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A Generalized Grothendieck-Riemann-Roch Theorem for Hirzebruch's χ_y-Characteristic and T_y-Characteristic

By

Shoji YOKURA *

§ 0. Introduction

The Grothendieck-Riemann-Roch (abbr. **GRR**) is a relative version of the Hirzebruch-Riemann-Roch (abbr. **HRR**), $\chi(X, E)=T(X, E)$. Hirzebruch's χ_y-characteristic $\chi_y(X, E)$ and T_y-characteristic $T_y(X, E)$ are a generalization of the Euler-Poincaré characterisitc $\chi(X, E)$ and the Todd characteristic $T(X, E)$ such that when $y=0$ $\chi_0(X, E)=\chi(X, E)$ and $T_0(X, E)=T(X, E)$, and they are equal; $\chi_y(X, E)=T_y(X, E)$, which is called the generalized Hirzebruch-Riemann-Roch (abbr. **g-HRR**). In this short note we show that we can get a generalized GRR version (abbr. **g-GRR**) such that when $y=0$ our **g-GRR** specializes to the original **GRR** and such that the Hirzebruch's **g-HRR** is induced from our **g-GRR** by mapping X to a point, just like **HRR** is induced from **GRR** by mapping X to a point. For the statements and the proofs of our main theorems see § 2.

Our **g-GRR** is entirely a formal consequence of the original **GRR**, achieved by some formal calculations with power series and Grothendieck's λ-rings. It remains to see whether there are some geometric constructions for **g-GRR**.

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§ 1. Hirzebruch's χ_y-Characteristic and T_y-Characteristic

Hirzebruch's *χ_y-characteristic* $\chi_y(X, E)$ and *T_y-characteristic* $T_y(X, E)$, for a non-singular complex projective variety X and a holomorphic vector bundle E on X,

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* Department of Mathematics, College of Liberal Arts, University of Kagoshima, Kagoshima, 890 Japan.

are a generalization of the original Euler-Poincaré characteristic $\chi(X, E) := \sum_{q\geq 0}(-1)^q \dim_{C} H^q(X; E)$ and Todd-characteristic $T(X, E) = \langle ch(E) td(T_X), [X] \rangle$, where $\langle\ ,\ \rangle$ is the Kronecker pairing, $[X]$ is the fundamental homology class of X, $ch(E)$ is the Chern character of the bundle E and $td(T_X)$ is the total Todd class of the tangent bundle T_X of X:

$$ch(E) := \sum_{i=1}^{\mathrm{rank} E} e^{\alpha_i} \text{ and } td(T_X) := \prod_{j=1}^{\dim X} \frac{\beta_j}{1-e^{-\beta_j}}$$

where α_i are the Chern roots of E and β_j are the Chern roots of T_X. For more details and related topics of these two characteristics see Hirzebruch's books [6] and [7].

Definition (1.1) (**χ_y-characteristic $\chi_y(X, E)$**).

$$\chi_y(X, E) := \sum_{p\geq 0}\left(\sum_{q\geq 0}(-1)^q \dim_{C} H^q(X, E\otimes \Lambda^p T_X^{\vee})\right) y^p$$

$$= \sum_{p\geq 0} \chi(X, E\otimes \Lambda^p T_X^{\vee}) y^p,$$

where $T_X^{\vee}$ is the dual of the tangent bundle T_X, i. e., the cotangent bundle of X.

For $E = \mathbb{1}$, the trivial line bundle, $\chi_y(X) := \chi_y(X, \mathbb{1})$ is called the *χ_y-genus*, which is a generalization of the *arithmetic genus* $\chi(X) := \chi(X, \mathbb{1})$.

Definition (1.2) (**T_y-characteristic $T_y(X, E)$**). Let us set: (as elements of $H^*(X; \mathbf{Q})\otimes \mathbf{Z}[y]$)

$$ch_{(1+y)}(E) := \sum_{i=1}^{\mathrm{rank}\, E} e^{\alpha_i(1+y)} \text{ and } \widetilde{td_{(y)}}(T_X) := \prod_{j=1}^{\dim X}\left(\frac{\beta_j(1+y)}{1-e^{-\beta_j(1+y)}} - \beta_j y\right),$$

where α_i are the Chern roots of E and β_j are the Chern roots of T_X. Then $T_y(X, E)$ is defined by

$$T_y(X, E) := \langle ch_{(1+y)}(E)\ \widetilde{td_{(y)}}(T_X), [X] \rangle.$$

For $E = \mathbb{1}$, the trivial line bundle, $T_y(X) := T_y(X, \mathbb{1}) = \langle \widetilde{td_{(y)}}(T_X), [X] \rangle$ is called the *generalized Todd genus* or the *T_y-genus*. For three distinguished values (i. e., 0, -1 and 1) of y, this T_y-characteristic becomes the following:

$y=0$: $T_0(X, E) = \langle ch(E)\ td(T_X), [X] \rangle = T(X, E)$, T-characteristic (or Todd-characteristic)

$y=-1$: $T_{-1}(X, E) = \langle \mathrm{rank}(E) \prod (1+\beta_j), [X] \rangle = \langle \mathrm{rank}(E) c(T_X), [X] \rangle = \mathrm{rank}(E) e(X)$, where $c(T_X)$ is the total Chern class of the tangent bundle T_X and $e(X)$ is the topological Euler-Poincaré characteristic of X,

$$y=1: T_1(X, E)=\left\langle \sum_{i=1}^{\mathrm{rank}E} e^{2\alpha_i}\prod_{j=1}^{\dim X} \frac{\beta_j}{\tanh\beta_j}, [X]\right\rangle=\mathrm{sign}(X, E),$$

the signature of X with values in the vector bundle E (see [7, Chapter 6]).

With the above definitions Hirzebruch showed the following generalized Hirzebruch-Riemann-Roch [6, § 21. 3] :

Theorem (1. 3). (**g-HRR**) $\chi_y(X, E)=T_y(X, E)$.

Note that $\chi_0(X)=T_0(X)=\chi(X)$, $\chi_{-1}(X)=T_{-1}(X)=e(X)$ and $\chi_1(X)=T_1(X)=\mathrm{sign}(X)$, the signature of X.

§ 2. A Generalized Grothendieck-Riemann-Roch

In this section we show our main results. We formulate our generalized Grothendieck-Riemann-Roch (**g-GRR**) in the same spirit as that for the formaulation of the original **GRR**.

A. Grothendieck generalized **HRR** to non-singular quasi-projective algebraic varieties over any field and proper morphisms with Chow cohomology ring theory instead of ordinary cohomology theory. For the complex case we can still take the ordinary cohomology theory. Since our argument is formal, it works in whatever context **GRR** is known. Here we stick ourselves to non-singular complex projective varieties and cohomology theory for the sake of simplicity.

For a variety X, let $K_0(X)$ be the Grothendieck group of algebraic coherent sheaves on X and for a morphism $f: X\to Y$ the Grothendieck pushforward $f_!: K_0(X)\to K_0(Y)$ is defined by

$$f_!(F) := \sum_{i\geq 0}(-1)^i R^i f_* F,$$

where $R^i f_* F$ is (the class of) the higher direct image sheaf of F. Then K_0 is a covariant functor with this pushforward. First we generalize this covariant functor K_0 as follows :

Theorem (2. 1). *Let $K_0{}^{[y]}(X) := K_0(X)\otimes \mathbf{Z}[y]$. For a morphism $f: X\to Y$, we define the pushforward $f^{[y]}{}_!: K_0{}^{[y]}(X)\to K_0{}^{[y]}(Y)$ as follows :*

$$f^{[y]}{}_! F := \sum_{p\geq 0} f_!(F\otimes \Lambda^p T_f^\vee)y^p$$

where $f_! : K_0(X)\to K_0(Y)$ *is the Grothendieck pushforwand and* $T_f^\vee := T_X^\vee - f^* T_Y^\vee$ *is a virtual "relative cotangent bundle" as an element of* $K^0(X)$*, and this pushforward is extended linearly with respect to the polynomial ring* $Z[y]$*. Then* $K_0{}^{[y]}$ *is a covariant functor with the above pushforward.*

Proof. Let $f : X\to Y$ and $g : Y\to Z$ be morphisms and let F be a coherent sheaf on X. By defintion

$$g^{[y]}{}_! f^{[y]}{}_! F = g^{[y]}{}_!\left(\sum_{p\geq 0} f_!(F\otimes \Lambda^p T_f^\vee)y^p\right)$$

$$= \sum_{p\geq 0}\left(\sum_{q\geq 0} g_!(f_! F\otimes \Lambda^p T_f^\vee)\otimes \Lambda^q T_g^\vee)y^q\right)y^p$$

$$= \sum_{p\geq 0}\left(\sum_{q\geq 0} g_! f_!(F\otimes \Lambda^p T_f^\vee \otimes \Lambda^q f^* T_g^\vee)y^q\right)y^p$$

(by the projection formula for $f_!$)

$$= \sum_{j\geq 0}(gf)_!(F\otimes \sum_{p+q=j} \Lambda^p T_f^\vee \otimes \Lambda^q f^* T_g^\vee)y^j$$

(by the additivity of $(gf)_!$)

$$= \sum_{j\geq 0}(gf)_!(F\otimes \Lambda^j (T_f^\vee + f^* T_g^\vee))y^i$$

(by the property of the exterior power Λ^j)

$$= \sum_{j\geq 0}(gf)_!(F\otimes \Lambda^j T_{gf}^\vee)y^j \qquad (\text{since } T_f^\vee + f^* T_g^\vee = T_{gf}^\vee)$$

$$= (gf)^{[y]}{}_! F.$$

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Remark (2. 2). For $y=0$, $f^{[y]}{}_!$ is nothing but the original Grothendieck pushforward $f_!$.

The original GRR (e. g., see [1, 2, 5, 6]) is the following :

Let $\tau : K_0(\;)\to H^*(\;;Q)$ *be the transformation defined by* $\tau(\circ) = ch(\circ)td(T_X)$ *for any variety X. Then* τ *is actually natural, i. e., for any morphism* $f : X\to Y$ *the following diagram commutes*

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\tau} & H^*(X;Q) \\ \downarrow f_! & & \downarrow f_* \\ K_0(Y) & \xrightarrow[\tau]{} & H^*(Y;Q) \end{array}$$

where $f_* : H^*(X;Q)\to H^*(Y;Q)$ *is the Gysin homomorphism.*

This **GRR** is generalized to our **g-GRR** as follows :

Theorem (2.3). *If we define* $\tau^{[y]}: K_0{}^{[y]}(X) \to H^*(X;Q)[y]$ *as follows*

$$\tau^{[y]}(F) := ch_{(1+y)}(F)\ \widetilde{td}_{(y)}(T_X),$$

which is extended linearly with respect to the polynomial ring $\mathbb{Z}[y]$, *then* $\tau^{[y]}: K_0{}^{[y]} \to H^*(\ ;Q)[y]$ *is a natural transformation, i. e., for any morphism* $f: X \to Y$ *the following diagram is commutative* :

$$\begin{array}{ccc} K_0{}^{[y]}(X) & \xrightarrow{\tau^{[y]}} & H^*(X;Q)[y] \\ \downarrow f^{[y]}{}_! & & \downarrow f_* \\ K_0{}^{[y]}(Y) & \xrightarrow[\tau^{[y]}]{} & H^*(Y;Q)[y]. \end{array}$$

Proof. It suffices to show that for a coherent sheaf F

$$\tau^{[y]}(f^{[y]}{}_!F) = f_*(\tau^{[y]}(F)),$$

i.e.,

$$ch_{(1+y)}(f^{[y]}{}_!F)\widetilde{td}_{(y)}(T_Y) = f_*(ch_{(1+y)}(F)\widetilde{td}_{(y)}(T_X)). \tag{2.3.1}$$

By the projection formula $f_*(a \circ f^*b) = f_*a \circ b$ and using the "virtual relative tangent bundle" $T_f := T_X - f^*T_Y$, we can see that (2.3.1) is equivalent to the following equation :

$$ch_{(1+y)}(f^{[y]}{}_!(F)) = f_*\left(ch_{(1+y)}(F)\frac{\widetilde{td}_{(y)}(T_x)}{\widetilde{td}_{(y)}(f^*T_Y)}\right) = f_*(ch_{(1+y)}(F)\widetilde{td}_{(y)}(T_f)) \tag{2.3.2}$$

i.e., $$ch_{(1+y)}\left(\textstyle\sum_{p\geq 0} f_!(F\otimes\Lambda^p T_f^\vee)y^p\right) = f_*(ch_{(1+y)}(F)\widetilde{td}_{(y)}(T_f)),$$

i.e.,

$$\textstyle\sum_{p\geq 0} ch_{(1+y)}(f_!(F\otimes\Lambda^p T_f^\vee))y^p = f_*(ch_{(1+y)}(F)\widetilde{td}_{(y)}(T_f)). \tag{2.3.3}$$

Before going on to the proof of (2.3.3), we recall a result form our previous paper [9] :

Proposition (2.3.4). *Let* q *be a variable and let*

$$ch_{(q)}(V) = \sum_{j=0}^{\mathrm{rank}V} e^{q\gamma_j} \text{ and } td_{(q)}(V) = \prod_{j=1}^{\mathrm{rank}V} \frac{q\gamma_j}{1-e^{-q\gamma_j}} = \sum_{i=0}^{\mathrm{rank}V} q^i td_i(V)$$

where γ_j are the Chern roots of a complex vector bundle V. Then for $f: X \to Y$

$$q^{\mathrm{reldim}(f)} ch_{(q)}(f_! F) td_{(q)}(T_Y) = f_*(ch_{(q)}(F) td_{(q)}(T_X))$$

or, equivalently,

$$q^{\mathrm{reldim}(f)} ch_{(q)}(f_! F) = f_*(ch_{(q)}(F) td_{(q)}(T_f)) \tag{2.3.5}$$

where $\mathrm{reldim}(f) := \dim X - \dim Y$ *is the relative dimension of a morphism $f: X \to Y$. (Note : (1) For a coherent sheaf F the definition of $ch_{(q)}(F)$ is similar to that of $ch(F)$. (2) This proposition is a formal translation of GRR, with the "q" doing nothing but keeping track of degrees.)*

Therefore, by (2.3.5), the left hand side of (2.3.3) becomes as follows :

$$\begin{aligned}
&\textstyle\sum_{p\geq 0} ch_{(1+y)}(f_!(F \otimes \Lambda^p T_f^\vee))y^p \\
&\quad = \textstyle\sum_{p\geq 0}(1+y)^{-\mathrm{reldim}(f)} f_*(ch_{(1+y)}(F\otimes \Lambda^p T_f^\vee) td_{(1+y)}(T_f))y^p \\
&\quad = f_*(\textstyle\sum_{p\geq 0}(1+y)^{-\mathrm{reldim}(f)} ch_{(1+y)}(F) ch_{(1+y)}(\Lambda^p T_f^\vee) td_{(1+y)}(T_f))y^p \\
&\quad = f_*(ch_{(1+y)}(F)(1+y)^{-\mathrm{reldim}(f)}(\textstyle\sum_{p\geq 0} ch_{(1+y)}(\Lambda^p T_f^\vee)y^p) td_{(1+y)}(T_f))
\end{aligned}$$

Hence, if we can show the following equation (2.3.6), then we get (2.3.3), thus (2.3.1) :

$$(1+y)^{-\mathrm{reldim}(f)}(\textstyle\sum_{p\geq 0} ch_{(1+y)}(\Lambda^p T_f^\vee)y^p) td_{(1+y)}(T_f) = \widetilde{td}_{(y)}(T_f). \tag{2.3.6}$$

In fact, this equation is a special case of the following more general equation :

Lemma (2. 3. 7). *For any virtual bundle E,*

$$(1+y)^{-\mathrm{rank}(E)}(\textstyle\sum_{p\geq 0} ch_{(1+y)}(\Lambda^p E^\vee)y^p) td_{(1+y)}(E) = \widetilde{td}_{(y)}(E).$$

Proof of Lemma (2. 3. 7). If E is a bundle of rank n and α_i are the Chern roots of E, then by definition we have

$$\begin{aligned}
&(1+y)^{-\mathrm{rank}(E)}(\textstyle\sum_{p\geq 0} ch_{(1+y)}(\Lambda^p E^\vee)y^p) td_{(1+y)}(E) \\
&\quad = \frac{1}{(1+y)^n}\prod_{i=1}^n (1+ye^{-\alpha_i(1+y)}) \prod_{i=1}^n \frac{\alpha_i(1+y)}{1-e^{-\alpha_i(1+y)}}
\end{aligned}$$

$$=\prod_{i=1}^{n}\Big(\frac{1+ye^{-\alpha_i(1+y)}}{1+y}\frac{\alpha_i(1+y)}{1-e^{-\alpha_i(1+y)}}\Big)$$

$$=\prod_{i=1}^{n}\Big(\frac{\alpha_i(1+y)}{1-e^{-\alpha_i(1+y)}}-\alpha_i y\Big)=\widetilde{td}_{(y)}(E).$$

To see that the equation also holds for any virtual bundle E, it suffices to observe that both sides of the equation are multiplicative. And for that we need the theory of Grothendieck's λ-*ring structure* on the contravariant functor K^0 (e. g., see [3] or [SGA 6] ; for our purpose we just recall the definition of $\lambda_y E=\sum_{p\geq 0}(\Lambda^p E)y^p$)). The multiplicativity of the middle term $\sum_{p\geq 0}ch_{(1+y)}(\Lambda^p E^\vee)y^p$, which can be expressed as $ch_{(1+y)}(\lambda_y(E^\vee))$, follows from the multiplicativity of $ch_{(1+y)}$ and λ_y.

Q. E. D.

Our **g-GRR** becomes the original **GRR** when $y=0$, and **g-HRR** follows from **g-GRR** by mapping X to a point.

Remark (2. 4). Now that we have established a generalized Grothendieck-Riemann-Roch theorem, it is quite natural or plausible to pose the question whether there is a "singular" extension of our generalized **GRR** theorem ($\boxed{?}$ in the diagram below), just like Baum-Fulton-MacPherson's Riemann-Roch theorem [1] (abbr. **BFM-RR**) is a "singular" extension of the original **GRR**.

$$\begin{array}{ccccc}
 & \text{"mapping to a point"} & & \text{"smooth"} & \\
\textbf{HRR} & \longleftarrow & \textbf{GRR} & \longleftarrow & \textbf{BFM-RR} \\
\uparrow \text{"}y=0\text{"} & & \uparrow \text{"}y=0\text{"} & & \uparrow \text{"}y=0\text{"} \\
\textbf{g-HRR} & \longleftarrow & \textbf{g-GRR} & \longleftarrow & \boxed{?} \\
 & \text{"mapping to a point"} & & \text{"smooth"} &
\end{array}$$

We do have a solution for $\boxed{?}$ such that when we restrict ourselves to the smooth category $\boxed{?}$ becomes our **g-GRR** and such that when "$y=0$" $\boxed{?}$ becomes **BFM-RR** ([10]). But our solution is still not satisfactory in the sense that it does not unify **BFM-RR**, Chern-MacPherson classes [8] and Goresky-MacPherson's homology L-classes [4]. Note that our **g-GRR** unifies **GRR** (in the case of $y=0$), Chern classes (in the case of $y=-1$) and Hirzebruch's cohomology L-classes (in the case of $y=1$). We hope to return to the question whether there is an ideal theory unifying **BFM-RR**, Chern-MacPherson classes and Goresky-MacPherson's homology L-classes.

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