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Small Resolutions of Schubert Varieties in Symplectic and Orthogonal Grassmannians

By

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§ 1. Introduction

Let G be a semisimple algebraic group over **C**, and B a Borel subgroup of G. Let P be a parabolic subgroup of G that contains B. Denote by W the Weyl group of G with respect to a fixed maximal torus $T \subset B$, and let $W_P \subset W$ be the Weyl group of P. We denote the set of minimal representatives of W/W_P by W^P. For $\omega \in W^P$, $X(\omega)$ denotes the Schubert variety in G/P corresponding to ω. $X(\omega)$ is the Zariski closure of the B-orbit of a unique T-fixed point e_ω of G/P. We call e_ω 'the centre' of $X(\omega)$. Our conventions for labelling the simple roots in W are the same as in [1].

Recall [3] that a resolution $p: \tilde{X} \to X$ of an irreducible complex variety X is said to be small if, for each $i > 0$, one has

$$\operatorname{codim}_X \{x \in X \mid \dim p^{-1}(x) \geq i\} > 2i.$$

If p is a small resolution, then for any $i \geq 0$, for the intersection cohomology sheaf $\mathscr{H}(X)$ (with respect to the middle perversity), the stalk $\mathscr{H}^i(X)_x$ is isomorphic to the singular cohomology group $H^i(p^{-1}(x); \mathbf{C})$. (See [3]).

If $p: \widetilde{X(\lambda)} \to X(\lambda)$ is a small resolution of a Schubert variety $X(\lambda)$ in G/P, then for $\tau \leq \lambda$, the Poincaré polynomial $P_t(p^{-1}(e_\tau))$, $q = t^2$, equals the Kazhdan-Lusztig polynomial $P_{\lambda w, \tau w}$ where $w = w_0(P)$. A.V. Zelevinskii [7] has constructed small resolutions for all Schubert varieties in the Grassmannian $G_{r,n} = SL(n, \mathbf{C})/P_r$, $1 \leq r < n$, where P_r is the maximal parabolic obtained by omitting the simple root α_r. In §2 of our paper we generalise Zelevinskii's

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construction to obtain resolutions of Schubert varieties in any G/P and address the question as to which of them are small.

Existence of Schubert varieties for which the known procedures of desingularisations do not yield small resolutions have been observed earlier (see [7]). However, to the best of our knowledge, examples of Schubert varieties not admitting any small resolutions at all are not to be found in the literature. In Theorem 1.2 below we give examples of such varieties in $Sp(2n)/Q$ for every parabolic $Q \subset P_n$. Here P_n denotes the maximal parabolic obtained by omitting the root α_n. Our proof of Theorem 1.2 is based on the following observation. If $p: \tilde{X} \to X$ is a resolution of a normal irreducible variety then by Zariski's Main Theorem, the fibre $p^{-1}(x)$ over any singular point x of X, has positive dimension. Therefore, any normal irreducible variety X with codimension 2 singular locus cannot have any small resolution. Using [6] we exhibit Schubert varieties with codimension two singular loci.

Let $G = Sp(2n, \mathbf{C})$ or $SO(2n, \mathbf{C})$, and let $P = P_n$. Recall, from [5], that the Schubert varieties in $Sp(2n, \mathbf{C})/P$ are indexed by $\bigcup_{0 \le r \le n} I_{n,r}$ where $I_{n,r} = \{(\lambda_1, \ldots, \lambda_r) \mid 1 \le \lambda_1 < \cdots < \lambda_r \le n\}$. There is a unique sequence of length 0, namely the empty sequence ( ). The Bruhat order on W/W_P agrees with the ordering on $\cup I_{n,r}$ where $\lambda = (\lambda_1, \cdots, \lambda_r) \ge \mu = (\mu_1, \cdots, \mu_s)$ if $r \le s$, $\lambda_i \ge \mu_i$ for $1 \le i \le r$. The dimension $\dim X(\lambda)$ is given as

$$\dim X(\lambda) = \sum_{i=1}^{r} \lambda_i + (n+1)(n-r) - \frac{1}{2} n(n+1), \tag{1.1}$$

where $\lambda \in I_{n,r}$.

Similarly, the Schubert varieties in $SO(2n)/P_n$ are labelled by the set $\bigcup_{\substack{0 \le r \le n \\ (n-r)\text{even}}} I_{n,r}$, with the Bruhat ordering on W/W_P exactly as in the symplectic case. Here, for $\lambda \in I_{n,r}$

$$\dim X(\lambda) = \sum_{i \le r} \lambda_i + n(n-r) - \frac{1}{2} n(n+1). \tag{1.2}$$

The main results of this paper are

Theorem 1.1. *Let $\lambda \in I_{n,r}$.*

(i) *The Schubert variety $X(\lambda) \subset Sp(2n)/P_n$ has a small resolution if $\lambda_r \le n - r$.*

(ii) *Assume $n - r$ is even so that λ gives rise to a Schubert variety $X(\lambda)$ in $SO(2n)/P_n$. $X(\lambda)$ has a small resolution if*

(a) *$\lambda_r < n - r$ or*

(b) *For $r \ge 2$, $\lambda_r = n$, $\lambda_{r-1} \le n - r$.*

Theorem 1.2. *Let $\lambda = (n)$, $n \ge 3$ and let Q be any parabolic subgroup* *contained in* $P_n \subset Sp(2n)$. *Let* $X(\Lambda)$ *be the inverse image of* $X(\lambda) \subset Sp(2n)/P_n$ *under the projection* $Sp(2n)/Q \to Sp(2n)/P_n$. *Then* $X(\Lambda)$ *does not admit any small resolution.*

The above theorems are proved in §4.

Actually, we construct small resolutions for a larger class of Schubert varieties than that considered in Theorem 1.1. See Theorem 4.2 for the precise statement. Incidentally Theorem 1.2 shows that the condition "$\lambda_r \leq n - r$" in Theorem 1.1 (i) cannot be dispensed with in general.

This paper-and in particular our proof of Theorem 1.1-was inspired by the work of Zelevinskii [7].

In our future work we plan to investigate the existence of small resolutions for Schubert varieties in the case of exceptional groups.

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§2. Bott-Samelson Resolution

Let G be any semisimple group, Q a parabolic subgroup containing a fixed Borel subgroup B and a maximal torus $T \subset B$.

Let $X(\lambda) \subset G/Q$ be any Schubert variety, and let P_λ be the largest subgroup of G which leaves $X(\lambda)$ invariant for the left action of G on G/Q. Clearly P_λ is a parabolic subgroup containing B. We refer to P_λ as the 'stabilizer' of $X(\lambda)$. Note that it is possible to find a parabolic subgroup $P \subset P_\lambda$ and a Schubert subvariety $X(\lambda')$, $\lambda' < \lambda$, such that $P_\lambda X(\lambda') = PX(\lambda') = X(\lambda)$. Let $R_0 = P_\lambda \cap P_{\lambda'}$. Then the map $\pi_0 \colon P_\lambda \times_{R_0} X(\lambda') \to X(\lambda)$ given by $[g, x] \mapsto gx$ is surjective and P_λ-equivariant, but not birational in general. However it is possible to choose P and $\lambda' < \lambda$ such that $\dim P/R_1$ equals the codimension of $X(\lambda')$ in $X(\lambda)$ and so $\pi_1 : P \times_{R_1} X(\lambda') \to X(\lambda)$ where $R_1 = P \cap P'_\lambda$, is P-equivariant and birational. For example one can choose $\lambda' = s_\alpha \lambda$ for a suitable simple root α and $P = P_\alpha$ the minimal parabolic corresponding to the simple root α.

Since any 1-dimensional Schubert variety is smooth, iterating this construction leads to a P-equivariant resolution

$$p \colon P_{(1)} \times_{R_1} P_{(2)} \times \cdots \times_{R_{r-2}} P_{(r-1)} \times_{R_{r-1}} X(\lambda^r) = \widetilde{X(\lambda)} \longrightarrow X(\lambda)$$

where $\lambda^1 = \lambda$, $\lambda^2 = \lambda', \cdots, P_{(i)} \subset P_{\lambda^i}$, $1 \leq i \leq r$, $P_{(r)} = P_{\lambda^r}$, $R_i = P_{(i)} \cap P_{(i+1)}$ and $X(\lambda^r)$ is smooth.

The above resolution is usually referred to as a Bott-Samelson resolution. When $X(\lambda) \subset G/B$ such resolutions were obtained by H. Hansen [4] and M. Demazure [2]. When $X(\lambda) \subset G_{r,n}$ is a Grassmannian Schubert variety the small resolutions constructed by Zelevinskii [7] are of the above type. Indeed one can check that his resolutions correspond to choosing $P_{(i)}$ to be *equal* to the stabilizer of $X(\lambda)$ at each step. In all our applications below it turns out that, as in Zelevinskii's work [7], at each step one can choose $P_{(i)}$ to be the stabilizer of λ^i, so we can iterate the construction $\pi_0: P_\lambda \times_{R_0} X(\lambda') \to X(\lambda)$ to obtain a desingularization of $X(\lambda)$ which is P_λ-equivariant.

We need a formula for the dimension of the fibre $p^{-1}(x)$ for a given Bott-Samelson resolution $p: \widetilde{X(\lambda)} \to X(\lambda)$. Suppose that $\widetilde{X(\lambda)} = P \times_R \widetilde{X(\lambda')}$ where $P = P_1 \subset P_\lambda$, $R = R_1$, and $p': \widetilde{X(\lambda')} \to X(\lambda')$ is a P'-equivariant Bott-Samelson resolution of $X(\lambda')$, $P' \subset P_{\lambda'}$. Since $p: \widetilde{X(\lambda)} = P \times_R \widetilde{X(\lambda')} \to X(\lambda)$ is P-equivariant, $p^{-1}(x) \cong p^{-1}(y)$ if x and y are in the same P-orbit. In fact if U is a P-orbit in $X(\lambda)$, then $p|p^{-1}(U): p^{-1}(U) \to U$ is a locally trivial bundle, and therefore $\dim p^{-1}(x) = \dim p^{-1}(U) - \dim U$ for all $x \in U$. If $\bar{U}$ denotes the Zariski closure of $U \subset X(\lambda)$, then $\bar{U}$ is a P-stable Schubert subvariety of $X(\lambda)$.

Conversely, if $X(\tau)$ is a P-stable Schubert subvariety of $X(\lambda)$, then it is the closure of the P-orbit of e_τ, the centre of $X(\tau)$. We shall denote the dimension of $p^{-1}(e_\tau)$ by $f_{p,\tau}$ and the codimension of $X(\tau)$ in $X(\lambda)$ by $\operatorname{codim}_\lambda \tau$.

Lemma 2.1. *Let $U(\tau) \subset X(\lambda)$ be the P-orbit of e_τ, and let $\pi: P \times_R X(\lambda') \to X(\lambda)$ and $p: P \times_R \widetilde{X(\lambda')} \to X(\lambda)$ be the birational morphisms constructed above. Then*

(a) $\pi^{-1}(U(\tau)) = P \times_R Z$ *where*

$$Z = \bigcup_\mu Re_\mu, \ \mu \in S(\tau, \lambda) = \{\sigma \le \lambda' \mid PX(\sigma) = X(\tau),\ RX(\sigma) = X(\sigma)\}.$$

(b) $p^{-1}(U(\tau)) = P \times_R p'^{-1}(Z)$, *where $p': \widetilde{X(\lambda')} \to X(\lambda')$ is a P'-equivariant Bott-Samelson resolution, $P' \subset P_{\lambda'}$.*
In particular

(c) $f_{p,\tau} = \operatorname{codim}_\lambda \lambda' - \operatorname{codim}_\tau \mu + f_{p',\mu}$ *for some* $\mu \in S(\tau, \lambda)$.

Proof. (a) Let $\mu \in S(\tau, \lambda)$. Suppose $ge_\mu = e_\tau$ for some $g \in P$. Then, as $U(\tau)$ is the orbit of e_τ, it follows that $Pe_\mu = U(\tau)$. Hence $\pi(P \times_R Re_\mu) = U(\tau)$. On the other hand if $\pi[g, x] = gx \in U(\tau)$, then clearly $\pi(P \times_R Rx) \subset U(\tau)$. Now the Zariski closure $\overline{Rx}$ is a Schubert variety $X(\sigma) \subset X(\lambda')$ which is R-stable, and $Rx = Re_\sigma$. Hence there exists an $h \in R$ such that $he_\sigma = x$, and an element $h' \in P$ such that $h'x = e_\tau$. Thus $h'he_\sigma = e_\tau$ and so $PX(\sigma) = X(\tau)$. This proves (a).

Part (b) follows from (a).

To prove (c), note that $\dim U(\tau) = \dim X(\tau)$. Since $p \mid p^{-1}(U(\tau)) : p^{-1}(U(\tau)) \to U(\tau)$ is a locally trivial bundle, denoting by $V(\sigma)$ the orbit of e_σ under R we see that $P \times_R p'^{-1}(V(\sigma))$ is the total space of a fibre bundle with base space P/R and fibre $p'^{-1}(V(\sigma))$ and so

$$\begin{aligned} f_{p,\tau} &= \dim p^{-1}(e_\tau) \\ &= \dim p^{-1}(U(\tau)) - \dim U(\tau) \\ &= \max_{\sigma \in S(\tau,\lambda)} \{\dim P/R + \dim p'^{-1}(V(\sigma))\} - \dim X(\tau) \\ &= \max_{\sigma \in S(\tau,\lambda)} \{\operatorname{codim}_\lambda \lambda' + \dim V(\sigma) + f_{p',\sigma}\} - \dim X(\tau) \\ &= \max_{\sigma \in S(\tau,\lambda)} \{\operatorname{codim}_\lambda \lambda' + f_{p',\sigma} - (\dim X(\tau) - \dim X(\sigma))\} \\ &= \max_{\sigma \in S(\tau,\lambda)} \{\operatorname{codim}_\lambda \lambda' - \operatorname{codim}_\tau \sigma + f_{p',\sigma}\} \\ &= \operatorname{codim}_\lambda \lambda' - \operatorname{codim}_\tau \mu + f_{p',\mu} \end{aligned}$$

for *some* $\mu \in S(\tau, \lambda)$, completing the proof.

Corollary 2.2. *If $X(\lambda')$ is smooth, and $p = \pi$ then*

$$f_{p,\tau} = \operatorname{codim}_\lambda \lambda' - \operatorname{codim}_{X(\tau)} X(\lambda') \cap X(\tau).$$

Proof. Since $f_{p',\sigma}$ in 2.1. (c) above is zero, the $\max_{\sigma \in S(\tau,\lambda)} \{\operatorname{codim}_\lambda \lambda' - \operatorname{codim}_\tau \sigma\}$ is attained when $\dim X(\sigma) = \dim X(\tau) \cap X(\lambda')$.

Remark 2.3. Since $\{x \in X(\lambda) \mid \dim p^{-1}(x) \geq i\}$ is P-stable, and since there are only finitely many P-orbits, $\operatorname{codim} \{x \in X(\lambda) \mid \dim p^{-1}(x) \geq i\} = \operatorname{codim} U(\tau) = \operatorname{codim}_\lambda \tau$ for some P-stable $X(\tau)$ where $\dim p^{-1}(e_\tau) = f_{p,\tau} = i$. Therefore, to check smallness of p, it suffices to verify that for each P-stable $X(\tau)$, and for each $\sigma \in S(\tau, \lambda)$, $\operatorname{codim}_\lambda \lambda' - \operatorname{codim}_\tau \sigma + f_{p',\sigma} < \frac{1}{2} \operatorname{codim}_\lambda \tau$, for $i > 0$.

Theorem 2.4. *Let $f: Y \to Z$ be a locally trivial bundle with smooth fibres between irreducible projective varieties. If $p: \tilde{Z} \to Z$ is a small resolution, then $q: Y \times_Z \tilde{Z} \to Y$, $q[y, z] = y$ is a small resolution.*

Proof. Let

$$Z_i = \{z \in Z \mid \dim p^{-1}(z) \geq i\} \subset Z.$$

Then $Y_i = \{y \in Y \mid \dim q^{-1}(y) \geq i\} = f^{-1}(Z_i)$, and $\operatorname{codim}_Y Y_i = \operatorname{codim}_Z Z_i$ since $f: Y \to Z$ is a locally trivial bundle. This completes the proof.

§3. Schubert Varieties in G/P_n

It is a well-known fact that, the Schubert subvarieties of $SL(n)/P_r$ are indexed by the set $I_{n,r} = \{(\lambda_1, \cdots, \lambda_r) \mid 1 \leq \lambda_1 < \cdots < \lambda_r \leq n\}$. When $G = Sp(2n)$ or $SO(2n)$, and $P = P_n$, the end parabolic, one identifies W/W_P with $\bigcup_{0 \leq r \leq n} I_{n,r}$ in the case $G = Sp(2n)$, and with $\bigcup_{\substack{0 \leq r \leq n \\ (n-r)-\text{even}}} I_{n,r}$ in the case $G = SO(2n)$, as described in the introduction (cf. [5]).

Thus any $\lambda \in I_{n,r}$ defines a Schubert variety in $Sp(2n)/P_n$, a Schubert variety in $SO(2n)/P_n$ if $n - r$ is even, and a Schubert variety in $SL(n)/P_r$ if $1 \leq r \leq n-1$. We denote any one of these varieties by the same symbol $X(\lambda)$ and it will be made explicit which one is meant whenever there is possibility of confusion.

Let $1 \leq r \leq n$, and let $\lambda = (\lambda_1, \cdots, \lambda_r) \in I_{n,r}$. A maximal subsequence of λ of consecutive integers will be referred to as a 'block' of λ. The element $\lambda \in I_{n,r}$ is simply the concatenation of the blocks of λ. In fact if a_i is the length and k_i is the last term of the i^{th} block, then λ is determined by the $2 \times m$ matrix $\begin{pmatrix} k_1 & \cdots & k_m \\ a_1 & \cdots & a_m \end{pmatrix}$ where m is the number of blocks in λ.

On the other hand, starting with a $2 \times p$ matrix $\begin{pmatrix} \ell_1 & \cdots & \ell_p \\ c_1 & \cdots & c_p \end{pmatrix}$ with $0 < \ell_1 < \ell_2 < \cdots < \ell_p \leq n$, $0 \leq c_i - \ell_i - \ell_{i-1}$, $(\ell_0 = 0)$, $\sum c_i = r$, we obtain a unique element μ of $I_{n,r}$ with at most p blocks. We often write

$$\mu = \begin{pmatrix} \ell_1 & \cdots & \ell_p \\ c_1 & \cdots & c_p \end{pmatrix}.$$

Theorem 3.1. *With the above notations, one has the following description of P_λ, the largest parabolic which leaves $X(\lambda)$ stable*:

(i) *Let $G = SL(n)$ or $Sp(2n)$. Then P_λ is the parabolic subgroup obtained by omitting the simple roots $\{\alpha_{k_1}, \cdots, \alpha_{k_m}\}$.*

(ii) *Let $G = SO(2n)$. Let $a_m \geq 2$ when $k_m = n$. Then P_λ is the parabolic subgroup obtained by omitting the simple roots $\{\alpha_{k_i} \mid 1 \leq i \leq m\}$. If $k_m = n$ and $a_m = 1$, then P_λ is obtained by omitting $\{\alpha_{k_i} \mid i \leq m-1\}$.*

Proof. We will only prove part (ii), part (i) being similar. Let $P_{\alpha_i} X(\lambda) = X(\mu)$. (Here P_{α_i} is the minimal parabolic subgroup of G and should not be confused with P_i, the latter being a maximal parabolic obtained by omitting the simple root α_i.) Write $\lambda = (\lambda_1, \cdots, \lambda_r)$. Let $1 \leq i \leq n-1$. Then $\mu = \max\{\lambda, \lambda'\}$ where

$$\lambda' = \begin{cases} \lambda \text{ if either both } i,\ i+1 \text{ occur in } \lambda \text{ or neither of them occur in } \lambda. \\ (\lambda_1, \cdots, \lambda_{t-1},\ i,\ \lambda_{t+1}, \cdots, \lambda_r) \text{ if } \lambda_t = i+1,\ \lambda_{t-1} \neq i. \\ (\lambda_1, \cdots, \lambda_{t-1},\ i+1,\ \lambda_{t+1}, \cdots, \lambda_r) \text{ if } \lambda_t = i,\ \lambda_{t+1} \neq i+1. \end{cases}$$

It follows that $\lambda' \le \lambda$ (equivalently $\mu = \lambda$), if and only if i is not the last term of a block in λ. Thus, in this case $P_{\alpha_i} X(\lambda) = X(\lambda)$. In case $i = n$, one has $\mu = \max\{\lambda, \lambda'\}$ where

$$\lambda' = \begin{cases} \lambda, & \text{if } \lambda_{r-1} \neq n-1 \\ (\lambda_1, \cdots, \lambda_{r-2}) & \text{if } \lambda_{r-1} = n-1. \end{cases}$$

Therefore $\lambda' \le \lambda$ except when $\lambda_{r-1} = n-1$ (and hence $\lambda_r = n$). It follows that $P_{\alpha_n} X(\lambda) = X(\lambda)$ unless $\lambda_{r-1} = n-1$, $\lambda_r = n$. This proves part (ii).

Example 3.2.

(i) Let $\lambda = (1, 2, 3, 6) \in I_{6,4}$. In case $G = SL(6)$, or $Sp(12)$, P_λ corresponds to omitting $\{\alpha_3, \alpha_6\}$. When $G = SO(12)$, P_λ is obtained by omitting $\{\alpha_3\}$.

(ii) Let $\mu = (2, 3, 5, 6) \in I_{6,2}$. Then for $G = SL(6)$, $Sp(12)$, or $SO(12)$, $P_\mu = \{\alpha_3, \alpha_6\}$.

Corollary 3.3. *Suppose $X(\tau)$ is a (Schubert) subvariety of $X(\lambda)$, $\lambda \in I_{n,r}$.*

(i) *Let $G = SL(n)$, or $Sp(2n)$. Then $X(\tau)$ is P_λ-stable if and only if there exists a sequence $c(\tau, \lambda) = (c_1, \cdots, c_m)$ of non-negative integers such that $0 \le a_i + c_i - c_{i-1} \le k_i - k_{i-1}$, with $c_m = 0$ when $G = SL(n)$, and*

$$\tau = \begin{pmatrix} k_1 & k_2 & \cdots & k_m \\ a_1 + c_1 & a_1 + c_2 - c_1 & \cdots & a_m + c_m - c_{m-1} \end{pmatrix}.$$

(ii) *Let $G = SO(2n)$. $X(\tau) \subset X(\lambda)$ is P_λ-stable if and only if there exists a sequence $c(\tau, \lambda) = (c_1, \cdots, c_m)$ such that $0 \le a_i + c_i - c_{i-1} \le k_i - k_{i-1}$, $c_m \equiv 0$ mod 2 when $k_m = n$, and, moreover, $0 \le a_m + c_m - c_{m-1} \le 1$ if $(k_m, a_m) = (n, 1)$ so that*

(a) *When $k_m \le n-2$, one has*

$$\tau = \begin{pmatrix} k_1 & k_2 & \cdots & k_m & n \\ a_1 + c_1 & a_2 + c_2 - c_1 & \cdots & a_m + c_m - c_{m-1} & \varepsilon \end{pmatrix}$$

where $\varepsilon = \frac{1}{2}(1 - (-1)^{c_m})$.

(b) *When $k_m = n-1$, one has*

$$\tau = \begin{pmatrix} k_1 & k_2 & \cdots & n-1 & n \\ a_1 + c_1 & a_2 + c_2 - c_1 & \cdots & a_m + c_m - c_{m-1} & \varepsilon \end{pmatrix}$$

with $\varepsilon = \frac{1}{2}(1 - (-1)^{c_m})$ and $a_m + c_m - c_{m-1} = 0$ when $\varepsilon = 1$.

(c) *when $k_m = n$, one has*

$$\tau = \begin{pmatrix} k_1 & k_2 & \cdots & k_m \\ a_1 + c_1 & a_2 + c_2 - c_1 & \cdots & a_m + c_m - c_{m-1} \end{pmatrix}.$$

In particular τ has at most m blocks.

Proof. Suppose $X(\tau) \subset X(\lambda)$ is P_λ-stable. Then the existence of the sequence $c(\tau, \lambda)$ such as in the corollary follows from Theorem 3.1 and the fact that

$$W(SL(n))/W_{P_r} \cong I_{n,r}$$

$$W(Sp(2n))/W_{P_n} \cong \bigcup_{0 \le r \le n} I_{n,r}$$

$$W(SO(2n))/W_{P_n} \cong \bigcup_{\substack{0 \le r \le n \\ (n-r)-\text{even}}} I_{n,r}.$$

Conversely, suppose $(c_1, \cdots, c_m)$ is a sequence of non-negative integers satisfying the conditions of the Corollary. Then $\tau \in W/W_{P_r}$ in case $G = SL(n)$, and $\tau \in W/W_{P_n}$ in case $G = Sp(2n)$ or $SO(2n)$. From Theorem 3.1 it follows that $X(\tau)$ is P_λ-stable, and so we need only show that $X(\tau) \subset X(\lambda)$. This follows from the observation that

$$(a_1 + c_1) + (a_1 + c_2 - c_i) + \cdots + (a_i + c_i - c_{i-1}) = \sum_{1 \le j \le i} a_j + c_i \ge \sum_{1 \le j \le i} a_j,$$

c_i being non–negative.

Definition 3.4. *For τ, λ as in the above theorem, we call $c(\tau, \lambda) = (c_1, \cdots, c_m)$ the 'depth' of τ in λ.* [cf. [7]]

Corollary 3.5. *Let $\lambda = (1, 2, \cdots, r) \in I_{n,r}$, with $(n - r)$ even when $G = SO(2n)$. Then $X(\lambda) \subset G/P_n$ is smooth for $G = Sp(2n)$ or $SO(2n)$. If $\mu = (1, 2, \cdots, r - 1, n) \in I_{n,r}$ with $n - r$ even, then $X(\mu)$ is smooth in $SO(2n)/P_n$.*

Proof. From Corollary 3.3, it follows that P_λ acts transitively on $X(\lambda)$. Hence $X(\lambda)$ is isomorphic to the homogeneous variety P_λ/I_λ where I_λ is the isotropy at e_λ. In fact for an obvious inclusion of $Sp(2(n - r))$ (respectively $SO(2(n - r))$) in $Sp(2n)$ (respectively $SO(2n)$) one has $X(\lambda) \cong Sp(2(n - r))/P_{n-r}$ (respectively $SO(2(n - r))/P_{n-r}$).

Similarly one proves $X(\mu)$ is smooth.

The next lemma gives a formula for the codimension $\text{codim}_\lambda \tau$ for a P_λ-stable subvariety $X(\tau) \subset X(\lambda) \subset G/P_n$ where $G = Sp(2n)$ or $SO(2n)$. If λ, $\tau \in I_{n,d}$ we write $X^{Gr}(\lambda)$ for the Schubert variety $X(\lambda) \subset SL(n)/P_d$ and we write $\text{codim}_\lambda^{Gr} \tau$ for the codimension of $X^{Gr}(\tau)$ in $X^{Gr}(\lambda)$.

Lemma 3.6. *Let $G = Sp(2n)$ or $SO(2n)$. Suppose $X(\tau)$ is a P_λ-stable subvariety of $X(\lambda) \subset G/P_n$ with depth $c(\tau, \lambda) = (c_1, \cdots, c_m)$. Then*

$$\text{codim}_\lambda \tau = \Gamma(\tau, \lambda) + q(\tau, \lambda)$$

where

$$\Gamma(\tau, \lambda) = \sum_{i=1}^{m} c_i(a_i + b_i),$$

$$q(\tau, \lambda) = \frac{1}{2} \sum_{i=1}^{m} (c_i - c_{i-1})^2,$$

where

$$b_i = \begin{cases} k_{i+1} - k_i - a_{i+1} & \text{if} \quad 0 \le i < m, \\ n - k_m + 1/2 & \text{if} \quad i = m, \ G = Sp(2n) \\ n - k_m - 1/2 & \text{if} \quad i = m, \ G = SO(2n), \end{cases}$$

$c_0 = 0 = k_0$.

Proof. Let $N = n + 1$ or n according as $G = Sp(2n)$ or $G = SO(2n)$. From equations 1.1 and 1.2,

$$\dim X(\lambda) = \sum_{1 \le i \le r} \lambda_i + N(n - r) - \frac{1}{2} n(n + 1),$$

and

$$\dim X(\tau) = \sum_{1 \le j \le r + \varepsilon + c_m} \tau_j + N(n - r - c_m - \varepsilon) - \frac{1}{2} n(n + 1).$$

Let $s = r + c_m + \varepsilon$, so that $\tau \in I_{n,s}$. Note that $\varepsilon = 1$ implies $N = n = \tau_{r+c_m+1}$. Thus,

$$\begin{aligned} \operatorname{codim}_\lambda \tau &= \sum_{1 \le i \le r} (\lambda_i - \tau_i) + N(c_m + \varepsilon) - \sum_{r+1 \le j \le s} \tau_j \\ &= \sum_{1 \le i \le r} (\lambda_i - \tau_i) + \sum_{r+1 \le j \le s-\varepsilon} (N - \tau_j) \\ &= \sum_{1 \le i \le r} (\lambda_i - \tau_i) + \sum_{1 \le k \le c_m} (N + k - \tau_{r+k}) - \frac{1}{2} c_m(c_m + 1) \\ &= \operatorname{codim}_\Lambda^{Gr} \tau' - \frac{c_m(c_m + 1)}{2}, \end{aligned} \tag{3.1}$$

where $\Lambda = (\lambda_1, \cdots, \lambda_r, N + 1, \cdots, N + c_m) \in I_{N+c_m, s-\varepsilon}$, $\tau' = (\tau_1, \cdots, \tau_{s-\varepsilon})$.

Writing

$$\Lambda = \begin{pmatrix} k_1 & \cdots & k_m & N + c_m \\ a_1 & \cdots & a_m & c_m \end{pmatrix},$$

$$\tau' = \begin{pmatrix} k_1 & k_2 & \cdots & k_m & N + c_m \\ a_1 + c_1 & a_2 + c_2 - c_1 & \cdots & a_m + c_m - c_{m-1} & 0 \end{pmatrix},$$

we see that (cf [7])

$$\begin{aligned}\operatorname{codim}_{\Lambda}^{Gr}\tau' &= \sum_{1\le i<m} c_i(a_i+b_i) + \sum_{1\le i\le m}(c_i^2 - c_ic_{i-1}) + c_m(a_m+N+c_m-(k_m+c_m))\\ &= \sum_{1\le i\le m} c_i(a_i+b_i) + \sum_{1\le i\le m}(c_i^2 - c_ic_{i-1}) + \frac{1}{2}c_m.\end{aligned}$$

Therefore, substituting in Equation 3.1,

$$\begin{aligned}\operatorname{codim}_{\lambda}\tau &= \sum_{1\le i\le m} c_i(a_i+b_i) + \sum_{1\le i\le m}(c_i^2 - c_ic_{i-1}) + \frac{1}{2}c_m - \frac{1}{2}c_m(c_m+1)\\ &= \sum_{1\le i\le m} c_i(a_i+b_i) + \frac{1}{2}\sum_{1\le i\le m}(c_i - c_{i-1})^2\end{aligned}$$

as required.

Remark 3.7. Note that in Lemma 3.6, $\operatorname{codim}_{\lambda}\tau \ge \Gamma(\tau, \lambda)$ where equality holds if and only if $q(\tau, \lambda) = 0$, equivalently, $\tau = \lambda$.

§4. Proof of the Main Theorems

Throughout this section we use the notations of §2 and §3. In particular, $\lambda \in I_{n,r}$ with m blocks,

$$\lambda = \begin{pmatrix} k_1 & \cdots & k_m \\ a_1 & \cdots & a_m \end{pmatrix}.$$

$P := P_{\lambda}$ is the stabilizer of $X(\lambda) \subset G/P_n$, $G = Sp(2n)$ or $G = SO(2n)$. The proof of Theorem 1.1 is based on induction on m. Unlike in the case of Schubert varieties in $SL(n)/P_r$ where $m = 1$ corresponds to a smooth variety namely a Grassmannian G_{a_1,k_1}, a Schubert variety in $Sp(2n)/P_n$ or $SO(2n)/P_n$ with $m = 1$ is not smooth in general. So we first try to construct a small resolution for $X(\lambda) \subset G/P_n$ when $m = 1$, in the theorem below.

We write $\ell(\lambda) = r$ (the "level" of λ) if $\lambda \in I_{n,r}$ for $X(\lambda) \subset G/P$. We caution the reader that $\ell(\lambda)$ is not the length $\ell_P(\lambda)$ in W/W_P. In particular, $\ell(\lambda) \ne \dim X(\lambda)$.

Theorem 4.1. *Let $\Gamma(\tau, \lambda)$ be as in Lemma 3.6.*

(i) *Suppose $X(\lambda) \subset G/P_n$, $\lambda = (k+1, \cdots, k+r) \in I_{n,r}$. Let $k < N - 2r$, where $N = n+1$ if $G = Sp(2n)$, $N = n$ if $G = SO(2n)$. Then there exists a P_{λ}-equivariant small resolution $p\colon \widetilde{X(\lambda)} \to X(\lambda)$ for $X(\lambda)$ such that $2f_{p,\tau} \le \Gamma(\tau, \lambda)$, with equality only when $\tau = \lambda$.*

(ii) *Let $X(\lambda) \subset SO(2n)/P_n$ where $\lambda = (k+1, \cdots, k+r, n)$, and $0 < r < n-1$,*

$k < n-2r$. Then there exists a P_λ-equivariant small resolution for $p\colon X(\lambda)\to X(\lambda)$ where for any P_λ-stable $X(\tau) \subset X(\lambda)$, $2f_{p,\tau} \leq \Gamma(\tau, \lambda)$ with equality only if $\lambda = \tau$.

Proof of (i). Let $\lambda' = (1, 2, \cdots, r)$. By Corollary 3.5 $X(\lambda')$ is smooth, and using 3.3, we see that $PX(\lambda') = X(\lambda)$. Thus $P \times_R X(\lambda')$ is smooth where $P = P_\lambda$, $P' = P_{\lambda'}$. $R = P_\lambda \cap P'_\lambda$. Consider the morphism $p\colon P \times_R X(\lambda') \to X(\lambda)$, $p[g, x] = gx$. Note that from Equations (1.1) and (1.2), $\text{codim}_\lambda \lambda' = kr$. Also a simple computation shows $\dim P/R = \dim G/R - \dim G/P = kr$. Hence $p\colon P \times_R X(\lambda') \to X(\lambda)$ is a Bott-Samelson resolution.

Let $X(\tau)$ be P-stable with depth $c(\tau, \lambda) = (c_1)$. Then

$$\tau = \begin{cases} (k - c_1 + 1, \cdots, k + r, n) \text{ if } c_1 \text{ is odd, } G = SO(2n) \\ (k - c_1 + 1, \cdots, k + r) \text{ otherwise.} \end{cases}$$

From 3.6,

$$\begin{aligned}\Gamma(\tau, \lambda) &= c_1(r + N - (k + r) - 1/2) \\ &= c_1(N - k - 1/2).\end{aligned}$$

Now, by Corollary 2.2

$$f_{p,\tau} = \text{codim}_\lambda \lambda' - \text{codim}_{X(\tau)}(X(\lambda') \cap X(\tau)).$$

Note that $X(\tau) \cap X(\lambda') = X(\tau \wedge \lambda')$ where

$$(\tau \wedge \lambda') = \begin{cases} \min\{\tau_i, \lambda'_i\}, & i \leq \ell(\lambda') \\ \tau_i, & i > \ell(\lambda'). \end{cases}$$

Hence $\tau \wedge \lambda' = (1, 2, \cdots, r, \tau_{r+1}, \cdots, \tau_s)$, where $s = \ell(\tau)$. In particular $\ell(\tau) = \ell(\tau \wedge \lambda')$ and so by Corollary 2.2,

$$\begin{aligned} f_{p,\tau} &= \text{codim}_\lambda \lambda' - \text{codim}_\tau(\tau \wedge \lambda') \\ &= \text{codim}_\lambda^{Gr} \lambda' - \text{codim}_\tau^{Gr} \tau \wedge \lambda' \\ &= kr - r(k - c_1) = rc_1. \end{aligned}$$

Therefore $\Gamma(\tau, \lambda) - 2f_{p,\tau} = c_1(N - k - 2r - (1/2))$. The RHS is non-negative since $k + 2r < N$, and $N - k - 2r$ is an integer. Also $2f_{p,\tau} = \Gamma(\tau, \lambda)$ implies $c_1 = 0$, in which case $f_{p,\tau} = 0 = \Gamma(\tau, \lambda)$ and $\tau = \lambda$.

Proof of (ii): In this case let $\lambda' = (1, 2, \cdots, r, n)$. By Corollary 3.5 $X(\lambda') \cong SO(2n)/P_{n-r}$ is smooth and as in case (i) above $p\colon P \times_R X(\lambda') \to X(\lambda)$ is a resolution. Proceeding as in (i), for the P-stable $X(\tau) \subset X(\lambda)$ with depth $c(\tau, \lambda) = (c_1, c_2)$ one has $c_2 = c_1 - \delta$, $\delta = \dfrac{1 - (-1)^{c_1}}{2}$. Also

$$\Gamma(\tau, \lambda) = c_1(n-k-1) + (c_1 - \delta)(1/2)$$
$$= c_1(n-k-(1/2)) - \delta/2,$$

and, as before

$$f_{p,\tau} = rc_1.$$

Hence $\Gamma(\tau, \lambda) - 2f_{p,\tau} \geq 0$ with equality if and only if $c_1 = 0 = c_2$, equivalently $\tau = \lambda$.

We now turn to the proof of Theorem 1.1. In fact we prove the following stronger result.

Theorem 4.2. *Suppose $X(\lambda) \subset G/P_n$, $G = Sp(2n)$ or $SO(2n)$. Then there is a P_λ-equivariant small resolution $p\colon \widetilde{X(\lambda)} \to X(\lambda)$ of the Bott-Samelson type such that for any P_λ-*stable subvariety *$X(\tau) \subset X(\lambda)$ one has $f_{p,\tau} \leq \frac{1}{2}\Gamma(\tau, \lambda)$, in the following cases:*

(i) *If $\lambda = \begin{pmatrix} k_1 & \cdots & k_m \\ a_1 & \cdots & a_m \end{pmatrix} \in I_{n,r}$, has exactly m blocks, then for all $i \geq 1$,*

$$k_m < N - a_m \quad \text{and} \quad k_m < N - (a_m + \cdots + a_i) + (b_{m-1} + \cdots + b_i), \tag{4.1}$$

where $N = n+1$ if $G = Sp(2n)$, $N = n$ if $G = SO(2n)$.

(ii) *For $\lambda = \begin{pmatrix} k_1 & \cdots & k_m & n \\ a_1 & \cdots & a_m & 1 \end{pmatrix} \in I_{n,r+1}$ with exactly $(m+1)$ blocks and $G = SO(2n)$, one has for all $i \geq 1$*

$$k_m < n - a_m \quad \text{and} \quad k_m < n - (a_m + \cdots + a_i) + (b_{m-1} + \cdots + b_i). \tag{4.2}$$

Proof. When $m = 1$, this is just Theorem 4.1. By induction assume that the theorem holds for any Schubert variety $X(\lambda')$ with fewer blocks than $X(\lambda)$. One can assume, without loss of generality, that $a_1 < k_1$. For, if $a_1 = k_1$, then $X(\lambda)$ is isomorphic to a Schubert variety $X(\mu)$ in $SO(n - a_1)/P_{n-a_1}$ where $\mu_i = \lambda_{a_1} + i - a_1$, with $(m-1)$ blocks in case (i) and m blocks in case (ii).

Writing $s = r - a_1$

$$\mu_s = \lambda_r - a_1 < N - (a_m + \cdots + a_i) + (b_{m-1} + \cdots + b_i) - a_1$$

and hence by induction hypothesis, there exists a small P_μ-equivariant resolution of $X(\mu)$ as stated in the theorem.

Therefore we assume that $a_1 < k_1$.

Proof of (i): As usual, we define

$$b_i = k_{i+1} - k_i - a_{i+1},\ 0 \leq i < m,\ (k_0 = 0)\ b_m = N - k_m - 1/2,\ a_0 = \infty.$$

Choose an i, $0 \le i < m$ such that $b_i \le a_i$, $a_{i+1} \le b_{i+1}$. Such an i exists because $a_0 = \infty$, and $b_m = N - k_m - 1/2 > a_m$.

We let

$$\lambda' = \begin{pmatrix} k_1 & \cdots & k_{i-1} & k_i + a_{i+1} & k_{i+2} & \cdots & k_m \\ a_1 & \cdots & a_{i-1} & a_i + a_{i+1} & a_{i+2} & \cdots & a_m \end{pmatrix}.$$

Then λ' has $(m-1)$ blocks, $\ell(\lambda') = \ell(\lambda) = r$, and $\lambda'_r = \lambda_r = k_m$ if $i < m-1$.

It is easy to verify that Equation 4.1 is satisfied for λ'. Therefore by induction hypothesis, there exists a $P_{\lambda'}$-equivariant Bott-Samelson resolution $p' : \widetilde{X(\lambda')} \to X(\lambda')$ such that for any $P' = P_{\lambda'}$-stable subvariety $X(\theta) \subset X(\lambda')$,

$$f_{p',\theta} \le \frac{1}{2}\Gamma(\theta, \lambda')$$

with equality only when $\theta = \lambda'$.

Note that with $P = P_\lambda$, one has $PX(\lambda') = X(\lambda)$. Let $R = P \cap P'$. One shows that $\operatorname{codim}_\lambda \lambda' = a_{i+1} b_i = \dim P/R$ and hence

$$p : \widetilde{X(\lambda)} = P \times_R \widetilde{X(\lambda')} \xrightarrow{q} P \times_R X(\lambda') \xrightarrow{\pi} X(\lambda)$$

is a P-equivariant Bott-Samelson resolution. We claim that for the resolution p, and for any P-stable subvariety $X(\tau) \subset X(\lambda)$, and for every $\sigma \in S(\tau, \lambda)$,

$$\operatorname{codim}_\lambda \tau - \operatorname{codim}_\tau \sigma \le \frac{1}{2}\Gamma(\tau, \lambda) - f_{p',\sigma}.$$

By Remark 2.3, this would then imply that,

$$f_{p,\tau} \le \frac{1}{2}\Gamma(\tau, \lambda).$$

Since, by induction hypothesis, writing $X(\theta) = P'X(\sigma)$,

$$f_{p',\sigma} = f_{p',\theta} \le \frac{1}{2}\Gamma(\theta, \lambda'),$$

we need only show that, for every $\sigma \in S(\tau, \lambda)$

$$\operatorname{codim}_\lambda \tau - \operatorname{codim}_\tau \sigma \le \frac{1}{2}(\Gamma(\tau, \lambda) - \Gamma(\theta, \lambda')).$$

Let $c(\tau, \lambda) = (c_1, c_2, \cdots, c_m)$, so that writing $a'_j = a_j + c_j - c_{j-1}$ $(c_0 = 0)$,

$$\tau = \begin{pmatrix} k_1 & k_2 & \cdots & k_m & n \\ a'_1 & a'_2 & \cdots & a'_m & \varepsilon \end{pmatrix}$$

where $\varepsilon = 0$ if $G = Sp(2n)$, and $\varepsilon = \frac{1}{2}(1 - (-1)^{c_m})$ if $G = SO(2n)$.

Using arguments similar to the proof of 3.3 it is easy to show that any $\sigma \in S(\tau, \lambda)$ must be of the form

$$\sigma = \begin{pmatrix} k_1 & k_2 & \cdots & k_i & k_i + a_{i+1} & k_{i+1} & k_{i+2} & \cdots & k_m & n \\ a'_1 & a'_2 & \cdots & a'_i & t & a'_{i+1} - t & a'_{i+2} & \cdots & a'_m & \varepsilon \end{pmatrix}.$$

Since $X(\sigma) \subset X(\lambda')$, one has $a'_1 + \cdots + a'_i + t \geq a_1 + \cdots + (a_i + a_{i+1})$ and hence $t \geq a_{i+1} - c_i$. Write $d = t - a_{i+1} + c_i$, so that $d \geq 0$, and $t = a_{i+1} + d - c_i \leq a_{i+1}$ implies $0 \leq d \leq c_i$. Also $a'_{i+1} - t \geq 0$ implies $d \leq c_{i+1}$. Note that $\ell(\tau) = \ell(\sigma) = c_m + \varepsilon$.

Since $X(\theta) = P'X(\sigma)$, one must have

$$\theta = \begin{pmatrix} k_1 & \cdots & k_{i-1} & k_i + a_{i+1} & k_{i+2} & \cdots & k_m & n \\ a'_1 & \cdots & a'_{i-1} & a_i + a_{i+1} + d - c_{i-1} & a_{i+2} + c_{i+2} - d & \cdots & a'_m & \varepsilon \end{pmatrix}.$$

Therefore the depth $c(\theta, \lambda')$ is

$$c(\theta, \lambda') = (c_1, \cdots, c_{i-1}, d, c_{i+2}, \cdots, c_m).$$

From equations 1.1 and 1.2, since $\ell(\tau) = \ell(\sigma)$, we get

$$\begin{aligned} \operatorname{codim}_\tau \sigma &= (a_{i+1} - d + c_i)(k_{i+1} - a'_{i+1} + t - (k_i + a_{i+1})) \\ &= (a_{i+1} - d + c_i)(b_i - c_{i+1} + d). \end{aligned}$$

Also since $\operatorname{codim}_\lambda \lambda' = a_{i+1} b_i$, we get

$$\begin{aligned} \operatorname{codim}_\lambda \lambda' - \operatorname{codim}_\tau \sigma &= a_{i+1}(c_{i+1} - d) + b_i(c_i - d) - (c_i - d)(c_{i+1} - d) \\ &\leq a_{i+1}(c_{i+1} - d) + b_i(c_i - d). \end{aligned}$$

Now, from 3.6,

$$\Gamma(\tau, \lambda) = \sum_{1 \leq j \leq m} c_j(a_j + b_j),$$

and

$$\Gamma(\theta, \lambda') = \sum_{\substack{1 \leq j \leq m \\ j \neq i, i+1}} c_j(a_j + b_j) + d(a_i + a_{i+1} + b_i + b_{i+1}).$$

Hence

$$\begin{aligned} \Gamma(\tau, \lambda) - \Gamma(\theta, \lambda') &= c_i(a_i + b_i) + c_{i+1}(a_{i+1} + b_{i+1}) - d(a_i + a_{i+1} + b_i + b_{i+1}) \\ &= (c_i - d)(a_i + b_i) + (c_{i+1} - d)(a_{i+1} + b_{i+1}). \end{aligned}$$

Since $b_i \leq a_i$ and $a_{i+1} \leq b_{i+1}$, we see that

$$\text{codim}_\lambda \lambda' - \text{codim}_\tau \sigma \leq \frac{1}{2}(\Gamma(\tau, \lambda) - \Gamma(\theta, \lambda')),$$

which completes the proof of (i). Proof of (ii) is similar.

Corollary 4.3. *Let Λ be the maximal representative in W/W_Q for $\lambda \in W/W_{P_n}$, for $B \subset Q \subset P_n$, with λ as in the above theorem. Then $X(\Lambda) \subset G/Q$ admits a small resolution which is P_λ-equivariant.*

Proof. Let $f: X(\Lambda) \to X(\lambda)$ be the map obtained from $\pi: G/Q \to G/P_n$. Since $X(\Lambda) = \pi^{-1}(X(\lambda))$, and since π is a locally trivial bundle with fibre P_n/Q, it follows from 2.4 that $X(\Lambda) \times_{X(\lambda)} \widetilde{X(\lambda)} \to X(\Lambda)$ is a small resolution, which is clearly P_λ-equivariant.

We single out an observation made in the introduction for the purpose of possible future reference as a

Remark 4.4. If $p: \tilde{X} \to X$ is a resolution of a normal irreducible variety then by Zariski's Main Theorem, the fibre $p^{-1}(x)$ over any singular point x of X, has positive dimension. Therefore, any normal irreducible variety X with codimension 2 singular locus cannot have any small resolution.

We now prove Theorem 1.2 showing the existence of Schubert varieties which do not have any small resolution.

Proof of Theorem 1.2. Let $X(\Theta)$ be the inverse image of $X(\tau) \subset X(\lambda)$, where $\tau = (n-1, n)$, under the projection $Sp(2n)/P \to Sp(2n)/P_n$. Then $\text{codim}_\Lambda \Theta = 2$. Therefore by the above remark it suffices to show that $X(\Theta)$ is contained in the singular locus of $X(\Lambda)$. Since the singular locus is B-stable, it suffices to show that the centre e_Θ is a singular point of $X(\Lambda)$. Equivalently, we show that $e_{\tilde{\Theta}}$ is a singular point for $X(\tilde{\Lambda}) \subset Sp(2n)/B$ where $\tilde{\Lambda}$ and $\tilde{\Theta}$ are the maximal lifts in W for Λ and Θ in W/W_P.

Using the results of [6] we now show that $e_{\tilde{\Theta}}$ is indeed a singular point of $X(\tilde{\Lambda})$. We follow the notations of [6]. From [6] one knows that $\dim T_{e_{\tilde{\Theta}}} X(\tilde{\Lambda}) = \# N(\tilde{\Lambda}, \tilde{\Theta})$. Now, if $\alpha = \epsilon_j - \epsilon_k$, $1 \leq j < k \leq n$, or if $\alpha = 2\epsilon_j$, $1 \leq j < n$, it is trivial to see that $\alpha \in N(\tilde{\Lambda}, \tilde{\Theta})$. An easy calculation shows that $2\epsilon_n \in N(\tilde{\Lambda}, \tilde{\Theta})$. Again, for $1 \leq j < k \leq n$, and $j < n-1$ one can show that $\epsilon_j + \epsilon_k \in N(\tilde{\Lambda}, \tilde{\Theta})$. When $j = n-1$, $k = n$, one has $r = n$, $s = n-1$, $a'_k = n+2$. It follows that condition (b) of Prop. C.1 of [6] is satisfied and hence $\epsilon_{n-1} + \epsilon_n \in N(\tilde{\Lambda}, \tilde{\Theta})$. Thus $\# N(\tilde{\Lambda}, \tilde{\Theta}) = \text{number of positive roots} = \dim Sp(2n)/B > \dim X(\tilde{\Lambda})$. Hence the proof.

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