$\begin{aligned} \Gamma_B\{\mu\circ(\alpha\times_B\beta)\} &= \Gamma_B\{\mu\circ(\alpha\times_B\beta)\}\circ\{(v\circ\Gamma_B q)\dotplus_B(\Gamma_B j\circ\rho)\} \\ &= \Gamma_B\{\mu\circ(\alpha\times_B\beta)\}\circ v\circ\Gamma_B q\dotplus_B\Gamma_B\{\mu\circ(\alpha\times_B\beta)\}\circ\Gamma_B j\circ\rho \\ &= J_{\Gamma,B}\{\mu\circ(\alpha\times_B\beta)\}\circ\Gamma_B q\dotplus_B\Gamma_B\{\mu\circ(\alpha\times_B\beta)\circ j\}\circ\rho \end{aligned}$$

by the definition of $J_{\Gamma,B}$. Now, the last term is:

$$\begin{aligned} &\Gamma_B\{\mu\circ(\alpha\times_B\beta)\circ j\}\circ\rho \\ &= \Gamma_B\{\mu\circ(\alpha\times_B\beta)\circ j\}\circ\{\Gamma_B(i_1\circ p_1)\dotplus_B\Gamma_B(i_2\circ p_2)\} \\ &= \Gamma_B\{\mu\circ(\alpha\times_B\beta)\circ j\circ i_1\circ p_1\}\dotplus_B\Gamma_B\{\mu\circ(\alpha\times_B\beta)\circ j\circ i_2\circ p_2\} \\ &= \Gamma_B\{\mu\circ(\alpha\times_B *_B)\circ\Delta_{A,B}\circ p_1\}\dotplus_B\Gamma_B\{\mu\circ(*_B\times_B\beta)\circ\Delta_{A,B}\circ p_2\} \\ &= \Gamma_B(f_*(\alpha)\circ p_1)\dotplus_B\Gamma_B(g_*(\beta)\circ p_2), \end{aligned}$$

since $\alpha \dotplus_B *_B = f_*(\alpha)$ and $*_B \dotplus_B \beta = g_*(\beta)$ in $[A, Z]^B_B$. We compose $\Gamma_B \Delta_{A,B}$ from the right and obtain

$$\begin{aligned}\Gamma_B(\alpha \dotplus_B \beta) &= \Gamma_B\{\mu \circ (\alpha \times_B \beta) \circ \Delta_{A,B}\} \\ &= J_{\Gamma,B}\{\mu \circ (\alpha \times_B \beta)\} \circ \Gamma_B(q \circ \Delta_{A,B}) \dotplus_B \{\Gamma_B(f_*(\alpha) \circ p_1 \circ \Delta_{A,B}) \dotplus_B \Gamma_B(g_*(\beta) \circ p_2 \circ \Delta_{A,B})\} \\ &= J_{\Gamma,B}\{\mu \circ (\alpha \times_B \beta)\} \circ \Gamma_B(q \circ \Delta_{A,B}) \dotplus_B \{\Gamma_B(f_*(\alpha)) \dotplus_B \Gamma_B(g_*(\beta))\} \\ &= J_{\Gamma,B}(\mu) \circ \Gamma_B(\alpha \wedge_B \beta) \circ \Gamma_B(q \circ \Delta_{A,B}) \dotplus_B \{\Gamma_B(f \circ \alpha) \dotplus_B \Gamma_B(g \circ \beta)\}.\end{aligned}$$

q.e.d.

Corollary 3.2. *Assume the conditions of Theorem* 3.1. *If* $\Gamma_B(q \circ \Delta_{A,B}) \simeq_B *_B$ *or* $\Gamma_B(\alpha \wedge_B \beta) \simeq_B *_B$ *or* $J_{\Gamma,B}(\mu) \simeq_B *_B$, *then the* Γ_B*-suspension map* $\Gamma_B : [A, Z]^B_B \to [\Gamma_B A, \Gamma_B Z]^B_B$ *satisfies*

$$\Gamma_B(\alpha \dotplus_B \beta) = \Gamma_B(f \circ \alpha) \dotplus_B \Gamma_B(g \circ \beta).$$

Proof. By the formula of Theorem 3.1, we have the result. *q.e.d.*

A space A is called a *co-grouplike space in* $\mathbf{Top}^B_B$ or a *fibrewise co-grouplike space over* B if it is a homotopy associative co-Hopf space in $\mathbf{Top}^B_B$ with a fibrewise pointed homotopy inverse $\nu : A \to A$, namely,

$$1_A \dotplus_B \nu \simeq_B *_B \simeq_B \nu \dotplus_B 1_A.$$

If Γ is a co-grouplike space in $\mathbf{Top}^B_B$, then so is $\Gamma_B X$ for any space X in $\mathbf{Top}^B_B$, and hence $[\Gamma_B X, Y]^B_B$ is a group for any spaces X and Y in $\mathbf{Top}^B_B$.

Theorem 3.3. *Let* Γ *be a co-grouplike space in* $\mathbf{Top}^B_B$. *Let* $\mu : X \times_B Y \to Z$ *be a pairing in* $\mathbf{Top}^B_B$ *with axes* $f : X \to Z$, $g : Y \to Z$, *and* (Z', μ') *a Hopf space in* $\mathbf{Top}^B_B$. *If the pairing induced by* μ *is denoted by* $\dotplus_B$ *and the one induced by* μ' *denoted by* $\dotplus'_B$, *then for any maps* $\zeta : Z \to Z'$, $\alpha : A \to X$ *and* $\beta : A \to Y$ *in* $\mathbf{Top}^B_B$, *the following formula holds in* $[\Gamma_B A, \Gamma_B(Z')]^B_B$:

$$\begin{aligned}&\Gamma_B\{\zeta \circ (\alpha \dotplus_B \beta)\} \\ &\quad = [\Gamma_B \zeta \circ J_{\Gamma,B}(\mu) \dot{-}_B J_{\Gamma,B}(\mu') \circ \Gamma_B\{(\zeta \circ f) \wedge_B (\zeta \circ g)\}] \circ \Gamma_B(\alpha \wedge_B \beta) \circ \Gamma_B(q \circ \Delta_{A,B}) \\ &\qquad \dotplus_B \Gamma_B\{(\zeta \circ f \circ \alpha) \dotplus'_B (\zeta \circ g \circ \beta)\}.\end{aligned}$$

Proof. We see by Theorem 3.1 that

$$\Gamma_B(\alpha \dotplus_B \beta) = J_{\Gamma,B}(\mu) \circ \Gamma_B(\alpha \wedge_B \beta) \circ \Gamma_B(q \circ \Delta_{A,B}) \dotplus_B \Gamma_B(f \circ \alpha) \dotplus_B \Gamma_B(g \circ \beta).$$

It follows that

$$\begin{aligned}\Gamma_B\{\zeta \circ (\alpha \dotplus_B \beta)\} &= \Gamma_B \zeta \circ \Gamma_B(\alpha \dotplus_B \beta) \\ &= \Gamma_B \zeta \circ J_{\Gamma,B}(\mu) \circ \Gamma_B(\alpha \wedge_B \beta) \circ \Gamma_B(q \circ \Delta_{A,B}) \\ &\quad \dotplus_B \Gamma_B(\zeta \circ f \circ \alpha) \dotplus_B \Gamma_B(\zeta \circ g \circ \beta).\end{aligned}$$

On the other hand, by Theorem 3.1, we see

$$\Gamma_B\{(\zeta\circ f\circ\alpha)\dotplus'_B(\zeta\circ g\circ\beta)\}$$
$$=J_{\Gamma,B}(\mu')\circ\Gamma_B\{(\zeta\circ f\circ\alpha)\wedge_B(\zeta\circ g\circ\beta)\}\circ\Gamma_B(q\circ\Delta_{A,B})\dotplus_B\Gamma_B(\zeta\circ f\circ\alpha)\dotplus_B\Gamma_B(\zeta\circ g\circ\beta).$$

It follows from the two equations above that

$$\Gamma_B\{\zeta\circ(\alpha\dotplus_B\beta)\}$$
$$=\Gamma_B\zeta\circ J_{\Gamma,B}(\mu)\circ\Gamma_B(\alpha\wedge_B\beta)\circ\Gamma_B(q\circ\Delta_{A,B})$$
$$\dot{-}_B J_{\Gamma,B}(\mu')\circ\Gamma_B\{(\zeta\circ f\circ\alpha)\wedge_B(\zeta\circ g\circ\beta)\}\circ\Gamma_B(q\circ\Delta_{A,B})\dotplus_B\Gamma_B\{(\zeta\circ f\circ\alpha)\dotplus'_B(\zeta\circ g\circ\beta)\}$$
$$=[\Gamma_B\zeta\circ J_{\Gamma,B}(\mu)\dot{-}_B J_{\Gamma,B}(\mu')\circ\Gamma_B\{(\zeta\circ f)\wedge_B(\zeta\circ g)\}]\circ\Gamma_B(\alpha\wedge_B\beta)\circ\Gamma_B(q\circ\Delta_{A,B})$$
$$\dotplus_B\Gamma_B\{(\zeta\circ f\circ\alpha)\dotplus'_B(\zeta\circ g\circ\beta)\}.$$

q. e. d.

Corollary 3.4. *Assume the conditions of Theorem* 3.3. *Then we have*

$$\Gamma_B\{\zeta\circ(\alpha\dotplus_B\beta)\}=\Gamma_B\{(\zeta\circ f\circ\alpha)\dotplus'_B(\zeta\circ g\circ\beta)\}$$

if one of the following conditions is satisfied;

(i) $\Gamma_B(q\circ\Delta_{A,B})\simeq_B *_B$,

(ii) $\Gamma_B(\alpha\wedge_B\beta)\simeq_B *_B$,

(iii) $\Gamma_B\zeta\circ J_{\Gamma,B}(\mu)\simeq_B J_{\Gamma,B}(\mu')\circ\Gamma_B\{(\zeta\circ f)\wedge_B(\zeta\circ g)\}$.

Proof. By the formula of Theorem 3.3, we have the result.

§4. Fibrewise Γ^*-Hopf Construction

We now study the dual constructions. Consider a fibrewise pointed fibration sequence

$$\cdots\longrightarrow\Omega_B(X\times_B Y)\xrightarrow{\partial}X\flat_B Y\xrightarrow{i}X\vee_B Y\xrightarrow{j}X\times_B Y,$$

where $X\flat_B Y$ is the fibrewise homotopy fibre of the inclusion map $j: X\vee_B Y\to X\times_B Y$ (cf. Crabb and James [1]). Then we have a fibrewise pointed fibration sequence

$$\cdots\longrightarrow\Omega_B\Gamma^*_B(X\times_B Y)\xrightarrow{\Gamma^*_B\partial}\Gamma^*_B(X\flat_B Y)\xrightarrow{\Gamma^*_B i}\Gamma^*_B(X\vee_B Y)\xrightarrow{\Gamma^*_B j}\Gamma^*_B(X\times_B Y)$$

(cf. (23.2) of [5]). Since $\Gamma^*_B(X\vee_B Y)$ is a Hopf space in $\mathbf{Top}^B_B$, we define a fibrewise pointed map $\sigma=\sigma(\Gamma, X, Y): \Gamma^*_B(X\times_B Y)\to\Gamma^*_B(X\vee_B Y)$ by

$$\sigma=\tilde{\gamma}\circ\{\Gamma^*_B(i_1\circ p_1)\times_B\Gamma^*_B(i_2\circ p_2)\}\circ\Delta_{C,B}=\Gamma^*_B(i_1\circ p_1)\dotplus_B\Gamma^*_B(i_2\circ p_2)$$

where $p_1: X\times_B Y\to X$, $i_1: X\to X\vee_B Y$, $p_2: X\times_B Y\to Y$, $i_2: Y\to X\vee_B Y$ are the fibrewise projections and the inclusions and $C=\Gamma^*_B(X\times_B Y)$. The map $\tilde{\gamma}$ is the fibrewise Hopf structure of $\Gamma^*_B(X\vee_B Y)$ induced by the fibrewise co-Hopf structure

$\gamma: \Gamma \to \Gamma \vee_B \Gamma$ of Γ.

Proposition 4.1. *Let Γ be a co-Hopf space in* $\mathbf{Top}_B^B$. *Let $j: X \vee_B Y \to X \times_B Y$ be the inclusion map. Then the map $\sigma: \Gamma_B^*(X \times_B Y) \to \Gamma_B^*(X \vee_B Y)$ defined above satisfies a relation*

$$(\Gamma_B^* j) \circ \sigma \simeq_B 1_{\Gamma_B^*(X \times_B Y)}.$$

Proof. We remark that $\Gamma_B^*(X \times_B Y) \cong_B \Gamma_B^* X \times_B \Gamma_B^* Y$ (cf. (9.9) of [5]). Then we have

$$\begin{aligned}(\Gamma_B^* p_1) \circ (\Gamma_B^* j) \circ \sigma &= \Gamma_B^* p_1 \circ \Gamma_B^* j \circ \{\Gamma_B^*(i_1 \circ p_1) \dotplus_B \Gamma_B^*(i_2 \circ p_2)\} \\ &= \Gamma_B^*(p_1 \circ j \circ i_1 \circ p_1) \dotplus_B \Gamma_B^*(p_1 \circ j \circ i_2 \circ p_2) \\ &= \Gamma_B^* p_1 \dotplus_B *_B \simeq_B \Gamma_B^* p_1 = (\Gamma_B^* p_1) \circ 1_{\Gamma_B^*(X \times_B Y)}.\end{aligned}$$

Similarly, we have $(\Gamma_B^* p_2) \circ (\Gamma_B^* j) \circ \sigma \simeq_B (\Gamma_B^* p_2) \circ 1_{\Gamma_B^*(X \times_B Y)}$. It follows that $(\Gamma_B^* j) \circ \sigma \simeq_B 1_{\Gamma_B^*(X \times_B Y)}$. *q.e.d.*

Now we prove naturality of fibrewise pointed map σ. Consider another fibrewise pointed fibration sequence:

$$\cdots \longrightarrow \Omega_B \Gamma_B^*(X' \times_B Y') \xrightarrow{\Gamma_B^* \partial'} \Gamma_B^*(X' \flat_B Y') \xrightarrow{\Gamma_B^* i'} \Gamma_B^*(X' \vee_B Y') \xrightarrow{\Gamma_B^* j'} \Gamma_B^*(X' \times_B Y').$$

Then we have another fibrewise pointed map

$$\sigma' = \Gamma_B^*(i_1' \circ p_1') \dotplus_B \Gamma_B^*(i_2' \circ p_2'): \Gamma_B^*(X' \times_B Y') \longrightarrow \Gamma_B^*(X' \vee_B Y').$$

Proposition 4.2. *Let Γ be a co-Hopf space in* $\mathbf{Top}_B^B$. *The following diagram is strictly commutative in* $\mathbf{Top}_B^B$ *for any maps $\alpha: X \to X'$ and $\beta: Y \to Y'$ in* $\mathbf{Top}_B^B$.

$$\begin{array}{ccc} \Gamma_B^*(X \times_B Y) & \xrightarrow{\sigma} & \Gamma_B^*(X \vee_B Y) \\ {\scriptstyle \Gamma_B^*(\alpha \times_B \beta)} \downarrow & & \downarrow {\scriptstyle \Gamma_B^*(\alpha \vee_B \beta)} \\ \Gamma_B^*(X' \times_B Y') & \xrightarrow{\sigma'} & \Gamma_B^*(X' \vee_B Y') \end{array}$$

Proof. We use the symbols i_1', p_1', σ' etc. for those maps corresponding to i_1, p_1, σ etc. respectively.

$$\begin{aligned}\sigma' \circ \Gamma_B^*(\alpha \times_B \beta) &= \{\Gamma_B^*(i_1' \circ p_1') \dotplus_B \Gamma_B^*(i_2' \circ p_2')\} \circ \Gamma_B^*(\alpha \times_B \beta) \\ &= \Gamma_B^*\{i_1' \circ p_1' \circ (\alpha \times_B \beta)\} \dotplus_B \Gamma_B^*\{i_2' \circ p_2' \circ (\alpha \times_B \beta)\} \\ &= \Gamma_B^*\{(\alpha \vee_B \beta) \circ i_1 \circ p_1\} \dotplus_B \Gamma_B^*\{(\alpha \vee_B \beta) \circ i_2 \circ p_2\} \\ &= \Gamma_B^*(\alpha \vee_B \beta) \circ \Gamma_B^*(i_1 \circ p_1) \dotplus_B \Gamma_B^*(\alpha \vee_B \beta) \circ \Gamma_B^*(i_2 \circ p_2) \\ &= \Gamma_B^*(\alpha \vee_B \beta) \circ \{\Gamma_B^*(i_1 \circ p_1) \dotplus_B \Gamma_B^*(i_2 \circ p_2)\} \\ &= \Gamma_B^*(\alpha \vee_B \beta) \circ \sigma.\end{aligned}$$

q.e.d.

Lemma 4.3. *Let Γ be a co-looplike space in* $\mathbf{Top}_B^B$. *Then*

$$(\Gamma_B^* i)_*: [A, \Gamma_B^*(X \flat_B Y)]_B^B \longrightarrow [A, \Gamma_B^*(X \vee_B Y)]_B^B$$

is a monomorphism for any space A in $\mathbf{Top}_B^B$.

Proof. By Proposition 4.1 and the long fibrewise pointed homotopy exact sequence, we have the result. *q.e.d.*

Theorem 4.4. *Let Γ be a co-looplike space in* $\mathbf{Top}_B^B$. *There exists a unique element $\omega = \omega(\Gamma, X, Y)$ of $[\Gamma_B^*(X \vee_B Y), \Gamma_B^*(X \flat_B Y)]_B^B$ such that*

$$1_{\Gamma_B^*(X \vee_B Y)} = (\Gamma_B^* i \circ \omega) \dotplus_B (\sigma \circ \Gamma_B^* j)$$

in $[\Gamma_B^*(X \vee_B Y), \Gamma_B^*(X \vee_B Y)]_B^B$.

Proof. Consider an exact sequence of algebraic loops and homomorphisms:

$$[\Gamma_B^*(X \vee_B Y), \Gamma_B^*(X \flat_B Y)]_B^B \xrightarrow{(\Gamma_B^* i)_*} [\Gamma_B^*(X \vee_B Y), \Gamma_B^*(X \vee_B Y)]_B^B$$
$$\xrightarrow{(\Gamma_B^* j)_*} [\Gamma_B^*(X \vee_B Y), \Gamma_B^*(X \times_B Y)]_B^B.$$

For elements $1_{\Gamma_B^*(X \vee_B Y)}$ and $\sigma \circ \Gamma_B^* j$ of $[\Gamma_B^*(X \vee_B Y), \Gamma_B^*(X \vee_B Y)]_B^B$ which is an algebraic loop, there exists a unique element $t \in [\Gamma_B^*(X \vee_B Y), \Gamma_B^*(X \vee_B Y)]_B^B$ such that

$$t \dotplus_B (\sigma \circ \Gamma_B^* j) = 1_{\Gamma_B^*(X \vee_B Y)}.$$

Then we see $(\Gamma_B^* j)_* \{t \dotplus_B (\sigma \circ \Gamma_B^* j)\} = (\Gamma_B^* j)_* \{1_{\Gamma_B^*(X \vee_B Y)}\} = \Gamma_B^* j$. We note that

$$(\Gamma_B^* j)_* \{t \dotplus_B (\sigma \circ \Gamma_B^* j)\} = (\Gamma_B^* j \circ t) \dotplus_B (\Gamma_B^* j \circ \sigma \circ \Gamma_B^* j)$$
$$\simeq_B (\Gamma_B^* j \circ t) \dotplus_B (1_{\Gamma_B^*(X \times_B Y)} \circ \Gamma_B^* j) \simeq_B (\Gamma_B^* j \circ t) \dotplus_B \Gamma_B^* j$$

by Proposition 4.1. It follows then that

$$(\Gamma_B^* j \circ t) \dotplus_B \Gamma_B^* j = \Gamma_B^* j$$

in $[\Gamma_B^*(X \vee_B Y), \Gamma_B^*(X \times_B Y)]_B^B$. Thus we have $\Gamma_B^* j \circ t = (\Gamma_B^* j)_*(t) = 0$. Hence there exists an element $\omega \in [\Gamma_B^*(X \vee_B Y), \Gamma_B^*(X \flat_B Y)]_B^B$ such that $(\Gamma_B^* i)_*(\omega) = t$. It follows that

$$\{(\Gamma_B^* i) \circ \omega\} \dotplus_B \{\sigma \circ (\Gamma_B^* j)\} = 1_{\Gamma_B^*(X \vee_B Y)}.$$

The uniqueness of ω is obtained by Lemma 4.3. *q.e.d.*

The element ω defined in Theorem 4.4 is natural. Consider another fibrewise pointed fibration sequence

$$\cdots \longrightarrow \Omega_B \Gamma_B^*(X' \times_B Y') \xrightarrow{\Gamma_B^* \partial'} \Gamma_B^*(X' \flat_B Y') \xrightarrow{\Gamma_B^* i'} \Gamma_B^*(X' \vee_B Y') \xrightarrow{\Gamma_B^* j'} \Gamma_B^*(X' \times_B Y').$$

Then we have another element ω' of $[\Gamma_B^*(X'\vee_B Y'),\ \Gamma_B^*(X'\flat_B Y')]_B^B$ such that

$$1_{\Gamma_B^*(X'\vee_B Y')}=(\Gamma_B^* i'\circ\omega')\dot{+}_B(\sigma'\circ\Gamma_B^* j')$$

in $[\Gamma_B^*(X'\vee_B Y'),\ \Gamma_B^*(X'\vee_B Y')]_B^B$.

Theorem 4.5. *Let* Γ *be a a co-looplike space in* $\mathbf{Top}_B^B$. *Let* $\alpha: X\to X'$ *and* $\beta: Y\to Y'$ *be any maps in* $\mathbf{Top}_B^B$. *Then the following diagram is homotopy commutative in* $\mathbf{Top}_B^B$.

$$\begin{array}{ccc} \Gamma_B^*(X\vee_B Y) & \xrightarrow{\ \omega\ } & \Gamma_B^*(X\flat_B Y) \\ {\scriptstyle \Gamma_B^*(\alpha\vee_B\beta)}\downarrow & & \downarrow{\scriptstyle \Gamma_B^*(\alpha\flat_B\beta)} \\ \Gamma_B^*(X'\vee_B Y') & \xrightarrow{\ \omega'\ } & \Gamma_B^*(X'\flat_B Y') \end{array}$$

Proof. Consider the monomorphism

$$(\Gamma_B^* i')_*:\ [\Gamma_B^*(X\vee_B Y),\ \Gamma_B^*(X'\flat_B Y')]_B^B \longrightarrow [\Gamma_B^*(X\vee_B Y),\ \Gamma_B^*(X'\vee_B Y')]_B^B$$

of Lemma 4.3. It is sufficient to show that

$$(\Gamma_B^* i')_*\{\omega'\circ\Gamma_B^*(\alpha\vee_B\beta)\}=(\Gamma_B^* i')_*\{\Gamma_B^*(\alpha\flat_B\beta)\circ\omega\}.$$

Now we see

$$\begin{aligned}
&(\Gamma_B^* i')_*\{\omega'\circ\Gamma_B^*(\alpha\vee_B\beta)\}\dot{+}_B\{\sigma'\circ\Gamma_B^* j'\circ\Gamma_B^*(\alpha\vee_B\beta)\}\\
&=\{\Gamma_B^* i'\circ\omega'\circ\Gamma_B^*(\alpha\vee_B\beta)\}\dot{+}_B\{\sigma'\circ\Gamma_B^* j'\circ\Gamma_B^*(\alpha\vee_B\beta)\}\\
&=\{(\Gamma_B^* i'\circ\omega')\dot{+}_B(\sigma'\circ\Gamma_B^* j')\}\circ\Gamma_B^*(\alpha\vee_B\beta)\\
&=1_{\Gamma_B^*(X'\vee_B Y')}\circ\Gamma_B^*(\alpha\vee_B\beta) \quad \text{by} \quad 4.4\\
&=\Gamma_B^*(\alpha\vee_B\beta)\circ 1_{\Gamma_B^*(X\vee_B Y)}\\
&=\Gamma_B^*(\alpha\vee_B\beta)\circ\{(\Gamma_B^* i\circ\omega)\dot{+}_B(\sigma\circ\Gamma_B^* j)\} \quad \text{by} \quad 4.4\\
&=\{\Gamma_B^*(\alpha\vee_B\beta)\circ\Gamma_B^* i\circ\omega\}\dot{+}_B\{\Gamma_B^*(\alpha\vee_B\beta)\circ\sigma\circ\Gamma_B^* j\}\\
&=\{\Gamma_B^*(\alpha\vee_B\beta)\circ\Gamma_B^* i\circ\omega\}\dot{+}_B\{\sigma'\circ\Gamma_B^*(\alpha\times_B\beta)\circ\Gamma_B^* j\} \quad \text{by} \quad 4.2\\
&=\{\Gamma_B^* i'\circ\Gamma_B^*(\alpha\flat_B\beta)\circ\omega\}\dot{+}_B\{\sigma'\circ\Gamma_B^* j'\circ\Gamma_B^*(\alpha\vee_B\beta)\}\\
&=(\Gamma_B^* i')_*\{\Gamma_B^*(\alpha\flat_B\beta)\circ\omega\}\dot{+}_B\{\sigma'\circ\Gamma_B^* j'\circ\Gamma_B^*(\alpha\vee_B\beta)\}.
\end{aligned}$$

Thus we have $(\Gamma_B^* i')_*\{\omega'\circ\Gamma_B^*(\alpha\vee_B\beta)\}=(\Gamma_B^* i')_*\{\Gamma_B^*(\alpha\flat_B\beta)\circ\omega\}$. *q.e.d.*

Consider the following diagram:

$$\Gamma_B^*(X\flat_B Y)\ \underset{\omega}{\overset{\Gamma_B^* i}{\rightleftarrows}}\ \Gamma_B^*(X\vee_B Y)\ \underset{\sigma}{\overset{\Gamma_B^* j}{\rightleftarrows}}\ \Gamma_B^*(X\times_B Y).$$

We have already shown that $\Gamma_B^* j\circ\sigma\simeq_B 1_{\Gamma_B^*(X\times_B Y)}$ (Proposition 4.1) and $1_{\Gamma_B^*(X\vee_B Y)}\simeq_B(\Gamma_B^* i\circ\omega)\dotplus_B(\sigma\circ\Gamma_B^* j)$ (Theorem 4.4).

Proposition 4.6. *Let Γ be a co-looplike space in $\mathbf{Top}_B^B$. Then the following relations hold.*

(i) $\omega\circ\Gamma_B^* i\simeq_B 1_{\Gamma_B^*(X\flat_B Y)}$

(ii) $\omega\circ\sigma\simeq_B *_B$.

Proof. (i) By Theorem 4.4, we have

$$1_{\Gamma_B^*(X\vee_B Y)}\circ\Gamma_B^* i=\{(\Gamma_B^* i\circ\omega)\dotplus_B(\sigma\circ\Gamma_B^* j)\}\circ\Gamma_B^* i.$$

It follows then that $\Gamma_B^* i=(\Gamma_B^* i\circ\omega\circ\Gamma_B^* i)\dotplus_B(\sigma\circ\Gamma_B^* j\circ\Gamma_B^* i)$ and hence

$$\Gamma_B^* i\circ 1_{\Gamma_B^*(X\flat_B Y)}=\Gamma_B^* i=(\Gamma_B^* i\circ\omega\circ\Gamma_B^* i)\dotplus_B *_B\simeq_B\Gamma_B^* i\circ\omega\circ\Gamma_B^* i.$$

Since $(\Gamma_B^* i)_*\colon[\Gamma_B^*(X\flat_B Y),\ \Gamma_B(X\flat_B Y)]_B^B\to[\Gamma_B(X\flat_B Y),\ \Gamma_B(X\vee_B Y)]_B^B$ is a monomorphism by Lemma 4.3, we have $1_{\Gamma_B^*(X\flat_B Y)}\simeq_B\omega\circ\Gamma_B^* i$.

(ii) By Proposition 4.1 and Theorem 4.4, we see

$$\begin{aligned}(\Gamma_B^* i\circ\omega\circ\sigma)\dotplus_B\sigma&\simeq_B(\Gamma_B^* i\circ\omega\circ\sigma)\dotplus_B(\sigma\circ\Gamma_B^* j\circ\sigma)\\&=\{(\Gamma_B^* i\circ\omega)\dotplus_B(\sigma\circ\Gamma_B^* j)\}\circ\sigma\simeq_B 1_{\Gamma_B^*(X\vee_B Y)}\circ\sigma=\sigma.\end{aligned}$$

It follows that $\Gamma_B^* i\circ\omega\circ\sigma\simeq_B *_B=\Gamma_B^* i\circ *_B$ and hence $\omega\circ\sigma\simeq_B *_B$. *q.e.d.*

Remark. The existence of such ω and σ with the relations mentioned above corresponds to the fibrewise pointed homotopy decomposition (cf. [7]):

$$\Gamma_B^*(X\vee_B Y)\simeq_B\Gamma_B^*(X\flat_B Y)\times_B\Gamma_B^*(X\times_B Y).$$

Definition 4.7. Let Γ be a co-looplike space in $\mathbf{Top}_B^B$. Let $\theta\colon A\to X\vee_B Y$ be a copairing in $\mathbf{Top}_B^B$. Then we define *Γ_B^*-Hopf construction*

$$J_{\Gamma,B}^*(\theta)\in[\Gamma_B^* A,\ \Gamma_B^*(X\flat_B Y)]_B^B$$

by $J_{\Gamma,B}^*(\theta)=\omega\circ\Gamma_B^*\theta$ for the element $\omega\in[\Gamma_B^*(X\vee_B Y),\ \Gamma_B^*(X\flat_B Y)]_B^B$ obtained in Theorem 4.4.

Proposition 4.8. *Let Γ be a co-looplike space in $\mathbf{Top}_B^B$. Let $\theta\colon A\to X\vee_B Y$ be a copairing in $\mathbf{Top}_B^B$. Then the following formulas hold.*

(i) *$J_{\Gamma,B}^*(\theta\circ\delta)=J_{\Gamma,B}^*(\theta)\circ\Gamma_B^*\delta$ for any map $\delta\colon D\to A$ in $\mathbf{Top}_B^B$.*

(ii) *$J_{\Gamma,B}^*\{(\alpha\vee_B\beta)\circ\theta\}=J_{\Gamma,B}^*(\alpha\vee_B\beta)\circ\Gamma_B^*\theta=\Gamma_B^*(\alpha\flat_B\beta)\circ J_{\Gamma,B}^*(\theta)$ for any maps $\alpha\colon X\to X'$ and $\beta\colon Y\to Y'$ in $\mathbf{Top}_B^B$.*

Proof. (i) $J_{\Gamma,B}^*(\theta\circ\delta)=\omega\circ\Gamma_B^*(\theta\circ\delta)=\omega\circ\Gamma_B^*\theta\circ\Gamma_B^*\delta=J_{\Gamma,B}^*(\theta)\circ\Gamma_B^*\delta$.

(ii) The first equation is the result of (i). We now prove the second one.

$$
\begin{aligned}
J^*_{\Gamma,B}\{(\alpha\vee_B\beta)\circ\theta\} &= \omega'\circ\Gamma^*_B\{(\alpha\vee_B\beta)\circ\theta\}\\
&= \omega'\circ\Gamma^*_B(\alpha\vee_B\beta)\circ\Gamma^*_B(\theta)\\
&= \Gamma^*_B(\alpha\flat_B\beta)\circ\omega\circ\Gamma^*_B(\theta) \quad \text{by} \quad 4.5\\
&= \Gamma^*_B(\alpha\flat_B\beta)\circ J^*_{\Gamma,B}(\theta).
\end{aligned}
$$

q. e. d.

§ 5. Fibrewise Γ^*-Suspension Formula

Consider a diagram

$$
\Gamma^*_B(Z\flat_B Z)\underset{\omega}{\overset{\Gamma^*_B i}{\rightleftarrows}}\Gamma^*_B(Z\vee_B Z)\underset{\sigma}{\overset{\Gamma^*_B j}{\rightleftarrows}}\Gamma^*_B(Z\times_B Z).
$$

In this situation we prove the following theorem.

Theorem 5.1. *Let Γ be a co-looplike space in* $\mathbf{Top}^B_B$. *Let $\theta: A\to X\vee_B Y$ be a copairing in* $\mathbf{Top}^B_B$ *with coaxes $h: A\to X$ and $r: A\to Y$. Let $\alpha: X\to Z$ and $\beta: Y\to Z$ be maps in* $\mathbf{Top}^B_B$. *Then we have the following formula in $[\Gamma^*_B A, \Gamma^*_B Z]^B_B$.*

$$
\Gamma^*_B(\alpha\dot{+}_B\beta)=\Gamma^*_B(\nabla_{Z,B}\circ i)\circ\Gamma^*_B(\alpha\flat_B\beta)\circ J^*_{\Gamma,B}(\theta)\dot{+}_B\{\Gamma^*_B(\alpha\circ h)\dot{+}_B\Gamma^*_B(\beta\circ r)\}.
$$

(*Remark.* $\Gamma^*_B(\nabla_{Z,B}\circ i)\circ\Gamma^*_B(\alpha\flat_B\beta)\circ J^*_{\Gamma,B}(\theta)=\Gamma^*_B(\nabla_{Z,B}\circ i)\circ J^*_{\Gamma,B}\{(\alpha\vee_B\beta)\circ\theta\}$

$=\Gamma^*_B\{\nabla_{Z,B}\circ i\circ(\alpha\flat_B\beta)\}\circ J^*_{\Gamma,B}(\theta)=\Gamma^*_B\{\nabla_{Z,B}\circ(\alpha\vee_B\beta)\circ i\}\circ J^*_{\Gamma,B}(\theta)$.)

Proof. We have $1_{\Gamma^*_B(Z\vee_B Z)}=(\Gamma^*_B i\circ\omega)\dot{+}_B(\sigma\circ\Gamma^*_B j)$ by Theorem 4.4, and hence

$$
\begin{aligned}
\Gamma^*_B\{(\alpha\vee_B\beta)\circ\theta\} &= \{(\Gamma^*_B i\circ\omega)\dot{+}_B(\sigma\circ\Gamma^*_B j)\}\circ\Gamma^*_B\{(\alpha\vee_B\beta)\circ\theta\}\\
&= \Gamma^*_B i\circ\omega\circ\Gamma^*_B\{(\alpha\vee_B\beta)\circ\theta\}\dot{+}_B\sigma\circ\Gamma^*_B j\circ\Gamma^*_B\{(\alpha\vee_B\beta)\circ\theta\}\\
&= \Gamma^*_B i\circ J^*_{\Gamma,B}\{(\alpha\vee_B\beta)\circ\theta\}\dot{+}_B\sigma\circ\Gamma^*_B\{j\circ(\alpha\vee_B\beta)\circ\theta\}.
\end{aligned}
$$

Now we see that the last term is:

$$
\begin{aligned}
\sigma\circ\Gamma^*_B\{j\circ(\alpha\vee_B\beta)\circ\theta\} &= \{\Gamma^*_B(i_1\circ p_1)\dot{+}_B\Gamma^*_B(i_2\circ p_2)\}\circ\Gamma^*_B\{j\circ(\alpha\vee_B\beta)\circ\theta\}\\
&= \Gamma^*_B\{i_1\circ p_1\circ j\circ(\alpha\vee_B\beta)\circ\theta\}\dot{+}_B\Gamma^*_B\{i_2\circ p_2\circ j\circ(\alpha\vee_B\beta)\circ\theta\}\\
&= \Gamma^*_B\{i_1\circ\nabla_{Z,B}\circ(\alpha\vee_B *_B)\circ\theta\}\dot{+}_B\Gamma^*_B\{i_2\circ\nabla_{Z,B}\circ(*_B\vee_B\beta)\circ\theta\}\\
&= \Gamma^*_B\{i_1\circ(\alpha\dot{+}_B *_B)\}\dot{+}_B\Gamma^*_B\{i_2\circ(*_B\dot{+}_B\beta)\}\\
&= \Gamma^*_B(i_1\circ h^*(\alpha))\dot{+}_B\Gamma^*_B(i_2\circ r^*(\beta)).
\end{aligned}
$$

Composing with $\Gamma^*_B\nabla_{Z,B}$ from the left, we have

$$
\Gamma^*_B(\alpha\dot{+}_B\beta)=\Gamma^*_B\{\nabla_{Z,B}\circ(\alpha\vee_B\beta)\circ\theta\}
$$

$$=\Gamma_B^*(\nabla_{Z,B}\circ i)\circ J_{\Gamma,B}^*\{(\alpha\vee_B\beta)\circ\theta\}\dotplus_B\{\Gamma_B^*(\nabla_{Z,B}\circ i_1\circ h^*(\alpha))\dotplus_B\Gamma_B^*(\nabla_{Z,B}\circ i_2\circ r^*(\beta))\}$$

$$=\Gamma_B^*(\nabla_{Z,B}\circ i)\circ\Gamma_B^*(\alpha\flat_B\beta)\circ J_{\Gamma,B}^*(\theta)\dotplus_B\{\Gamma_B^*(\alpha\circ h)\dotplus_B\Gamma_B^*(\beta\circ r)\}.$$

q. e. d.

Corollary 5.2. *Assume the conditions of Theorem* 5.1. *If* $\Gamma_B^*(\nabla_{Z,B}\circ i)\simeq_B *_B$ *or* $\Gamma_B^*(\alpha\flat_B\beta)\simeq_B *_B$ *or* $J_{\Gamma,B}^*(\theta)\simeq_B *_B$, *then the* Γ*-loop map* $\Gamma_B^*: [A, Z]_B^B\to[\Gamma_B^*A, \Gamma_B^*Z]_B^B$ *satisfies*

$$\Gamma_B^*(\alpha\dotplus_B\beta)=\Gamma_B^*(\alpha\circ h)\dotplus_B\Gamma_B^*(\beta\circ r).$$

Proof. By the formula of Theorem 5.1, we have the result. *q. e. d.*

Theorem 5.3. *Let* Γ *be a co-grouplike space in* $\mathbf{Top}_B^B$. *Let* $\theta: A\to X\vee_B Y$ *be a copairing in* $\mathbf{Top}_B^B$ *with coaxes* $h: A\to X$ *and* $r: A\to Y$. *Let* (A', θ') *be a co-Hopf space in* $\mathbf{Top}_B^B$. *If the pairing induced by* θ *is denoted by* $\dotplus_B$ *and the one induced by* θ' *denoted by* $\dotplus'_B$, *then for any maps* $\delta: A'\to A$, $\alpha: X\to Z$ *and* $\beta: Y\to Z$ *in* $\mathbf{Top}_B^B$, *the following formula holds in* $[\Gamma_B^*(A'), \Gamma_B^*Z]_B^B$.

$$\Gamma_B^*\{(\alpha\dotplus_B\beta)\circ\delta\}$$
$$=\Gamma_B^*(\nabla_{Z,B}\circ i)\circ\Gamma_B^*(\alpha\flat_B\beta)\circ[J_{\Gamma,B}^*(\theta)\circ\Gamma_B^*\delta\dot{-}_B\Gamma_B^*\{(h\circ\delta)\flat_B(r\circ\delta)\}\circ J_{\Gamma,B}^*(\theta')]$$
$$\dotplus_B\Gamma_B^*\{(\alpha\circ h\circ\delta)\dotplus'_B(\beta\circ r\circ\delta)\}.$$

Proof. We see

$$\Gamma_B^*(\alpha\dotplus_B\beta)=\Gamma_B^*(\nabla_{Z,B}\circ i)\circ\Gamma_B^*(\alpha\flat_B\beta)\circ J_{\Gamma,B}^*(\theta)\dotplus_B\Gamma_B^*(\alpha\circ h)\dotplus_B\Gamma_B^*(\beta\circ r)$$

by Theorem 5.1. Then composing with $\Gamma_B^*\delta$ from the right, we have

$$\Gamma_B^*\{(\alpha\dotplus_B\beta)\circ\delta\}$$
$$=\Gamma_B^*(\nabla_{Z,B}\circ i)\circ\Gamma_B^*(\alpha\flat_B\beta)\circ J_{\Gamma,B}^*(\theta)\circ\Gamma_B^*\delta\dotplus_B\Gamma_B^*(\alpha\circ h\circ\delta)\dotplus_B\Gamma_B^*(\beta\circ r\circ\delta).$$

On the other hand, by Theorem 5.1, we have

$$\Gamma_B^*\{(\alpha\circ h\circ\delta)\dotplus'_B(\beta\circ r\circ\delta)\}$$
$$=\Gamma_B^*(\nabla_{Z,B}\circ i)\circ\Gamma_B^*\{(\alpha\circ h\circ\delta)\flat_B(\beta\circ r\circ\delta)\}\circ J_{\Gamma,B}^*(\theta')\dotplus_B\Gamma_B^*(\alpha\circ h\circ\delta)\dotplus_B\Gamma_B^*(\beta\circ r\circ\delta).$$

It follows that

$$\Gamma_B^*\{(\alpha\dotplus_B\beta)\circ\delta\}$$
$$=\Gamma_B^*(\nabla_{Z,B}\circ i)\circ\Gamma_B^*(\alpha\flat_B\beta)\circ J_{\Gamma,B}^*(\theta)\circ\Gamma_B^*\delta$$
$$\dot{-}_B\Gamma_B^*(\nabla_{Z,B}\circ i)\circ\Gamma_B^*\{(\alpha\circ h\circ\delta)\flat_B(\beta\circ r\circ\delta)\}\circ J_{\Gamma,B}^*(\theta')\dotplus_B\Gamma_B^*\{(\alpha\circ h\circ\delta)\dotplus'_B(\beta\circ r\circ\delta)\}$$
$$=\Gamma_B^*(\nabla_{Z,B}\circ i)\circ\Gamma_B^*(\alpha\flat_B\beta)\circ[J_{\Gamma,B}^*(\theta)\circ\Gamma_B^*\delta\dot{-}_B\Gamma_B^*\{(h\circ\delta)\flat_B(r\circ\delta)\}\circ J_{\Gamma,B}^*(\theta')]$$
$$\dotplus_B\Gamma_B^*\{(\alpha\circ h\circ\delta)\dotplus'_B(\beta\circ r\circ\delta)\}.$$

q. e. d.

Corollary 5.4. *Assume the conditions of Theorem* 5.3. *Then the formula*

$$\Gamma_B^*\{(\alpha \dotplus_B \beta)\circ\delta\} = \Gamma_B^*\{(\alpha\circ h\circ\delta)\dotplus'_B(\beta\circ r\circ\delta)\}$$

holds if one of the following conditions is satisfied:

(i) $\Gamma_B^*(\nabla_{Z,B}\circ i)\simeq_B *_B$,

(ii) $\Gamma_B^*(\alpha\flat_B\beta)\simeq_B *_B$,

(iii) $J_{\Gamma,B}^*(\theta)\circ\Gamma_B^*\delta\simeq_B\Gamma_B^*\{(h\circ\delta)\flat_B(r\circ\delta)\}\circ J_{\Gamma,B}^*(\theta')$.

Proof. By the formula of Theorem 5.3, we have the result.

References

[1] Crabb, M. and James, I. M., *Fibrewise homotopy theory*, to be published.

[2] Hoo, C. S., A note on a theorem of Ganea, Hilton and Peterson, *Proc. Amer. Math. Soc.*, **19** (1968), 909-911.

[3] ———, On the suspension of an H-space, *Duke Math. J.*, **36** (1969), 315-324.

[4] James, I. M., *General topology and homotopy theory*, Springer-Verlag, New York, 1984.

[5] ———, *Fibrewise topology*, Cambridge University Press, Cambridge, 1989.

[6] Oda, N., Pairings of homotopy sets over and under *B*, *Canad. Math. Bull.*, **36** (1993), 231-240.

[7] ———, Fibrewise decomposition of generalized suspension spaces and loop spaces, *Publ. RIMS, Kyoto Univ.*, **30** (1994), 281-295.

[8] Whitehead, G. W., Elements of homotopy theory, *Graduate Texts in Math.*, **61**, Springer-Verlag, New York Heidelberg Berlin, 1978.