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Indecomposable Restricted Representations of Quantum sl_2

By

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Abstract

We construct and classify all the indecomposable restricted representations of $U_q(sl_2)$ when q is a root of unity.

§ 1. Introduction

Let $U_q(sl_2)$ be the quantum group associated to the complex simple Lie algebra $sl(2, \boldsymbol{C})$. The irreducible representations of $U_q(sl_2)$ are well-understood [9], [10], essentially with a small restriction, there is upto isomorphism, exactly one irreducible representation V_n for each non-negative integer n. If q is not a root of unity then it is known that any finite-dimensional representation of $U_q(sl_2)$ is completely reducible [9], [14] and hence the indecomposable finite-dimensional representations of $U_q(sl_2)$ are just the irreducible ones. If q is a root of unity, the finite-dimensional representations are no longer completely reducible and the study of indecomposable representations becomes an interesting and natural problem [16].

The representations V_n for $0 \leqq n < l$ remain irreducible when regarded as a representation of the first Frobenius kernel of quantum sl_2 which was introduced in [10]. They are called the restricted irreducible representations of quantum sl_2. In this paper we study the restricted indecomposable representations of $U_q(sl_2)$ when $q=\epsilon$ is a primitive l^{th} root of unity. Thus we classify all indecomposable representations of the first Frobenius kernel of quantum sl_2. We show that any indecomposable reducible retsricted module is either projective or isomorphic to a Weyl module or to a dual Weyl module or to a maximal submodule of a Weyl module. The representation theory of quantum groups at roots of unity is closely related to the representation theory of Lie algebras in characteristic p. Our results are analogous to the results for modular Lie

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algebras [2], [12], [15], although some of our techniques are different. The results of [12] used the action of the corresponding algebraic group and the support varieties of restricted modules introduced in [3]. In this paper we give simpler proofs which in fact 'specialize' to the case of modular Lie algebras.

The paper is organized as follows. Section 2 is of a preliminary nature. In Section 3 we give explicit constructions of the indecomposable modules. Finally in Section 4 we prove our classification theorem.

§ 2. Preliminaries

In this section we recall the basic definitions and properties of the restricted finite-dimensional Hopf algebra U_ϵ^{red}.

2.1. Let q be an indeterminate. For $n, r \in \boldsymbol{N}$, let

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q,$$

$$[n; r]_q = \frac{[n]_q!}{[r]_q![n-r]_q!}.$$

It is known that these are all elements of $\boldsymbol{Z}[q, q^{-1}]$ and can be specialized by letting $q=\epsilon$ where ϵ is a primitive l^{th} root of unity, with l odd and greater than 1. We denote the corresponding complex numbers by $[n]$ etc.

2.2.

Definition. $\tilde{U}_\epsilon^{red}(sl_2)$ is the associative algebra over $\boldsymbol{C}$ with generators e, f, k and the following defining relations:

$$kek^{-1} = \epsilon^2 e,$$

$$kfk^{-1} = \epsilon^{-2} f,$$

$$[e, f] = \frac{k - k^{-1}}{\epsilon - \epsilon^{-1}},$$

$$e^l = 0, \qquad f^l = 0, \qquad k^{2l} = 1.$$

Notice that k^l is central in $\tilde{U}_\epsilon^{red}(sl_2)$ and hence acts as ± 1 on any indecomposable $\tilde{U}_\epsilon^{red}(sl_2)$-module. It suffices to study the indecomposable representations on which $k^l=1$ since the other case is obtained by twisting these with the automorphism $e \to -e$, $k \to -k$ and $f \to f$.

Denote by U_ϵ^{red} the quotient of $\tilde{U}_\epsilon^{red}(sl_2)$ by the two-sided ideal generated by $k^l - 1$. Let U_ϵ^+ (resp. U_ϵ^-) be the subalgebra of U_ϵ^{red} generated by e (resp. f) and U_ϵ^0 the (semisimple) subalgebra generated by $k^{\pm 1}$. As vector spaces we have

$$U_\epsilon^{red}=U_\epsilon^- U_\epsilon^0 U_\epsilon^+ ,$$

and hence the elements $f^r k^n e^s$, $0 \leqq r, s, n \leqq l-1$ form a basis of U_ϵ^{red}. The Cartan involution ω of U_ϵ^{red} is defined by extending,

$$\omega(e)=f, \qquad \omega(f)=e, \qquad \omega(k)=k^{-1},$$

to an algebra automorphism.

2.3. It is well-known that U_ϵ^{red} is a Hopf algebra with comultiplication given by,

$$\Delta(e)=e\otimes k+1\otimes e ,$$
$$\Delta(f)=f\otimes 1+k^{-1}\otimes f ,$$
$$\Delta(k)=k\otimes k .$$

The antipode S is the anti-automorphism of U_ϵ^{red} defined by extending,

$$S(k)=k^{-1}, \qquad S(e)=-ek^{-1}, \qquad S(f)=-kf .$$

The counit is the algebra homomorphism that sends k to 1 and e and f to zero.

2.4. The quantum Casimir element of U_ϵ^{red} is defined by,

$$\Omega=fe+\frac{\epsilon k+\epsilon^{-1}k^{-1}-2}{(\epsilon-\epsilon^{-1})^2} .$$

It is easy to check that Ω is in the centre of U_ϵ^{red}. The following Lemma can be proved by a simple induction.

Lemma. *For any $i \geqq 1$, we have,*

$$f^i e^i=\prod_{j=0}^{i-1}\left(\Omega-\frac{\epsilon^{2j+1}k+\epsilon^{-2j-1}k^{-1}-2}{(\epsilon-\epsilon^{-1})^2}\right).$$

2.5. For any non-zero complex 'number μ, let $T_\mu : U_\epsilon^{red} \to U_\epsilon^{red}$ be the automorphism defined by extending,

$$T_\mu(k)=k, \qquad T_\mu(e)=\mu e, \qquad T_\mu(f)=\mu^{-1}f .$$

Clearly, $T_\mu \cdot T_\lambda = T_{\mu\lambda}$. Let T be the group $\{T_\mu : \mu \in \boldsymbol{C}^\times\}$. The action of T on U_ϵ^{red} defines a $\boldsymbol{Z}$-gradation on U_ϵ^{red}. The subalgebras $B^\pm=U_\epsilon^0 U_\epsilon^\pm$ are T-invariant subalgebras of U_ϵ^{red}. Let σ be the anti-graded anti-involution of U_ϵ^{red} induced by,

$$\sigma(e)=f, \qquad \sigma(f)=e, \qquad \sigma(k)=k .$$

If M is a left U_ϵ^{red}-module then σ defines a U_ϵ^{red}-module structure on the dual vector space M^* as follows,

$$(gf)(m)=f(\sigma(g)\cdot m), \quad g\in U_\epsilon^{red}, \quad f\in M^*, \quad m\in M.$$

Any irreducible representation of U_ϵ^0 is one-dimensional and so is determined by a character, $\lambda: U_\epsilon^0\to \boldsymbol{C}$. It is clear from the definition that $\lambda^\sigma\cong\lambda$. Thus U_ϵ^{red} together with the grading induced by T and the anti-graded anti-automorphism σ satisfies the conditions of [5].

2.6. The Hopf algebra structure on U_ϵ^{red} implies that U_ϵ^{red} is a Frobenius algebra [8], i.e. U_ϵ^{red} admits a non-degenerate bilinear form $\langle\,,\rangle$ satisfying,

$$\langle uv, w\rangle=\langle u, vw\rangle, \qquad \text{for all } u, v, w\in U_\epsilon^{red}.$$

As a consequence we have,

Proposition [1, Thm 62.11]. *Every projective module for U_ϵ^{red} is injective.*

2.7. We conclude this section with some results on indecomposable pairs of linear maps $A, B: V\to W$ where V and W are distinct non-zero finite-dimensional vector spaces.

Definition. We say that (A, B) is an indecomposable pair of linear maps if there do not exist subspaces V_1, V_2 of V and subspaces W_1, W_2 of W such that,

(i) $V=V_1\oplus V_2$, $W=W_1\oplus W_2$,
(ii) $A(V_i)\subset W_i$, $B(V_i)\subset W_i$, $i=1, 2$,
(iii) at least one of V_1 or W_1 is non-zero.

Suppose that $\dim(V)=n+1$ and $\dim(W)=n$. Choose a basis $v_0, v_1, \cdots, v_n$ of V and a basis $w_1, w_2, \cdots, w_n$ of W. It is easy to see that the maps ϕ_n, ψ_n: $V\to W$ defined by,

$$\phi_n(v_0)=0\,,$$
$$\phi_n(v_i)=w_i\,, \qquad i\neq 0\,,$$
$$\psi_n(v_i)=w_{i+1}\,, \qquad i\neq n\,,$$
$$\psi_n(v_n)=0\,,$$

are indecomposable.

Another example of an indecomposable pair of maps exists in the case when $\dim(V)=n$, $\dim(W)=n+1$. Choose a basis $v_1, \cdots, v_n$ of V and a basis $w_0, w_1, w_2, \cdots, w_n$ of W. The pair $\iota_n, \eta_n: V\to W$ defined by,

$$\iota_n(v_i)=w_i\,,$$
$$\eta_n(v_i)=w_{i-1}\,,$$

for all $1\leqq i\leqq n$ is indecomposable.

The next result is a direct consequence of the Kronecker-Weierstrass theorem [4, Ch. XII].

Theorem. *Let (A, B) be an indecomposable pair of linear maps from V to W. Assume that the dimension of V is m and that of W is n. Then exactly one of the following statements is true:*

(i) *$m-n=1$ and $A=\phi_n$, $B=\psi_n$,*

(ii) *$m-n=-1$ and $A=\eta_n$, $B=\iota_n$,*

(iii) *$m=n$ and either A and B are bijective or A (resp. B) is bijective and $ker(B)$ (resp. $ker(A)$) is one-dimensional.*

§ 3. Construction of Indecomposable Representations

In this section we give explicit constructions of some indecomposable representations of U_ϵ^{red}.

3.1. For any non-negative integer n and for any $0 \leqq r \leqq l-1$, let $V(n, r)$ denote the Weyl module of dimension $nl+r$. More precisely, if $(n, r) \neq (0, 0)$ and $m=nl+r-1$, then $V(n, r)$ has a basis $v_0, v_1, \cdots, v_m$, on which the action of the generators of U_ϵ^{red} is given by,

$$k \cdot v_i = \epsilon^{m-2i} v_i \,, \tag{1}$$

$$e \cdot v_i = [m-i+1] v_{i-1} \,, \tag{2}$$

$$f \cdot v_i = [i+1] v_{i+1} \,, \tag{3}$$

where we set $v_{-1}=0$ and $v_{m+1}=0$. Notice that the group T introduced in (2.5) acts on $V(n, r)$ as follows,

$$T_\mu \cdot v_i = \mu^{m-2i} v_i \,, \qquad i=0, \cdots, m \,.$$

The following lemma is trivial.

Lemma. *Let ρ denote the representation of U_ϵ^{red} on $V(n, r)$ defined above. Then*

$$T_\mu \cdot \rho(e) \cdot T_\mu^{-1} = \mu^2 \rho(e) \,,$$

$$T_\mu \cdot \rho(k) \cdot T_\mu^{-1} = \rho(k) \,,$$

$$T_\mu \cdot \rho(f) \cdot T_\mu^{-1} = \mu^{-2} \rho(f) \,.$$

3.2. For $0 \leqq i \leqq l-1$, let

$$V(n, r)_i = \{ v \in V(n, r) : k \cdot v = \epsilon^{m-2i} v \} \,.$$

Proposition.

(i)

$$\dim V(n, r)_i = n+1 \quad \textit{if} \quad 0 \leqq i \leqq r-1,$$
$$= n \quad \textit{otherwise}.$$

(ii) *The modules* $V(0, r)$ *are irreducible and each irreducible* U_ϵ^{red}*-module is isomorphic either to* $V(0, r)$ *for some* $1 \leqq r \leqq l-1$ *or to* $V(1, 0)$.

(iii)

$$(\Omega - [r/2]^2) \cdot V(n, r) = 0,$$

where $[r/2] = [(l+1)r/2]$.

Proof. To prove (i) observe that if $0 \leqq i \leqq r-1$ the elements $\{v_i, v_{l+i}, \cdots, v_{nl+i}\}$ form a basis of $V(n, r)_i$ and that if $r \leqq i \leqq l-1$ then the corresponding basis of $V(n, r)$ is $\{v_i, v_{l+i}, \cdots, v_{(n-1)l+i}\}$.

Part (ii) is well-known, (cf. [9]). Part (iii) is a simple calculation.

3.3. For any U_ϵ^{red}-module M, the maximal semisimple submodule of M is called the socle of M and is denoted by $\mathrm{soc}(M)$.

Theorem. *Let* $n > 0$.

(i) $V(n, r)$ *is indecomposable if* $r > 0$.

(ii) *For* $1 \leqq r \leqq l-1$ *we have,*

$$\mathrm{soc}(V(n, r)) \cong V(0, l-r)^{\oplus n}.$$

If $r=0$ *then* $V(n, 0) \cong V(1, 0)^{\oplus n}$.

Proof. To prove (i), assume first that $r \neq 0$. Let $\mathcal{A}$ be the subalgebra of $\mathrm{End}(V(n, r))$ consisting of operators that commute with the action of U_ϵ^{red} on $V(n, r)$. Using Lemma 3.1 it is easy to see that $\mathcal{A}$ is T-stable. Since T is a one-dimensional algebraic torus we can write,

$$\mathcal{A} = \bigoplus_{i \in Z} \mathcal{A}_i$$

where $\mathcal{A}_i = \{A \in \mathcal{A} : T_\mu A T_\mu^{-1} = \mu^i A\}$. It is immediate that $\mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j}$. Thus $\mathcal{A}_i$ consists of nilpotent endomorphisms for all $i \neq 0$.

We now show that $\mathcal{A}_0$ consists of scalar endomorphisms. Let $A \in \mathcal{A}_0$. Since $[A, T_\mu] = 0$ we have

$$Av_i = \alpha_i v_i,$$

for some scalars $\alpha_i \in \boldsymbol{C}$. The conditions $[A, e] = 0 = [A, f]$ imply that

$$[m-i+1]\alpha_{i-1} = [m-i+1]\alpha_i, \qquad [i+1]\alpha_{i+1} = [i+1]\alpha_i,$$

for all $i = 1, \cdots, m-1$. This forces,

$$\alpha_0 = \cdots = \alpha_{l-1},\ \alpha_l = \cdots = \alpha_{2l-1},\ \cdots\ \alpha_{nl} = \cdots = \alpha_{nl+r-1}.$$

In addition, since

$$[m-kl+1]\alpha_{kl-1}=[m-kl+1]\alpha_{kl}\neq 0$$

for $k=1, \cdots, n$ we get,

$$\alpha_{kl-1}=\alpha_{kl}\,.$$

Thus $A=\alpha_0\cdot id$ and so $\mathcal{A}_0$ consists of scalars as asserted.

It is now convenient to consider $\mathcal{A}$ as a Lie algebra with the Lie bracket,

$$[x, y]=xy-yx\,, \qquad x, y\in\mathcal{A}\,.$$

Clearly, $[\mathcal{A}_i, \mathcal{A}_j]\subset\mathcal{A}_{i+j}$ and $[\mathcal{A}_0, \mathcal{A}_j]=0$. Since $\mathcal{A}$ is finite-dimensional it follows that all T-homogeneous elements of $\mathcal{A}$ are ad-nilpotent. This implies that the Lie algebra $\mathcal{A}$ is nilpotent by the Engel-Jacobson theorem [6, Ch. 2].

Let N denote the Jacobson radical of the associative algebra $\mathcal{A}$. By Wedderburn's theorem the algebra $\mathcal{A}/N$ must be commutative since $\mathcal{A}$ is a nilpotent Lie algebra. Hence N coincides with the set of nilpotent elements of $\mathcal{A}$. Thus $\mathcal{A}_i\subset N$ for all i, forcing

$$\mathcal{A}=\boldsymbol{C}\oplus N\,.$$

If the Weyl module $V(n, r)$ is decomposable, then $\mathcal{A}$ has a non-zero non-invertible idempotent. This contradicts the fact that all non-invertible elements in $\mathcal{A}$ are nilpotent.

For part (ii) notice that for each $0\leqq i\leqq n-1$, the elements $v_{il+r}, \cdots, v_{il+l-1}$ span a submodule of $V(n, r)$ isomorphic to $V(0, l-r)$. It is not difficult to see that for any other element $w\in V(n, r)$, with $e\cdot w=0$ one has $f^{l-1}\cdot w\neq 0$. Thus the socle of $V(n, r)$ is the direct sum of n copies of $V(0, l-r)$. Since the dimension of $V(n, 0)$ is nl it follows also that $V(n, 0)$ is completely reducible.

3.4. The dual M^* of a U_ϵ^{red}-module M is defined by using the antipode:

$$(gf)(m)=f(S(g))m\,, \qquad \text{for all } g\in U_\epsilon^{red},\ f\in M^*,\ m\in M.$$

Fix a basis of M. Then the action of $g\in U_\epsilon^{red}$ on M^* in the dual basis is the transpose of the action of $S(g)$ on M in the original basis. Clearly the dual of an indecomposable representation is again indecomposable. Thus, the dual Weyl modules form another class of indecompossable modules for U_ϵ^{red}.

Lemma. *The dual Weyl module $V(n, r)^*$ is not isomorphic to $V(m, s)$ for any $m\geqq 0$ and $0\leqq s\leqq l-1$ if $n\neq 0$. The modules $V(0, r)$ and $V(1, 0)$ are self-dual.*

Proof. It suffices for dimension reasons to show that the modules $V(n, r)$ and $V(n, r)^*$ are not isomorphic. Therefore it suffices to observe that,

$$\operatorname{soc}(V(n, r))\cong V(0, l-r)^{\oplus n}\,, \qquad \operatorname{soc}(V(n, r)^*)\cong V(0, r)^{\oplus(n+1)}\,.$$

The first isomorphism was proved in the preceding theorem and the second can be proved similarly. That the modules $V(0, r)$ and $V(1, 0)$ are self-dual is immediate from Proposition 3.2 (ii).

3.5. We now give a construction of a one-parameter family of non-isomorphic indecomposable modules of U_ϵ^{red}. These modules can be identified with a family of maximal submodules of the Weyl modules, we shall prove this in Section 4.

Let $\{v_0, v_1, \cdots, v_{l-1}\}$ be a basis of $\boldsymbol{C}^l$. Let X, Z be elements of $\mathrm{End}(\boldsymbol{C}^l)$ defined by:

$$X\cdot v_i = v_{i-1}, \qquad Z\cdot v_i = \epsilon^{-2i}\cdot v_i,$$

where we set $v_{-1}=v_{l-1}$. Clearly,

$$X^l = Z^l = 1, \qquad ZX = \epsilon^2 XZ.$$

The elements $X^i Z^j$, $0 \leqq i, j \leqq l-1$ form a basis of $\mathrm{End}(\boldsymbol{C}^l)$. We denote by $Z^{1/2}$ the operator $Z^{(l+1/2)}$.

Proposition. (i) *Let $A \in \mathrm{End}(\boldsymbol{C}^n)$ and $1 \leqq r \leqq l-1$. The following formulas define an action of U_ϵ^{red} on $\boldsymbol{C}^l \otimes \boldsymbol{C}^n$:*

$$k \longrightarrow \epsilon^{r-1} Z \otimes 1 .$$

$$f \longrightarrow -\frac{Z^{1/2}-Z^{-(1/2)}}{\epsilon-\epsilon^{-1}} X^{-1} \otimes 1,$$

$$e \longrightarrow X \frac{\epsilon^r Z^{1/2}-\epsilon^{-r} Z^{-(1/2)}}{\epsilon-\epsilon^{-1}} \otimes 1 + \frac{[r]}{l} X \sum_{i=0}^{l-1} Z^i \otimes A .$$

(ii) *The module $V(A, n, r)$ defined in* (i) *is indecomposable if and only if $\boldsymbol{C}^n$ is indecomposable as a $\boldsymbol{C}A$-module.*

(iii) *$V(A, n, r) \cong V(B, n, r)$ if and only if A and B are conjugate.*

(iv) *$V(A, n, r)^* \cong V(A, n, l-r)$.*

(v) *$(\Omega - [r/2]^2)\cdot V(\lambda, n, r) = 0$.*

Proof. A simple checking shows that the formulas given in (i) do define a representation of U_ϵ^{red}. If $\boldsymbol{C}^n$ is decomposable as a $\boldsymbol{C}A$-module then it is clear that the representation of U_ϵ^{red} is decomposable. Conversely assume that the representation of U_ϵ^{red} is decomposable. Let $P: V(A, n, r) \to V(A, n, r)$ be the projection onto one of the submodules. Writing P as a polynomial in the non-commuting variables X, Z with coefficients in $\mathrm{End}(\boldsymbol{C}^n)$, we find by using the fact that $[P, k]=0$ and $[P, f]=0$ that $P=1\otimes Q$ for some $Q \in \mathrm{End}(\boldsymbol{C}^n)$. Now using the fact that $[Q, e]=0$ we see that $[Q, A]=0$. Hence A preserves both the image and the kernel of Q and so $\boldsymbol{C}^n$ is decomposable as a $\boldsymbol{C}A$-module.

If A and B are conjugate, say $B=CAC^{-1}$, then it is easy to check that $1\otimes C$ defines an U_ϵ^{red}-module isomorphism from $V(A, n. r)$ onto $V(B, n, r)$. Conversely by using the methods involved in proving part (ii) one can show that any U_ϵ^{red}-module isomorphism from $V(A, n, r)$ onto $V(B, n, r)$ must be of type $1\otimes C$ for some $C\in \mathrm{End}(\boldsymbol{C}^n)$. This forces $B=CAC^{-1}$.

Using the remarks in (3.4) one can write down the explicit formulas for the action of the generators of U_ϵ^{red} on the dual module $V(A, n, r)^*$. These formulas show that $V(A, n, r)^*\cong V(A^t, n, r)$. Since A and A^t are always conjugate, (iv) now follows from (iii). Part (v) is a simple calculation.

3.6. If A is a single Jordan block with eigenvalue λ, denote by $V(\lambda, n, r)$ the module $V(A, n, r)$. Let $V(\lambda, n, r)^\omega$ denote the module obtained by twisting the module structure on $V(\lambda, n, r)$ by the Cartan involution $\boldsymbol{\omega}$.

Proposition. *If $\lambda\neq -1$ we have,*

$$V(\lambda, n, r)^\omega \cong V\left(-\frac{\lambda}{\lambda+1}, n, r\right).$$

This can be proved by a direct calculation which we omit. The importance of this proposition is that $V(-1, n, r)^\omega$ is not a module of type $V(\mu, n, s)$ where $\mu\neq -1$. To see that $V(-1, n, r)^\omega$ is not isomorphic to $V(-1, n, s)$ it suffices to notice that:

$$\dim(\{v\in V(-1, n, s): e\cdot v=0\})=n+1\,,$$

$$\dim(\{v\in V(-1, n, s): f\cdot v=0\})=n\,.$$

Since it is obviously not a Weyl module, this is a new indecomposable module which we denote by $V(\infty, n, r)$.

3.7. The modules $V(0, 1, r)$ can be identified with the Verma modules over U_ϵ^{red}. Recall that for $0\leqq r\leqq l-1$ the Verma module $M(r)$ is the quotient of U_ϵ^{red} by the left ideal generated by e and $k-\epsilon^{r-1}$. Clearly $M(r)$ is l-dimensional. Further, if M is any other U_ϵ^{red}-module generated by an element m satisfying,

$$e\cdot m=0\,, \qquad k\cdot m=\epsilon^{r-1}\cdot m\,,$$

then M is a quotient of $M(r)$. With these comments it is now easy to check that $V(0, 1, r)\cong M(l-r)$ for any $1\leqq r\leqq l-1$. Hence $M(r)^*\cong M(l-r)$ by Proposition 3.5(iv). The module $M(0)$ is isomorphic to $V(1, 0)$ and so is irreducible. The module $V(0, r)$ is the unique irreducible quotient of $M(r)$ for $1\leqq r\leqq l-1$. The following lemma can now be proved easily.

Lemma. *For $1\leqq r\leqq l-1$, there exists a non-split exact sequence.*

$$0\longrightarrow V(0, l-r)\longrightarrow M(r)\longrightarrow V(0, r)\longrightarrow 0\,.$$

3.8. Our final set of examples are the indecomposable projective covers $X(r)$ of the irreducible modules $V(0, r)$ for $1\leqq r\leqq l-1$. Such projective covers exist by (cf. [11, § 6.3]).

Proposition. *Let* $1 \leqq r \leqq l-1$.
(i)

$$[X(r): M(j)] = 1 \quad \text{if } j=r \text{ or } l-r,$$
$$= 0 \quad \text{otherwise.}$$

(ii) $\dim X(r) = 2l$.
(iii) $\mathrm{soc}\,(X(r)) = V(0, r)$.
(iv) *The following short exact sequence of* U_ϵ^{red}*-modules is non-split:*

$$0 \longrightarrow M(l-r) \longrightarrow X(r) \longrightarrow M(r) \longrightarrow 0 .$$

Proof. By (2.5) the algebra U_ϵ^{red} satisfies all the conditions of [5]. Theorems 4.5 and 5.1 of [5] now imply that each $X(r)$ admits a filtration in which the corresponding quotients are Verma modules and the following formula holds:

$$[X(r): M(j)] = [M(j): V(0, r)] .$$

Parts (i) and (ii) are now immediate from (3.7).

By Proposition 2.6, $X(r)$ is an injective module over U_ϵ^{red}. Using the arguments of [7, pp. 50–52] one can show that an injective U_ϵ^{red}-module is indecomposable if and only if its socle is simple and that two injective modules are isomorphic if and only if their socles are isomorphic. This yields $\mathrm{soc}\,(X(r)) = V(0, r)$. Part (iv) is now immediate.

3.9. We now give an explicit basis of the modules $X(r)$.

Proposition. $\dim \mathrm{Ext}^1(M(r), M(l-r)) = 1$.

Proof. By Proposition 3.8 we know that $\mathrm{Ext}^1(M(r), M(l-r))$ has dimension greater than 0. Consider a short exact sequence of U_ϵ^{red}-modules,

$$0 \longrightarrow M(l-r) \xrightarrow{\alpha} N \xrightarrow{\beta} M(r) \longrightarrow 0 .$$

Let $v_0, v_1, \cdots, v_{l-1}$ be a basis of $M(l-r)$ such that

$$k \cdot v_i = \epsilon^{l-r-1-2i} v_i , \quad f \cdot v_i = [i+1] v_{i+1} , \quad e \cdot v_0 = 0 .$$

Choose $w_0 \in N$ such that,

$$k \cdot w_0 = \epsilon^{r-1} w_0 , \quad \beta(w_0) \neq 0 , \quad e \cdot \beta(w_0) = 0 .$$

Since the k-eigenspaces of $M(l-r)$ are one-dimensional and $e \cdot w_0 \in M(l-r)$ it follows that,

$$e \cdot w_0 = \mu_0 v_{l-r-1} .$$

If $\mu_0 = 0$ then it is clear that the subspace spanned by the elements $\{w_i = (f^i/[i]!) \cdot w_0 : 0 \leqq i \leqq l-1\}$ is preserved by U_ϵ^{red}. Since $\beta(w_i) \neq 0$ we get, $N \cong$

$M(r)\oplus M(l-r)$ contradicting the fact that our sequence is non-split. Thus $\mu_0\neq 0$ and by rescaling the w_i we can and do assume that $\mu_0=1$. It is now easy to calculate the action of e on the w_i, namely,

$$e\cdot w_i=[r-i]w_{i-1}+\mu_i v_{l-r-1+i}\ ,$$

where $\mu_i=[l-r-1+i\,;\,i]$ and as usual v_i and w_i are zero if $i<0$ or $i>l-1$.

Corollary. (i) *$X(r)$ is a $2l$ dimensional module with basis $\{v_0, v_1, \cdots, v_{l-1}, w_0, w_1, \cdots, w_{l-1}\}$ on which the generators of U_ϵ^{red} act as follows,*

$$k\cdot v_i=\epsilon^{l-r-2i-1}v_i\ ,\qquad k\cdot w_i=\epsilon^{r-2i-1}w_i\ ,$$

$$f\cdot v_i=[i+1]v_{i+1}\ ,\qquad f\cdot w_i=[i+1]w_{i+1}\ ,$$

$$e\cdot v_i=[l-r-i]v_{i-1}\ ,\qquad e\cdot w_i=[r-i]w_{i-1}+\mu_i v_{i-r-1+i}\ ,$$

where $\mu_i=[l-r-1+i\,;\,i]$ and we assume as usual that v_i and w_i are zero if $i<0$ or $i>l-1$.

(ii) *The action of Ω on $X(r)$ is not semisimple.*

(iii) *$X(r)^*\cong X(r)$.*

Proof. Part (i) is immediate from Proposition 3.9. Part (ii) is a computation. To prove (iii), observe that the non-split exact sequence in Proposition 3.8(iv) induces a non-split exact sequence of the dual U_ϵ^{red}-modules,

$$0\longrightarrow M(r)^*\longrightarrow X(r)^*\longrightarrow M(l-r)^*\longrightarrow 0\ .$$

Since $M(r)^*\cong M(l-r)$, (cf. (3.7)) we have a non-split exact sequence,

$$0\longrightarrow M(l-r)\longrightarrow X(r)^*\longrightarrow M(r)\longrightarrow 0\ .$$

Applying Proposition (3.8)(iv) and (3.9) yields the desired isomorphism $X(r)\cong X(r)^*$.

Remark. The modules $X(r)$ were defined in [13].

§4. Classification of Indecomposable Representations

We state and prove our main theorem in this section. We begin with the following simple proposition.

4.1.

Proposition. *Let M be an indecomposable representation of U_ϵ^{red}. There exists $0\leqq r\leqq (l-1)/2$ such that,*

$$(\Omega-[r/2]^2)^2\cdot M=0\ .$$

If $r=0$, Ω is zero on M.

Proof. Since $k^l=1$ on any U_ϵ^{red}-module M, the action of k on M is semisimple and the eigenvalues of k are contained in $\{\epsilon^i : 0\leqq i\leqq l-1\}$. Using Lemma 2.4 we get,

$$\Omega\cdot\prod_{i=1}^{(l-1)/2}(\Omega-[i/2]^2)^2\cdot M=0\,. \tag{4}$$

As a result any eigenvalue of Ω must be of the form $[i/2]^2$ for some $0\leqq i\leqq (l-1)/2$. Further since M is indecomposable Ω has only one eigenvalue on M, say $[r/2]^2$, for some $0\leqq r\leqq l-1$. In particular the operators $(\Omega-[i/2]^2)$ are invertible on M for all $0\leqq i\leqq l-1$, $i\neq r$, $l-r$, The proposition now follows.

4.2. For $0\leqq r\leqq(l-1/2)$ let C_r denote the category of U_ϵ^{red}-modules M with the property,

$$(\Omega-[r/2]^2)^2\cdot M=0\,.$$

For $0\leqq i\leqq l-1$ set, $M_i=\{m\in M : km=\epsilon^{r-2i-1}\cdot m\}$. The main result of this paper is,

Theorem. *Let M be an indecomposable object of C_r.*

(i) *If $r=0$ then M is isomorphic to $V(1, 0)$.*

(ii) *If $r>0$ and Ω is semisimple then M or M^* is isomorphic to precisely one of $V(n, i)$, $V(\lambda, m, i)$ where $i=l-r$ or r, n is any non-negative integer, m is any positive integer and $\lambda\in\boldsymbol{C}\cup\{\infty\}$.*

(iii) *If Ω is not semisimple on M then $r>0$ and M is isomorphic to $X(r)$.*

The rest of the section is devoted to proving this theorem.

4.3.

Lemma. *Let M be an indecomposable object in C_r.*

(i) *The restriction of fe (resp. ef) to M_i is invertible if $i\neq 0$, r (resp. $i\neq r-1$, $l-1$). If in addition Ω acts semisimply on M, the restriction of fe (resp. ef) to M_0 and M_r (resp. M_{r-1} and M_{l-1}) is identically zero.*

(ii) *The map $f: M_i\to M_{i+1}$ (resp. $e: M_i\to M_{i-1}$) is injective if $i\neq r-1$, $l-1$ (resp. $i\neq 0$, r).*

(iii)

$$\begin{aligned}\dim(M_i)&=\dim(M_0) && \text{if } 0\leqq i\leqq r-1\,,\\ &=\dim(M_{l-1}) && \text{if } r\leqq i\leqq l-1\,.\end{aligned}$$

Proof. Since $M\in C_r$ and the restriction of fe to M_i is $(\Omega-[(r-2i)/2]^2)$, the proof of part (i) follows. Parts (ii) and (iii) are now immediate.

4.4. We now prove part (i) of Theorem 4.2. Let $r=0$. By Lemma 4.3 (ii) e is injective on every eigenspace M_i, $i\neq 0$. Since $e^l=0$ it follows that $e\cdot M_0=0$. Let $\{m_1, \cdots, m_n\}$ be a basis of M_0. Clearly for each $0\leqq s\leqq n$ the subspace $M(s)$ of M spanned by $\{f^i \cdot m_s : 0 \leqq i \leqq l-1\}$ is an irreducible submodule of M. Applying Lemma 4.3(ii) again we see that $M=\bigoplus_{s=1}^n M(s)$. Since M is indecomposable this means that $n=1$ and hence $M \cong V(1, 0)$ by Proposition 3.2(ii).

4.5. Assume now that $(\Omega-[r/2]^2)\cdot M=0$ for some $r>0$ and that M is indecomposable and reducible.

Proposition. *The pair of maps* $e, f^{l-1}: M_0 \to M_{l-1}$ *is non-zero if and only if the pair* $f, e^{l-1}: M_{l-1} \to M_0$ *is zero.*

Proof. Suppose first that both pairs are zero. Using Lemma 4.3 one deduces easily that the subspaces $\bigoplus_{i=0}^{r-1} M_i$ and $\bigoplus_{i=r}^{l-1} M_i$ are submodules of M. Since M is indecomposable this means that $M_i=0$ either for all $i \in \{1, \cdots, r-1\}$ or for all $i \in \{r, \cdots, l-1\}$. Proceeding as in the proof of Theorem 4.2(i) we see that M must be irreducible contradicting our assumption.

For the converse assume that the pair e, f^{l-1} is non-zero when restricted to M_0. Let $V_1=\ker(e) \cap \ker(f^{l-1}) \cap M_0$ and let W_1 be a subspace of M_0 complementary to V_1 (notice that $W_1 \neq 0$). Let V_2 and W_2 be subspaces of M_{l-1} defined similarly by interchanging e and f. Set,

$$N=\sum_{i=0}^{r-1} f^i V_1 \oplus \sum_{i=0}^{l-r-1} e^i W_2 , \tag{5}$$

$$N'=\sum_{i=0}^{r-1} f^i W_1 \oplus \sum_{i=0}^{l-r-1} e^i V_2 . \tag{6}$$

By Lemma 4.3 $M=N \oplus N'$. Suppose in addition that N and N' are submodules of M. Since M is indecomposable and $N' \neq 0$ it follows that $N=0$. In particular $W_2=0$ and so,

$$M_{l-1}=V_2=\ker(f) \cap \ker(e^{l-1}) \cap M_{l-1} .$$

Thus to complete the proof of the proposition, we must show that N and N' are submodules. We show now that N is a submodule, the proof for N' is similar and left to the reader. By definition V_1 and W_2 are contained in eigenspaces of k. As Ω acts semi-simply on M one checks easily that,

$$ef^i V_1 \subset N , \qquad fe^j W_2 \subset N ,$$

for all $i \in \{0, \cdots, r-1\}$, $j \in \{0, \cdots, l-r-1\}$. Thus, to prove that e and f preserve N it is enough to show that,

$$e^{l-r} W_2 \subset f^{r-1} V_1 , \qquad f^r V_1 \subset e^{l-r-1} W_2 .$$

By Lemma 4.3 this is equivalent to,

$$e^{l-1} W_2 \subset V_1 , \qquad f^{l-1} V_1 \subset W_2 .$$

The second inclusion is obvious since $f^{l-1} V_1=0$. The first can be deduced from the fact that $e^l=0$ and the following easy consequence of Lemma 2.4,

$$f^{l-1}e^{l-1}W_2=\prod_{i=0}^{l-2}([r/2]^2-[(2j+r+2)/2]^2)\cdot W_2=0\,.$$

The proof of the proposition is now complete.

Corollary. *If the pair e, f^{l-1} (resp. f, e^{l-1}) is non-zero on M_0 (resp. M_{l-1}) then it is indecomposable in the sense of Definition* 2.7.

Proof. Assume that the restriction of e, f^{l-1} to M_0 is non-zero and that $M_0=V_1\oplus V_2$, $M_{l-1}=W_1\oplus W_2$, with e, $f^{l-1}(V_i)\subset W_j$, $i\neq j$. Let N and N' be defined as in equations (5) and (6) above. Since the pair f, e^{l-1} is zero on M_{l-1}, it follows as before that N and N' are submodules of M. Hence either N or N' is zero proving that the pair e, f^{l-1} is indecomposable. The case when f, e^{l-1} is a non-zero pair can be treated similarly.

4.6.

Lemma. *Let M and N be two modules in $\mathcal{C}_r$. Then $M\cong N$ if and only if there exist isomorphisms of vector spaces $\pi_i: M_i\to N_i$ $i=0$, $l-1$ such that,*

$$e\cdot\pi_0=\pi_{l-1}\cdot e\,,\qquad f^{l-1}\cdot\pi_0=\pi_{l-1}\cdot f^{l-1}\,,$$

and

$$f\cdot\pi_{l-1}=\pi_0\cdot f\,,\qquad e^{l-1}\cdot\pi_{l-1}=\pi_0\cdot e^{l-1}\,.$$

Proof. Given π_0 and π_{l-1} define maps $\pi_i: M_i\to N_i$ by setting,

$$\pi_i(f^i m)=f^i\pi_0(m)\,,\qquad \text{for } 0\leqq i\leqq r-1,\ m\in M_0\,,$$

and

$$\pi_i(e^{l-i-1}m')=e^{l-i-1}\pi_{l-1}(m')\,,\qquad \text{for } r+1\leqq i\leqq l-1,\ m'\in M_{l-1}\,.$$

Using Lemma 4.3 it is easy to check that $\oplus_i\pi_i: M\to N$ is an isomorphism of U_ϵ^{red}-modules. The converse statement is trivial.

4.7. We can now prove part (ii) of the Theorem. By the results of Section 4.5 exactly one of the pairs of maps e, $f^{l-1}: M_0\to M_{l-1}$, f, $e^{l-1}: M_{l-1}\to M_0$ is non-zero and the non-zero pair is indecomposable. Consider the case when the first pair of maps is non-zero, the other case is similar and we omit the details. By Theorem 2.7 $\dim M_0-\dim M_{l-1}$ is either ± 1 or 0. If the difference is one then by Theorem 2.7(i) there exist bases of M_0 and M_{l-1} such that,

$$e|_{M_0}=\phi_n\,,\qquad f^{l-1}|_{M_0}=\psi_n\,,$$

where $n=\dim M_{l-1}$. The Weyl module $V(n, r)$ is indecomposable and satisfies all the above assumptions. Thus one can define linear maps π_0 and π_{l-1} satisfying the assumptions of Lemma 4.6 and so one may conclude that $V(n, r)\cong M$ as U_ϵ^{red}-modules.

The case when the difference is -1 can be dealt with similarly by using Theorem 2.7(ii) and we find that $V(n, l-r)^* \cong M$.

Now suppose that $\dim M_0 = \dim M_{l-1} = n$. By Theorem 2.7(iii), either f^{l-1} or e is bijective on M_0. If f^{l-1} is bijective, choose a basis $m_1, \cdots, m_n$ of M_0. Since $f: M_i \to M_{i+1}$ is injective if $i \neq l-1$ the elements,

$$\left\{\frac{f^i}{[i]!} m_j : 0 \leqq i \leqq l-1,\ j=1, \cdots, n\right\}$$

from a basis of M. In other words $M \cong \boldsymbol{C}^l \otimes \boldsymbol{C}^n$ with the action of k and f given by:

$$k = \epsilon^{r-1} Z \otimes 1\,, \qquad f = \frac{(Z^{1/2} - Z^{-(1/2)})}{\epsilon - \epsilon^{-1}} \otimes 1\,,$$

where X, Z are the elements of $\mathrm{End}(\boldsymbol{C}^l)$ defined in Section 3. The action of e on M can be determined by writing e as a polynomial in the non-commuting variables X and Z and imposing the defining relations of U_ϵ^{red}. It is not hard to see that the action of e is exactly as in Proposition 3.5. Hence we see that M is isomorphic to $V(\lambda, n, r)$ for some λ. If the restriction of f^{l-1} to M_0 has a non-zero kernel then the restriction of e^{l-1} to M_{l-1} is injective. Now using the Cartan involution ω one shows that $M \cong V(\infty, n, r)$.

The proof of part (ii) is now complete.

4.8. The proof of part (iii) proceeds as follows. We first prove that if Ω does not act semisimply on M, then M contains a $2l$-dimensional submodule N. Next we show that N corresponds to a non-trivial element of either $\mathrm{Ext}^1(M(r), M(l-r))$ or $\mathrm{Ext}^1(M(l-r), M(r))$. Propositions 3.8(iv) and 3.9 then imply that $N \cong X(r)$ or $N \cong X(l-r)$. By Proposition 2.6 N is projective and injective and so a direct summand of M. Since M is indecomposable, $M=N$ and the Theorem follows.

Choose N to be any submodule of M of minimal possible dimension on which Ω does not act semisimply. Notice that this implies that Ω acts semisimply on every submodule of N. Since Ω does not act semisimply on N it follows from Lemma 4.3 that Ω does not act semisimply on either N_0 or N_{l-1}, say on N_0. Let $m \in N_0$ be such that,

$$(\Omega - [r/2]^2) \cdot m \neq 0\,, \qquad (\Omega - [r/2]^2)^2 \cdot m = 0\,, \tag{7}$$

or equivalently,

$$fe \cdot m \neq 0\,, \qquad (fe)^2 \cdot m = 0\,. \tag{8}$$

By applying Lemma 2.4 it is easy to see that:

(i) for $i > 1$, the element $f^i e^i \cdot m$ is a linear combination of m and $fe \cdot m$,

(ii) $f^{l-1} \cdot m \neq 0$.

Since $U_\epsilon^{red} \cdot m \subset N$ is a submodule on which Ω does not act semisimply, we have $N = U_\epsilon^{red} \cdot m$ (by the minimality of N). We now prove that $\dim N = 2l$.

By Lemma 4.3 it is enough to prove that N_0 and N_{l-1} have dimension 2. Since N_0 is spanned by $f^i e^i \cdot m$ it follows from (i) that $\dim N_0 = 2$. As an immediate consequence of this, we find by using Lemma 4.3 that,

$$(\Omega - [r/2]^2) \cdot f^{r-1} m \neq 0 . \tag{9}$$

The elements $e \cdot m$, $f^{l-1} \cdot m$ of N_{l-1} must be linearly independent since $f^l \cdot m = 0$ whereas $fe \cdot m \neq 0$. So $\dim N_{l-1} \geqq 2$. If Ω acts semisimply on N_{l-1} then $(ef)e \cdot m = 0$ by Lemma 4.3. It is now easy to check that the span of $\{f^i \cdot m, f^i e \cdot m : 0 \leqq i \leqq l-1\}$ is a submodule of N on which Ω does not act semisimply and hence is equal to N. Since $\dim N \geqq 2l$, we conclude that $\dim N = 2l$.

We claim that Ω must act semisimply on N_{l-1}. Assume for a contradiction that it does not. Then reasoning as before one can prove $\dim N_{l-1} = 2$. Since the maps e, $f^{l-1} : N_0 \to N_{l-1}$ are non-zero, one of the following possibilities must occur:

(a) $im(e) \subset im(f^{l-1})$,

(b) $im(f^{l-1}) \cap im(e) = 0$,

(c) $im(f^{l-1}) \subset im(e)$.

The first case cannot occur since $fe \cdot m \neq 0$. The scond implies that

$$N_{l-1} = im(e) \oplus im(f^{l-1}) .$$

Since Ω preserves each summand this would force Ω to be semisimple on N_{l-1} contradicting our assumption. Since $e \cdot m$ and $f^{l-1} \cdot m$ are linearly independent the third possibility implies that e is an isomorphism from N_0 onto N_{l-1}. One proves similarly that $f : N_{l-1} \to N_0$ is injective. By Lemma 4.3 Ω does not act semisimply on N_{r-1} and N_r. Working with these subspaces we conclude as before that $e : N_r \to N_{r-1}$ and $f : N_{r-1} \to N_r$ are injective. But then Lemma 4.3 implies that e and f are injective on the entire module contradicting $e^l = f^l = 0$. Hence Ω must be semisimple on N_{l-1}.

Consider the element $m' = e^{l-r} \cdot m$. By Lemma 4.3 $m' \neq 0$ and $e \cdot m' = e^{l-r+1} \cdot m = 0$. Further using Lemma 2.4 and equation (9) we get,

$$f^{l-1} \cdot m' = f^{r-1} f^{l-r} e^{l-r} \cdot m = \prod_{j=0}^{l-r-1} (\Omega - [(2j+r)/2]^2) \cdot f^{r-1} \cdot m \neq 0 .$$

The results of Section 3.7 imply that the U_ϵ^{red}-submodule generated by m' is isomorphic to $M(l-r)$. By Lemma 2.4,

$$f^{l-r-1} \cdot m' = \prod_{j=0}^{l-r-2} (\Omega - [(2j+r+1)/2]^2) \cdot em ,$$

and the operators in the product are invertible on N_{l-1}. Therefore $e \cdot m \in U_\epsilon^{red} \cdot m' \cong M(l-r)$. Since all the k-eigenspaces of $M(l-r)$ are one-dimensional, this forces $f^{l-1} \cdot m \notin U_\epsilon^{red} \cdot m'$. Hence, the quotient module $N/M(l-r)$ has dimension l and is generated by the image m'' of m with $e \cdot m'' = 0$ and $f^{l-1} \cdot m'' \neq 0$. Thus we may conclude as before that $N/M(l-r)\cong M(r)$. Since Ω does not act semisimply on N the short exact sequence,

$$0 \longrightarrow M(l-r) \longrightarrow N \longrightarrow M(r) \longrightarrow 0,$$

is non-split and so N corresponds to a non-trivial element of $\mathrm{Ext}^1(M(r), M(l-r))$. The proof of the Theorem is now complete.

4.9. To conclude the paper we now use our classification theorem to prove that the modules $V(\lambda, n, r)$ can be identified with maximal submodules of the Weyl module $V(n, r)$.

Let $\{v_i : 0\leqq i\leqq l-1\}$ denote the basis of C^l from Section 3.5. It is easy to see that one can choose a basis $\{w_i : 0\leqq j\leqq l-1\}$ of C^n such that, the action of U_ϵ^{red} on $V(\lambda, n, r)$ is given by extending,

$$k\cdot v_i\otimes w_j=\epsilon^{r-1-2i}v_i\otimes w_j,$$

$$f\cdot v_i\otimes w_j=[i+1]v_{i+1}\otimes w_j,$$

$$e\cdot v_i\otimes w_j=[r-i]v_{i-1}\otimes w_j, \qquad i\neq 0,$$

$$e\cdot v_0\otimes w_j=[r]v_{l-1}\otimes w_{j-1}+[r](1+\lambda)v_{l-1}\otimes w_j,$$

where $0\leqq i\leqq l-1$, $0\leqq j\leqq n-1$ and $w_{-1}=0$.

The following Lemma can be proved by a straightforward verification.

Lemma. *Let $n\in Z$, $n\geqq 0$, $0<r\leqq l-1$, $\lambda\in C$.*

(i) *The following formulas define a deformed action of U_ϵ^{red} on $V(n, r)$:*

$$k\cdot v_i=\epsilon^{m-2i},$$

$$f\cdot v_i=[i+1]v_{i+1},$$

$$e\cdot v_i=[r-i]v_{i-1}, \qquad i\neq 0 \bmod l,$$

$$e\cdot v_{jl}=[r]v_{jl-1}+[r](1+\lambda)v_{(j+1)l-1},$$

(ii) *Let $V^\lambda(n, r)$ be the deformed Weyl module defined in* (i). *The assignment $v_i\otimes w_j\to v_{i+jl}$ defines an isomorphism of $V(\lambda, n, r)$ onto the submodule of $V^\lambda(n, r)$ spanned by $\{v_0, v_1, \cdots, v_{nl-1}\}$.*

(iii) *Ω act on $V^\lambda(n, r)$ as $[r/2]^2\cdot id$.*

(iv) $\dim V^\lambda(n, r)_0=n+1$ *and* $\dim V^\lambda(n, r)_{l-1}=n$.

Proposition.

(i) *$V^\lambda(n, r)\cong V(n, r)$.*

(ii) *$V(\infty, n, r)$ is isomorphic to the submodule of $V(n, r)$ spanned by $\{v_r, v_{r+1}, \cdots, v_m\}$.*

Proof. It is clear from the definition of $V^{\lambda}(n, r)$ that the pair of maps $f, e^{l-1}: V^{\lambda}(n, r)_{l-1} \to V^{\lambda}(n, r)$ is zero. Also, it is not hard to check that the subspace of $\mathrm{Hom}(V^{\lambda}(n, r)_0, V^{\lambda}(n, r)_{l-1})$ spanned by the maps $e, f^{l-1}: V^{\lambda}(n, r)_0 \to V^{\lambda}(n, r)_{l-1}$ contains the maps ϕ_n, ψ_n defined in Section 2. This implies that the pair e, f^{l-1} is indecomposable.

By Lemma 4.9 and Theorem 4.2 we can conclude that $V^{\lambda}(n, r)$ is isomorphic to either $V(n, r)$ or to $V(n, r)^*$. Since the submodule of $V^{\lambda}(n, r)$ spanned by $v_r, v_{r+1}, \cdots, v_{l-1}$ is isomorphic to $V(0, l-r)$ it follows from Lemma 3.4 that $V^{\lambda}(n, r) \cong V(n, r)$.

Recall that $V(\infty, n, r) = V(-1, n, r)^{\omega}$. By Lemma 4.9(i), $V(-1, n, r)$ is isomorphic to the submodule of $V(n, r)$ spanned by $\{v_0, \cdots, v_{nl-1}\}$. Hence $V(\infty, n. r)$ is isomorphic to a submodule of $V(n, r)^{\omega}$. To complete the proof of (ii), it is enough to note that there exists a U_{ϵ}^{red}-module isomorphism $\eta: V(n, r)^{\omega} \to V(n, r)$ such that, $\eta(Cv_i) = Cv_{m-i}$ for all $0 \leqq i \leqq m$.

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