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Yangians and Gelfand-Zetlin Bases

To Professor I. M. Gelfand on his 80th birthday

By

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Abstract

We establish a connection between the modern theory of Yangians and the classical construction of the Gelfand-Zetlin bases for the Lie algebra $\mathfrak{gl}_n$. Our approach allows us to produce the q-analogues of the Gelfand-Zetlin formulae in a straightforward way.

Let V be an irreducible finite-dimensional module over the complex Lie algebra $\mathfrak{gl}_n$. There is a canonical basis in the space of V associated with the chain of subalgebras $\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_n$. It is called the Gelfand-Zetlin basis, and the action of $\mathfrak{gl}_n$ on its vectors was explicitly described in [GZ] for the first time. Since then several authors provided alternative proofs of the original Gelfand-Zetlin formulae; see [Z2] and references therein.

Denote by $\mathrm{Z}(\mathfrak{gl}_n)$ the centre of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$. The subalgebra in $\mathrm{U}(\mathfrak{gl}_n)$ generated by $\mathrm{Z}(\mathfrak{gl}_1), \mathrm{Z}(\mathfrak{gl}_2), \dots, \mathrm{Z}(\mathfrak{gl}_n)$ is evidently commutative. The Gelfand-Zetlin basis in V consists of the eigenvectors of this subalgebra, and the corresponding eigenvalues are pairwise distinct. These properties suggest that for the given module V, an explicit description of $\mathrm{Z}(\mathfrak{gl}_1), \mathrm{Z}(\mathfrak{gl}_2), \dots, \mathrm{Z}(\mathfrak{gl}_n)$ should be used to construct the Gelfand-Zetlin basis. It shall be done in the present paper. Namely, for any vector v of the Gelfand-Zetlin basis we point out an element $b \in \mathrm{U}(\mathfrak{gl}_n)$ such that $v = b \cdot \xi$ where $\xi \in V$ is the highest weight vector (Section 2). Moreover, our construction implies the Gelfand-Zetlin formulae (Section 3).

We will employ the following description of $\mathrm{Z}(\mathfrak{gl}_n)$. Let e_{ij} be the standard generators of the algebra $\mathrm{U}(\mathfrak{gl}_n)$. Consider the sum over all permutations g of $1, 2, \dots, n$

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$$\sum_{g}(-1)^{\ell(g)} \prod_{i=1,\ldots,n}^{\rightarrow} (\delta_{g(i),i}(u-i+1)+e_{g(i),i})$$

where $\ell(g)$ denotes the length of the permutation g. The factors in the above product do not commute in general. Put them in the natural order: the factor indexed by i stands on the left of that indexed by j if $i<j$. The sum is a polynomial in u and the coefficients of this polynomial generate $Z(\mathfrak{gl}_n)$ [Z1], cf. [N].

Our construction is based on the chain of subalgebras

$$Y(\mathfrak{gl}_1) \subset Y(\mathfrak{gl}_2) \subset \cdots \subset Y(\mathfrak{gl}_n)$$

where $Y(\mathfrak{gl}_n)$ denotes the Yangian [D1] of the Lie algebra $\mathfrak{gl}_n$, cf. [C2]. In our construction we use the same generators of the algebra $Y(\mathfrak{gl}_n)$ as were introduced in [D2], see also [T]. There exists an algebra homomorphism $Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$. The generators of $Z(\mathfrak{gl}_n)$ mentioned above arise as the images of canonical central elements of $Y(\mathfrak{gl}_n)$ with respect to that homomorhism (Section 1), cf. [O].

Our construction for $U(\mathfrak{gl}_n)$ also admits a natural generalization to the quantum universal enveloping algebra $U_q(\mathfrak{gl}_n)$, cf. [C1]. This generalization produces the same q-analogues of the Gelfand-Zetlin formulae as were given in [J2]. To make our presentation clearer, we only formulate (Section 4) the main statements for $U_q(\mathfrak{gl}_n)$ and provide detailed proofs for $U(\mathfrak{gl}_n)$.

§1. In this section we state several known facts about the *Yangian* $Y(\mathfrak{gl}_n)$ of the complex Lie algebra $\mathfrak{gl}_n$. This is an associative algebra generated by the elements $T_{ij}^{(s)}$ where $i,j=1,\ldots,n$ and $s=1,2,\ldots$ subjected to the following relations. Introduce the formal Laurent series in u^{-1}

$$T_{ij}(u)=\delta_{ij}u+T_{ij}^{(1)}+T_{ij}^{(2)}u^{-1}+T_{ij}^{(3)}u^{-2}+\cdots$$

and form the matrix

$$T(u)=[T_{ij}(u)]_{i,j=1}^{n}.$$

Let P be the permutation map in $(\mathbf{C}^n)^{\otimes 2}$. Consider the *Yang R-matrix*, it is the $\mathrm{End}((\mathbf{C}^n)^{\otimes 2})$-valued function

$$R(u,v)=\mathrm{id}+\frac{P}{u-v}.$$

Put $\check{R}(u,v)=P\cdot R(u,v)$. Then the relations for $T_{ij}^{(s)}$ can be written as

$$\check{R}(u,v)\cdot T(u)\otimes T(v)=T(v)\otimes T(u)\cdot \check{R}(u,v). \tag{1.1}$$

Observe that the generators $T_{ij}^{(s)}$ with $i,j=1,\ldots,m$ obey exactly the same relations as the corresponding generators of $\mathrm{Y}(\mathfrak{gl}_m)$. Thus we have the chain of subalgebras

$$\mathrm{Y}(\mathfrak{gl}_1) \subset \mathrm{Y}(\mathfrak{gl}_2) \subset \cdots \subset \mathrm{Y}(\mathfrak{gl}_n).$$

The relations (1.1) also imply that for any $h \in \mathbf{C}$ the map

$$T_{ij}(u) \longmapsto T_{ij}(u+h)$$

defines an automorhism of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$; here the series in $(u+h)^{-1}$ should be re-expanded in u^{-1}.

We will use the following definition. Let $X(u) = [X_{ij}(u)]_{i,j=1}^m$ be an arbitrary matrix whose entries are formal Laurent series in u^{-1} with coefficients in $\mathrm{Y}(\mathfrak{gl}_n)$. Define the *quantum determinant* of this matrix to be the sum over all permutations g of $1, 2, \ldots, m$

$$\operatorname{qdet} X(u) = \sum_g (-1)^{\ell(g)} \cdot X_{1g(1)}(u) X_{2g(2)}(u-1) \cdots X_{m,g(m)}(u-m+1);$$

here $\ell(g)$ denotes the length of the permutation g. We will also denote by $\operatorname{pdet} X(u)$ the sum

$$\sum_g (-1)^{\ell(g)} \cdot X_{1g(1)}(u-m+1) X_{2g(2)}(u-m+2) \cdots X_{m,g(m)}(u).$$

Consider the formal series

$$A_m(u) = \operatorname{qdet} [T_{ij}(u)]_{i,j=1}^m; \qquad m = 1, \ldots, n.$$

Proposition 1.1. *a) The coefficients of $A_n(u)$ belong to the centre of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$. b) All the coefficients of $A_1(u), \ldots, A_n(u)$ pairwise commute.*

Proof. The part a) is well known and its proof can be found for instance in [KS]. Since the generators $T_{ij}^{(s)}$ with $i, j = 1, \ldots, m$ obey the same relations as the corresponding generators of $\mathrm{Y}(\mathfrak{gl}_m)$, we obtain from a) that

$$[A_m(u), T_{ij}(v)] = 0; \qquad i, j = 1, \ldots, m. \tag{1.2}$$

The part b) follows directly from the above commutation relations □

It is convenient to assume $A_0(u) = 1$. Now we introduce the formal series with coefficients in $\mathrm{Y}(\mathfrak{gl}_n)$ which together with $A_1(u), \ldots, A_n(u)$ play the main role in this paper. For any $m = 1, \ldots, n-1$ denote by $B_m(u)$, $C_m(u)$, $D_m(u)$ respectively the quantum determinants of the submatrices in $T(u)$ specified by the rows $1, \ldots, m$ and the columns $1, \ldots, m-1, m+1$; by the rows $1, \ldots, m-1, m+1$ and the columns $1, \ldots, m$; by rows $1, \ldots, m-1, m+1$ and the same columns. These quantum determinants have been used in [D2].

Proposition 1.2. *The following commutation relations hold in* $\mathrm{Y}(\mathfrak{gl}_n)$:

$$[A_m(u),\, B_l(v)] = 0 \qquad \text{if } l \neq m, \tag{1.3}$$

$$[C_m(u),\, B_l(v)] = 0 \qquad \text{if } l \neq m, \tag{1.4}$$

$$[B_m(u),\, B_l(v)] = 0 \qquad \text{if } |l-m| \neq 1, \tag{1.5}$$

$$(u-v)\cdot[A_m(u),\, B_m(v)] = B_m(u)A_m(v) - B_m(v)A_m(u), \tag{1.6}$$

$$(u-v)\cdot[C_m(u),\, B_m(v)] = D_m(u)A_m(v) - D_m(v)A_m(u). \tag{1.7}$$

Proof. It follows from (1.1) that the entries of any $m \times m$ submatrix in $T(u)$ obey the same relations as the corresponding entries of the matrix $[T_{ij}(u)]_{i,j=1}^m$. Therefore if we have two square submatrices $X(u)$, $Y(u)$ in $T(u)$ and one of them contains the other, then due to (1.2)

$$[\operatorname{qdet} X(u),\ \operatorname{qdet} Y(v)] = 0.$$

This observation provides the relations (1.3), (1.4), (1.5).

It suffices to prove the relations (1.6), (1.7) only for $m = n-1$. Introduce the matrix

$$\hat{T}(u) = [\hat{T}_{ij}(u)]_{i,j=1}^n$$

where $\hat{T}_{ij}(u)$ is equal to $(-1)^{i-j}$ times the quantum determinant of the matrix obtained from $T(u)$ by removing the row j and the column i. Then

$$T(u)\hat{T}(u-1) = \operatorname{qdet} T(u), \tag{1.8}$$

$$\hat{T}^t(u)T^t(u-n+1) = \operatorname{qdet} T(u) \tag{1.9}$$

where the superscript t denotes the matrix transposition; see [KS] for the proof of these equalities. The matrix $T(u)$ is invertible as a formal Laurent series in u^{-1}; denote by $\tilde{T}(u)$ the inverse matrix. Then from (1.1) we get the equality

$$\tilde{T}(u) \otimes \tilde{T}(v) \cdot \check{R}(u, v) = \check{R}(u, v) \cdot \tilde{T}(v) \otimes \tilde{T}(u).$$

The series $\operatorname{qdet} T(u)$ is also invertible and commutes with each entry of the matrix $\hat{T}(u)$. Therefore from the last equality, from (1.8) and from

$$\check{R}(u+1, v+1) = \check{R}(u, v),$$

we obtain the matrix relation

$$\hat{T}(u) \otimes \hat{T}(v) \cdot \check{R}(u, v) = \check{R}(u, v) \cdot \hat{T}(v) \otimes \hat{T}(u).$$

By the definition of the matrix $\hat{T}(u)$ we have the equalities

$$A_{n-1}(u) = \hat{T}_{nn}(u), \quad D_{n-1}(u) = \hat{T}_{n-1,n-1}(u), \tag{1.10}$$

$$B_{n-1}(u) = -\hat{T}_{n-1,n}(u), \quad C_{n-1}(u) = -\hat{T}_{n,n-1}(u).$$

The commutation relations (1.6), (1.7) with $m = n - 1$ are contained in the above matrix relation □

We will keep using the matrices $\hat{T}(u)$ and $\tilde{T}(u)$ introduced in the proof of Proposition 1.2.

Lemma 1.3. *For any* $m = 1, \dots, n-1$ *we have the equality*

$$\operatorname{qdet}[T_{ij}(u-n+m)]_{i,j=1}^{m} = \operatorname{pdet}[\tilde{T}_{ij}(u)]_{i,j=m+1}^{n} \cdot \operatorname{qdet} T(u).$$

Proof. Let $w_1, w_2, \dots, w_n$ be the standard basis in $\mathbf{C}^n$. Let I be the $n \times n$ matrix unit and $J \in \operatorname{End}((\mathbf{C}^n)^{\otimes n})$ be the antisymmetrization map. Then

$$T(u) \otimes T(u-1) \otimes \cdots \otimes T(u-n+1) \cdot J = J \cdot \operatorname{qdet} T(u);$$

see [KS] for the proof of this equality. It implies that

$$\begin{aligned} &I^{\otimes(n-m)} \otimes T(u-n+m) \otimes \cdots \otimes T(u-n+2) \otimes T(u-n+1) \cdot J \\ &\quad = \tilde{T}(u-n+m+1) \otimes \cdots \otimes \tilde{T}(u-1) \otimes \tilde{T}(u) \otimes I^{\otimes m} \cdot J \cdot \operatorname{qdet} T(u). \end{aligned}$$

Thus we have a matrix equality over the space $(\mathbf{C}^n)^{\otimes n}$. Taking its diagonal entry corresponding to the vector

$$w_{m+1} \otimes \cdots \otimes w_{n-1} \otimes w_n \otimes w_1 \otimes \cdots \otimes w_{m-1} \otimes w_m,$$

we get the equality claimed by Lemma 1.3 □

Proposition 1.4. *The following relation holds in* $\mathrm{Y}(\mathfrak{gl}_n)$:

$$C_m(u) B_m(u-1) = D_m(u) A_m(u-1) - A_{m+1}(u) A_{m-1}(u-1).$$

Proof. It suffices to prove Proposition 1.4 only for $m = n - 1$. Applying Lemma 1.3 to $m = n - 2$ and using the equalities (1.8), (1.10) along with Proposition 1.1 (a), we get

$$\begin{aligned} A_n(u) A_{n-2}(u-1) &= \operatorname{qdet} T(u) \cdot \operatorname{qdet}[T_{ij}(u-1)]_{i,j=1}^{n-2} \\ &= \operatorname{qdet} T(u) \cdot \operatorname{pdet}[\tilde{T}_{ij}(u+1)]_{i,j=n-1}^{n} \cdot \operatorname{qdet} T(u+1) \\ &= \operatorname{qdet} T(u) \cdot (\tilde{T}_{n-1,n-1}(u) \tilde{T}_{nn}(u+1) \\ &\qquad - \tilde{T}_{n-1,n}(u) \tilde{T}_{n-1,n}(u+1)) \cdot \operatorname{qdet} T(u+1) \\ &= \hat{T}_{n-1,n-1}(u-1) \hat{T}_{nn}(u) - \hat{T}_{n-1,n}(u-1) \hat{T}_{n,n-1}(u) \\ &= D_{n-1}(u-1) A_{n-1}(u) - B_{n-1}(u-1) C_{n-1}(u). \end{aligned}$$

Due to the relation (1.7) the right hand side of the above equalities coincides with

$$D_{n-1}(u) A_{n-1}(u-1) - C_{n-1}(u) B_{n-1}(u-1).$$

Thus Proposition 1.4 for $m = n - 1$ is proved □

§2. Let e_{ij} be the standard generators of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$. The algebra $\mathrm{Y}(\mathfrak{gl}_n)$ contains $\mathrm{U}(\mathfrak{gl}_n)$ as a subalgebra: the maps $e_{ji} \mapsto t_{ij}^{(1)}$ define the imbedding. One can also define a homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n)$ by

$$T_{ij}(u) \longmapsto \delta_{ij} u + e_{ji}.$$

Denote the images of the series $A_m(u)$, $B_m(u)$, $C_m(u)$ and $D_m(u)$ under this homomorphism by $a_m(u)$, $b_m(u)$, $c_m(u)$ and $d_m(u)$ respectively. These images are polynomial in u and

$$a_m(u) = u^m + (e_{11} + \cdots + e_{mm} - m(m-1)/2)u^{m-1} + \cdots, \tag{2.1}$$
$$b_m(u) = e_{m+1,m} u^{m-1} + \cdots, \quad c_m(u) = e_{m,m+1} u^{m-1} + \cdots.$$

The above equalities show that the coefficients of the polynomials $a_m(u)$, $b_m(u)$ and $c_m(u)$ generate the algebra $\mathrm{U}(\mathfrak{gl}_n)$. We will explicitly describe the action of these polynomials in each irreducible finite-dimensional module of the Lie algebra $\mathfrak{gl}_n$. We will use Proposition 1.2 and Proposition 1.4 along with the following observation: if $\mathfrak{n}$ is the subalgebra in $\mathfrak{gl}_m$ spanned by the elements e_{ij}, $1 \leqslant i < j \leqslant m$ then by the definition of the quantum determinant

$$a_m(u) \in \prod_{i=1}^{m} (u + e_{ii} - i + 1) + \mathrm{U}(\mathfrak{gl}_m)\mathfrak{n}, \tag{2.2}$$

$$c_m(u) \in \mathrm{U}(\mathfrak{gl}_m)\mathfrak{n}. \tag{2.3}$$

Let V be an irreducible finite dimensional $\mathfrak{gl}_n$-module. Denote by ξ its highest weight vector:

$$e_{ii} \cdot \xi = \lambda_i \xi; \quad e_{ij} \cdot \xi = 0, \quad i < j.$$

Then each difference $\lambda_i - \lambda_{i+1}$ is a non-negative integer. For any $h \in \mathbf{C}$ the mappings

$$e_{ii} \longmapsto e_{ii} + h; \quad e_{ij} \longmapsto e_{ij}, \quad i \neq j$$

define an automorphism of the algebra $\mathrm{U}(\mathfrak{gl}_n)$. So we will assume that each λ_i is also an integer. Denote by $\mathscr{T}$ the set of all arrays Λ with integral entries of the form

$$\begin{array}{ccc}
\lambda_{n1} \ \lambda_{n2} & \cdots\cdots\cdots\cdots & \lambda_{nn} \\
\lambda_{n-1,1} \ \lambda_{n-1,2} & \cdots\cdots & \lambda_{n-1,n-1} \\
\ddots & & \therefore \\
\lambda_{21} & & \lambda_{22} \\
& \lambda_{11} &
\end{array}$$

where $\lambda_{ni} = \lambda_i$ and $\lambda_i \geqslant \lambda_{mi}$ for all i and m. The array Λ is called a *Gelfand-Zetlin scheme* if

$$\lambda_{mi} \geqslant \lambda_{m-1,i} \geqslant \lambda_{m,i+1}$$

for all possible m and i. Denote by $\mathscr{S}$ the subset in $\mathscr{T}$ consisting of the Gelfand-Zetlin schemes.

There is a canonical decomposition of the space V into the direct sum of one-dimensional subspaces associated with the chain of subalgebras

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_n.$$

These subspaces are parametrized by the elements $\Lambda \in \mathscr{S}$. The subspace $V_\Lambda \subset V$ corresponding to $\Lambda \in \mathscr{S}$ is contained in an irreducible $\mathfrak{gl}_m$-submodule of the highest weight $(\lambda_{m1}, \lambda_{m2}, \ldots, \lambda_{mm})$ for each $m = n-1, n-2, \ldots, 1$. These conditions define V_Λ uniquely, cf. [GZ]. Denote by Λ° the array where $\lambda_{mi} = \lambda_i$ for any m; then $\Lambda^\circ \in \mathscr{S}$ and $\xi \in V_{\Lambda^\circ}$.

For any $\Lambda \in \mathscr{T}$ put

$$\alpha_{m\Lambda}(u) = \prod_{i=1}^{m} (u + \lambda_{mi} - i + 1).$$

Proposition 2.1. *The subspace $V_\Lambda \subset V$ is an eigenspace of $a_m(u)$ with the eigenvalue $\alpha_{m\Lambda}(u)$.*

Proof. The coefficients of the polynomial $a_m(u)$ belong to the center of the algebra $\mathrm{U}(\mathfrak{gl}_m)$ and act in any irreducible $\mathfrak{gl}_m$-submodule of V via scalars. Applying $a_m(u)$ to the highest weight vector in this submodule and using (2.2) we get Proposition 2.1 by the definition of the subspace V_Λ □

Endow the set of the pairs (m, i) with the following relation of precedence: $(m, i) \prec (l, j)$ if $i < j$ or $i = j$ and $m > l$. This relation corresponds to reading $\Lambda \in \mathscr{T}$ by diagonals from the left to the right, downwards in each diagonal. Let $\nu_{mi} = i - \lambda_{mi} - 1$, it is a root of the polynomial $\alpha_{m\Lambda}(u)$. Note that if $\Lambda \in \mathscr{S}$ then $\nu_{m1} < \nu_{m2} < \cdots < \nu_{mm}$. Put $\nu_i = i - \lambda_i - 1$, then $\nu_{mi} \geqslant \nu_i$. Consider the vector in V

$$\xi_\Lambda = \overrightarrow{\prod_{(l,j)}} \left(\prod_{s=\nu_j}^{\nu_{lj}-1} b_l(s) \right) \cdot \xi; \tag{2.4}$$

here for each fixed l the elements $b_l(s) \in \mathrm{U}(\mathfrak{gl}_n)$ commute because of the relation (1.5). The products in brackets do not commute with each other in general. We arrange them from the left to the right according to the above relation of precedence for the pairs (m, i).

Theorem 2.2. *For any $\Lambda \in \mathscr{T}$ we have the equality*

$$a_m(u) \cdot \xi_\Lambda = \alpha_{m\Lambda}(u) \xi_\Lambda.$$

Proof. We will employ the induction on the number of the factors $b_l(s)$ in (2.4). If there is no factors then $\Lambda = \Lambda^\circ$ and $\xi_\Lambda = \xi \in V_{\Lambda^\circ}$. In particular, the required equality then holds by Proposition 2.1.

Assume that $\Lambda \neq \Lambda^\circ$. Let (l, j) be the minimal pair such that $\lambda_{lj} \neq \lambda_j$. Let Ω be the array obtained from Λ by increasing the (l, j)-entry by 1. Then $\Omega \in \mathscr{T}$ and

$$\xi_\Lambda = b_l(v_{lj} - 1) \cdot \xi_\Omega. \tag{2.5}$$

If $l \neq m$ then $\alpha_{m\Omega}(u) = \alpha_{m\Lambda}(u)$. By the relation (1.3) and by the inductive assumption we get

$$\begin{aligned} a_m(u) \cdot \xi_\Lambda &= a_m(u) b_l(v_{lj} - 1) \cdot \xi_\Omega = b_l(v_{lj} - 1) a_m(u) \cdot \xi_\Omega \\ &= b_l(v_{lj} - 1) \cdot \alpha_{m\Omega}(u) \xi_\Omega = \alpha_{m\Lambda}(u) \xi_\Lambda. \end{aligned}$$

Now suppose that $l = m$; then by the definition of Ω we have

$$\alpha_{m\Omega}(u) = \frac{u - v_{mj} + 1}{u - v_{mj}} \alpha_{m\Lambda}(u).$$

In particular, by the inductive assumption we then have

$$a_m(v_{mj} - 1) \cdot \xi_\Omega = \alpha_{m\Omega}(v_{mj} - 1) \xi_\Omega = 0.$$

Therefore by the relation (1.6) and again by the inductive assumption we get

$$\begin{aligned} a_m(u) \cdot \xi_\Lambda &= a_m(u) b_m(v_{mj} - 1) \cdot \xi_\Omega \\ &= \frac{u - v_{mj}}{u - v_{mj} + 1} b_m(v_{mj} - 1) a_m(u) \cdot \xi_\Omega \\ &= \frac{u - v_{mj}}{u - v_{mj} + 1} b_m(v_{mj} - 1) \cdot \alpha_{m\Omega}(u) \xi_\Omega = \alpha_{m\Lambda}(u) \xi_\Lambda. \end{aligned}$$

Thus Theorem 2.2 is proved for any m □

The subspaces $V_\Lambda \subset V$ are separated by the corresponding eigenvalues of $a_1(u), \dots, a_{n-1}(u)$. Therefore by comparing Proposition 2.1 and Theorem 2.2 we get

Corollary 2.3. *For any $\Lambda \in \mathscr{S}$ we have $\xi_\Lambda \in V_\Lambda$.*

In the next section we will describe the action of the polynomials $b_m(u)$ and $c_m(u)$ on the vectors ξ_Λ with $\Lambda \in \mathscr{S}$. Then we will prove that all these vectors do not vanish.

Proposition 2.4. *If $\Lambda \in \mathscr{T} \setminus \mathscr{S}$ then $\xi_\Lambda = 0$.*

Proof. As well as in the proof of Theorem 2.2 we will employ the induction on the number of the factors $b_l(s)$ in (2.4). If there is no factors then $\Lambda = \Lambda^\circ \in \mathscr{S}$ and we have nothing to prove.

Assume that $\Lambda \neq \Lambda^\circ$. Let (l, j) be the minimal pair such that $\lambda_{lj} \neq \lambda_j$. Denote by Ω the array obtained from increasing the (l, j)-entry of Λ by 1, then $\Omega \in \mathscr{T}$ and we have the equality (2.5). If $\Omega \notin \mathscr{S}$ then $\xi_\Omega = 0$ by the inductive assumption, so that $\xi_\Lambda = 0$ due to (2.5).

Now suppose that $\Omega \in \mathscr{S}$. We will prove that either $\Lambda \in \mathscr{S}$ or $\xi_\Lambda = 0$. By Theorem 2.2 the vector ξ_Λ is an eigenvector for the polynomials $a_1(u), \ldots, a_{n-1}(u)$. Their eigenvalues separate the subspaces $V_\Upsilon \subset V$ with $\Upsilon \in \mathscr{S}$. Therefore $\xi_\Lambda \in V_\Upsilon$ for some $\Upsilon \in \mathscr{S}$. Suppose that $\xi_\Lambda \neq 0$, then

$$\alpha_{m\Lambda}(u) = \alpha_{m\Upsilon}(u), \qquad m = 1, \ldots, n-1.$$

Consider the roots $\nu_{mi} = i - \lambda_{mi} - 1$ of the polynomial $\alpha_{m\Lambda}(u)$. Since $\Omega \in \mathscr{S}$, we have the inequalities

$$\nu_{m1} < \nu_{m2} < \cdots < \nu_{mm} \qquad \text{if } m \neq l;$$

$$\nu_{l1} < \cdots < \nu_{lj} \leqslant \nu_{l,j+1} < \cdots < \nu_{ll}.$$

Therefore the array Λ can be uniquely restored from the collection of the polynomials $\alpha_{1\Lambda}(u), \ldots, \alpha_{n-1,\Lambda}(u)$. Thus $\Lambda = \Upsilon \in \mathscr{S}$ and the Proposition 2.4 is proved □

Remark 2.5. Let us form the matrix $E = [-e_{ji}]_{i,j=1}^n$. The coefficients of the polynomial $a_n(u) = \operatorname{qdet}(u - E)$ belong to the center of the algebra $\mathrm{U}(\mathfrak{gl}_n)$ and from (1.8), (1.9) we obtain the matrix identities

$$a_n(E) = 0, \qquad a_n(E^t + n - 1) = 0. \tag{2.6}$$

Now consider the matrix $S = [S_{ij}]_{i,j=1}^n$ where S_{ij} denotes the image of the element $-e_{ji}$ in the module V. Replacing the coefficients of the polynomial $a_n(u)$ by their eigenvalues in the module V and taking into account Proposition 2.1, we get the *characteristic identities* [G] for the Lie algebra $\mathfrak{gl}_n$:

$$\prod_{i=1}^n (S + \lambda_i - i + 1) = 0, \qquad \prod_{i=1}^n (S^t + \lambda_i - i + n) = 0.$$

§3. Let $\Lambda \in \mathscr{S}$ be fixed. The functions $b_m(u) \cdot \xi_\Lambda$ and $c_m(u) \cdot \xi_\Lambda$ are polynomial in u of the degree $m - 1$. Since $\nu_{m1} < \nu_{m2} < \cdots < \nu_{mm}$, to describe these functions it suffices to determine their values at $u = \nu_{mi}$ for each $i = 1, \ldots, m$. This will be done in the present section. Put

$$\gamma_{mi\Lambda} = \prod_{j=1}^{i} (\nu_{mi} - \nu_j) \prod_{j=1}^{i-1} (\nu_{mi} - \nu_j - 1) \times$$

$$\times \prod_{j=i+1}^{m+1} (\nu_{m+1,j} - \nu_{mi}) \prod_{j=i}^{m-1} (\nu_{m-1,j} - \nu_{mi} + 1).$$

Let the indices $m < n$ and $i \leqslant m$ be fixed. Denote by Λ^+ the array obtained from Λ by increasing the (m, i)-entry by 1.

Theorem 3.1. *We have*

$$c_m(\nu_{mi}) \cdot \xi_\Lambda = \begin{cases} \gamma_{mi\Lambda} \xi_{\Lambda^+} & \text{if } \Lambda^+ \in \mathcal{S}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $\Lambda = \Lambda^\circ$ then $\Lambda^+ \notin \mathcal{S}$. On the other hand then $\xi_\Lambda = \xi$ while due to (2.3)

$$c_m(\nu_{mi}) \cdot \xi = 0.$$

Assume that $\Lambda \neq \Lambda^\circ$. Consider the minimal pair (l, j) such that $\lambda_{lj} \neq \lambda_j$. As well as in the proof of Theorem 2.2 let Ω be the array obtained from Λ by increasing the (l, j)-entry by 1. Then $\Omega \in \mathcal{S}$ and we have the equality (2.5). We will prove first that

$$c_m(\nu_{mi}) b_l(\nu_{lj} - 1) \cdot \xi_\Omega = b_l(\nu_{lj} - 1) c_m(\nu_{mi}) \cdot \xi_\Omega \tag{3.1}$$

for $(l, j) \neq (m, i)$. If $l \neq m$ then we obtain (3.1) directly from the relation (1.4). If $l = m$ but $j \neq i$ then $\nu_{mj} - 1 \neq \nu_{mi}$ since $\Omega \in \mathcal{S}$. Due to Theorem 2.2 we then also have the equalities

$$a_m(\nu_{mj} - 1) \cdot \xi_\Omega = \alpha_{m\Omega}(\nu_{mj} - 1) \xi_\Omega = 0,$$
$$a_m(\nu_{mi}) \cdot \xi_\Omega = \alpha_{m\Omega}(\nu_{mi}) \xi_\Omega = 0.$$

Therefore by the relation (1.7) we again obtain that

$$c_m(\nu_{mi}) b_l(\nu_{mj} - 1) \cdot \xi_\Omega = b_m(\nu_{mj} - 1) c_m(\nu_{mi}) \cdot \xi_\Omega.$$

If $\lambda_{mi} = \lambda_i$ then $\Lambda^+ \notin \mathcal{S}$. On the other hand, applying the equality (3.1) repeatedly we then get

$$c_m(\nu_{mi}) \cdot \xi_\Lambda = c_m(\nu_{mi}) \overset{\rightarrow}{\prod_{(l,j)}} \left(\prod_{s=\nu_j}^{\nu_{lj}-1} b_l(s) \right) \cdot \xi$$
$$= \overset{\rightarrow}{\prod_{(l,j)}} \left(\prod_{s=\nu_j}^{\nu_{lj}-1} b_l(s) \right) c_m(\nu_{mi}) \cdot \xi = 0,$$

as we have claimed. Now we assume that $\lambda_{mi} < \lambda_i$.

Consider the array Υ obtained from Λ by changing each entry corresponding to $(l, j) \prec (m, i)$ for λ_j, and by increasing the (m, i)-entry by 1. Then $\Upsilon \in \mathcal{T}$ and due to Theorem 2.2 we have

$$a_m(\nu_{mi}-1)\cdot\xi_\wr = \alpha_{m\wr}(\nu_{mi}-1)\xi_\wr = 0.$$

We then also have

$$\xi_\Lambda = pb_m(\nu_{mi}-1)\cdot\xi_\wr$$

where

$$p = \prod_{(l,j)\prec(m,i)}^{\rightarrow}\left(\prod_{s=\nu_j}^{\nu_{lj}-1} b_l(s)\right).$$

Therefore applying the equality (3.1) repeatedly and using Proposition 1.4 we get

$$\begin{aligned} c_m(\nu_{mi})\cdot\xi_\Lambda &= pc_m(\nu_{mi})b_m(\nu_{mi}-1)\cdot\xi_\wr \\ &= -\,pa_{m+1}(\nu_{mi})a_{m-1}(\nu_{mi}-1)\cdot\xi_\wr \\ &= -\,\alpha_{m+1,\wr}(\nu_{mi})\alpha_{m-1,\wr}(\nu_{mi}-1)p\cdot\xi_\wr \\ &= -\,\alpha_{m+1,\wr}(\nu_{mi})\alpha_{m-1,\wr}(\nu_{mi}-1)\xi_{\Lambda^+} = \gamma_{mi\Lambda}\xi_{\Lambda^+}. \end{aligned}$$

The last equality proves Theorem 3.1 when $\Lambda^+\in\mathscr{S}$. If $\Lambda^+\notin\mathscr{S}$ then

$$c_m(\nu_{mi})\cdot\xi_\Lambda = 0$$

by the same equality and by Proposition 2.4 □

Remark 3.2. If $\Lambda^+\in\mathscr{S}$ then $\gamma_{mi\Lambda}>0$. Indeed, if $\Lambda\in\mathscr{S}$ then we have the inequalities

$$\nu_{mi} < \nu_{m+1,i+1} < \nu_{m+1,i+2} < \cdots < \nu_{m+1,m+1};$$

$$\nu_{mi} \leqslant \nu_{m-1,i} < \nu_{m-1,i+1} < \cdots < \nu_{m-1,m-1}.$$

If $\Lambda^+\in\mathscr{S}$ then we also have

$$\nu_{mi}-1 \geqslant \nu_i > \nu_{i-1} > \cdots > \nu_1.$$

Thus all the factors in the product $\gamma_{mi\Lambda}$ are positive.

Proposition 3.3. *If* $\Lambda\in\mathscr{S}$ *then* $\xi_\Lambda\neq 0$.

Proof. As well as in the proofs of Theorem 2.2 and Proposition 2.4 we will employ the induction on the number of the factors $b_l(s)$ in (2.4). If there is no factors then $\Lambda=\Lambda^\circ$ and $\xi_\Lambda=\xi\neq 0$.

Assume that $\Lambda\neq\Lambda^\circ$. Let (l,j) be the minimal pair such that $\lambda_{lj}\neq\lambda_j$. Let Ω be the array obtained from Λ by increasing the (l,j)-entry by 1. Since $\Lambda\in\mathscr{S}$, we also have $\Omega\in\mathscr{S}$. Then $\xi_\Omega\neq 0$ by the inductive assumption. On the other hand, by Theorem 3.1 we then have

$$c_l(\nu_{lj})\cdot\xi_\Lambda = \gamma_{lj\Lambda}\xi_\Omega$$

where $\gamma_{lj\Lambda} \neq 0$ due to Remark 3.2. Therefore $\xi_\Lambda \neq 0$ □

Now consider the array Λ^- obtained from Λ by decreasing the (m, i)-entry of Λ by 1; then $\Lambda^- \in \mathcal{T}$. Since

$$\nu_{mi} \geqslant \nu_i > \nu_{i-1} > \cdots > \nu_1, \tag{3.2}$$

one can define

$$\beta_{mi\Lambda} = \prod_{j=1}^{i} \frac{\nu_{mi} - \nu_{m+1,j} + 1}{\nu_{mi} - \nu_j + 1} \prod_{j=1}^{i-1} \frac{\nu_{mi} - \nu_{m-1,j}}{\nu_{mi} - \nu_j}$$

Theorem 3.4. *We have*

$$b_m(\nu_{mi}) \cdot \xi_\Lambda = \begin{cases} \beta_{mi\Lambda} \xi_{\Lambda^-} & \text{if } \Lambda^- \in \mathcal{S}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We will use again some of the arguments which appeared in the proofs of Theorem 2.2 and Proposition 2.4. Consider the vector $b_m(\nu_{mi}) \cdot \xi_\Lambda \in V$. It is an eigenvector of the element $a_l(u)$ with the eigenvalue $\alpha_{l\Lambda^-}(u)$ for any l. Indeed, if $l \neq m$ then $\alpha_{l\Lambda^-}(u) = \alpha_{l\Lambda}(u)$. On the other hand, by the relation (1.3) and by Theorem 2.2 we then get

$$a_l(u) b_m(\nu_{mi}) \cdot \xi_\Lambda = b_m(\nu_{mi}) a_l(u) \cdot \xi_\Lambda = \alpha_{l\Lambda}(u) b_m(\nu_{mi}) \cdot \xi_\Lambda.$$

Now suppose that $l = m$; then by the definition of Λ^- we have

$$\alpha_{m\Lambda^-}(u) = \frac{u - \nu_{mi} - 1}{u - \nu_{mi}} \alpha_{m\Lambda}(u).$$

Since $a_m(\nu_{mi}) \cdot \xi_\Lambda = 0$ due to Theorem 2.2, by the relation (1.6) and again by Theorem 2.2 we get

$$\begin{aligned} a_m(u) b_m(\nu_{mi}) \cdot \xi_\Lambda &= \frac{u - \nu_{mi} - 1}{u - \nu_{mi}} b_m(\nu_{mi}) a_m(u) \cdot \xi_\Lambda \\ &= \alpha_{m\Lambda^-}(u) b_m(\nu_{mi}) \cdot \xi_\Lambda. \end{aligned}$$

The subspaces $V_\Omega \subset V$ with $\Omega \in \mathcal{S}$ are separated by the eigenvalues of $a_1(u), \ldots, a_{n-1}(u)$. Therefore $b_m(\nu_{mi}) \cdot \xi_\Lambda \in V_\Omega$ for some $\Omega \in \mathcal{S}$. Since $\Lambda \in \mathcal{S}$, the array Λ^- can be uniquely restored from the collection of the polynomials $\alpha_{1\Lambda^-}(u), \ldots, \alpha_{n-1,\Lambda^-}(u)$. Thus if $\Lambda^- \notin \mathcal{S}$ then

$$b_m(\nu_{mi}) \cdot \xi_\Lambda = 0$$

as we have claimed.

Assume that $\Lambda^- \in \mathcal{S}$. Then the above consideration implies that

$$b_m(\nu_{mi}) \cdot \xi_\Lambda \in V_{\Lambda^-}.$$

Therefore by Corollary 2.3 and Proposition 3.3 we obtain that

$$b_m(v_{mi}) \cdot \xi_\Lambda = \beta \xi_{\Lambda^-} \tag{3.3}$$

for some $\beta \in \mathbf{C}$. We will prove that $\beta = \beta_{mi\Lambda}$.

Let us compare the action of the element $c_m(v_{mi}+1)$ on both sides of the equality (3.3). By Theorem 3.1 we have

$$c_m(v_{mi}+1) \cdot \beta \xi_{\Lambda^-} = \beta \gamma_{mi\Lambda^-} \xi_\Lambda$$

where $\gamma_{mi\Lambda^-} \neq 0$ due to Remark 3.2. On the other hand, applying Proposition 1.4 and using the equality

$$a_m(v_{mi}) \cdot \xi_\Lambda = \alpha_{m\Lambda}(v_{mi}) \xi_\Lambda = 0,$$

we get

$$\begin{aligned} c_m(v_{mi}+1) b_m(v_{mi}) \cdot \xi_\Lambda &= -\, a_{m+1}(v_{mi}+1) a_{m-1}(v_{mi}) \cdot \xi_\Lambda \\ &= -\, \alpha_{m-1,\Lambda}(v_{mi}+1) \alpha_{m-1,\Lambda}(v_{mi}) \xi_\Lambda. \end{aligned}$$

Since $\xi_\Lambda \neq 0$ by Proposition 3.3, we finally obtain that

$$\beta = -\, \alpha_{m+1,\Lambda}(v_{mi}+1) \alpha_{m-1,\Lambda}(v_{mi}) \gamma^{-1}_{mi\Lambda^-} = \beta_{mi\Lambda}.$$

Thus we have proved Theorem 3.4 □

Remark 3.5. If $\Lambda \in \mathscr{S}$ then $\beta_{mi\Lambda} > 0$ for any indices m and i. Indeed, then

$$v_{mi} \geqslant v_{m+1,i} > v_{m+1,i-1} > \cdots > v_{m+1,1};$$

$$v_{mi} > v_{m-1,i-1} > v_{m-1,i-2} > \cdots > v_{m-1,1}.$$

These inequalities along with (3.2) show that all the factors in the product $\beta_{mi\Lambda}$ are positive.

Theorems 2.2, 3.1, 3.4 and Proposition 3.3 along with Corollary 2.3 completely describe the action of the Lie algebra $\mathfrak{gl}_n$ in the module V. In particular, they provide explicit formulae for the action of generators e_{mm}, $e_{m,m+1}$ and $e_{m+1,m}$ on the vectors ξ_Λ with $\Lambda \in \mathscr{S}$. The first equality in (2.1) and Theorem 2.2 imply that

$$e_{mm} \cdot \xi_\Lambda = \left(\sum_{i=1}^{m} \lambda_{mi} - \sum_{i=1}^{m-1} \lambda_{m-1,i} \right) \xi_\Lambda. \tag{3.4}$$

Put

$$\tau_{mi\Lambda} = \prod_{\substack{j=1 \\ j \neq i}}^{m} (v_{mi} - v_{mj})^{-1}.$$

Then using the Lagrange interpolation formula we obtain that

$$e_{m,m+1}\cdot\xi_\Lambda=\sum_{\Lambda^+}\gamma_{mi\Lambda}\tau_{mi\Lambda}\xi_{\Lambda^+},\tag{3.5}$$

$$e_{m+1,m}\cdot\xi_\Lambda=\sum_{\Lambda^-}\beta_{mi\Lambda}\tau_{mi\Lambda}\xi_{\Lambda^-}$$

where Λ^+ and Λ^- are Gelfand-Zetlin schemes obtained from Λ by increasing and decreasing the (m, i)-entry by 1 respectively.

Remark 3.6. The equalities (3.5) are not the Gelfand-Zetlin formulae in their canonical form [GZ]. To obtain the latter, one should employ vectors which differ from ξ_Λ by certain scalar factors. Namely, one should replace the factor $b_l(s)$ in the definition (2.4) of ξ_Λ by

$$b_l(s)\cdot\left(\prod_{k=1}^{j}(s+1-v_k)\prod_{k=1}^{j-1}(s-v_k)\prod_{k=j+1}^{l+1}(v_{l+1,j}-s-1)\prod_{k=j}^{l-1}(v_{l-1,j}-s)\right)^{-1/2}.$$

Since $\Lambda\in\mathscr{S}$ and $v_j\leqslant s<v_{lj}$, all factors in the four above products are positive.

§4. The construction given above admits a natural generalization to the case of the quantum universal enveloping algebra $\mathrm{U}_q(\mathfrak{gl}_n)$. We will point out here only the main statements. The proofs are quite similar to those in the case of $\mathrm{U}(\mathfrak{gl}_n)$ and will be omitted. Some of them are contained in [T].

Let us introduce the *quantum Yangian* $\mathrm{Y}_q(\mathfrak{gl}_n)$, cf. [C1]. This is an associative algebra over the field $\mathbf{F}=\mathbf{Q}(q)$, generated by the elements $T_{ij}^{(s)}$ where $i,j=1,\dots,n$ and $s=0,1,\dots$ such that $T_{ii}^{(0)}$ are invertible and $T_{ij}^{(0)}=0$ for $i>j$. These elements are subjected to the following relations. Introduce the formal Laurent series in x^{-1}

$$T_{ij}(x)=T_{ij}^{(0)}x+T_{ij}^{(1)}+T_{ij}^{(2)}x^{-1}+T_{ij}^{(3)}x^{-2}+\cdots$$

and form the matrix

$$T(x)=[T_{ij}(x)]_{i,j=1}^{n}.$$

Let $w_1,\dots,w_n$ be the standard basis in $\mathbf{F}^n$. Consider the *Cherednik R-matrix*, it is the $\mathrm{End}\,((\mathbf{F}^n)^{\otimes 2})$-valued function $R_q(x,y)$ such that

$$R_q(x,y)\cdot w_i\otimes w_j=\begin{cases}(xq-yq^{-1})w_i\otimes w_i, & i=j;\\ (x-y)w_i\otimes w_j+x(q-q^{-1})w_j\otimes w_i, & i>j;\\ (x-y)w_i\otimes w_j+y(q-q^{-1})w_j\otimes w_i, & i<j.\end{cases}$$

Let P be the permutation map in $(\mathbf{F}^n)^{\otimes 2}$; put $\check{R}(x,y)=P\cdot R(x,y)$. Then the relations for $T_{ij}^{(s)}$ are of the same form as (1.1) above:

$$\check{R}(x,y)\cdot T(x)\otimes T(y)=T(y)\otimes T(x)\cdot\check{R}(x,y).\tag{4.1}$$

The generators $T_{ij}^{(s)}$ with $i, j = 1, \dots, m$ obey exactly the same relations as the corresponding generators of $Y_q(\mathfrak{gl}_m)$. Thus we have the chain of subalgebras

$$Y_q(\mathfrak{gl}_1) \subset Y_2(\mathfrak{gl}_2) \subset \cdots \subset Y_q(\mathfrak{gl}_n).$$

The relations (4.1) also imply that for any $h \in \mathbf{F} \setminus \{0\}$ the map

$$T_{ij}(x) \longmapsto T_{ij}(xh)$$

defines an automorhism of the algebra $Y_q(\mathfrak{gl}_n)$.

Let $X(x) = [X_{ij}(x)]_{i,j=1}^m$ be an arbitrary matrix whose entries are formal Laurent series in x^{-1} with coefficients from $Y_q(\mathfrak{gl}_n)$. Define the *quantum determinant* of this matrix to be the sum over all permutations g of $1, 2, \dots, m$

$$\operatorname{qdet} X(u) = \sum_g (-q)^{-\ell(g)} \cdot X_{1g(1)}(x) X_{2g(2)}(xq^{-2}) \cdots X_{m,g(m)}(xq^{2-2m}).$$

We will also denote by $\operatorname{pdet} X(u)$ the sum

$$\sum_g (-q)^{\ell(g)} \cdot X_{1g(1)}(xq^{2-2m}) X_{2g(2)}(xq^{4-2m}) \cdots X_{m,g(m)}(x).$$

Define the formal series $A_m(x)$, $B_m(x)$, $C_m(x)$ and $D_m(x)$ by the matrix $T(x)$ in the same way as the formal series $A_m(u)$, $B_m(u)$, $C_m(u)$ and $D_m(u)$ were defined by the matrix $T(u)$ in Section 1.

Proposition 4.1. *a) The coefficients of $A_n(x)$ belong to the centre of the algebra $Y_q(\mathfrak{gl}_n)$. b) All the coefficients of $A_1(x), \dots, A_n(x)$ pairwise commute.*

Define the q-commutator $[X, Y]_q = XY - qYX$ as usual. Then we have

Proposition 4.2. *The following commutation relations hold in $Y_q(\mathfrak{gl}_n)$:*

$$[A_m(x), B_l(y)] = 0 \qquad \text{if } l \neq m,$$

$$[C_m(x), B_l(y)] = 0 \qquad \text{if } l \neq m,$$

$$[B_m(x), B_l(y)] = 0 \qquad \text{if } |l - m| \neq 1,$$

$$\frac{x - y}{q - q^{-1}} [A_m(x), B_m(y)]_q = yB_m(x)A_m(y) - xB_m(y)A_m(x),$$

$$\frac{x - y}{q - q^{-1}} [C_m(x), B_m(y)] = y(D_m(x)A_m(y) - D_m(y)A_m(x)).$$

The matrix $T(x)$ is invertible as a formal Laurent series in x^{-1}; denote by $\widetilde{T}(x)$ the inverse matrix.

Lemma 4.3. *For any $m = 1, \dots, n-1$ we have the equality*

$$\operatorname{qdet} [T_{ij}(xq^{2(m-n)})]_{i,j=1}^m = \operatorname{pdet} [\widetilde{T}_{ij}(x)]_{i,j=m+1}^n \cdot \operatorname{qdet} T(x).$$

Proposition 4.4. *The following relation holds in* $\mathrm{Y}_q(\mathfrak{gl}_n)$:

$$qC_m(x)B_m(xq^{-2}) = D_m(x)A_m(xq^{-2}) - A_{m+1}(x)A_{m-1}(xq^{-2}).$$

By definition, the *quantum universal enveloping algebra* $\mathrm{U}_q(\mathfrak{gl}_n)$ is an associative algebra over F generated by the elements t_i, t_i^{-1} with $i = 1,\dots,n$ and e_i, f_i with $i = 1,\dots,n-1$. These elements are subjected to the following relations [J1]:

$$t_it_i^{-1} = t_i^{-1}t_i = 1, \quad [t_i, t_j] = 0,$$

$$t_ie_j = e_jt_iq^{\delta_{ij}-\delta_{i,j+1}}, \quad t_if_j = f_jt_iq^{\delta_{i,j+1}-\delta_{ij}},$$

$$[e_i, f_j] = \frac{t_it_{i+1}^{-1} - t_{i+1}t_i^{-1}}{q - q^{-1}}\,\delta_{ij},$$

$$[e_i, e_j] = [f_i, f_j] = 0 \quad \text{if } |i-j| > 1,$$

$$[e_i, [e_{i\pm 1}, e_i]_q]_q = [f_i, [f_{i\pm 1}, f_i]_q]_q = 0.$$

Introduce the q-analogues of the root vectors in $\mathfrak{gl}_n$ by induction:

$$e_{i,i+1} = e_i, \quad e_{i+1,i} = f_i,$$

$$e_{ij} = [e_{ik}, e_{kj}]_q \qquad i < k < j,$$

$$e_{ij} = [e_{ik}, e_{kj}]_{q^{-1}} \qquad i > k > j.$$

One can define a homomorphism $\mathrm{Y}_q(\mathfrak{gl}_n) \to \mathrm{U}_q(\mathfrak{gl}_n)$ as follows [J1]:

$$T_{ii}(x) \longmapsto \frac{xt_i - t_i^{-1}}{q - q^{-1}}, \tag{4.2}$$

$$T_{ij}(x) \longmapsto xt_ie_{ji}, \qquad i < j;$$

$$T_{ij}(x) \longmapsto e_{ji}t_j^{-1}, \qquad i > j.$$

Denote the images of the series $A_m(x)$, $B_m(x)$, $C_m(x)$ and $D_m(x)$ under this homomorhism by $a_m(x)$, $b_m(x)$, $c_m(x)$ and $d_m(x)$ respectively. These images are polynomial in x and

$$a_m(x) = (x^mq^{m(1-m)}t_1\cdots t_m + \cdots + (-1)^mt_1^{-1}\cdots t_m^{-1})\cdot(q - q^{-1})^{-m},$$

$$b_m(u) = (x^mq^{m(1-m)}t_1\cdots t_mf_m + \cdots + xb)\cdot(q - q^{-1})^{1-m},$$

$$c_m(u) = (x^{m-1}c + \cdots + (-1)^me_mt_1^{-1}\cdots t_m^{-1})\cdot(q - q^{-1})^{1-m}$$

for some $b, c \in \mathrm{U}_q(\mathfrak{gl}_n)$. The above equalities show that the coefficients of the polynomials $a_m(x)$, $b_m(x)$ and $c_m(x)$ generate the algebra $\mathrm{U}_q(\mathfrak{gl}_n)$.

Let us recall several known facts about finite-dimensional $\mathrm{U}_q(\mathfrak{gl}_n)$-modules [J1, L, R]. It is known that any such module is completely reducible and all the irreducible modules are uniquely characterized by their highest weights. Let V be an irreducible finite-dimensional $\mathrm{U}_q(\mathfrak{gl}_n)$-module of the highest weight

$(\kappa_1, \kappa_2, \ldots, \kappa_n)$. Denote by ξ the highest weight vector:

$$t_i \cdot \xi = \kappa_i \xi; \quad e_{ij} \cdot \xi = 0, \quad i < j.$$

Then $\kappa_i = \varepsilon_i q^{\lambda_i}$ where $\varepsilon_i = \pm 1$, $\lambda_i \in \mathbf{Z}$ and $\lambda_i \geqslant \lambda_{i+1}$. The maps

$$t_i \longmapsto \varepsilon_i t_i, \quad e_i \longmapsto \varepsilon_i e_i, \quad f_i \longmapsto \varepsilon_{i+1} f_i$$

define an automorphism of $\mathrm{U}_q(\mathfrak{gl}_n)$. So we will assume that each $\varepsilon_i = 1$.

There is a canonical decomposition of the space V into the direct sum of one-dimensional subspaces associated with the chain of subalgebras

$$\mathrm{U}_q(\mathfrak{gl}_1) \subset \mathrm{U}_q(\mathfrak{gl}_2) \subset \cdots \subset \mathrm{U}_q(\mathfrak{gl}_n).$$

These subspaces are parametrized by the Gelfand-Zetlin schemes Λ. The subspace $V_\Lambda \subset V$ corresponding to $\Lambda \in \mathscr{S}$ is contained in an irreducible $\mathrm{U}_q(\mathfrak{gl}_m)$-submodule of the highest weight $(q^{\lambda_{m1}}, q^{\lambda_{m2}}, \ldots, q^{\lambda_{mm}})$ for each $m = n-1, n-2, \ldots, 1$. These conditions define V_Λ uniquely.

Let again $\nu_{mi} = i - \lambda_{mi} - 1$. For any $\Lambda \in \mathscr{T}$ put

$$\alpha_{m\Lambda}(x) = \prod_{i=1}^{m} \frac{x q^{-\nu_{mi}} - q^{\nu_{mi}}}{q - q^{-1}}.$$

Proposition 4.5. *The subspace $V_\Lambda \subset V$ is an eigenspace of $a_m(x)$ with the eigenvalue $q^{m(1-m)/2} \alpha_{m\Lambda}(x)$.*

For any $\Lambda \in \mathscr{T}$ define the vector ξ_Λ in a way similar to (2.4):

$$\xi_\Lambda = \prod_{(l,j)}^{\rightarrow} \left(\prod_{s=\nu_j}^{\nu_{lj}-1} q^{l(l-1)/2 - sl + 1} b_l(q^{2s}) \right) \cdot \xi, \tag{4.3}$$

here for each fixed l the elements $b_l(q^{2s}) \in \mathrm{U}_q(\mathfrak{gl}_n)$ commute because of Proposition 4.2.

Theorem 4.6. *a) For any $\Lambda \in \mathscr{T}$ we have the equality*

$$a_m(x) \cdot \xi_\Lambda = q^{m(1-m)/2} \alpha_{m\Lambda}(x) \xi_\Lambda.$$

b) For any $\Lambda \in \mathscr{S}$ we have $\xi_\Lambda \in V_\Lambda$.
c) If $\Lambda \in \mathscr{T} \setminus \mathscr{S}$ then $\xi_\Lambda = 0$.

For any $k \in \mathbf{Z}$ put $[k] = \dfrac{q^k - q^{-k}}{q - q^{-1}}$ and define

$$\beta_{mi\Lambda} = \prod_{j=1}^{i} \frac{[\nu_{mi} - \nu_{m+1,j} + 1]}{[\nu_{mi} - \nu_j + 1]} \prod_{j=1}^{i-1} \frac{[\nu_{mi} - \nu_{m-1,j}]}{[\nu_{mi} - \nu_j]},$$

$$\gamma_{mi\Lambda} = \prod_{j=1}^{i} [\nu_{mi} - \nu_j] \prod_{j=1}^{i-1} [\nu_{mi} - \nu_j - 1] \times$$

$$\times \prod_{j=i+1}^{m+1} [\nu_{m+1,j} - \nu_{mi}] \prod_{j=i}^{m-1} [\nu_{m-1,j} - \nu_{mi} + 1],$$

$$\tau_{mi\Lambda} = \prod_{\substack{j=1 \\ j \neq i}}^{m} [\nu_{mi} - \nu_{mj}]^{-1}.$$

Let the indices $m < n$ and $i \leqslant m$ be fixed. Denote by Λ^+ and Λ^- the arrays obtained from Λ by increasing and decreasing the (m, i)-entry by 1 respectively.

Theorem 4.7. *a) We have*

$$c_m(q^{2\nu_{mi}}) \cdot \xi_\Lambda = \begin{cases} q^{m(1-m)/2 + m\nu_{mi}} \gamma_{mi\Lambda} \xi_{\Lambda^+} & \textit{if } \Lambda^+ \in \mathcal{S}; \\ 0 & \textit{otherwise.} \end{cases}$$

b) If $\Lambda \in \mathcal{S}$ then $\xi_\Lambda \neq 0$.

c) We have

$$b_m(q^{2\nu_{mi}}) \cdot \xi_\Lambda = \begin{cases} q^{m(1-m)/2 + m\nu_{mi} - 1} \beta_{mi\Lambda} \xi_{\Lambda^-} & \textit{if } \Lambda^- \in \mathcal{S}; \\ 0 & \textit{otherwise.} \end{cases}$$

Theorems 4.6, 4.7 completely describe the action of the algebra $\mathrm{U}_q(\mathfrak{gl}_n)$ in the module V. In particular, they provide explicit formulae for the action of generators t_m, e_m and f_m on the vectors ξ_Λ with $\Lambda \in \mathcal{T}$, parallel to the formulae (3.4), (3.5):

$$t_m \cdot \xi_\Lambda = \prod_{i=1}^{m} q^{\lambda_{mi}} \prod_{i=1}^{m-1} q^{-\lambda_{m-1,i}} \xi_\Lambda,$$

$$e_m \cdot \xi_\Lambda = \sum_{\Lambda^+} \gamma_{mi\Lambda} \tau_{mi\Lambda} \xi_{\Lambda^+},$$

$$f_m \cdot \xi_\Lambda = \sum_{\Lambda^-} \beta_{mi\Lambda} \tau_{mi\Lambda} \xi_{\Lambda^-}$$

where Λ^+ and Λ^- are Gelfand-Zetlin schemes obtained from Λ by increasing and decreasing the (m, i)-entry by 1 respectively. The last two formulae look exactly as (3.5).

To obtain the q-analogues of the canonical Gelfand-Zetlin formulae given in [J2] for the first time and rederived in [UTS], one should extend the basic field $\mathbf{F}$ and to rescale the vectors ξ_Λ in a way similar to that at the end of Section 3. Namely, one should replace the factor $b_l(q^{2s})$ in the definition (4.3) of ξ_Λ by

$$b_l(q^{2s}) \cdot \left(\prod_{k=1}^{j} [s + 1 - \nu_k] \prod_{k=1}^{j-1} [s - \nu_k] \prod_{k=j+1}^{l+1} [\nu_{l+1,j} - s - 1] \prod_{k=j}^{l-1} [\nu_{l-1,j} - s] \right)^{-1/2}.$$

Remark 4.8. Introduce the matrices

$$Q = [q^{-2i}\delta_{ij}]_{i,j=1}^{n}, \qquad \hat{T}(x) = [\hat{T}_{ij}(x)]_{i,j=1}^{n}$$

where $\hat{T}_{ij}(u)$ is equal to $(-q)^{j-i}$ times the quantum determinant of the matrix obtained from $T(x)$ by removing the row j and the column i. Then

$$T(x)\hat{T}(xq^{-2}) = Q\hat{T}^{t}(x)Q^{-1}T^{t}(xq^{2-2n}) = \text{qdet}\, T(u) \tag{4.4}$$

where the superscript t denotes the usual matrix transposition; see [C1], [T] for the proof of these equalities. Let $E_{\pm}$ be the matrices taking part in the homomorphism (4.2):

$$T(x) \longmapsto xE_{+} - E_{-},$$

cf. the matrices $L^{(\pm)}$ from [RTF]. The coefficients of the polynomial $a_n(u) = \text{qdet}\,(xE_{+} - E_{-})$ belong to the center of the algebra $\mathrm{U}_q(\mathfrak{gl}_n)$ and from (4.4) we obtain the matrix identities

$$a_n(E_{-}E_{+}^{-1}) = 0, \qquad a_n((E_{+}^{t})^{-1}E_{-}^{+}q^{2n-2}) = 0$$

which are the q-analogues of (2.6).

Remark 4.9. We can also treat q as a complex number rather than an indeterminate and consider $\mathrm{U}_q(\mathfrak{gl}_n)$, $\mathrm{Y}_q(\mathfrak{gl}_n)$ as algebras over **C**. If q is generic then the results of this section remain valid. In the peculiar case of q being a root of unit it is easy to generalize these results to the irreducible highest weight modules. Moreover the technique works for the periodic and semiperiodic modules [T] as well. This allows us to determine the branching rules corresponding to the restriction from $\mathrm{U}_q(\mathfrak{gl}_n)$ to $\mathrm{U}_q(\mathfrak{gl}_{n-1})$ and to define the Gelfand-Zetlin bases for $\mathrm{U}_q(\mathfrak{gl}_n)$-modules without classical analogues. It will be done in details in the forthcoming paper.

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References

[C1] Cherednik, I. V., A new interpretation of Gelfand-Zetlin bases, *Duke Math. J.*, **54** (1987), 563–577.

[C2] ———, Quantum groups as hidden symmetries of classic representation theory, in *Differential geometric methods in theoretical physics*, World Scientific, Singapore, (1989), 47–54.

[D1] Drinfeld, V. G., Hopf algebras and the quantum Yang-Baxter equation, *Soviet Math. Dokl.*, **32** (1985), 254–258.

[D2] Drinfeld, V. G., A new realization of Yangians and quantized affine algebras, *Soviet Math. Dokl.*, **36** (1988), 212–216.

[G] Gould, M. D., On the matrix elements of the $U(n)$ generators, *J. Math. Phys.*, **22** (1981), 15–22.

[GZ] Gelfand, I. M. and Zetlin, M. L., Finite-dimensional representations of the unimodular group, *Dokl. Akad. Nauk SSSR*, **71** (1950), 825–828.

[J1] Jimbo, M., A q-analogue of $U(gl(N+1))$, Hecke algebra and the Yang-Baxter equation, *Lett. Math. Phys.*, **11** (1986), 247–252.

[J2] ———, Quantum R-matrix for the generalized Toda system, *Commun. Math. Phys.*, **102** (1986), 537–547.

[KS] Kulish, P. P. and Sklyanin, E. K., Quantum spectral transform method: recent developments, in *Integrable quantum field theories, Lecture Notes in Phys.*, Springer, Berlin-Heidelberg, **151** (1982), 61–119.

[L] Lusztig, G., On deformations of certain simple modules over enveloping algebras, *Adv. Math.*, **70** (1988), 237–249.

[N] Nazarov, M. L., Quantum Berezinian and the classical Capelli identity, *Lett. Math. Phys.*, **21** (1991), 123–131.

[O] Olshanskiĭ, G. I., Representations of infinite-dimensional classical groups, limits of enveloping algebras, and Yangians, in *Topics in Representation Theory*, Adv. Soviet Math., AMS, Providence RI, **2** (1991), 1–66.

[R] Rosso, M., Finite-dimensional representations of the q-analogues of the universal enveloping algebras of complex simple Lie algebras, *Commun. Math. Phys.*, **117** (1988), 581–593.

[RTF] Reshetikhin, N. Yu, Takhtajan, L. A. and Faddeev, L. D., Quantization of Lie groups and Lie algebras, *Leningrad Math. J.*, **1** (1990), 193–225.

[T] Tarasov, V., Cyclic monodromy matrices for $sl(n)$ trigonometric R-matrices, *Comm. Math. Phys.*, **158** (1993), 459–483.

[UTS] Ueno, K., Takebayashi, T. and Shibukawa, Y., Gelfand-Zetlin basis for $U_q(gl(N+1))$-modules, *Lett. Math. Phys.*, **18** (1989), 215–221.

[Z1] Zhelobenko, D. P., *Compact Lie groups and their representations*, Transl. of Math. Monographs, **40**, AMS, Providence RI, 1973.

[Z2] ———, On Gelfand-Zetlin bases for classical Lie algebras, in *Representations of Lie groups and Lie algebras*, Akadémiai Kiadó, Budapest, (1985), 79–106.