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The Structure of Hilbert Flag Varieties

Dedicated to the memory of our father

By

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Abstract

In this paper we present a geometric realization of infinite dimensional analogues of the finite dimensional representations of the general linear group. This requires a detailed analysis of the structure of the flag varieties involved and the line bundles over them. In general the action of the restricted linear group can not be lifted to the line bundles and thus leads to central extensions of this group. It is determined exactly when these extensions are non-trivial. These representations are of importance in quantum field theory and in the framework of integrable systems. As an application, it is shown how the flag varieties occur in the latter context.

§ 1. Introduction

Let H be a complex Hilbert space. If H is finite dimensional, then it is a classical result that the finite dimensional irreducible representations of the general linear group $GL(H)$ can be realized geometrically as the natural action of the group $GL(H)$ on the space of global holomorphic sections of a holomorphic line bundle over a space of flags in H. By choosing a basis of H, one can identify this space of holomorphic sections with a space of holomorphic functions on $GL(H)$ that are certain polynomial expressions in minors of the matrices corresponding to the elements of $GL(H)$. Infinite dimensional analogues of some of these representations occur in quantum field theory, see e.g. [5]. Infinite dimensional Grassmann manifolds play an important role in the framework of integrable systems. The first person to realize this was Sato, see [24].

In this paper we will give an infinite dimensional analogue of all these representations. Thereto we take a separable Hilbert space H. In H we consider a collection of flags that generalizes the Grassmanian from chapter 7 in [23]. This flag variety carries a natural Hilbert space structure and there

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exist line bundles over it that are similar to the finite dimensional ones. This includes the determinant bundle and its dual from [23]. In the "dominant" case the space of global holomorphic sections of such a line bundle turns out to be non-trivial. However, the action of the analogue of the general linear group can, in general, not be lifted to the line bundle under consideration and one has to pass to a central extension of this group. Besides of the introduction, this paper consists of three sections. In the first section we give the definition of the flag variety $\mathfrak{F}$ and we treat some properties of $\mathfrak{F}$. The second section is devoted to the construction of the holomorphic line bundles, to a description of the corresponding central extensions and to the analysis of the space of global holomorphic sections. As an application, we show in the final section what role the geometry plays in the context of some integrable systems. A more detailed description of the content of the different sections is as follows.

The first subsection of section 2 discusses the type of flags in H that will be considered. Here the model for the size of the flags is the basic flag $F^{(0)}$ corresponding to a finite orthogonal decomposition of H. The flag variety $\mathfrak{F}$ is a homogeneous space for a certain unitary group $U_{\mathrm{res}}(H)$. As in the finite dimensional case it is convenient to see $\mathfrak{F}$ also as a homogeneous space for a larger group of automorphisms of H, namely $GL_{\mathrm{res}}(H)$. This is the analogue of the general linear group in this framework. Analogously to the finite dimensional case the group $U_{\mathrm{res}}(H)$ is the unitary component in the polar decomposition of $GL_{\mathrm{res}}(H)$. In the second subsection we give an explicit description of the manifold structure on $\mathfrak{F}$ and we discuss decompositions of certain open subsets of $GL_{\mathrm{res}}(H)$. A first difference with the finite dimensional situation appears at the description of the connected components of $\mathfrak{F}$ in the third subsection. The fourth subsection contains the technical prerequisites for the construction of the line bundles. First we choose a suitable orthonormal basis of H, we order its index set conveniently and we introduce the Weyl group W of $GL_{\mathrm{res}}(H)$. Next we show that the charts around the points in the W-orbit through $F^{(0)}$ cover $\mathfrak{F}$. By using this covering one obtains a stratification of $\mathfrak{F}$ into parts that are all homeomorphic to a Hilbert space. On the group level this gives you the Birkhoff decomposition for $GL_{\mathrm{res}}(H)$.

Let $\mathfrak{F}^{(0)}$ be the connected component of $\mathfrak{F}$ containing $F^{(0)}$. From the foregoing results one deduces that $\mathfrak{F}^{(0)}$ is a homogeneous space for a Banach Lie group $\mathfrak{G}$ that permits you to take suitable minors. As a group the group $\mathfrak{G}$ is a subgroup of $GL_{\mathrm{res}}(H)$, but its topology is stronger than the one induced by $GL_{\mathrm{res}}(H)$. The description of $\mathfrak{G}$ and its topology can be found in the first subsection of section 3. There we introduce also the maximal torus $T($[illegible]$)$ of $\mathfrak{G}$ and its group of analytic characters $\hat{T}$. In the second subsection we introduce a dense tower of finite dimensional flag varieties in $\mathfrak{F}^{(0)}$. The next subsection shows how you can associate to certain elements $\psi_{\underline{k}}$, $k\in\mathbf{Z}^m$, of $\hat{T}$ a holomorphic line bundle $L(\underline{k})$ over $\mathfrak{F}^{(0)}$. Further it is shown there that, if one tries to lift the action of the connected component $GL^{(0)}_{\mathrm{res}}(H)$ of $GL_{\mathrm{res}}(H)$, one might meet obstructions and that one can only lift the action of a central extension of $GL^{(0)}_{\mathrm{res}}(H)$. The natural question that comes up then is "how essential is this extension". This question is treated fully in the next subsection. Then one has come to the final subsection of this section. There we determine, when $L(\underline{k})$ has global non-trivial sections and we show that the action of $\mathfrak{G}$ on this space defines an irreducible $\mathfrak{G}$-module of highest weight $\psi_{\underline{k}}$.

Section 4 is an illustration of the fact that the geometry of the foregoing sections plays a role in the theory of integrable systems. The system we will consider is the multicomponent KP-hierarchy. The first subsection describes the flows in $GL_{\mathrm{res}}(H)$ corresponding to this system. The algebraic framework for this system of equations is given in the second subsection. In the final subsection of this paper we indicate how the flag varieties form the starting point of the construction of solutions to the equations of the multicomponent KP-hierarchy and the modified versions of the KP-hierarchy.

We would like to thank the referee for bringing to our attention papers by Faltings [10] and Kashiwara [19], who give an algebraic geometrical approach to infinite dimensional flag varieties.

§2. Properties of Hilbert Flag Varieties

2.1. The flag variety. Let H be a separable complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. We will consider certain finite chains of subspaces in H and we will call them flags as in the finite dimensional case. First one has to specify the "size" of the components of the flag. Therefore we start with an orthogonal decomposition of H,

$$H=H_1\oplus\cdots\oplus H_m,\quad\text{where}\quad H_i\perp H_j\ \text{for}\ i\neq j. \tag{1}$$

We assume that $m_i=\dim(H_i)$ satisfies $1\leq m_i\leq\infty$. An example of a decomposition occurring in quantum field theory is the one corresponding to the positive and negative spectrum of the Dirac operator, see [5] and [21]. In the context of integrable systems we have:

Example 2.1.1. Let $(\cdot,\cdot)$ be the standard inner product on $\mathbf{C}^r$. The Grassmann manifold $Gr(H)$ that is crucial at the construction of solutions of KP-type hierarchies in [23], [13] and [25] corresponds to the case that H is the space of power series

$$H=L^2(S^1,\mathbf{C}^r)=\{\sum_{n\in\mathbf{Z}}a_nz^n,\ a_n\in\mathbf{C}^r,\ \sum_{n\in\mathbf{Z}}(a_n,a_n)<\infty\},$$
$$H_1=\{\sum_{n\geq 0}a_nz^n\in H\}\quad\text{and}\quad H_2=\{\sum_{n<0}a_nz^n\in H\}.$$

If one takes $r = 1$ and k and l in $\mathbf{Z}$ with $k > l$, then the basic manifold corresponding to the (k, l)-modified KP-hierarchy is the flag variety corresponding to $H = L^2(S^1, \mathbf{C}) = H_1 \oplus H_2 \oplus H_3$, with

$$H_1 = \{\sum_{n \geq k} a_n z^n \in H\},\ H_2 = \{\sum_{n \geq l}^{k-1} a_n z^n \in H\} \quad \text{and} \quad H_3 = \{\sum_{n < l} a_n z^n \in H\}.$$

This correspondence is described in the fourth section.

Let p_i, $1 \leq i \leq m$, be the orthogonal projection of H onto H_i. Then we will use throughout this paper the following

Notation 2.1.2. If g belongs to $\mathscr{B}(H)$, the space of bounded linear operators from H to H, then $g = (g_{ij})$, $1 \leq i \leq m$ and $1 \leq j \leq m$ denotes the decomposition of g w.r.t. the $\{H_i | 1 \leq i \leq m\}$. That is to say $g_{ij} = p_i \circ g | H_j$.

Remark 2.1.3. Let K_i, $i = 1, 2$, be Hilbert spaces with inner products $\langle\ ,\ \rangle_i$, $i = 1, 2$. If A belongs to $\mathscr{B}(K_1, K_2)$, the space of bounded linear operators from K_1 to K_2, then its adjoint $A^*\colon K_2 \to K_1$ is defined by

$$\langle A(k_1), k_2 \rangle_2 = \langle k_1, A^*(k_2) \rangle_1.$$

If $g = (g_{ij})$ as in notation 2.1.2, then we have for its adjoint g^* the decomposition $(g^*)_{ij} = (g_{ji})^*$, for all i and j.

2.1.4. To the decomposition (1) we associate the basic flag $F^{(0)}$ given by

$$0 \subset H_1 \subset \cdots \subset \bigoplus_{j=1}^{r} H_j \subset \cdots \subset H.$$

Now we consider in H flags $F = \{F(0), \ldots, F(m)\}$, that is to say chains of closed subspaces of H,

$$\{0\} = F(0) \subset F(1) \subset \cdots \subset F(m) = H,$$

that are of the same "size" as the basic flag $F^{(0)}$, i.e. for all i, $1 \leq i \leq m$,

$$\dim (F(i)/F(i-1)) = \dim (H_i).$$

To such a flag F is associated an orthogonal decomposition of H,

$$H = F_1 \oplus \cdots \oplus F_m, \quad \text{where} \quad F_i = F(i) \cap F(i-1)^{\perp}.$$

We will denote such a flag F by $F = \{F(0), \ldots, F(m)\}$ as well as $F = \{F_1, \ldots F_m\}$.

The class of flags one obtains in this way is still too wide and we will require that our flags do not differ too much from the basic flag. One can express this "nearness" in various ways. Our choice is a natural generalization of that used in [23] for the Grassmann manifold. However, a lot of the constructions given here for that case can be carried out with some minor adjustments also for other choices. We start by introducing notations for some spaces of compact operators that occur in the sequel.

Notation 2.1.5. If K_1 and K_2 are Hilbert spaces, then we denote the space of Hilbert-Schmidt operators from K_1 to K_2 by $\mathscr{HS}(K_1, K_2)$ and the Hilbert-Schmidt norm by $\|\cdot\|_{\mathscr{HS}}$. We will write $\mathscr{N}(K_1, K_2)$ for the space of nuclear operators from K_1 to K_2 and the trace norm on it will be denoted by $\|\cdot\|_{\mathscr{N}}$. The space $\mathscr{C}(K_1, K_2)$ of compact operators from K_1 to K_2 will be assumed to have been equipped with the operator norm. Then we have the following chain of continuous inclusions:

$$\mathscr{N}(K_1, K_2) \subset \mathscr{HS}(K_1, K_2) \subset \mathscr{C}(K_1, K_2).$$

In each of these spaces the collection of finite dimensional operators $\mathscr{F}(K_1, K_2)$ lies dense. If K_2 is equal to K_1, then we simply write $\mathscr{F}(K_1)$, $\mathscr{N}(K_1)$, $\mathscr{HS}(K_1)$ and $\mathscr{C}(K_1)$ for respectively $\mathscr{F}(K_1, K_1)$, $\mathscr{N}(K_1, K_1)$, $\mathscr{HS}(K_1, K_1)$ and $\mathscr{C}(K_1, K_1)$.

Definition 2.1.6. Let $\mathfrak{F}$ be the collection of flags $F = \{F_1, \dots, F_m\}$, satisfying $\dim(F_i) = \dim(H_i)$, and for all i and j with $j \neq i$, the orthogonal projection $p_j \colon F_i \to H_j$ is a Hilbert-Schmidt operator. We call $\mathfrak{F}$ the *flag variety* corresponding to the decomposition (1).

Remark 2.1.7. If only one m_i is infinite, then the Hilbert-Schmidt condition is superfluous. E.g. the space of flags with $m_i < \infty$ for all $i < m$, plays a role in [2] at the construction of irreducible representations of the Hilbert Lie group $U(\mathfrak{H})_2$. This is the unitary part of the group of invertible transformations of the form "identity + a Hilbert-Schmidt operator".

Remark 2.1.8. Instead of the condition $p_j \colon F_i \to H_j$, $i \neq j$, belongs to $\mathscr{HS}(F_i, H_j)$, one could also consider flags such that this map belongs to $\mathscr{N}(F_i, H_j)$ or $\mathscr{C}(F_i, H_j)$. The flag varieties one obtains in this way we denote by $\mathfrak{F}(\mathscr{N})$ respectively $\mathfrak{F}(\mathscr{C})$. A more asymmetric condition is considered in [14] where it is required that merely for $i < j$ the projection $p_i \colon F_j \to H_i$ is Hilbert-Schmidt. In this way we get the flag manifold $\mathfrak{F}(\mathscr{B})$. Because of the inclusions mentioned above, we have a chain of injections

$$\mathfrak{F}(\mathscr{N}) \subset \mathfrak{F} \subset \mathfrak{F}(\mathscr{C}) \subset \mathfrak{F}(\mathscr{B}).$$

For $m = 2$ more general versions of flag spaces are considered in [9].

Remark 2.1.9. In [8], they associate a Banach Grassmann manifold to each Banach Jordan pair. It would be interesting to see if, and if so, how the flag varieties introduced here fit into their framework.

2.1.10. The space $\mathfrak{F}$ is a natural generalization of the Grassmann manifold introduced in section 7.1 of [23]. The flag variety $\mathfrak{F}$ is a homogeneous space for an analogue adapted to this situation, of the general linear group. The Banach structure of this group follows directly from that of its Lie algebra. Therefore we start with the analogue of the Lie algebra of the general linear group.

Definition 2.1.11. A *restricted endomorphism* of H is a $u = (u_{ij})$ in $\mathscr{B}(H)$ such that u_{ij} is a Hilbert-Schmidt operator for all $i \neq j$. We denote the space of all restricted endomorphisms of H by $\mathscr{B}_{\text{res}}(H)$.

For all i and j, we extend the elements of $\mathscr{HS}(H_i, H_j)$ outside H_i by zero and obtain thus a natural embedding of $\mathscr{HS}(H_i, H_j)$ into $\mathscr{HS}(H)$. The space $\mathscr{B}_{\text{res}}(H)$ is a subalgebra of $\mathscr{B}(H)$ since the collection of Hilbert-Schmidt operators is a 2-sided ideal in $\mathscr{B}(H)$. Hence it is also a Lie subalgebra of the Lie algebra $\mathscr{B}(H)$. The algebra $\mathscr{B}_{\text{res}}(H)$ becomes a Banach algebra if we equip it with the norm $\|\cdot\|_2$ defined by

$$\|u\|_2 = \|u\| + \sum_{i \neq j} \|u_{ij}\|_{\mathscr{HS}}.$$

Since the adjoint of a Hilbert-Schmidt operator is again Hilbert-Schmidt, it is clear that $\mathscr{B}_{\text{res}}(H)$ is stable under "taking adjoints". If $GL(H)$ denotes the group of invertible elements in $\mathscr{B}(H)$, then we consider

Definition 2.1.12. The *restricted linear group*, $GL_{\text{res}}(H)$, consists of $\{g \mid g \in GL(H) \cap \mathscr{B}_{\text{res}}(H)\}$.

To see that $GL_{\text{res}}(H)$ is indeed a group, one merely has to show that if $g = (g_{ij})$ belongs to $GL_{\text{res}}(H)$ then its inverse $g^{-1} = ((g^{-1})_{ij})$ also belongs to $GL_{\text{res}}(H)$. Now, the relation

$$g_{ii}(g^{-1})_{ii} = \text{Id}_{H_i} - \sum_{j \neq i} g_{ij}(g^{-1})_{ji},$$

shows first of all that for all i, $1 \leq i \leq m$, both g_{ii} and $(g^{-1})_{ii}$ are Fredholm operators, i.e. they have a finite dimensional kernel and cokernel. Next one considers the relation

$$g_{21}(g^{-1})_{11} + g_{22}(g^{-1})_{21} + \sum_{j>2} g_{2j}(g^{-1})_{j1} = 0.$$

Since the operator g_{21} is Hilbert-Schmidt and the operators $(g^{-1})_{11}$ and g_{22} are Fredholm, the operator $(g^{-1})_{21}$ has to be Hilbert-Schmidt too. Continuing in this fashion, one shows that all $(g^{-1})_{ij}$ with $i \neq j$ are Hilbert-Schmidt. In other words, $GL_{\text{res}}(H)$ consists of the invertible elements of $B_{\text{res}}(H)$. As such, it is in a natural way a Banach Lie group with Lie algebra $B_{\text{res}}(H)$.

The analogue of the unitary group $U(H)$ in this context is:

Definition 2.1.13. The *restricted unitary group*, $U_{\mathrm{res}}(H) = GL_{\mathrm{res}}(H) \cap U(H)$.

Both $U_{\mathrm{res}}(H)$ and $GL_{\mathrm{res}}(H)$ are natural generalizations of the restricted unitary and general linear group, introduced in chapter 6 of [23]. The Lie algebra of $U_{\mathrm{res}}(H)$ consists of

$$\mathfrak{u}_{\mathrm{res}}(H) = \{X \mid X \in \mathfrak{B}_{\mathrm{res}}(H),\ X^* = -X\}.$$

This is a real Lie subalgebra of $\mathfrak{B}_{\mathrm{res}}(H)$ and the Lie algebra $\mathfrak{B}_{\mathrm{res}}(H)$ can be written as

$$\mathfrak{B}_{\mathrm{res}}(H) = \mathfrak{u}_{\mathrm{res}}(H) \oplus i.\mathfrak{u}_{\mathrm{res}}(H).$$

In other words $\mathfrak{B}_{\mathrm{res}}(H)$ is the complexification of $\mathfrak{u}_{\mathrm{res}}(H)$. On the group level this corresponds to the fact that the group $GL_{\mathrm{res}}(H)$ possesses a "polar decomposition" of which $U_{\mathrm{res}}(H)$ forms the unitary component. For, consider the sets

$$P(H) = \{A \mid A \in GL(H),\ A = A^* \text{ and } A > 0\} \text{ and}$$
$$P_{\mathrm{res}}(H) = \mathfrak{B}_{\mathrm{res}}(H) \cap P(H).$$

On $P_{\mathrm{res}}(H)$ we put the topology induced by $\mathfrak{B}_{\mathrm{res}}(H)$. Since the map $A \mapsto \sqrt{A}$ from $P_{\mathrm{res}}(H)$ to $P(H)$ is locally given by a convergent power series in A, this map is in fact a continuous map from $P_{\mathrm{res}}(H)$ to itself. Thus we get

Proposition 2.1.14. *The map $(u, p) \mapsto up$ from $U_{\mathrm{res}}(H) \times P_{\mathrm{res}}(H)$ to $GL_{\mathrm{res}}(H)$ is a homeomorphism.*

Proof. The inverse of this map is

$$g \longmapsto (g(\sqrt{g^*g})^{-1}, \sqrt{g^*g})$$

and we have just seen that it is continuous. □

With each g in $GL_{\mathrm{res}}(H)$ we can associate the flag

$$0 \subset gH_1 \subset g(H_1 \oplus H_2) \subset \cdots g(H_1 \oplus \cdots \oplus H_i) \subset \cdots \subset H.$$

From the definition of $GL_{\mathrm{res}}(H)$ one sees directly that this flag belongs to $\mathfrak{F}$.

The group $U_{\mathrm{res}}(H)$ acts already transitively on $\mathfrak{F}$. Let $F = \{F_1, \ldots, F_m\}$ belong to $\mathfrak{F}$. From the definition of $\mathfrak{F}$ we know that there is for each i, $1 \leq i \leq m$, an isometry u_i between H_i and F_i. If we put $u = u_1 \oplus \cdots \oplus u_m$, then the condition defining $\mathfrak{F}$ implies that u belongs to the group $U_{\mathrm{res}}(H)$ and that $F = u(F^{(0)})$.

The stabilizer in $GL_{\mathrm{res}}(H)$ of the basic flag is the "parabolic subgroup"

$$P = \left\{ g \mid g \in GL_{\text{res}}(H),\ g = \begin{pmatrix} g_{11} & \cdots & \cdots & g_{1m} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & g_{mm} \end{pmatrix}, \text{ with } g_{ii} \in GL_{\text{res}}(H_i),\ 1 \le i \le m \right\}$$

Thus we can identify $\mathfrak{F}$ also with the homogeneous space $GL_{\text{res}}(H)/P$. Let $\tau\colon GL_{\text{res}}(H) \to \mathfrak{F}$ be the projection $\tau(g) = g \cdot F^{(0)}$. On $\mathfrak{F}$ we will put a Hilbert manifold structure that makes τ into an open submersion. This will be discussed in the next subsection.

Remark 2.1.15. It will be clear that for the spaces $\mathfrak{F}(\mathcal{N})$, $\mathfrak{F}(\mathcal{C})$ and $\mathfrak{F}(\mathcal{B})$ the corresponding general linear group consists of those $g = (g_{ij})$ in $GL(H)$ such that respectively

(2) $$g_{ij} \in \mathcal{N}(H_j, H_i) \quad \text{for all } i \neq j,$$

(3) $$g_{ij} \in \mathcal{C}(H_j, H_i) \quad \text{for all } i \neq j,$$

(4) $$g_{ij} \in \mathcal{HS}(H_j, H_i) \quad \text{for all } i < j.$$

2.2. The manifold structure of $\mathfrak{F}$. In this subsection we discuss the Hilbert manifold structure on $\mathfrak{F}$ and some decompositions of open subsets in $GL_{\text{res}}(H)$. From the definition of the parabolic group P one sees directly that the Lie algebra of P is given by

$$L(P) = \{g \mid g = (g_{ij}) \in B_{\text{res}}(H),\ g_{ij} = 0 \text{ for all } i > j\}$$

and that a complement of $L(P)$ in $B_{\text{res}}(H)$ is the Hilbert space $(E, \|\cdot\|_{\mathcal{HS}})$ with

$$E = \bigoplus_{\substack{1 \le j \le m-1 \\ i > j}} \mathcal{HS}(H_j, H_i).$$

From section 6.1 in [3], we know then that the homogeneous space $\mathfrak{F} = GL_{\text{res}}(H)/P$ carries an analytic E-manifold structure for which τ is a submersion and for which the natural action of $GL_{\text{res}}(H)$ on $\mathfrak{F}$ is analytic.

Next we give descriptions of some open subsets in $GL_{\text{res}}(H)$ that will be needed later on. Consider for each k, $1 \le k \le m-1$, the set $\Omega(k)$ in $GL_{\text{res}}(H)$ given by

$$\Omega(k) = \left\{ g \in GL_{\text{res}}(H) \mid \begin{pmatrix} g_{11} & \cdots & g_{1i} \\ \vdots & & \vdots \\ g_{i1} & \cdots & g_{ii} \end{pmatrix} \in GL_{\text{res}}(H_1 \oplus \cdots \oplus H_i) \quad \text{for all } i \le k \right\}.$$

Since we have for each i, $1 \le i \le m$, a continuous surjection from $\mathcal{B}_{\text{res}}(H)$ onto $\mathcal{B}_{\text{res}}(H_1 \oplus \cdots \oplus H_i)$ given by

$$b \longmapsto \begin{pmatrix} b_{11} & \cdots & b_{1i} \\ \vdots & & \vdots \\ b_{i1} & \cdots & b_{ii} \end{pmatrix},$$

the set $\Omega(k)$ is open and, as in the finite dimensional case, it can be decomposed. For, let $U_-(k)$ and $P(k)$ be the Lie subgroups of $GL_{\text{res}}(H)$ defined by

$$U_-(k) = \left\{ g = (g_{ij}) \in GL_{\text{res}}(H) \;\middle|\; \begin{array}{ll} g_{ii} = \text{Id}_{H_i} & \text{for all } i \\ g_{ij} = 0 & \text{for } j > i \\ g_{ij} = 0 & \text{for } i > j \text{ and } j > k \end{array} \right\}$$

and

$$P(k) = \{g = (g_{ij}) \in GL_{\text{res}}(H) | g_{ij} = 0 \text{ if } i > j \text{ and } j \leq k\}.$$

Clearly $P(k) \cap U_-(k) = \{\text{Id}_H\}$ and this gives you the uniqueness in

Lemma 2.2.1. *The map $(u, p) \mapsto up$ from $U_-(k) \times P(k) \to GL_{\text{res}}(H)$ determines a homeomorphism between $U_-(k) \times P(k)$ and $\Omega(k)$.*

Proof. We use induction on k to show the result. Let g be an element in $\Omega(1)$. Then we know that g_{11} is invertible and if we define $u(1) \in U_-(1)$ by $u(1)_{r1} = -g_{r1}g_{11}^{-1}$ for all $r \geq 2$, then one sees directly that $u(1)g$ belong to $P(1)$. Assume now that we know $\Omega(l) = U_-(l)P(l)$. Since we have $\Omega(l) \supset \Omega(l+1)$ and $U_-(l) < U_-(l+1)$, we may assume that $g \in \Omega(l+1)$ belongs to $P(l)$. Hence the condition $g \in \Omega(l+1)$ means that $g_{l+1,l+1}$ is invertible. Define $u(l+1) = (u_{ij})$ in $U_-(l+1)$ by $u_{jl+1} = -g_{jl+1}(g_{l+1,l+1})^{-1}$ for $j > l+1$ and $u_{ij} = 0$ if $i > j$ and $j \neq l+1$. Then $u(l+1) \cdot g$ belongs to the parabolic group $P(l+1)$. This proves the lemma. □

As in the finite dimensional case we call $\Omega(m-1) = U_-(m-1) \cdot P$ the big cell of $\mathfrak{F}$ and we also write Ω and U_- instead of $\Omega(m-1)$ and $U_-(m-1)$.

From this lemma we see that the restriction of τ to U_- gives you a diffeomorphism $u \mapsto uF^{(0)}$ between U_- and the open neighborhood $\tau(\Omega)$ of $F^{(0)}$. Clearly the group U_- is diffeomorphic to the Hilbert space E. Note that from the definition of Ω one can conclude directly that

$$\tau(\Omega) = \{F = (F_i) \in \mathfrak{F} | \bigoplus_{j \leq l} p_j : \bigoplus_{j \leq l} F_j \longrightarrow \bigoplus_{j \leq l} H_j \text{ is a bijection for all } l \leq m\}.$$

This characterization of $\tau(\Omega)$ tells you how to choose around a general point of $\mathfrak{F}$ a concrete neighborhood diffeomorphic to E. This requires, however, the introduction of the following notation.

Notation 2.2.2. If W is closed subspace of H, then we denote the orthogonal projection on W by p_W.

Consider a $F = (F_1, \ldots, F_m)$ in $\mathfrak{F}$. Then the analogue of $\tau(\Omega)$ for F is

$$U_F = \{V = (V_i) \text{ in } \mathfrak{F} \mid \bigoplus_{i \le l} p_{F_i} : \bigoplus_{i \le l} V_i \longrightarrow \bigoplus_{i \le l} F_i \text{ is a bijection for all } 1 \le l \le m\}.$$

Since $\mathfrak{F} = U_{\text{res}}(H) \cdot F^{(0)}$, we have for all F and G in $\mathfrak{F}$ that, if $i \neq j$, the map $p_{F_i} : G_j \to F_i$ is a Hilbert-Schmidt operator. Hence, if V belongs to U_F, then there is a unique operator A in $\bigoplus_{\substack{1 \le j \le m-1 \\ i > j}} \mathscr{H}\mathscr{S}(F_j, F_i)$ such that for all i, $1 \le i \le m$,

$$V(i) = \{x + A(x) \mid x \in F(i)\}.$$

This is why we call V also the graph of A and we write $V = \text{graph}(A)$.

It is convenient to have a special name for the flags in U_F.

Definition 2.2.3. A flag V in U_F is called *transversal* to F.

Let g_F be an element of $U_{\text{res}}(H)$ such that $g_F \cdot F^{(0)} = F$. Instead of the big cell Ω in $GL_{\text{res}}(H)$ with respect to the decomposition $H = H_1 \oplus \cdots \oplus H_m$, we could also have introduced a big cell with respect to $H = F_1 \oplus \cdots \oplus F_m$ and it will be clear that this set can be written as

$$g_F U_- P(g_F^{-1}).$$

Consequently, we get for U_F that

$$U_F = \{g_F u p (g_F)^{-1} F \mid \text{with } u \in U_- \text{ and } p \in P\} = \tau(g_F U_- P).$$

Then we can define for each F in $\mathfrak{F}$ a diffeomorphism $\varphi_F : U_F \to E$ by

$$\varphi_F(g_F u F^{(0)}) = u - \text{Id}.$$

Each (U_F, φ_F) is a concrete chart around F for the E-manifold structure on $\mathfrak{F}$.

We have obtained now a concrete description of the manifold structure on $\mathfrak{F}$:

Proposition 2.2.4. *The (U_F, φ_F) are the charts of the analytic E-manifold structure on $\mathfrak{F}$.*

Proof. It is sufficient to show for each $U_{F(1)}$ and $U_{F(2)}$ with $U_{F(1)} \cap U_{F(2)} \neq \emptyset$ that

$$\varphi_{F(2)} \circ \varphi_{F(1)}^{-1} : \varphi_{F(1)}(U_{F(1)} \cap U_{F(2)}) \longrightarrow \varphi_{F(2)}(U_{F(1)} \cap U_{F(2)})$$

is an analytic map. From the step by step decomposition described in Lemma 2.2.1 follows that the U_--component of $(g_{F(2)})^{-1} g_{F(1)} u$ actually depends analytically on u. This proves the proposition. $\square$

2.3. The connected components of $GL_{\mathrm{res}}(H)$. Let $g = (g_{ij})$ be an element of $GL_{\mathrm{res}}(H)$. Recall that in the proof that $GL_{\mathrm{res}}(H)$ consists of the invertible elements in $\mathscr{B}_{\mathrm{res}}(H)$, we have shown that each g_{ii} is a Fredholm operator. The collection of Fredholm operators on a Hilbert space K is an open part of the space $\mathscr{B}(K)$. Its connected components are given by the index, which is defined as

$$\operatorname{ind}(B) = \dim(\ker(B)) - \dim(\operatorname{coker}(B)),$$

where B is a Fredholm operator on K. Since all off-diagonal operators are Hilbert-Schmidt and hence compact, the operator

$$\tilde{g} = \begin{pmatrix} g_{11} & & 0 \\ & \ddots & \\ 0 & & g_{mm} \end{pmatrix}, \text{ where } g = (g_{ij}) \in GL_{\mathrm{res}}(H),$$

is a Fredholm operator of index zero. Hence we have that the indices of the $\{g_{ii} | 1 \le i \le m\}$ satisfy

$$\sum_{i=1}^{m} \operatorname{ind}(g_{ii}) = 0 \text{ and } \operatorname{ind}(g_{kk}) = 0 \text{ if } m_k < \infty.$$

These relations lead to the introduction of the subgroup Z of $\mathbf{Z}^m$ defined by

$$Z = \{z = (z_i) \in \mathbf{Z}^m \mid \sum_{i=1}^{m} z_i = 0,\ z_k = 0 \text{ if } m_k < \infty\}.$$

The standard properties of the index imply that the map $i\colon GL_{\mathrm{res}}(H) \to Z$,

$$g \longmapsto (\operatorname{ind}(g_{11}), \ldots, \operatorname{ind}(g_{mm})),$$

is a continuous group homomorphism. Hence the sets

$$GL_{\mathrm{res}}^{(z)}(H) = \{g \mid g \in GL_{\mathrm{res}}(H),\ i(g) = z\}, \text{ with } z \in Z,$$

are open. In fact, they are exactly the connected components of $GL_{\mathrm{res}}(H)$, for

Proposition 2.3.1. *For each $z \in Z$, the set $GL_{\mathrm{res}}^{(z)}(H)$ is non-empty and connected.*

Proof. Let $z = (z_i)$ be in Z and let $h_i \in \Phi(H_i)$ be such that $\operatorname{ind}(h_i) = z_i$. Then

$$h = \begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_m \end{pmatrix}$$

belongs to $\Phi(H)$ and has index zero. Therefore one can add to h an isomorphism between the kernel of h and the orthogonal complement of the image of h to obtain an element of $GL_{\text{res}}^{(z)}(H)$. This shows that i is surjective. As for the connectedness it suffices to show that $GL_{\text{res}}^{(0)}(H)$ is connected. First one notes that, since P is homeomorphic to

$$\prod_{i=1}^{m} GL(H_i) \times \prod_{j<i} \mathscr{H}\mathscr{S}(H_j, H_i)$$

and all the $GL(H_i)$ are connected, (see [20]), the group P is connected. Next we show that each element of $GL_{\text{res}}^{(0)}(H)$ can be joined by a continuous path to an element of P. For an element $g = up$ in $\Omega = U_-P$ it is clear how to proceed: the map $t \mapsto \{\text{Id} + (1-t)(u - \text{Id})\}p$ joins g with p. A general element g is first joined with an element in $\Omega(1)$. For, if g_{11} is not invertible, then there is a bijection E between $\ker(g_{11})$ and $\mathfrak{I}(g_{11})^{\perp} \cap H_1$ and we extend E by zero on $\ker(g_{11})^{\perp}$ to get an element E of $\mathscr{B}_{\text{res}}(H)$. It is no restriction to assume $\|E\|_2 < \|g\|_2$. Then we know that $g + tE$ belongs to $GL_{\text{res}}^{(0)}(H)$ for all $t \in [0, 1]$ and by construction $g + E$ belongs to $\Omega(1)$ and we can write $g + E = u_1 p_1$. The map $t \mapsto \{\text{Id} + (1-t)(u_1 - \text{Id})\}p_1$ joins $g + E$ with p_1. By adding a small finite dimensional operator in $\mathscr{B}(H_2)$, one reduces the case to an element in $\Omega(2)$ that can be linked in the same way to an element of $P(2)$. Continuing in this fashion one finds a continuous path from g to an element of P. This proves the assertion. □

This Proposition is the extension to flag varieties of Proposition 6.2.4 in [23].

Since the parabolic group P is connected, we see that

Corollary 2.3.2. *The connected components of* $\mathfrak{F}$ *are given by*

$$\mathfrak{F}^{(z)} = \{g \cdot F^{(0)} \mid g \in GL_{\text{res}}^{(z)}(H)\}.$$

Remark 2.3.3. A holomorphic line bundle L over $\mathfrak{F}$ consists simply of a collection of holomorphic line bundles $\{L_z \to \mathfrak{F}^{(z)} \mid z \in Z\}$. Therefore we restrict our attention to holomorphic line bundles over $\mathfrak{F}^{(0)}$ in the third section.

2.4. A special covering of $\mathfrak{F}$. In this subsection we choose a suitable orthonormal basis of H and we introduce a collection of charts of $\mathfrak{F}$ that can be described completely in terms of the index set of this orthonormal basis. In particular these charts cover $\mathfrak{F}$ and enable you to give a combinatorial description of the Birkhoff decomposition of $GL_{\text{res}}(H)$ and to construct concretely a collection of holomorphic line bundles over $\mathfrak{F}^{(0)}$.

Let $\{e_s \mid s \in S_i\}$, $1 \le i \le m$, be an orthonormal basis of H_i. Recall that $\dim(H_i) = m_i$ for all i, $1 \le i \le m$. Hence we can write

$$S_i = \{s_i(k) \mid 1 \le k < m_i + 1\}$$

On S_i we define a total order by

$$s_i(k) \geq s_i(l) \Longleftrightarrow k \geq l.$$

These orders we compose to a total order on the index set $S = \bigcup_{i=1}^{m} S_i$ by requiring that

$$s_j < s_i \text{ for all } s_i \in S_i \text{ and all } s_j \in S_j \text{ with } j > i.$$

Now that we have an orthonormal basis $\{e_s | s \in S\}$ of H with a totally ordered index set S, we can associate to each bounded g in $\mathscr{B}(H)$ an $S \times S$-matrix $[g] = (g_{st})$ with matrix coefficients

$$g_{st} = \langle g(e_t), e_s \rangle, \text{ where } s \text{ and } t \text{ are in } S.$$

Notation 2.4.1. Let $\mathfrak{gl}(S)$ be the collection of $S \times S$-matrices corresponding to operators in $\mathscr{B}_{\text{res}}(H)$.

The context has been chosen such that the product of two elements in $\mathfrak{gl}(S)$ is again in $\mathfrak{gl}(S)$ and therefore $\mathfrak{gl}(S)$ is a Lie algebra. In $\mathfrak{gl}(S)$ we have the Lie subalgebra $\mathfrak{gl}(\infty)$ corresponding to the matrices of the operators

Definition 2.4.2. An operator g in $\mathscr{B}(H)$ is called a "finite-size" operator if it has only a finite number of non-zero matrix coefficients w.r.t. the $\{e_s | s \in S\}$.

Remark 2.4.3. If $m = 2$ and $m_1 = m_2 = \infty$, then $S = \mathbf{Z}$. In [18] it was shown that the Lie algebra A_∞ can be realized as a central extension of the collection $\overline{\mathfrak{gl}(\infty)}$ of $\mathbf{Z} \times \mathbf{Z}$-matrices $g = (g_{ij})$ of "finite-width", i.e. satisfying $g_{ij} = 0$ if $|i - j| > N$ for some N. The composition of such matrices is always defined and from this point of view $\mathfrak{gl}(S)$ can be seen as a complete bounded version of $\overline{\mathfrak{gl}(\infty)}$. The central extension defining A_∞ also occurs naturally in our geometric framework, see subsection 4 of the next section.

In the sequel we will frequently use some notations related to subsets of S.

Notation 2.4.4. The number of elements in a subset A of S is denoted by $\# A$.

Notation 2.4.5. If A is a non-empty subset of S, then we denote the closure of the span of the $\{e_s | s \in A\}$ by H_A. If A is empty, then H_A denotes the space $\{0\}$. It is convenient to denote the orthogonal projection onto H_A by p_A.

Maps between subsets of S have a direct translation to partial isometries between closed subspaces of H, i.e.

Notation 2.4.6. If A and B are subsets of S and $\tau: A \to B$ is a map with uniformly bounded finite fibers, then we denote by $\underline{\tau}$ the mapping from H_A to H_B given by

$$\underline{\tau}\,(\sum_{t\in A}\lambda_t e_t)=\sum_{t\in A}\lambda_t e_{\tau(t)}.$$

Now that we have chosen the orthonormal basis $\{e_s | s\in S\}$, we can introduce "diagonal operators" in $\mathscr{B}_{\mathrm{res}}(H)$. Suppose that we have a set of bounded complex numbers

$$\{\delta_s | s\in S,\ \delta_s\in \mathbf{C} \text{ and } |\delta_s|\le M \text{ for all } s\in S\}.$$

Then we can associate to it a diagonal operator $\operatorname{diag}(\delta_s)$ in $\mathscr{B}(H)$ by

$$\operatorname{diag}(\delta_s)(\sum_{t\in S}\lambda_t e_t)=\sum_{t\in S}\delta_t\lambda_t e_t.$$

Inside $GL_{\mathrm{res}}(H)$ we have then the "maximal torus"

$$T=\{g | g\in GL_{\mathrm{res}}(H),\ g=\operatorname{diag}(\delta_s)\}.$$

Clearly T is commutative and it is a straightforward verification to show that the centralizer of T inside $GL(H)$ is equal to T. Hence we have

Lemma 2.4.7. *The centralizer $Z(T)$ of T in $GL_{\mathrm{res}}(H)$ is equal to T.*

Each permutation σ of S determines a unitary map $\underline{\sigma}: H\to H$ as in notation 2.4.6. With the help of the matrix, one shows that the normalizer of T in $GL(H)$ consists of

$$\{t\cdot\underline{\sigma} | t\in T,\ \sigma \text{ a permutation of } S\}.$$

Hence, if we define the subgroup W of $U_{\mathrm{res}}(H)$ as

$$W=\{\underline{\sigma} | \underline{\sigma}\in U_{\mathrm{res}}(H),\ \sigma \text{ a permutation of } S\},$$

then we have

Corollary 2.4.8. *The normalizer $N(T)$ of T in $GL_{\mathrm{res}}(H)$ is the semi-direct product of W and T. In particular, we see that W is isomorphic to $N(T)/Z(T)$ and we call W the Weyl group of T.*

To each $\underline{\sigma}$ in W, corresponds a partition $\Sigma=\bigcup_{i\ge 1}\Sigma_i$ of S, where $\Sigma_i=\sigma(S_i)$. The concrete description of which partitions occur in this way, brings one in a natural way to the consideration of subsets of S that are "equal up to a finite set". Therefore we define

Definition 2.4.9. If A and B are subsets of S, then we call A and B *commensurable* (notation $A\approx B$) if $A-\{A\cap B\}$ and $B-\{A\cap B\}$ are finite. We write $i(A, B)$ for the number

$$\#\{A - \{A \cap B\}\} - \#\{B - \{A \cap B\}\}.$$

Thus commensurability is equivalent to: the orthogonal projection p_{H_B}: $H_A \to H_B$ is a Fredholm operator with index $i(A, B)$. Let $\Sigma = \{\Sigma_i | 1 \leq i \leq m\}$ be an arbitrary partition of S into m disjoint parts. Then this partition corresponds to an element of W, if and only if the following two conditions hold:

(5) $$\#\Sigma_i = \#S_i \qquad \text{for all } i,\ 1 \leq i \leq m, \text{ and}$$

(6) $$\Sigma_i \approx S_i \qquad \text{for all } i,\ 1 \leq i \leq m$$

To any partition Σ satisfying these conditions there corresponds a flag F_Σ in $\mathfrak{F}$ given by

$$0 \subset H_{\Sigma_1} \subset H_{\Sigma_1 \cup \Sigma_2} \subset \cdots \subset H_S = H.$$

For simplicity we denote for each Σ satisfying (5) and (6) the open set U_{F_Σ} in $\mathfrak{F}$ also by U_Σ.

In the sequel we will make use of the following notion

Definition 2.4.10. An element in H is said to be of *order* s, $s \in S$, if it has the form

$$h = a_s e_s + \sum_{\substack{t \in S \\ t < s}} a_t e_t, \text{ with } a_s \neq 0,$$

Notation 2.4.11. If W is a subspace of H then the union of all the elements in W of some order s in S and $\{0\}$ is called the space of elements of *finite order* in W and is denoted by W_{fin}.

For each z in Z, we denote the collection of partitions Σ of the index set S such that F_Σ belongs to $\mathfrak{F}^{(z)}$, by $\mathscr{S}(z)$. The basic property of the $\{F_\Sigma | \Sigma \in \mathscr{S}(z)\}$ is

Proposition 2.4.12. *For each flag $F = (F(1), \ldots, F(m))$ in $\mathfrak{F}^{(z)}$ there is a Σ in $\mathscr{S}(z)$ such that F is transversal to F_Σ.*

Proof. Let $g \in GL_{\text{res}}(H)$ be such that $F = g \cdot F^{(0)}$. First we show that there is a Σ_1 commensurable with S_1 and with $\#S_1 = \#\Sigma_1$, such that $p_{H_{\Sigma_1}} \circ g|_{H_1}$ is an isomorphism between H_1 and H_{Σ_1}. Since $p_1(g(H_1))$ has finite codimension in H_1, we can find a $S_1(n) = \{s_1(k),\ k \geq n\}$, $n \geq 0$, such that $F(1) = g(H_1)$ projects surjectively onto $H_{S_1(n)}$. The kernel of this projection has a basis $\{h_j | 1 \leq j \leq N\}$ of elements of finite order, i.e.

$$h_j = e_{s_j} + \sum_{\substack{t \in S \\ t < s_j}} \alpha_j(t) e_t, \text{ where } s_i \neq s_j \text{ for } i \neq j.$$

It is clear that we can take $\Sigma_1 = S_1(n) \cup \{s_j | 1 \leq j \leq N\}$. The other parts of the desired partition Σ of S are constructed step by step from Σ_1. For, assume that we have found disjoint $\{\Sigma_j | j \leq i\}$ with $\Sigma_j \approx S_j$ and $\#S_j = \#\Sigma_j$ such that the orthogonal projection of $F(j)$ onto $\bigotimes_{\ell \leq j} H_{\Sigma_\ell}$ is a bijection for all $j \leq i$. Then we know that $p_{i+1}(g(H_{i+1}))$ has finite codimension in H_{i+1}. So there exists a subset $\tilde{S}_{i+1}$ of S_{i+1}, commensurable with S_{i+1} and disjoint of $\Sigma_1 \cup \cdots \Sigma_i$, such that $F(i+1)$ projects surjectively onto $H_{\Sigma_1} \oplus \cdots \oplus H_{\Sigma_i} \oplus H_{\tilde{S}_{i+1}}$. The kernel of this projection is again finite dimensional and has a basis of elements of different order. As Σ_{i+1} one takes then the union of $\tilde{S}_{i+1}$ and the orders of the elements in this basis. In this way we obtain after a finite number of steps the desired partition Σ of S. $\square$

This proposition is a generalization of Proposition 7.16 in [23].

Remark 2.4.13. Since F_Σ is transversal to F_Π if and only if $\Sigma = \Pi$, we can conclude from this proposition directly that $\mathfrak{F}^{(0)}$ is no longer compact if H is infinite dimensional.

For each Σ in $\mathscr{S} = \bigcup_{z \in Z} \mathscr{S}(z)$, the elements of finite order in $F_\Sigma(j)$ span a dense subspace of $F_\Sigma(j)$ for all j, $1 \leq j \leq m$. By combining this with Proposition 2.4.12 we get the following generalization of proposition 7.3.2 in [23].

Corollary 2.4.14. *For each flag F in $\mathfrak{F}$ and for each j, $1 \leq j \leq m$, the space $F(j)_{fin}$ forms a dense subspace of $F(j)$.*

For each flag $F = (F(0), \ldots, F(m))$ in $\mathfrak{F}^{(z)}$ we can concretely describe a $\Sigma(F)$ in $\mathscr{S}(z)$ such that F is transversal to $F_{\Sigma(F)}$. Namely for each $1 \leq i \leq m$ we put

$$\Sigma(F)(i) = \{s | s \in S,\ F(i) \text{ contains an element of order } s\}$$

and

$$\Sigma(F)_i = \Sigma(F)(i) - \Sigma(F)(i-1) \text{ for } i > 1.$$

Clearly each $F(i)$ projects bijectively onto $H_{\Sigma(F)(i)}$ and therefore $\Sigma(F) = \{\Sigma(F)_i\}$ belongs to $\mathscr{S}(z)$. Next we consider flags that give the same partition in this way.

If $\Sigma \in \mathscr{S}(z)$, then we write

$$\mathfrak{F}_\Sigma = \{F | F \in \mathfrak{F},\ \Sigma(F) = \Sigma\}.$$

Let U_0 be the subgroup of $GL_{\text{res}}(H)$ of all operators with a unipotent lower triangular matrix, i.e.

$$U_0 = \{u | u \in GL_{\text{res}}(H), \text{ for all } s \in S,\ u(e_s) = e_s + \sum_{t<s} u_{ts} e_t\}.$$

Then we want to show

Proposition 2.4.15. *The subset* $\mathfrak{F}_\Sigma$ *is exactly the* U_0*-orbit through* F_Σ.

Proof. From the form of an operator u in U_0, one sees directly that for each element h of order s, the element $u(h)$ has also order s. Thus we have that $\Sigma(uF) = \Sigma(F)$ for each F in Σ.

Assume now that F belongs to $\mathfrak{F}_\Sigma$. Since $F(i)$ projects bijectively onto $F_\Sigma(i)$ for all i, $1 \leq i \leq m$, the flag F is the graph of an operator T in

$$\bigoplus_{j=1}^{m-1} \bigoplus_{j<i} \mathscr{H}\mathscr{S}(H_{\Sigma_j}, H_{\Sigma_i}).$$

In particular this means that there is an u in $GL_{\text{res}}(H)$ such that $u(F_\Sigma) = F$ and for all $s_i \in \Sigma_i$, $1 \leq i \leq m$,

$$u(e_{s_i}) = e_{s_i} + \sum_{j>i} \sum_{s_j \in \Sigma_j} T_{s_j s_i} e_{s_j}, \text{ if } i < m \text{ and } u(e_{s_i}) = e_{s_i} \text{ if } i = m. \tag{7}$$

The fact that F belongs to $\mathfrak{F}_\Sigma$ can be expressed completely in term of the coefficients $\{T_{s_j s_i} | s_i \in \Sigma_i, s_j \in \Sigma_j, j > i\}$. Namely, it is equivalent to

$$T_{s_j s_i} = 0, \text{ if } s_j > s_i. \tag{8}$$

If namely $T_{s_j s_i} \neq 0$ for some s_j with $s_j > s_i$, then the element $u(e_{s_i})$ will be of some order $s \notin \Sigma(i)$ and hence $\Sigma(F)(i) \neq \Sigma(i)$. This contradicts the fact that F belongs to $\mathfrak{F}_\Sigma$. By definition, the operator u defined by (7) belongs to U_0, if condition (8) is satisfied. This concludes the proof of the proposition. □

Thus we have obtained a subdivision of each connected component of $\mathfrak{F}$,

$$\mathfrak{F}^{(z)} = \bigcup_{\Sigma \in \mathscr{S}(z)} \mathfrak{F}_\Sigma,$$

into parts that are homeomorphic to a Hilbert space, thanks to property (8). This is a generalization to flag varieties of the stratification in section 7.3 of [23]. Let σ_Σ, for each $\Sigma \in \mathscr{S}(z)$, be a permutation of S such that $\underline{\sigma}_\Sigma(S_i) = \Sigma_i$, for all i, $1 \leq i \leq m$. Then this decomposition of $\mathfrak{F}$ translates directly to the group $GL_{\text{res}}(H)$ and results in

Proposition 2.4.16. (*Birkhoff decomposition.*) *Each connected component of the group* $GL_{\text{res}}(H)$ *decomposes as*

$$GL^{(z)}_{\text{res}} H = \bigcup_{\Sigma \in \mathscr{S}(z)} U_0 \, \underline{\sigma}_\Sigma P.$$

Remark 2.4.17. The decomposition derived here is the analytic equivalent of the algebraic decomposition from [22].

§3. Holomorphic Line Bundles over $\mathfrak{F}^{(0)}$

3.1. Another description of $\mathfrak{F}^{(0)}$. For each Σ in $\mathscr{S}(z)$, there are numerous $\underline{\sigma} \in W$ such that $F_\Sigma = \underline{\sigma}\, F^{(0)}$. We start by describing a special choice that is useful at the description of $\mathfrak{F}^{(0)}$ as the homogeneous space of another group.

We construct a bijection $\sigma_i\colon S_i \to \Sigma_i$, as follows: since S_i and Σ_i are commensurable, there is a $N \geq 0$ such that

$$\{s_i(k) | k > N\} \subset \Sigma_i \cap S_i.$$

Consider the finite set $\Sigma_i - \{s_i(k) | k > N\}$. If it is empty, then we define $\sigma_i\colon S_i \to \Sigma_i$ as follows:

$$\sigma_i(s_i(k)) = s_i(k + N) \qquad \text{for all } k,\ 1 \leq k < m_i + 1.$$

In this case we put $\ell_i = -N$.

If Σ_i is not equal to $\{s_i(k) | k > N\}$, then we write $\Sigma_i - \{s_i(k) | k > N\} = \{t_1, \dots t_{N+\ell_i}\}$ for some ℓ_i in $\mathbf{Z}$, $\ell_i > -N$, and we define σ_i by

$$\sigma_i(s_i(k)) = t_k \qquad \text{for all } k,\ 1 \leq k \leq N + \ell_i,$$

$$\sigma_i(s_i(k)) = s_i(k - \ell_i) \qquad \text{for all } k > N + \ell_i.$$

For all i, $1 \leq i \leq m$, consider the map $\tau_i\colon S_i \to S_i$ defined by

$$\tau_i(s_i(k)) = s_i(N + 1) \qquad \text{for all } k,\ 1 \leq k \leq N + \ell_i$$

$$\tau_i(s_i(k)) = s_i(k - \ell_i) \qquad \text{for all } k > N + \ell_i.$$

Since $p_i \circ \underline{\sigma}_i - \underline{\tau}_i$ is a finite dimensional operator, $p_i \circ \underline{\sigma}_i$ and $\underline{\tau}_i$ have the same index and for $\underline{\tau}_i$ one clearly has $\operatorname{ind}(\underline{\tau}_i) = N + \ell_i - N = \ell_i$. In other words, the number ℓ_i is equal to z_i. The $\{\sigma_i | 1 \leq i \leq m\}$ compose to a bijection $\sigma\colon S \to S$ such that $\underline{\sigma}\, F^{(0)} = F_\Sigma$. We will introduce a special term for this type of permutations.

Definition 3.1.1. Let Σ be a partition of S in $\mathscr{S}(z)$. A permutation σ of S such that $\sigma(S_i) = \Sigma_i$ is called *admissible* of level N if the following property holds

(i) For each i and for all $k > N + z_i$, $\sigma(s_i(k)) = s_i(k - z_i)$.

One easily verifies that the collection of admissible permutations of S of all levels forms a subgroup W_a of W. It has a normal subgroup $W_a^{(0)} = W_a \cap GL_{\text{res}}^{(0)}(H)$ such that the quotient $W_a / W_a^{(0)}$ is isomorphic to Z. The elements of $W_a^{(0)}$ can be described in a direct concrete way. If G is a finite subset of S and if ρ is a permutation of G, then we denote the extension of ρ by the identity to a permutation of S, by $\tilde{\rho}$. Then we have

$$W_a^{(0)} = \varinjlim_N W_a^{(0)}(N) = \varinjlim_N \{ \underline{\sigma} \mid \underline{\sigma} \in W_a^{(0)} \text{ is of level } N \}$$
$$= \{ \underline{\tilde{\rho}} \mid \rho \text{ a permutation of } G,\ G \text{ a finite subset of } S \}.$$

Now we can introduce another group that acts transitively on $\mathfrak{F}^{(0)}$. Its advantage is that it enables you to construct in a simple way holomorphic line bundles over $\mathfrak{F}^{(0)}$. Let Σ be in $\mathscr{S}(0)$ and let σ be an admissible permutation of S such that $\sigma(S_i) = \Sigma_i$. From the definition of admissibility we know that $\underline{\sigma}$ decomposes in operators $(\underline{\sigma}_{ij})$ with the properties

(i) For each $1 \leq i \leq m$, $\underline{\sigma}_{ii} = \mathrm{Id}_{H_i} +$ a "finite-size" operator.

(ii) For all i and j, $i \neq j$, $\underline{\sigma}_{ij}$ is a "finite-size" operator.

Since every flag F in $\mathfrak{F}^{(0)}$ is transversal to some F_Σ, with Σ in $\mathscr{S}(0)$, we may conclude that each F in $\mathfrak{F}^{(0)}$ is equal to $g \cdot F^{(0)}$ with $g \in GL_{\text{res}}(H)$ of the form

(a) For each i, $1 \leq i \leq m$, $g_{ii} = \mathrm{Id}_{H_i} +$ a "finite-size" operator.

(b) For all i and j, $i < j$, g_{ij} is a "finite-size" operator.

(c) For all i and $j, j < i$, g_{ij} belongs to $\mathscr{HS}(H_j, H_i)$.

Note that for all the operators g_{ii} from (a) $\det(g_{ii})$ is defined. Since we are working in an analytic setting we will consider a somewhat wider class of operators such that on one hand we work in a Banach framework and on the other we can take determinants of certain minors. Recall, see [12], that the determinant is defined for each operator of the form "identity $+ a$ nuclear operator". Therefore we introduce

$$B_2(H) = \left\{ g \mid g \in \mathscr{B}_{\text{res}}(H), \quad \begin{matrix} g_{ii} - \mathrm{Id}_{H_i} \in \mathscr{N}(H_i) \\ g_{ij} \in \mathscr{HS}(H_j, H_i) \text{ for } i \neq j \end{matrix} \right\}.$$

On $B_2(H)$ we put a different topology than the one induced by $\mathscr{B}_{\text{res}}(H)$. For, let $\mathscr{X}$ be the subspace of $\mathscr{B}_{\text{res}}(H)$ defined by

$$\mathscr{X} = \left\{ b \mid b \in \mathscr{B}_{\text{res}}(H), \quad \begin{matrix} b_{ij} \in \mathscr{HS}(H_j, H_i) \text{ for } i \neq j \\ b_{ii} \in \mathscr{N}(H_i) \end{matrix} \right\}.$$

Then $\mathscr{X}$ is a Banach space if we equip it with the norm $\|\cdot\|_{\mathscr{X}}$ given by

$$\|b\|_{\mathscr{X}} = \sum_{i \neq j} \|b_{ij}\|_{\mathscr{HS}} + \sum_{i=1}^{m} \|b_{ii}\|_1 .$$

The collection $B_2(H)$ is nothing but $\mathscr{X}$ shifted by the identity and we transfer the Banach structure on $\mathscr{X}$ to $B_2(H)$ by means of the map $g \mapsto g + \mathrm{Id}$. Since the product of two Hilbert-Schmidt operators is nuclear, one sees that $B_2(H)$ is closed under multiplication. Moreover the multiplication with an element of $B_2(H)$ is an analytic map from $B_2(H)$ to itself. In $B_2(H)$ we have the subgroup U_- and its "adjoint" the group

$$U_+ = \{u^* | u \in U_-\}.$$

Consider an element b in $B_2(H)$. Now we define $u = (u_{ij})$ in U_- and $v = (v_{ij})$ in U_+ by

$$u_{ii} = v_{ii} = \mathrm{Id}_{H_i},\ u_{ij} = -b_{ij} \text{ if } i > j,\ u_{ij} = 0 \text{ if } j > i,$$
$$v_{ij} = -b_{ij} \text{ if } i < j \text{ and } v_{ij} = 0 \text{ if } i > j.$$

A direct verification shows that ubv belongs to $\mathrm{Id} + \mathcal{N}(H)$. Since $B_2(H)$ is closed w.r.t. taking adjoints, we have

Lemma 3.1.2. *Every $b \in B_2(H)$ can be written in the form $b = u_1 b_1 v_1$ or $b = v_2 b_2 u_2$, where u_1 and u_2 belong to U_-, v_2 and v_1 belong to U_+ and b_1 and b_2 lie in* $\mathrm{Id} + \mathcal{N}(H)$.

The decompositions in lemma 3.1.2 are clearly not unique, but they suffice to define a determinant map $\det: B_2(H) \to \mathbf{C}$. Namely, for $b = u_1 b_1 v_1$ as in lemma 3.1.2, we put $\det(b) = \det(u_1 b_1 v_1) = \det(b_1)$.

To see that this is well-defined, we note first of all that for $u \in U_- \cap \{\mathrm{Id} + \mathcal{N}(H)\}$ and $v \in U_+ \cap \{\mathrm{Id} + \mathcal{N}(H)\}$ we have $\det(u) = \det(v) = 1$, since $u - \mathrm{Id}$ and $v - \mathrm{Id}$ have zero trace. Now, assume $b \in B_2(H)$ can be written as $b = u_1 b_1 v_1 = u_2 b_2 v_2$ with $b_i \in \mathrm{Id} + \mathcal{N}(H)$, $u_i \in U_-$ and $v_i \in U_+$. Then $b_2 = (u_2^{-1} u_1) b_1 (v_1 v_2^{-1})$ and, since both b_1 and b_2 belong to $\mathrm{Id} + \mathcal{N}(H)$, this implies that $u_2^{-1} u_1 \in U_- \cap \{\mathrm{Id} + \mathcal{N}(H)\}$ and $v_1 v_2^{-1} \in U_+ \cap \{\mathrm{Id} + \mathcal{N}(H)\}$. By the multiplicativity of the determinant on $\mathrm{Id} + \mathcal{N}(H)$, we get $\det(b_2) = \det(u_2^{-1} u_1)$ $\det(b_1) \det(v_1 v_2^{-1}) = \det(b_1)$.

Remark 3.1.3. Since the operators in $\mathrm{Id} + \mathcal{N}(H)$ lie dense in $B_2(H)$ and since det is multiplicative on $\mathrm{Id} + \mathcal{N}(H)$ we get that for each b_1 and b_2 in $B_2(H)$

$$\det(b_1 b_2) = \det(b_1) \det(b_2)$$

From the fact that an operator g of the form $\mathrm{Id} + \mathcal{N}(H)$ is invertible if and only if $\det(g)$ is non-zero, we see that the invertible elements of $B_2(H)$ form a group $\mathfrak{G}$ and are given by

$$\mathfrak{G} = \{b | b \in B_2(H),\ \det(b) \neq 0\}.$$

Clearly $\mathfrak{G}$ is a Banach Lie group with Lie algebra $\mathscr{L}$ and it acts analytically and transitively on $\mathfrak{F}^{(0)}$. The stabilizer of $F^{(0)}$ in $\mathfrak{G}$ has the form

$$\mathscr{T} = \left\{ t = \begin{pmatrix} t_{11} & \cdots & & t_{1m} \\ 0 & & & \vdots \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 0 & t_{mm} \end{pmatrix} \begin{array}{l} t_{ii} \in \{\mathrm{Id} + \mathcal{N}(H_i)\} \cap GL(H_i) \\ t_{ij} \in \mathscr{HS}(H_j, H_i) \text{ for } j > i \end{array} \right\}.$$

Thus we can identity $\mathfrak{F}^{(0)}$ with the homogeneous space $\mathfrak{G}/\mathscr{T}$.

Remark 3.1.4. If we would work with $\mathfrak{F}(\mathscr{N})$ instead of $\mathfrak{F}$, then we could simply take instead of $B_2(H)$ the collection $\mathrm{Id} + \mathscr{N}(H)$ and instead of $\mathfrak{G}$ the group of invertible operators of the form $\mathrm{Id} + \mathscr{N}(H)$. If one likes to work with $\mathfrak{F}(\mathscr{C})$ then the group

$$G_{\mathscr{C}} = \left\{ b \mid b = (b_{ij}) \in GL(H), \begin{array}{l} b_{ij} \in \mathscr{N}(H_j, H_i) \text{ if } j > i \\ b_{ii} \in \mathrm{Id} + \mathscr{N}(H_i) \\ b_{ij} \in \mathscr{C}(H_j, H_i) \text{ if } j < i \end{array} \right\}$$

acts transitively on the connected component of $\mathfrak{F}(\mathscr{C})$ containing the basic flag and allows you to take determinants of suitable minors.

3.1.5. Next we consider the maximal torus $T(\mathscr{N}) = T \cap \mathfrak{G}$ in $\mathfrak{G}$. It consists of all operators of the form $\mathrm{diag}(\{1 + t_s\})$, with $1 + t_s \neq 0$ and $\sum_{s \in S} |t_s| < \infty$. In $T(\mathscr{N})$ we have the dense subgroup T_f given by

$$T_f = \{t \mid t = \mathrm{diag}(\{1 + t_s\}) \in T(\mathscr{N}),\ t_s \neq 0 \text{ for only finitely many } s \text{ in } S\}$$

Any analytic group homomorphism of T_f into $\mathbf{C}^*$ has the form

$$t = \mathrm{diag}((1 + t_s)) \longmapsto \prod_{s \in S} (1 + t_s)^{m_s} =: \chi_{\underline{m}}(t),$$

where $\underline{m} = \{m_s\}$, with $m_s \in \mathbf{Z}$ for all $s \in S$. This character $\chi_{\underline{m}}$ can be continued to an analytic character of $T(\mathscr{N})$ if and only if there are only finitely many different m_s, $s \in S$. This extension of $\chi_{\underline{m}}$ is also denoted by $\chi_{\underline{m}}$ and we write $\hat{T}$ for the group of analytic characters of $T(\mathscr{N})$. Following the finite dimensional terminology, we will speak, when $T(\mathscr{N})$ acts on a vector according to a $\chi \in \hat{T}$, of "v is a vector of weight χ".

For each s and r in S, let E_{sr} be the operator in $\mathscr{B}_{\mathrm{res}}(H)$ given by

$$E_{sr}(e_l) = \delta_{rl} e_s \qquad \text{for all } l \in S.$$

The adjoint action of $T(\mathscr{N})$ on $\mathscr{B}_{\mathrm{res}}(H)$ gives for these elements

$$tE_{sr}t^{-1} = \frac{1 + t_s}{1 + t_r} E_{sr} =: \chi_{sr}(t) E_{sr}.$$

A character χ of $T(\mathscr{N})$ is called *positive*, notation $\chi \geq 0$, if it belongs to the semigroup generated by the $\{\chi_{sr} \mid s \in S_i,\ r \in S_j,\ i > j\}$. This enables you to define a partial order on $\hat{T}$ by

$$\chi \leq \psi \Longleftrightarrow 0 \leq \chi^{-1}\psi.$$

Clearly we call χ in $\hat{T}$ *negative* if and only if $\chi^{-1} \geq 0$.

Remark 3.1.6. One can see the space $\mathfrak{F}^{(0)}$ also as a homogeneous space for the group of invertible operators in "Id + $\mathscr{HS}(H)$", which in its turn can be identified with an open part of the Hilbert-Schmidt operators on H. This group, however, does not permit you to take suitable minors.

Remark 3.1.7. One can give the same type of description for the other components $\mathfrak{F}^{(z)}$ by taking some $\Sigma \in \mathscr{S}(z)$ and by introducing the group $\mathfrak{G}$ as the operators that decompose w.r.t. $H = H_{\Sigma_1} \oplus \cdots \oplus H_{\Sigma_m}$ in the above way.

3.2. Finite dimensional subvarieties. In this subsection we consider some finite dimensional flag subvarieties contained in $\mathfrak{F}^{(0)}$. Let K be a finite subset of S. For simplicity we assume that K contains all the S_i that are finite. The general linear group $GL(H_K)$ embeds into $GL^{(0)}_{\text{res}}(H)$ by extending $u \in GL(H_K)$ on H_{S-K} by the identity. We write $\mathfrak{F}_K$ for the subvariety of $\mathfrak{F}^{(0)}$ given by

$$\mathfrak{F}_K = \{uF^{(0)} \mid u \in GL(H_K)\} = \{uF^{(0)} \mid u \in U(H_K)\}.$$

If $K_1 \subset K_2$, then we have a natural embedding of $GL(H_{K_1})$ into $GL(H_{K_2})$ and of $\mathfrak{F}_{K_1}$ in $\mathfrak{F}_{K_2}$. Now one considers a collection of finite subsets $\{K_n \mid n \in \mathbf{N}\}$ of S given by

$$K_n = \bigcup_{i=1}^{m} \{s_i(k) \mid k \leq n + \max_{m_j < \infty} (m_j)\}.$$

Then $S = \bigcup_{n \in \mathbf{N}} K_n$ and with the identifications mentioned above we write

$$GL(\infty) = \bigcup_{n \in \mathbf{N}} GL(H_{K_n}), \ U(\infty) = \bigcup_{n \in \mathbf{N}} U(H_{K_n}) \text{ and } \mathfrak{F}(\infty) = \bigcup_{n \in \mathbf{N}} \mathfrak{F}_{K_n}.$$

Since, for all $i \neq j$, the "finite-size" operators are dense in $\mathscr{HS}(H_j, H_i)$, and the U_Σ, $\Sigma \in \mathscr{S}(0)$, cover $\mathfrak{F}^{(0)}$, we get

Lemma 3.2.1. *The space $\mathfrak{F}(\infty)$ lies dense in $\mathfrak{F}^{(0)}$.*

Consider now a holomorphic function f on $\mathfrak{F}^{(0)}$. The restriction of f to some $\mathfrak{F}_{K_n}$ must be a constant, since $\mathfrak{F}_{K_n}$ is a compact complex manifold. Hence f is a constant on $\mathfrak{F}(\infty)$ and the lemma implies then

Corollary 3.2.2. *The only holomorphic functions from $\mathfrak{F}^{(0)}$ to $\mathbf{C}$ are the constants.*

This is a generalization for flag varieties of Proposition 7.2.2 in [23].

Remark 3.2.3. The results of this subsection remain true if one would work with the nuclear flag space $\mathfrak{F}(\mathscr{N})$ or the "compact" flag variety $\mathfrak{F}(\mathscr{C})$. However· the "finite-size" flags from $\mathfrak{F}(\infty)$ are not lying dense in the space of bounded flags $\mathfrak{F}(\mathscr{B})$.

3.3. The line bundles and the central extension. For each $\underline{k} = (k_1, \dots, k_m)$ in $\mathbf{Z}^m$, we define $\psi_{\underline{k}}$ in $\hat{T}$ by

$$\psi_{\underline{k}}(\operatorname{diag}\{1+t_s\}) = \prod_{s_1 \in S_1} (1+t_{s_1})^{k_1} \prod_{s_2 \in S_2} (1+t_{s_2})^{k_2} \cdots \prod_{s_m \in S_m} (1+t_{s_m})^{k_m}$$

Clearly $\psi_{\underline{k}}$ extends to an analytic character of $\mathscr{T}$ by means of the formula

$$\psi_{\underline{k}}(t) = \det(t_{11})^{k_1} \cdots \det(t_{mm})^{k_m}.$$

According to section 6.5 in [4], there exists for each analytic character $\psi_{\underline{k}}$ of $\mathscr{T}$, a holomorphic line bundle $L(\underline{k}) = \mathfrak{G} \otimes^{\mathscr{T}} \mathbf{C}$ over $\mathscr{F}^{(0)} = \mathfrak{G}/\mathscr{T}$. It is concretely defined as follows: consider on the space $\mathfrak{G} \times \mathbf{C}$ the equivalence relation

$$(g_1, \lambda_1) \sim (g_2, \lambda_2) \Longleftrightarrow g_1 = g_2 \circ t, \text{ with } t \in \mathscr{T} \text{ and } \lambda_2 = \lambda_1 \psi_{\underline{k}}(t).$$

The space $\mathfrak{G} \times \mathbf{C}$ modulo this equivalence relation is $L(\underline{k})$. For each $g \in \mathfrak{G}$ and each λ in $\mathbf{C}$, we denote the equivalence class to which the pair (g, λ) belongs by $[g, \lambda]$. There is a natural projection $\pi_{\underline{k}}\colon L(\underline{k}) \to \mathfrak{F}^{(0)}$ given by

$$\pi_{\underline{k}}([g, \lambda]) = g \cdot F^{(0)}.$$

The space $L(\underline{k})$ is a Hilbert manifold based on the Hilbert space $E \oplus \mathbf{C}$. For each $\Sigma \in \mathscr{S}(0)$ one can give a concrete trivialization of $L(\underline{k})$ above U_Σ. Let σ be an admissible permutation of S such, that $\Sigma_i = \sigma(S_i)$. Then we define $\varphi_\Sigma\colon E \oplus \mathbf{C} \to \pi_{\underline{k}}^{-1}(U_\Sigma)$ by

$$\varphi_\Sigma(A, \lambda) = [\underline{\sigma}(\mathrm{Id} + A), \lambda].$$

Assume we have a Σ and R in $\mathscr{S}(0)$ such that $\pi_{\underline{k}}^{-1}(U_\Sigma) \cap \pi_{\underline{k}}^{-1}(U_R)$ is non-empty. Let σ and ρ be admissible permutations of S with $\sigma(S_i) = \Sigma_i$ and $\rho(S_i) = R_i$. If (A, λ) is such that $\varphi_\Sigma(A, \lambda)$ belongs to $\pi_{\underline{k}}^{-1}(U_R)$ then we know that $\underline{\rho}^{-1}\, \underline{\sigma}(\mathrm{Id} + A)$ belongs to $\Omega \cap \mathfrak{G}$ and because of Lemma 2.2.1

$$\underline{\rho}^{-1}\, \underline{\sigma}(\mathrm{Id} + A) = u(A)p(A), \text{ with } u(A) \in U_-,\ p(A) \in \mathscr{T}$$

Here $u(A)$ and $p(A)$ depend analytically of A and thus we get

$$\varphi_R^{-1} \varphi_\Sigma(A, \lambda) = (u(A) - \mathrm{Id}, \lambda \psi_{\underline{k}}(p(A)))$$

and this map is clearly analytic. This proves that

Lemma 3.3.1. *The* $\{(\pi_{\underline{k}}^{-1}(U_\Sigma), \varphi_\Sigma^{-1}) \mid \Sigma \in \mathscr{S}(0)\}$ *are the charts of an analytic* $E \oplus \mathbf{C}_-$ *structure on* $L(\underline{k})$.

Remark 3.3.2. For the case $m = 2$, $m_1 = m_2 = \infty$, the bundles $L((+1, 0))$ and $L((-1, 0))$ are the determinant bundle Det and its dual Det* as introduced in section 7.7 of [23].

There is a natural analytic action of the group $\mathfrak{G}$ on the space $L(\underline{k})$ by left translations

$$g_1 \cdot [g_2, \lambda] = [g_1 g_2, \lambda].$$

This is a lifting of the natural action of $\mathfrak{G}$ on $\mathfrak{F}^{(0)}$ to one on $L(\underline{k})$. However, the natural action of $GL_{\mathrm{res}}^{(0)}(H)$ can, in general, not be lifted to one on $L(\underline{k})$. Such an attempt may lead to nontrivial central extensions of $GL_{\mathrm{res}}^{(0)}(H)$ as we will show.

Note that each g in $GL_{\mathrm{res}}^{(0)}(H)$ can be written as $g = dg_2$, with $g_2 \in \mathfrak{G}$ and d belonging to the subgroup

$$D = \{g \mid g = (g_{ij}) \in GL_{\mathrm{res}}^{(0)}(H),\ g_{ij} = 0 \text{ if } i \neq j\}.$$

of $GL_{\mathrm{res}}^{(0)}(H)$. Clearly the group D normalizes the group $\mathfrak{G}$. Since the determinant of an operator of the form "identity + nuclear" is invariant under conjugation with an invertible operator, we get that D centralizes each $\psi_{\underline{k}}$, i.e. for each t in $\mathscr{T}$ and each d in D we have

$$\psi_{\underline{k}}(dtd^{-1}) = \psi_{\underline{k}}(t).$$

This fact permits you to lift the action of D on $\mathfrak{F}^{(0)}$ to one on $L(\underline{k})$ by means of

$$d \cdot [g, \lambda] = [dgd^{-1}, \lambda].$$

For an element d from $D \cap \mathfrak{G}$, this action differs by a factor $\psi_{\underline{k}}(d^{-1})$ from the action induced by that of $\mathfrak{G}$. Hence we cannot combine them to an action of $GL_{\mathrm{res}}^{(0)}(H)$ on $\mathfrak{F}^{(0)}$. To overcome this problem we build a group extension G of $GL_{\mathrm{res}}^{(0)}(H)$. It is defined by

$$G = \{(g, d) \mid g \in GL_{\mathrm{res}}^{(0)}(H),\ d \in D \text{ and } gd^{-1} \in \mathfrak{G}\}.$$

As one verifies directly this group acts on $L(\underline{k})$ by means of

$$(g, d)[g_1, \lambda_1] = [gg_1 d^{-1}, \lambda_1].$$

It is simply the combination of the $\mathfrak{G}$-action and the D-action given above. Let $\pi\colon G \to GL_{\mathrm{res}}^{(0)}(H)$ be the canonical projection, i.e. $\pi((g, d)) = g$ for all $(g, d) \in G$. For certain subgroups of $GL_{\mathrm{res}}^{(0)}(H)$ there exist several ways to embed them into G. Therefore we introduce special notations for two of them. Let $\underline{i}$ resp.

$\underline{j}$ be the embedding of $\mathfrak{G}$ resp. D into G given by

$$\underline{i}\,(g) = (g, \mathrm{Id}) \text{ and } \underline{j}\,(d) = (d, d).$$

As a group G is the semi-direct product of $\underline{i}\,(\mathfrak{G})$ and $\underline{j}\,(D)$. We equip each $GL(H_i)$ with the operator norm topology and we put on $\underline{j}\,(D)$ the product Banach Lie group structure. On $\underline{i}\,(\mathfrak{G})$ we take the Banach structure based on $\mathscr{X}$. The conjugation with an element d of D defines an analytic diffeomorphism of $\mathfrak{G}$. Hence if we put on G the product topology of $\underline{i}\,(\mathfrak{G})$ and $\underline{j}\,(D)$, it becomes a Banach Lie group based on

$$(\bigoplus_{i=1}^{m} \mathscr{B}(H_i)) \oplus \mathscr{X}.$$

The group G is a fiber bundle over $GL_{\mathrm{res}}^{(0)}(H)$, with fiber $\mathscr{T} \cap D$. This is clear from the following useful trivializations. For each $\Sigma = (\Sigma_i)$ in $\mathscr{S}(0)$ consider the open set $G(\Sigma)$ of G given by

$$G(\Sigma) = \{(g, d) \,|\, (g, d) \in G,\ p_{\Sigma_i} \circ g \,|\, H_i \text{ is a bijection for all } i,\ 1 \le i \le m\}.$$

The group G is the union of these open sets. If σ is an admissible permutation with $\sigma(S_i) = \Sigma_i$, then we define an analytic bijection from $\pi(G(\Sigma)) \times \{\mathscr{T} \cap D\}$ to $G(\Sigma)$ by

$$(g, t) \longmapsto (g, \tilde{d})(\mathrm{Id}, t),$$

where $\tilde{d}$ in D is determined by

$$\tilde{d}_{ii} = \underline{\sigma}^{-1} \circ p_{\Sigma_i} \circ g \,|\, H_i. \tag{9}$$

Next we try to minimalize the extension of $GL_{\mathrm{res}}^{(0)}(H)$ that acts on $\mathfrak{F}^{(0)}$ and $L(\underline{k})$. Thereto we consider the action of the kernel of π on $L(\underline{k})$

$$(\mathrm{Id}, d) \cdot [g, \lambda] = [gd^{-1}, \lambda] = [g, \psi_{\underline{k}}(d^{-1})\lambda].$$

In particular the group $D(\underline{k}) = \{(\mathrm{Id}, d) \,|\, (\mathrm{Id}, d) \in G \text{ and } \psi_{\underline{k}}(d) = 1\}$ acts trivially on $L(\underline{k})$ and we see that it suffices to consider the extension $G(\underline{k}) = G/D(\underline{k})$ of $GL_{\mathrm{res}}^{(0)}(H)$. If the character $\psi_{\underline{k}}$ is trivial, i.e. $\underline{k} = 0$, then $G(\underline{k})$ is just $GL_{\mathrm{res}}^{(0)}(H)$. For $\underline{k} \neq 0$, one computes directly that $G(\underline{k})$ is a central extension of $GL_{\mathrm{res}}^{(0)}(H)$ with $\mathrm{Ker}(\pi)/D(\underline{k}) \cong \mathbf{C}^*$. For $m = 2$, $m_1 = m_2 = \infty$, the extension $G((-1, 0))$ is the one introduced in section 6.6 of [23].

One can describe such an extension with a Borel 2-cocycle $\alpha\colon GL_{\mathrm{res}}^{(0)}(H) \times GL_{\mathrm{res}}^{(0)}(H) \to \mathbf{C}^*$. It can be constructed as follows: take a section ρ of the fiber bundle $G \xrightarrow{\pi} GL_{\mathrm{res}}^{(0)}(H)$, i.e. for each g in $GL_{\mathrm{res}}^{(0)}(H)$ we have

$$\rho(g) = (g, q(g)) \text{ with } q(g) \in D.$$

By definition we have for each g_1 and g_2 in $GL_{\mathrm{res}}^{(0)}(H)$ that

$$q(g_1)q(g_2)q(g_1g_2)^{-1} \in D \cap \mathfrak{G}.$$

Thus we get for the action on $L(\underline{k})$ the relation

$$\begin{aligned}\rho(g_1g_2)\cdot[g, \lambda] &= \rho(g_1)\cdot\{\rho(g_2)\cdot[g, \lambda\psi_{\underline{k}}(q(g_1)q(g_2)q(g_1g_2)^{-1})]\}\\ &=: \rho(g_1)\cdot\{\rho(g_2)\cdot[g, \lambda\alpha(g_1, g_2)^{-1}]\}.\end{aligned}$$

The group $G(\underline{k})$ is then isomorphic as a group to the product space $GL_{\mathrm{res}}^{(0)}(H) \times \mathbf{C}^*$ with the multiplication

$$(g_1, \lambda_1) * (g_2, \lambda_2) = (g_1g_2, \lambda_1\lambda_2\alpha(g_1, g_2)).$$

If $\tilde{\rho}$ is another section of $G \xrightarrow{\pi} GL_{\mathrm{res}}^{(0)}(H)$ with $\tilde{\rho}(g) = (g, \tilde{q}(g))$, then we have by definition for each g in $GL_{\mathrm{res}}^{(0)}(H)$ that $\tilde{q}(g) = q(g)t(g)$ with $t(g)$ in $\mathscr{T} \cap D$. The corresponding 2-cocycle $\tilde{\alpha}$ satisfies

$$\tilde{\alpha}(g_1, g_2) = \frac{\psi_{\underline{k}}(t(g_1g_2))}{\psi_{\underline{k}}(t(g_1))\psi_{\underline{k}}(t(g_2))} \quad \alpha(g_1, g_2).$$

In other words, it differs by a trivial 2-cocycle and we merely have to consider one section ρ.

A section ρ can be composed from the local trivializations of $\pi: G \to GL_{\mathrm{res}}^{(0)}(H)$ defined above. First we number the elements of $\mathscr{S}(0)$: $\mathscr{S}(0) = \{\Sigma^{(i)} \mid i \geq 0\}$, such that $\Sigma^{(0)}$ is the partition corresponding to the basic flag. For $g \in G(\Sigma^{(0)})$ we choose $q(g)$ according to the trivialization (9) with $\sigma = \mathrm{Id}$. Next we define $q(g)$ inductively by: if g belongs to $\bigcup_{i=0}^{m} G(\Sigma^{(i)})$ and not to $\bigcup_{i=0}^{m-1} G(\Sigma^{(i)})$, then we take $q(g)$ according to the trivialization of $G(\Sigma^{(m)})$ given by (9). In particular if g, h and gh belong to $G(\Sigma^{(0)})$, then the 2-cocycle α is given by

$$\alpha(g, h) = \prod_{i=1}^{m} \det\,(\mathrm{Id} + \sum_{j \neq i} g_{ij}h_{ji}h_{ii}^{-1}g_{ii}^{-1})^{k_i}. \tag{10}$$

From this formula we will compute in the next subsection the corresponding Lie algebra 2-cocycle.

3.4. The non-triviality of the extension $G(\underline{k})$. First we consider the case that $k_i = l$, $l \in \mathbf{Z}$, for all i. Then we have for each $g \in D \cap \mathscr{T}$ that $\psi_{\underline{k}}(g) = \det\,(g)^l$. We can adjust the $\mathfrak{G}$-action on $L(\underline{k})$ as follows:

$$g * [x, \lambda] = [gx, \det\,(g)^{-l}\lambda].$$

Hence for elements $d \in D \cap \mathfrak{G}$, we get

$$d * [x, \lambda] = \psi_{\underline{k}}(d^{-1})[dxd^{-1}d, \lambda] = [dxd^{-1}, \lambda].$$

Now we combine this new $\mathfrak{G}$-action with that of D and we define for $g = d_1 g_1$ in $GL_{\mathrm{res}}^{(0)}(H)$, where $d_1 \in D$ and $g - 1 \in \mathfrak{G}$,

$$g * [x, \lambda] := [d_1 g_1 x d_1^{-1}, \det(g_1)^{-l}\lambda].$$

It is a straightforward verification to show that this is well-defined and that it defines an action of $GL_{\mathrm{res}}^{(0)}(H)$ on $L(\underline{k})$. This implies that $G(\underline{k})$ is a trivial central extension of $GL_{\mathrm{res}}^{(0)}(H)$.

Secondly we consider the case where at most one of the m_i is infinite. Then $GL_{\mathrm{res}}^{(0)}(H)$ is simply $GL(H)$ and we know from [20] that this group is contractible. In particular the fiber bundle $G \xrightarrow{\pi} GL(H)$ is then topologically trivial and the group $G(\underline{k})$ is the direct product of $GL(H)$ and $\operatorname{Ker}(\pi)/D(\underline{k})$. Hence we may assume in the sequel that there are at least two infinite m_i's.

The next case we have a look at is that $\underline{k}$ satisfies

$$k_i \neq 0 \Longrightarrow m_i < \infty. \tag{11}$$

Let $g \mapsto (g, q(g))$ be a section of $G \to GL_{\mathrm{res}}^{(0)}(H)$. We will adjust $q(g)$ such that the 2-cocycle determining $G(\underline{k})$ becomes trivial. Namely, we define $\tilde{q}(g)$ in D by

$$\begin{aligned} &\tilde{q}(g)_{ii} = q(g)_{ii} \quad \text{if } m_i = \infty \text{ and} \\ &\tilde{q}(g)_{ii} = \operatorname{diag}(1, \ldots, 1, \det(q(g)_{ii}^{-1}))\, q(g)_{ii}, \quad \text{if } m_i < \infty. \end{aligned} \tag{12}$$

Then $g \mapsto (g, \tilde{q}(g))$ defines another section $\tilde{\rho}$ of $G \to GL_{\mathrm{res}}^{(0)}(H)$ and the corresponding 2-cocycle $\tilde{\alpha}$ is trivial

$$\tilde{\alpha}(g_1, g_2) = \psi_{\underline{k}}(\tilde{q}(g_1 g_2)\tilde{q}(g_2)^{-1}\tilde{q}(g_1)^{-1}) = 1.$$

In the cases considered so far we have seen that the fiber bundle $G \xrightarrow{\pi} GL_{\mathrm{res}}^{(0)}(H)$ is trivial and hence also the central extension $G(\underline{k})$ of $GL_{\mathrm{res}}^{(0)}(H)$. We will show now that the extension $G(\underline{k})$ can be non-trivial.

Note that the 2-cocycle α is given close to the identity by an analytic expression. Hence we can consider the corresponding Lie algebra 2-cocycle $d\alpha$. We consider the elements of $\mathscr{B}_{\mathrm{res}}(H)$ as left invariant vector field on $GL_{\mathrm{res}}^{(0)}(H)$. Then $d\alpha \colon \mathscr{B}_{\mathrm{res}}(H) \times \mathscr{B}_{\mathrm{res}}(H) \to \mathbf{C}$ is given by

$$d\alpha(X, Y) = \frac{d}{dt}\frac{d}{ds}\alpha(\exp(tX), \exp(sY))\Big|_{\substack{t=0\\s=0}} - \frac{d}{ds}\frac{d}{dt}\alpha(\exp(sY), \exp(tX))\Big|_{\substack{t=0\\s=0}}.$$

For $X = (X_{ij})$ in $\mathscr{B}_{\mathrm{res}}(H)$, we write $g = \exp(tX) = (g_{ij})$. With respect to the parameter t we have

$$g_{ii} = \mathrm{Id}_{H_i} + tX_{ii} + \text{“higher order in } t\text{”}$$

$$g_{ij} = tX_{ij} + \text{“higher order in } t\text{” for } i \neq j.$$

If $h = (h_{ij}) = \exp(sY)$, then we are interested in the ts-term in

$$\begin{aligned}\det\Big(\mathrm{Id}_{H_i} + \sum_{j \neq i} g_{ij} h_{ji} h_{ii}^{-1} g_{ii}^{-1}\Big) \\ &= \det(\mathrm{Id}_{H_i} + N) = 1 + \sum_{k=1}^{\infty} \operatorname{Trace}(\Lambda^k N) \\ &= 1 + ts \sum_{j \neq i} \operatorname{Trace}(X_{ij} Y_{ji}) + \text{“at least 2nd order in } t \text{ or } s\text{”}.\end{aligned}$$

By combining this expression with the local formula (10) for α, we obtain the following formula for $d\alpha$:

$$d\alpha(X, Y) = \sum_{i=1}^{m} k_i \operatorname{Trace}\Big\{\sum_{j \neq i} X_{ij} Y_{ji} - \sum_{j \neq i} Y_{ij} X_{ji}\Big\}.$$

This Lie algebra cocycle is trivial if it has the form $f([X, Y])$ with $f\colon \mathscr{B}_{\mathrm{res}}(H) \to \mathbf{C}$ some linear map. The element $[X, Y]$ in $\mathscr{B}_{\mathrm{res}}(H)$ has the form

$$[X, Y] = \begin{pmatrix} [X_{11}, Y_{11}] + \Sigma_{j \neq 1} X_{1j} Y_{j1} - \Sigma_{j \neq 1} Y_{1j} X_{j1} & & * \\ & \ddots & \\ * & & [X_{mm}, Y_{mm}] + \Sigma_{j \neq m} X_{mj} Y_{jm} - \Sigma_{j \neq m} Y_{mj} X_{jm} \end{pmatrix}.$$

Note that if $\underline{k}$ satisfies (11), then we can directly define such an f. For, in that case, we have for all i with $k_i \neq 0$ that Trace $[X_{ii}, Y_{ii}]$ is well-defined and equal to zero and we can take

$$f(X) = \sum_{i, k_i \neq 0} k_i \operatorname{Trace}(X_{ii}).$$

This is the infinitesimal version of the trivialization described at the beginning of this subsection. There is, however, no well-defined trace function for general elements of $\mathscr{B}_{\mathrm{res}}(H)$ so that this formula makes no sense in the general case.

Now, let i and j be such that $i < j$, $m_i = m_j = +\infty$ and $k_i \neq k_j$. Then we have an element Λ in $GL_{\mathrm{res}}(H)$ given by

$$\begin{aligned} \Lambda(e_{s_i(k)}) &= e_{s_i(k+1)}, \\ \Lambda(e_{s_j(k)}) &= e_{s_j(k-1)} \text{ if } k > 1, \\ \Lambda(e_{s_l(k)}) &= e_{s_l(k)} \text{ if } l \neq i \text{ and } l \neq j. \\ \Lambda(e_{s_j(1)}) &= e_{s_i(1)} \end{aligned}$$

Now we have that $d\alpha(\Lambda, \Lambda^{-1})$ is equal to

$$k_i - k_j = k_i \text{ Trace } \Lambda_{ij}(\Lambda^{-1})_{ji} - k_j \text{ Trace } (\Lambda_{ji}^{-1}\Lambda_{ij}) \neq 0$$

In particular $d\alpha$ is a non-trivial Lie algebra 2-cocycle. This implies that also the group 2-cocycle α is non-trivial. For, consider the commuting elements $g_1 = \exp(t\Lambda)$ and $g_2 = \exp(s\Lambda^{-1})$. In case that α was trivial we would have $\alpha(g_1, g_2) = \alpha(g_2, g_1)$. However, for sufficiently small t and s, the map $(t, s) \mapsto \alpha(g_1, g_2)$ is a non-constant holomorphic function, since $d\alpha(\Lambda, \Lambda^{-1}) \neq 0$. On the other hand one computes directly that for all i, $1 \leq i \leq m$,

$$(g_2 g_1)_{ii} = (g_2)_{ii}(g_1)_{ii} \Longrightarrow \alpha(g_2, g_1) = 1.$$

This shows that $\alpha(g_1, g_2) \neq \alpha(g_2, g_1)$ and hence α is a non-trivial 2-cocycle. We summarize this result in a

Theorem 3.4.1.

(a) *The extension $G(\underline{k})$ is always trivial if there is at most one infinite m_i.*

(b) *If there are at least two infinite dimensional components in the basic flag, then $G(\underline{k})$ is trivial if and only if for all i and j,*

$$m_i = m_j = \infty \Longrightarrow k_i = k_j.$$

(c) *If $k_i \neq k_j$ for infinite dimensional H_i and H_j, then the corresponding Lie algebra 2-cocycle for the extension $G(\underline{k})$ is given by*

$$d\alpha(X, Y) = \sum_{i=1}^{m} k_i \text{ Trace } \{\sum_{j \neq i} X_{ij} Y_{ji} - \sum_{j \neq i} Y_{ij} X_{ji}\}.$$

Remark 3.4.2. Consider the case $m = 2$, $m_1 = m_2 = \infty$, $k_1 = -1$ and $k_2 = 0$ and restrict $d\alpha$ to $\overline{\mathfrak{gl}_{(\infty)}} \cap \mathfrak{gl}(S)$. Then we have the 2-cocycle defining the Lie algebra A_∞.

3.5. The holomorphic sections of $L(\underline{k})$. Let $\mathfrak{L}(\underline{k})$ denote the space of global holomorphic sections of $L(\underline{k})$. The space $\mathfrak{L}(\underline{k})$ is given the topology of uniform convergence on compact subsets of $\mathfrak{F}^{(0)}$. It becomes then a complete locally convex space, see [16]. Let $\underline{f} : \mathfrak{F}^{(0)} \to L(\underline{k})$ belong to $\mathfrak{L}(\underline{k})$. Then it can be written as

$$\underline{f}\ (g \cdot F^{(0)}) = [g, f(g)], \qquad \text{for all } g \in \mathfrak{G},$$

where $f \colon \mathfrak{G} \to \mathbf{C}$ is a holomorphic function satisfying

$$f(gt) = f(g)\psi_{\underline{k}}(t)^{-1} \qquad \text{for all } g \in \mathfrak{G} \text{ and all } t \in \mathscr{T} \tag{13}$$

Thus we can identify $\mathfrak{L}(\underline{k})$ with the space of holomorphic functions on $\mathfrak{G}$ that satisfy this condition. Since each (g, d) in G acts as an analytic diffeomorphism on $\mathfrak{F}^{(0)}$ as well as $L(\underline{k})$, we get a natural action of G on $\mathfrak{L}(\underline{k})$ that corresponds on the functions on $\mathfrak{G}$ satisfying (13) to

$$(g, d)(f)(g_1) = f(g^{-1}g_1 d), \text{ with } g_1 \in \mathfrak{G} \text{ and } (g, d) \in G.$$

Let K_n be the finite subset of S introduced in subsection 3.2. By restricting the elements of $\mathfrak{L}(\underline{k})$ to $GL(H_{K_n})$ one obtains a space of holomorphic functions on $GL(H_{K_n})$ satisfying

(14)
$$f(gt) = f\left(\begin{pmatrix} g_{11} & \cdots & g_{1m} \\ \vdots & & \vdots \\ g_{m1} & \cdots & g_{mm} \end{pmatrix}\begin{pmatrix} t_{11} & \cdots & & t_{1m} \\ 0 & & & \\ \vdots & \ddots & & \\ 0 & \cdots & 0 & t_{mm} \end{pmatrix}\right) = \prod_{i=1}^{m} \det(t_{ii})^{-k_i} f(g),$$

where $g \in GL(H_{K_n})$, $t \in GL(H_{K_n}) \cap \mathscr{T}$ and the decomposition of g and t is w.r.t. $H_{K_n} = \bigoplus_{i=1}^{m} H_{K_n \cap S_i}$. The Borel-Weil theorem says that such functions $\neq 0$ exist if and only if $\underline{k}$ satisfies

(15) $$k_1 \leq k_2 \cdots \leq k_{m-1} \leq k_m.$$

Since $\mathfrak{F}(\infty)$ is dense in $\mathfrak{F}^{(0)}$, the restriction of some non-zero f in $\mathfrak{L}(\underline{k})$ must be non-zero for a sufficiently large n. Hence this condition from the finite dimensional situation is also necessary in this Hilbert context. We will show that it is sufficient too. So we assume from now on that $\underline{k} \in \mathbf{Z}^m$ satisfies condition (15).

Before we will construct concrete non-zero elements of $\mathfrak{L}(\underline{k})$, we will first introduce some basic building blocks. If $\Sigma = \{\Sigma_j\}$ belongs to $\mathscr{S}(0)$, then we write

$$\Sigma(i) = \bigcup_{j \leq i} \Sigma_j \quad \text{and} \quad \mathscr{S}_i = \{\Sigma(i) \mid \Sigma \in \mathscr{S}(0)\}.$$

Let $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_m$ be an admissible permutation of S corresponding to Σ. Consider for $g \in \mathfrak{G}$ the operator $(\underline{\sigma}_1 \oplus \cdots \oplus \underline{\sigma}_i)^{-1} \circ p_{\Sigma(i)} \circ g \mid \bigoplus_{j \leq i} H_j$ from $\bigoplus_{j \leq i} H_j$ to itself. It decomposes as

$$\begin{pmatrix} h_{11} & \cdots & h_{1i} \\ \vdots & \ddots & \vdots \\ h_{i1} & \cdots & h_{ii} \end{pmatrix}, \text{ with } h_{jj} - \mathrm{Id}_{H_j} \in \mathscr{N}(H_j),\ h_{ij} \in \mathscr{H}\mathscr{S}(H_j, H_i) \qquad \text{for all } j \neq i.$$

In particular we can take the determinant of this operator. The function $f_{\Sigma(i)}\colon \mathfrak{G} \to \mathbf{C}$ defined by

$$f_{\Sigma(i)}(g) = \det\left((\underline{\sigma}_1 \oplus \cdots \oplus \underline{\sigma}_i)^{-1} \circ p_{\Sigma(i)} \circ g \,\Big|\, \bigoplus_{j \le i} H_j\right)$$

satisfies for each $t = (t_{ij})$ in $\mathscr{T}$ the condition

$$f_{\Sigma(i)}(gt) = f_{\Sigma(i)}(g) \det(t_{11}) \cdots \det(t_{ii})$$

In other words $f_{\Sigma(i)}$ belongs to $\mathfrak{L}((-1, \ldots, -1, 0, \ldots))$. Now we consider the action of $T(\mathcal{N})$ on such a function $f_{\Sigma(i)}$. Let g be in $\mathfrak{G}$ and $t = \operatorname{diag}(\{1 + t_s\})$ in $T(\mathcal{N})$. By definition $f_{\Sigma(i)}(t^{-1}g)$ is equal to the determinant of the operator

$$e_l \xmapsto{\;g\;} \sum_{r \in S} g_{rl} e_r \xmapsto{\;t^{-1}\;} \sum_{r \in S} (1 + t_r)^{-1} g_{rl} e_r$$

$$\xmapsto{\;pH_{\Sigma(i)}\;} \sum_{r \in \Sigma(i)} (1 + t_r)^{-1} g_{rl} e_r \xmapsto{\;\sigma^{-1}\;} \sum_{c \in \Sigma^{(0)}(i)} (1 + t_{\sigma(c)})^{-1} g_{\sigma(c) l} e_c.$$

Hence each $f_{\Sigma(i)}$ is an eigenvector for the $T(\mathcal{N})$-action and we have

$$t \cdot f_{\Sigma(i)} = \prod_{c \in \Sigma^{(0)}(i)} (1 + t_{\sigma(c)})^{-1} f_{\Sigma(i)}.$$

If we define the character ψ_i of $T(\mathcal{N})$ by

$$\psi_i(t) = \prod_{c \in \Sigma^{(0)}(i)} (1 + t_c)^{-1},$$

then we get in general

$$t \cdot f_{\Sigma(i)} = \prod_{\substack{\sigma^{-1}(c) \notin \Sigma^{(0)}(i) \\ c \in \Sigma^{(0)}(i)}} (1 + t_c) \cdot \prod_{\substack{c \in \Sigma^{(0)}(i) \\ \sigma(c) \notin \Sigma^{(0)}(i)}} (1 + t_{\sigma(c)})^{-1} \cdot \psi_i(t) f_{\Sigma(i)}.$$

Because $\underline{\sigma}$ has index zero, the products in the right hand side are over the same number of elements. Since we have by definition, for each $c_1 \in \Sigma^{(0)}(i)$ with $\sigma^{-1}(c_1) \notin \Sigma^{(0)}(i)$ and for each $c_2 \in \Sigma^{(0)}(i)$ with $\sigma(c_2) \notin \Sigma^{(0)}(i)$, that $\chi_{c_1 \sigma(c_2)} \le 0$, it is clear that the weight of $f_{\Sigma(i)}$ is less then or equal to ψ_i. In other words, among the weights of the $\{f_{\Sigma(i)}, \Sigma(i) \in \mathscr{S}_i\}$, ψ_i is maximal. Note that condition (15) on $\underline{k}$ allows you to decompose $\psi_{\underline{k}}$ as follows

$$\begin{aligned}\psi_{\underline{k}}(t) &= \prod_{i=1}^{m} \det(t_{ii})^{k_i} \\ &= \Big\{\prod_{i=1}^{m} \det(t_{ii})\Big\}^{k_m} \cdot \Big\{\prod_{i=1}^{m-1} \det(t_{ii})\Big\}^{-k_m + k_{m-1}} \cdots \{\det(t_{11})^{-k_2 + k_1}\}\end{aligned}$$

$$= \psi_m(t)^{-k_m} \cdot \prod_{i<m} \psi_i(t)^{k_{i+1}-k_i}$$

Now we choose for each j, $1 \leq j \leq m-1$ a non-zero homogeneous polynomial P_j in the $\{f_{\Sigma(j)} \mid \Sigma(j) \in \mathscr{S}_j\}$ of degree $k_{j+1} - k_j$. Let $P_m \colon \mathfrak{G} \to \mathbf{C}^*$ be the function

$$g \longmapsto \det(g)^{-k_m}.$$

From the foregoing formulae it will be clear that $P = \prod_{i=1}^{m} P_i$ is a non-zero holomorphic function on $\mathfrak{G}$ that belongs to $\mathfrak{L}(\underline{k})$. If we consider the special choice $\tilde{P} = \prod_{i=1}^{m} \tilde{P}_i$

$$\tilde{P}_i = (f_{\Sigma^{(0)}(i)})^{k_{i+1}-k_i}, \quad 1 \leq i \leq m-1, \quad \tilde{P}_m = P_m,$$

then the $T(\mathscr{N})$-action on this element of $\mathfrak{L}(\underline{k})$ is given by the character

$$t \longmapsto \prod_{i=1}^{m-1} \psi_i(t)^{k_{i+1}-k_i} \cdot \psi_m(t)^{-k_m} = \psi_{\underline{k}}(t).$$

Let $\mathfrak{L}_f(\underline{k})$ be the span of the functions P described above. From the $T(\mathscr{N})$-action on the $\{f_{\Sigma(i)}\}$ one concludes that this is the highest weight occurring in $\mathfrak{L}_f(\underline{k})$. If H is finite dimensional then it is known that $\mathfrak{L}_f(\underline{k}) = \mathfrak{L}(\underline{k})$. By using this and the fact that $\mathfrak{F}(\infty)$ is dense in $\mathfrak{F}^{(0)}$, we get

Theorem 3.5.1.

(a) *The space $\mathfrak{L}(\underline{k})$ is non-zero if and only if $k_1 \leq \cdots \leq k_m$.*

(b) *The subspace $\mathfrak{L}_f(\underline{k})$ lies dense in $\mathfrak{L}(\underline{k})$.*

Next we consider the representation of G on $\mathfrak{L}(\underline{k})$. Let V be a closed subspace of $\mathfrak{L}_f(\underline{k})$ and let v be a non-zero element of V. Then there is an n such that the restriction of v to $GL(H_{K_n})$ is non-zero. Since the representation of $U(H_{K_n})$ on the holomorphic functions on $GL(H_{K_n})$ satisfying (14) is irreducible, we get

$$\operatorname{span}\{u \cdot v \mid u \in U(H_{K_n})\} = \{f|_{GL(H_{K_n})} \mid f \in \mathfrak{L}(\underline{k})\}.$$

So, if we define W as the closure of the span of the $\{u \cdot v \mid u \in U(\infty)\}$ then we have for each v_1 in V a sequence $\{w_n\}$ in W such that $v_1 \mid GL(H_{K_n}) = w_n \mid GL(H_{K_n})$. Since $U(\infty) \cdot F^{(0)}$ is dense in $\mathfrak{F}^{(0)}$, this implies that $\{w_n\}$ converges to v_1 and we get that $V = W$. Hence we can say

Theorem 3.5.2. *Let $\underline{k} = (k_1, \ldots, k_m) \in \mathbf{Z}^m$ satisfy $k_1 \leq \cdots \leq k_m$. The repre-*

sentation of $\mathfrak{G}$ *(and hence of* G*) on* $\mathfrak{L}(\underline{k})$ *is topologically irreducible.*

This Theorem is a generalization of Theorem 10.4.6 in [23].

For each $n \in \mathbf{N}$, we know that the representation of $GL(H_{K_n})$ on $\{f | GL(H_{K_n}) | f \in \mathfrak{L}(\underline{k})\}$ has up to a constant a unique vector that is w.r.t. the $\{T(\mathcal{N}) \cap GL(H_{K_n})\}$ action of highest weight

$$t \longmapsto \prod_{i=1}^{m-1} \{\psi_i(t)\}^{k_{i+1}-k_i} (\det(t))^{k_m}.$$

In view of the foregoing results we may conclude now for the space $\mathfrak{L}(\underline{k})$:

Theorem 3.5.3. *All* $T(\mathcal{N})$*-weights* ψ *occurring in* $\mathfrak{L}(\underline{k})$ *satisfy*

$$\psi \leq \prod_{i=1}^{m-1} (\psi_i)^{k_{i+1}-k_i} \psi_m^{-k_m} = \psi_{\underline{k}}$$

The vector $\tilde{P}$ *spans the subspace of vectors with* $T(\mathcal{N})$*-weight* $\psi_{\underline{k}}$.

In view of the results in the Theorems 3.5.2 and 3.5.3, one could call the characters $\psi_{\underline{k}}$ satisfying (15) dominant.

§ 4. Applications to Integrable Systems

4.1. A group of commuting flows. In the first subsection we discuss the flows that form the basis of the equations of the multicomponent KP-hierarchy and of the modified equations. Let H be the Hilbert space $L^2(S^1, \mathbf{C}^r)$ with the usual norm. Let $\{f_\ell \mid 1 \leq \ell \leq r\}$ be the standard basis of $\mathbf{C}^r$. Then the elements of H can be written as

$$\sum_{i \in \mathbf{Z}} \sum_{k=1}^{r} \alpha_{ik} f_k \lambda^i, \quad \text{with } \alpha_{ik} \in \mathbf{C}.$$

The space H is decomposed as $H = H_1 \oplus H_2$ with

$$H_1 = \{\sum_{i \geq 0} h_i \lambda^i \mid h_i \in \mathbf{C}^r\} \quad \text{and} \quad H_2 = H_1^\perp.$$

The elements $\{f_k \lambda^i \mid 1 \leq k \leq r,\ i \geq 0\}$ are an orthonormal basis of H_1 and the $\{f_k \lambda^i \mid i < 0,\ 1 \leq k \leq r\}$ one of H_2. To get a numbering like in the foregoing section, one defines $e_{k+ir-1} = f_k \lambda^i$. In the present context it is also convenient to see the matrix $[g]$ of an operator g in $\mathscr{B}_{\text{res}}(H)$ as an $\mathbf{Z} \times \mathbf{Z}$-matrix with entries in $\mathfrak{gl}_r(\mathbf{C})$, i.e.

$$[g] = \begin{pmatrix} \ddots & & & \\ & G_{k+1\ell+1} & G_{k+1\ell} & \\ & G_{k\ell+1} & G_{k\ell} & \\ & & & \ddots \end{pmatrix}, \quad \text{with } G_{st} \in \mathfrak{gl}_r(\mathbf{C}).$$

An important operator in $\mathscr{B}_{\mathrm{res}}(H)$ is the multiplication Λ on each factor with λ. It has the matrix $[\Lambda]$ with $\Lambda_{ii-1} = \mathrm{Id}$ and $\Lambda_{ij} = 0$, if $j \neq i - 1$. One verifies directly that the centralizer $Z(\Lambda)$ of Λ in $\mathscr{B}_{\mathrm{res}}(H)$ consists of all g in $\mathscr{B}_{\mathrm{res}}(H)$ such that the matrix of g looks like

$$[g] = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & G_{00} & G_{10} & & \\ & G_{-10} & G_{00} & G_{10} & \ddots \\ & & G_{-10} & G_{00} & \ddots \\ & & & \ddots & \end{pmatrix}.$$

Clearly, multiplying with an A from $\mathfrak{gl}_r(\mathbf{C})$, defines an element of $\mathscr{B}_{\mathrm{res}}(H)$. Let $\mathfrak{h}$ be the diagonal matrices in $\mathfrak{gl}_r(\mathbf{C})$. It is obvious then that

$$\{\sum_{i \in \mathbf{Z}} H_i \Lambda^i \in \mathscr{B}_{\mathrm{res}}(H),\ H_i \in \mathfrak{h} \qquad \text{for all } i\}$$

is a maximal commutative subalgebra of $\mathscr{B}_{\mathrm{res}}(H)$. The group of commuting flows that we will consider is contained in this algebra and takes care of essentially all independent directions. To be more precise, let U be a connected neighborhood of S^1 in $\mathbf{C}$ and let $\Gamma(U)$ be the space of all analytic maps $\gamma: U \to \mathfrak{h}$ such that $\det(\gamma(u)) \neq 0$ for all $u \in U$. In a natural way $\Gamma(U)$ is a group. If $U_1 \supset U_2$ then we get an embedding of $\Gamma(U_1)$ into $\Gamma(U_2)$ by restricting functions to U_2. We write Γ for the inverse limit of the $\{\Gamma(U)\}$. Each $\gamma \in \Gamma$ has a Fourier series

$$\gamma = \sum_{i \in \mathbf{Z}} \gamma_i \lambda^i, \ \gamma_i \in \mathfrak{h}.$$

The multiplication with γ, defines the element $\sum_{i \in \mathbf{Z}} \gamma_i \Lambda^i$ in $\mathscr{B}_{\mathrm{res}}(H)$. Let E_α, $1 \leq \alpha \leq r$, be the diagonal matrix in $\mathfrak{gl}_r(\mathbf{C})$ with (α, α)-entry equal to 1 and the other entries equal to zero. At the consideration of the flows from Γ on $\mathfrak{F}$ we make use of a decomposition of the elements of Γ. In Γ we consider namely the following subgroups

$$\Gamma_+ = \{\gamma \mid \gamma \in \Gamma,\ \gamma = \exp(\sum_{\substack{i>0 \\ 1 \leq \alpha \leq r}} t_{i\alpha} E_\alpha \lambda^i)\},$$

$$\Gamma_- = \{\gamma \mid \gamma = \sum_{j \leq 0} \gamma_j \lambda^j \in \Gamma\} \text{ and}$$

$$\Delta = \{\delta \mid \delta = \mathrm{diag}(\lambda^{k_1}, \ldots, \lambda^{k_r}) \text{ with } k_i \in \mathbf{Z} \text{ for all } i\}.$$

Then there holds

Lemma 4.1.1. *The group Γ decomposes as $\Gamma = \Gamma_+ \Delta \Gamma_-$.*

This lemma is a direct consequence of the decomposition of holomorphic line bundles over $\mathbf{P}^1(\mathbf{C})$, see [11]. As we will see in the third subsection the flows from Γ_- do not contribute to the system. Hence there is no need for a description in coordinates for elements from Γ_-.

4.2. The multicomponent KP-hierarchy. We present here a formal algebraic set-up of this system of equations in which the formulae from the appendix of [27] make sense. It offers one also the possibility to consider these equations from an algebraic point of view. Let R be a complex commutative differential algebra with a collection of commuting derivations $\{\partial_{i\alpha} \mid i \geq 1,\ 1 \leq \alpha \leq r\}$ of R. In the geometric picture R will be an algebra of meromorphic functions on Γ_+ and $\partial_{i\alpha}$ will be taking the partial derivative w.r.t. $t_{i\alpha}$. Let ∂ be the derivation $\sum_{\alpha=1}^{r} \partial_{1\alpha}$. The equations of the hierarchy can be formulated conveniently in terms of relations for certain elements from the ring $\mathfrak{gl}_r(R)((\partial, \partial^{-1}))$ of pseudo differential operators in ∂ with coefficients from $\mathfrak{gl}_r(R)$. We extend the $\partial_{i\alpha}$ to derivations of $\mathfrak{gl}_r(R)((\partial, \partial^{-1}))$ by letting it act coefficient wise on elements of $\mathfrak{gl}_r(R)$ and on an element $\sum_{j \leq N} p_j \partial^j$ of $\mathfrak{gl}_r(R)((\partial, \partial^{-1}))$ by

$$\partial_{i\alpha}\Big(\sum_{j \leq N} p_j \partial^j\Big) = \sum_{j \leq N} \partial_{i\alpha}(p_j) \partial^j$$

In the ring $\mathfrak{gl}_r(R)((\partial, \partial^{-1}))$ we denote the differential operator part $\sum_{j \geq 0} p_j \partial^j$ of $P = \sum_j p_j \partial^j$ by P_+ and we write P_- for $P - P_+$. Let E_α, $1 \leq \alpha \leq r$, be as in the foregoing subsection. In the ring $\mathfrak{gl}_r(R)((\partial, \partial^{-1}))$ we consider elements of the form

$$L = \partial + \sum_{j>0} \ell_j \partial^{-j} \quad \text{and} \quad U_\alpha = E_\alpha + \sum_{j>0} u_{\alpha j} \partial^{-j} \tag{16}$$

Examples of this type of operators can be obtained as follows: take the trivial example $L = \mathrm{Id}\,\partial = \partial$ and $U_\alpha = E_\alpha$ and choose some $K = \mathrm{Id} + \sum_{j>0} k_j \partial^{-j}$ in $\mathfrak{gl}_r(R)((\partial, \partial^{-1}))$. Such a K is invertible and

$$L = K \partial K^{-1} \quad \text{and} \quad U_\alpha = K E_\alpha K^{-1} \tag{17}$$

have the form (16). Following [27], the equations of the multicomponent KP-hierarchy are

$$[L, U_\alpha] = [U_\alpha, U_\beta] = 0 \tag{18}$$

$$U_\alpha U_\beta = \delta_{\alpha\beta} U_\beta \tag{19}$$

$$\partial_{i\alpha}(L) = [(L^i U_\alpha)_+, L] = [B_{i\alpha}, L] \tag{20}$$

$$\partial_{i\alpha}(U_\beta) = [B_{i\alpha}, U_\beta]. \tag{21}$$

The equations (18) and (19) are satisfied by all elements L and $\{U_\alpha\}$ of the form (17). The equations (20) and (21) boil down to non-linear differential equations for the $\{u_{\alpha j}\}$ and $\{\ell_j\}$. Since all the solutions of the multicomponent KP-hierarchy that we will construct are of the form (17), we merely have to focus on (20) and (21). These last equations can be seen as compatibility conditions for a linear system. This requires the introduction of a $\mathrm{gl}_r(R)((\partial, \partial^{-1})$-module. Let M consist of the formal products

$$\{\sum_{j=-\infty}^{N} \beta_j \lambda^j\} \exp(\sum_{\substack{i>1 \\ 1\le\alpha\le r}} t_{i\alpha} E_\alpha \lambda^i) := \{\sum_{j=-\infty}^{N} \beta_j \lambda^j\} g(\lambda),$$

with $\beta_j \in \mathrm{gl}_r(R)$. For $\beta \in \mathrm{gl}_r(R)$, the action of β on M is defined by

$$\beta\{\sum_{j=-\infty}^{N} \beta_j \lambda^j\} g(\lambda) = \{\sum_{j=-\infty}^{N} \beta\beta_j \lambda^j\} g(\lambda).$$

The action of $\partial_{i\alpha}$ on M is defined such that it corresponds to "differentiating" this formal product w.r.t. the variable $t_{i\alpha}$, i.e.

$$\partial_{i\alpha}(\{\sum_j b_j \lambda^j\} g(\lambda)) = \{\sum_j \partial_{i\alpha}(b_j)\lambda^j + \sum_j b_j E_\alpha \lambda^{i+j}\} g(\lambda).$$

In particular we see that the action of ∂ on M

$$\partial\{\sum b_j \lambda^j\} g(\lambda) = \sum\{\partial(b_j)\lambda^j + \sum b_j \lambda^{j+1}\} g(\lambda)$$

is invertible with the inverse ∂^{-1} given by

$$\partial^{-1}\{\sum b_j \lambda^j\} g(\lambda) = \{\sum_{i=0}^{\infty} \sum_j (-1)^i \partial^i(b_j)\lambda^{j-i-1}\} g(\lambda).$$

These actions compose to a $\mathrm{gl}_r(R)((\partial, \partial^{-1}))$-module structure on M. In fact, M is a free $\mathrm{gl}_r(R)((\partial, \partial^{-1})$-module with generator $g(\lambda)$, for, if $P = \Sigma P_j \partial^j \in \mathrm{gl}_r(R)((\partial, \partial^{-1}))$, then a direct computation shows

$$P \cdot g(\lambda) = \{\sum P_j \lambda^j\} g(\lambda).$$

Let Δ be the subgroup of $\mathrm{gl}_r(\mathbf{C}(\lambda, \lambda^{-1}))$ given by

$$\Delta = \{\delta \mid \delta = \mathrm{diag}(\lambda^{k_1}, \dots, \lambda^{k_r}) \text{ with } (k_1, \dots, k_r) \in \mathbf{Z}^r\}.$$

Take any $\delta = \Sigma d_j \lambda^j$, $d_j \in \mathfrak{gl}_r(\mathbf{C})$, in Δ. To δ corresponds the element $\underline{\delta} = \Sigma d_j \partial^j$ in $\mathfrak{gl}_r(R)((\partial, \partial^{-1}))$. Then we have the notion

Definition 4.2.1. A function of *type* δ is an element of ψ of M that has the form

$$\psi = \{(\mathrm{Id} + \sum_{j<0} \psi_j \lambda^j)(\sum_k d_k \lambda^k)\} g(\lambda).$$

To any function ψ of type δ we associate the operator K_ψ in $\mathfrak{gl}_r(R)((\partial, \partial^{-1}))$ given by

$$K_\psi = \mathrm{Id} + \sum_j \psi_j \partial^j.$$

Next we assume that we have been given operators L and $\{U_\alpha\}$ of the form (17). Then we introduce the following notion:

Definition 4.2.2. A *wavefunction of type* δ for L and the $\{U_\alpha\}$ is a function ψ of type δ satisfying

(a) $L(\psi) = \lambda\psi$

(b) $U_\alpha \psi = \psi E_\alpha$

(c) $\partial_{i\alpha}(\psi) = P_{i\alpha} \cdot \psi$ with $P_{i\alpha} \in \mathfrak{gl}_r(R)[\partial]$.

The first two properties translate respectively into

$$L = K_\psi \partial K_\psi^{-1} \quad \text{and} \quad U_\alpha = K_\psi E_\alpha K_\psi^{-1}. \tag{22}$$

Hence L and the $\{U_\alpha\}$ are completely determined by ψ. One computes directly that (c) implies $P_{i\alpha} = (L^i U_\alpha)_+$ and by applying the operators $\partial_{i\alpha}$ to the equations (a) and (b) and by substituting (c) one shows

Theorem 4.2.3. *If ψ is a wavefunction of type δ, then the operators $K_\psi \partial K_\psi^{-1}$ and $\{K_\psi E_\alpha K_\psi^{-1}\}$ satisfy the equations of the multicomponent KP-hierarchy.*

The equations from definition 4.2.2 are called a linearization of the system and from theorem 4.2.3 we see that we merely have to show (*c*) if L and the $\{U_\alpha\}$ are defined by (22).

4.3. The solutions. First we consider the space H and its decomposition as in subsection 4.1. Since $m = 2$, all flags in $\mathfrak{F}$ correspond to subspaces W of H. For each W in $\mathfrak{F}$, consider

$$\Delta_W = \{\delta \mid \delta \in \Delta, \text{ there is a } \gamma \in \Gamma_+ \text{ such that } \gamma^{-1}\delta^{-1}W \text{ is transversal to } H_1\}$$

The first property of Δ_W is

Lemma 4.3.1. *The collection Δ_W is non-empty.*

For each δ in Δ_W we consider the open subset $\Gamma(\delta, W)$ of Γ_+ given by

$$\Gamma(\delta, W) = \{\gamma \mid \in \Gamma_+, \ \gamma^{-1}\delta^{-1}W \text{ is transversal to } H_1\}.$$

Let R be the ring of analytic functions on $\Gamma(\delta, W)$ and let $\partial_{i\alpha}$ be the derivation of R consisting of partial differentiation w.r.t. $t_{i\alpha}$. Then there holds

Theorem 4.3.2.
(a) *For each* $W \in \mathfrak{F}$ *and each* $\delta \in \Delta_W$ *there is a unique function* $\psi_W^\delta = \hat{\psi}_W^\delta \cdot \delta \cdot g(\lambda)$ *of type* δ, *such that* $\psi_W^\delta(\gamma) \in W$ *for all* $\gamma \in \Gamma(\delta, W)$.
(b) *The function* ψ_W^δ *from* (a) *is a wavefunction of type* δ.

For a proof, we refer to [13]. If we write $\psi_W^\delta = K_W^\delta \cdot \delta \cdot g(\lambda)$, then we know from theorem 4.2.3 that

$$L_W^\delta = K_W^\delta \partial (K_W^\delta)^{-1} \quad \text{and} \quad U_{\alpha,W}^\delta = K_W^\delta E_\alpha (K_W^\delta)^{-1}$$

are solutions of the multicomponent KP-hierarchy. The following theorem makes clear why we did not consider the commuting flows from Γ_-. Its proof can also be found in [13].

Theorem 4.3.3. *For each* $g = \sum_{j \le 0} \gamma_j \lambda^j$ *in* Γ_-, *we have* $L_{gW}^\delta = L_W^\delta$ *and* $U_{\alpha,gW}^\delta = U_{\alpha,W}^\delta$.

Remark 4.3.4. If $r > 1$, then Δ_W may contain several elements. If δ_1 and δ_2 are in Δ_W, then the solutions $\{L_W^{\delta_1}, U_{W,\alpha}^{\delta_1}\}$ and $\{L_W^{\delta_2}, U_{W,\alpha}^{\delta_2}\}$ are related by so-called differential difference equations that reduce in a specific case to the equations of the Toda-lattice, see [13]. These differential difference equations are a generalization to the KP-level of equations considered in [1].

Also in the multicomponent setting the coefficients of $\hat{\psi}_W^\delta$ can be expressed in terms of Fredholm determinants related to the line bundle $L((-1, 0))$. If $W = g_1 F^{(0)}$, with $g_1 \in \mathfrak{G}$, then we define $\tau_{g_1|H_1}: G \to \mathbf{C}$ by

$$\tau_{g_1|H_1}((g, q)) = (g, q) \cdot (f_{\Sigma_1^{(0)}})(g_1) = \det(p_{\Sigma_1^{(0)}} \circ g^{-1} \circ g_1 \circ q \mid H_1).$$

It measures the failure of G-equivariance of the section corresponding to $f_{\Sigma_1^{(0)}}$. If one takes another element $\tilde{g}_1$ of $\mathfrak{G}$ with $W = \tilde{g}_1 F^{(0)}$, then $\tau_{\tilde{g}_1|H_1}$ and $\tau_{g_1|H_1}$ differ by a non-zero constant. In Δ we consider the elements $\Delta_{i/j}$ given by

$$\Delta_{i/j} = \operatorname{diag}(\ldots \lambda, \ldots, \lambda^{-1}, \ldots),$$

where the λ-factor stands at the i-th place, the λ^{-1}-factor at the j-th place and the resulting factors are equal to 1.

If $k \in \mathbf{C}$, $|k| > 1$, then we still need the element $q_k^{(i)}$ from Γ_+ given by

$$q_k^{(i)} = \operatorname{diag}\left(\ldots 1, 1 - \frac{\lambda}{k}, 1, \ldots\right),$$

where the factor $1-\frac{\lambda}{k}$ stands at the i-th place. Then there holds

Theorem 4.3.5. *Consider a W in $\mathfrak{F}$ and a δ in Δ_W. Then $\delta^{-1}(W)=gF^{(0)}$, with $g\in\mathfrak{G}$ and we have*

(a) For all $1\leq i\leq r$, the (i,i)-entry of $\hat{\psi}_W^\delta$, is the L^2-boundary value of

$$k\longmapsto\frac{\tau_{g|H_1}(\gamma q_k^{(i)})}{\tau_{g|H_1}(\gamma)}$$

(b) For $j\neq i$, there is a lifting $\tilde{\Delta}_{i/j}$ of $\Delta_{i/j}$ to G such that the (i,j) entry of $\hat{\psi}_W^\delta$ is the L^2-boundary value of

$$k\longmapsto k^{-1}\frac{\tau_{g|H_1}(\gamma\tilde{\Delta}_{i|j}q_k^i)}{\tau_{g|H_1}(\gamma)}.$$

This theorem gives a geometric interpretation of formulae, stated in the appendix of [27] and generalizes the one component interpretation given in [25]. For the one component case a representation theoretic interpretation of the τ depending polynomially of the $\{t_{i_\alpha}\}$ was given in [18]. For a proof of theorem 4.3.5 we refer the reader to [13]. There one can also find more equations that fit in the framework just described.

Next we consider the one component case somewhat more in detail. For convenience we denote the set of independent variables simply as $t=\{t_i\,|\,i\geq 1\}$ and we see the elements of R as functions in t. Further we restrict each $\tau_{g_1|H_1}$ to Γ_+ and we write simply τ or $\tau(t)$. If $k\in\mathbf{Z}$ and $W\in\mathfrak{F}^{(k,-k)}$, then $\Delta_W=\{\lambda^{-k}\}$ and we have exactly one solution $L_W^{\lambda^{-k}}$ to the KP-hierarchy. In this way every component of $\mathfrak{F}$ leads to the same bunch of solutions of the KP-hierarchy. Hence, for the construction of solutions, it suffices to consider only one component of $\mathfrak{F}$. The different components are, however, essential for the modified systems as we will see in a moment.

In the one component case the Japanese school, see [17], translates the equations that are satisfied by the wavefunction to equations for the τ-function and these can be written in the so-called bilinear form

$$\oint\tau\left(\left(t_i-\frac{1}{ik^i}\right)\right)\tau\left(\left(s_i+\frac{1}{ik^i}\right)\right)e^{\sum_{i\geq 1}(t_i-s_i)k^i}\,dk=0, \tag{23}$$

for all (t_i) and (s_i) and with dk such that

$$\oint\frac{dk}{2\pi ik}=-1.$$

Also this formula can be given a geometric interpretation. For if $W\in\mathfrak{F}^{(0)}$ then one can consider $W^\perp$ as an element of the Grassmann manifold corresponding to $H = H_2 \oplus H_1$. If ψ_W is linked to τ by theorem 4.3.5, then one can show

Theorem 4.3.6. *The wavefunction $\psi_{W^\perp}$ can be expressed in τ by*

$$\hat{\psi}_{W^\perp}(\lambda) = \lambda^{-1} \frac{\tau\left(\left(t_i + \frac{1}{i\lambda^i}\right)\right)}{\tau((t_i))}$$

If we consider instead of Γ_+ the group of flows consisting of the adjoints of the elements of Γ_+, then relation (23) boils down to the orthogonality relations for ψ_W and $\psi_{W^\perp}$.

For $\ell \in \mathbf{Z}$, let W_ℓ be a general element of $\mathfrak{F}^{(\ell,-\ell)}$. According to the theorems 4.3.2 and 4.3.5 there corresponds a wavefunction of type λ^ℓ to W_ℓ and a τ-function τ_ℓ to W_ℓ. Consider an increasing set of integers $\ell_1 < \ell_2 \cdots < \ell_s$ and denote it by $\hat{\ell}$. To $\hat{\ell}$ corresponds a decomposition of H by

$$H = H_1 \oplus \cdots \oplus H_{s+1}, \text{ where } H_1 = \{ \sum_{i \geq \ell_s} a_i \lambda^i \in H \},$$

$$H_j = \{ \sum_{\substack{i \geq \ell_j \\ i < \ell_{j+1}}} a_i \lambda^i \in H \} \text{ for } j,\ 1 \leq j < s \text{ and, } H_{s+1} = \{ \sum_{i < \ell_1} a_i \lambda^i \in H \}.$$

Denote the flagvariety corresponding to this decomposition by $\mathfrak{F}_{\hat{\ell}}$. Then we can describe the elements of $\mathfrak{F}_{\hat{\ell}}^{(0)}$ in terms of nonlinear equations for the corresponding τ functions.

Theorem 4.3.7. *For $\ell_1 < \cdots < \ell_s$, let W_{ℓ_i} be a general elements of $\mathfrak{F}^{(\ell_i,-\ell_i)}$ and let τ_{ℓ_i} be a with W_{ℓ_i} corresponding τ-function. Then the $\{W_{\ell_i}\}$ determine an element of $\mathfrak{F}_{\hat{\ell}}^{(0)}$ if and only if the $\{\tau_{\ell_i}\}$ satisfy the following bilinear equations*

$$\oint \tau_{\ell_{i+1}}\left(\left(t_j - \frac{1}{j\lambda^j}\right)\right) \tau_{\ell_i}\left(\left(s_j + \frac{1}{j\lambda^j}\right)\right) \lambda^{\ell_{i+1}-\ell_i} e^{\sum_{j \geq 1}(t_j - s_j)\lambda^j} d\lambda = 0,$$

for all $\{t_j\}$ and $\{s_j\}$ and for i, $1 \leq i < s$.

The simplest case of these equations $\ell_1 < \ell_2$, gives a relation between 2 τ-functions. It has been considered in [17] and is called the (ℓ_2, ℓ_1)-modified KP-hierarchy there.

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