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Remarks on the Bifurcation of Two Dimensional Capillary-Gravity Waves of Finite Depth*

By

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Abstract

We consider the problem to determine the two dimensional water waves of permanent configuration. The problem is a bifurcation problem ([**16**]). In this paper, we define a mapping G between two Hilbert spaces and prove that the solutions of the above problem are in one-to-one correspondence to the zeros of the mapping G. Our most important contribution in this paper is to clarify the role of the aspect ratio, i.e., the ratio between the mean depth of the flow and the wave length. In particular, we prove that there is no degenerate bifurcation point, whatever the aspect ratio may be.

Key words: progressive, capillary-gravity wave, bifurcation from double eigenvalue with O(2)-symmetry, degeneracy

§ 1. Introduction

We consider two dimensional progressive water waves on an incompressible inviscid fluid. By definition, progressive waves move at constant speeds and do not change their profiles during the motion. Therefore the flows are stationary when we observe them in a moving coordinate system. In the moving frame, we consider the wave profile, which is a free boundary to be sought. The fluid flow beneath the free boundary is determined by the differential equations but the motion of the air above the free boundary is neglected. The mathematical formulation of the problem goes back to 1847 by Stokes. We, however, consider this famous and old problem from the viewpoint of [**8, 9**].

The problem is known to be a bifurcation problem ([**12, 13, 16, 22, 23**]). In

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[16], we formulated the problem as to find zeros of a mapping

$$F : \mathbb{R}^2 \times [0, 1) \times X \to Y,$$

where X and Y are certain Banach spaces. The detail will be recalled in the next section. What is important is to notice that our formulation in § 2 contains three nondimensional parameters : $p \in \mathbb{R}$, $q \in \mathbb{R}$, and $\eta \in [0, 1)$:

- p is the nondimensional gravity constant ;
- q is the nondimensional capillarity constant ;
- η is the aspect ratio of the flow. The depth is finite and infinite according as $\eta \in (0, 1)$ and $\eta = 0$, respectively

Thus we solve a bifurcation problem with three parameters. This suggests us the possibility of the existence of *degenerate bifurcation points of multiplicity two, a singularity of codimension three.*

Our goal in this paper is to discuss the degeneracy. The degenerate bifurcation points of multiplicity two are well expected by the number of the parameters, but we explain the situation by a simple account to those readers who are fluid-mechanics-oriented and are less familiar to the bifurcation theory. Without entering the details of the equation or seeing the physical meaning, we can say the following by the generic bifurcation theory : Our problem is to find $(p, q, \eta ; x)$ satisfying

$$F(p, q, \eta ; x) = 0,$$

Assume that $F(p, q, \eta ; 0) \equiv 0$. (We actually prove in § 2 that our problem satisfy this.) In order to study the bifurcation from the trivial solution $x = 0$, we consider the Fréchet derivative of F at $x = 0$. Let $D_x F(p, q, \eta ; 0)$ denote it. Generically we can expect that the set of (p, q, η) at which $D_x F(p, q, \eta ; 0)$ fails to be isomorphic is codimension one in the three dimensional (p, q, η) space. Let us denote by B this two dimensional subset such that $D_x F(p, q, \eta ; 0)$ fails to be isomorphic if and only if $(p, q, \eta) \in B$. Then B is the set of points from which bifurcations occur. Usually B consists of an infinite number of subsets, each of which is of two dimension :

$$B = \bigcup_{n=1}^{\infty} B_n. \tag{1.1}$$

For instance, it is frequently the case that X is spanned by some elementary functions $\{\phi_n\}_{n=1}^{\infty}$. Putting

$$B_n = \{(p, q, \eta) ; D_x F(p, q, \eta ; 0)\phi_n = 0\},$$

we assume (1.1). A typical situation is $\phi_n(s) = \sin ns$. If this is the case, we say that "solutions of mode n" bifurcate from B_n. Since each B_n are two dimensional, we may assume generically that $B_n \cap B_m$ are non-empty one dimensional subsets in (p, q, η) space, if n and m are different integers. Thus we have double bifurcation points (=bifurcation points of multiplicity two) on $B_{mn} = B_n \cap B_m$ in the sense that the null space of $D_x F(p, q, \eta; 0)$ is spanned by ϕ_m and ϕ_n if $(p, q, \eta) \in B_{mn}$. On the set B_{mn} we may apply generic bifurcation theory ([**3**, **8**, **9**]) and prove the existence of the non-trivial solutions. However, since B_{mn} is of one dimension, it may well be possible that there exists a point on B_{mn} such that the genericity assumption fails to be met at the point. It is known ([**8**, **9**]) that the zero set has complex structure near such degenerate bifurcation points.

The above argument is of pure mathematical nature. On the other hand, there is a numerical evidence that a degenerate bifurcation point exists. [**18**], [**19**] and [**21**] performed numerical computations of the bifurcations of water waves of infinite depth. Namely they solved $F(p, q, 0; x) = 0$. The numerical computation strongly suggests the existence of a certain degeneracy, which will be explained in § 5 in detail. It is easy to show that there is no degenerate bifurcation point in the plane $\{(p, q, 0)\}$. Therefore it must be sought in the three dimensional space $\{(p, q, \eta)\}$. However, to our surprise, [**18**] proved that there is no such degenerate bifurcation. Thus we have reached a kind of paradox.

In this paper we take another look at the progressive waves of finite depth. Since formulation in this paper is different from those in [**16**, **18**], we here try to look for the degenerate bifurcation points again. Our conclusion is the same as before : there is no degenerate bifurcation point.

This paper consists of six sections. We define in § 2 an integro-differential equations, which is a master equation. § 3 is concerned with the reduction to a four dimensional subspace by the Lyapunov-Schmidt method and obtain the bifurcation equation. Then in § 4 we prove that the bifurcation equations satisfy a relation called O(2)-equivariance. We also prove that the bifurcation equation is of a simple form. § 5 is the core of this paper : we prove nonexistence of degenerate bifurcation points. Concluding remarks are given in § 6. The relation of the present paper and [**18**] will be explained at the end of § 2.

§ 2. The Fundamental Equation

In this section, we derive the fundamental equation which describes all the properties of the free boundaries. It consists of two subsections, 2.1 and 2.2. The first half, 2.1, is abstracted from [**16**]. It is included here in order to make the paper self-contained. The second half, 2.2, is a remark about the aspect ratio of the flow.

2.1. Levi-Civita's formulation

We consider progressive water waves. We assume that the density of the fluid is constant and we neglect the viscosity. The flows are assumed to be two dimensional and irrotational. In a reference frame moving at the propagating speed of the wave we take (x, y) coordinate system with x horizontally to the right and y vertically upward. We let $y=h(x)$ represent the wave profile, which is stationary in our coordinate system. We further assume that the wave profile is periodic in x with a period, say L, and that the wave profile is symmetric with respect to the y-axis. We consider two cases: the case where the flow is infinitely deep and the case where the depth of the flow is finite. By the assumption, we have only to consider the fluid in

$$R\equiv\{(x, y)\ ;\ -L/2<x<L/2,\ y_0<y<h(x)\},$$

where y_0 is either a finite constant or $-\infty$ (see Figure 1). Since the fluid motion is described by the velocity potential and the stream function we denote by

$$f=U+iV$$

the complex potential of the flow. Here U is the velocity potential and V is the stream function.

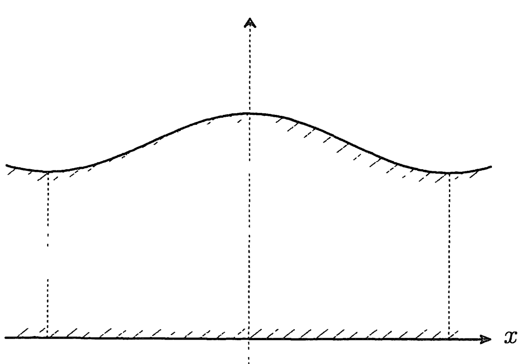

Figure 1. A progressive wave of permanent configuration.

We first consider the case where the flow is infinitely deep. Then the problem is to find a wave profile function $y=h(x)$ and complex potentials such that f is analytic function of $z\equiv x+iy$ in $-\infty<y<h(x)$, and satisfies the following (2.1-4).

$$U\left(\pm\frac{L}{2}, y\right)=\pm\frac{cL}{2}, \qquad \text{on } -\infty<y<h\left(\pm\frac{L}{2}\right), \text{ respectively,} \tag{2.1}$$

$$V=0 \qquad \text{on } y=h(x) \tag{2.2}$$

(2.3) $$\frac{1}{2}\left|\frac{df}{dz}\right|^2+gy+\frac{T}{m}K=\text{constant} \qquad \text{on } y=h(x)$$

(2.4) $$\frac{df}{dz}\to c \qquad \text{as } y\to -\infty$$

where g is a constant called the gravity acceleration ; m is the mass density of the fluid ; c is a constant, which we call mean velocity ; T is a constant called the surface tension coefficient or the capillarity constant ; K is the curvature of the curve $y=h(x)$, which is represented by

$$K=-\left(\frac{h_x}{\sqrt{1+h_x^2}}\right)_x,$$

where the subscript x implies the differentiation. We remark that df/dz is periodic in $x=\mathrm{Re}[z]$ with the period L.

In the case where the depth is finite, the conditions (2.1–3) are retained while (2.4) is replaced by the condition that the bottom bed $\{y=y_0\}$ is a stream line. We write this in the following way :

(2.4′) $$V(x, y_0)\equiv -a \qquad (-L/2<x<L/2)$$

where a is a constant. The derivation of (2.1–4) and (2.4′) are found in Crapper [5], Milne-Thomson [**14**], Zeitler [**25**]. Note that the constants a and c has the same sign. For, it follows from (2.2) that U satisfies Neumann's boundary condition on the bottom bed and the free boundary. We then remark that we may assume $c\neq 0$. In fact, if $c=0$, then $U=0$ and $df/dz\equiv 0$. Consequently, we have

$$gh(x)-\frac{T}{m}\left(\frac{h_x}{\sqrt{1+h_x^2}}\right)_x=s_0$$

where s_0 is a constant. Putting $h_1(x)=h(x)-s_0/g$ and multiplying the equality by h_1, we obtain $h_1\equiv 0$. This implies that the free boundary is flat and the fluid is at rest. Thus we may assume that $c\neq 0$ without losing generality.

It follows from (2.1) that U/c takes its maximum on the side boundary $x=L/2$ and its minimum on $-L/2$. On the other hand, we have

$$a=\int_{y_0}^{h(L/2)}\frac{\partial V}{\partial y}(L/2, y)dy=\int_{y_0}^{h(L/2)}\frac{\partial U}{\partial x}(L/2, y)dy.$$

The conclusion now follows from the maximum principle. ∎

Since both cases are treated similarly, we henceforth assume that both a and c are positive constants.

We now introduce a device which was invented by Levi-Civita [13] in the case of water waves of infinite depth and later used by Struik [22] in the case of finite depth. We define the following functions :

$$\zeta=\exp\Big(-\frac{2\pi i f}{cL}\Big), \qquad \omega=i\log\Big(c^{-1}\frac{df}{dz}\Big). \tag{2.5}$$

We regard ζ as the independent variable and ω as the dependent variable : specifically, by the relation $\zeta\leftrightarrow f\leftrightarrow z\leftrightarrow\omega$, we regard ω as a function of ζ. Note that (2.5) gives us

$$\frac{dz}{d\zeta}=-\frac{L}{2\pi i\zeta}e^{i\omega}. \tag{2.6}$$

In the case where the flow is infinitely deep, ζ runs in the punctured disk $0<|\zeta|<1$ and ω is analytic in ζ by the periodicity of df/dz. As $\zeta\to 0$, ω converges zero by (2.4). Thus the origin is a removable singularity and ω is an analytic function in the entire disk : $|\zeta|<1$. Let θ and τ denote, respectively, the real and the imaginary part of ω. Let (ρ, σ) be the polar coordinates of ζ, i.e., $\zeta=\rho e^{i\sigma}$. We further define θ and τ as the real and imaginary parts of $\omega(\zeta)$, respectively. Levi-Civita [13] succeeded in writing (2.1–4) in terms of ζ and ω.

Find a function $\omega=\omega(\zeta)$ which is continuous on $\{|\zeta|\le 1\}$, is analytic in $\{|\zeta|<1\}$ and satisfies $\omega(0)=0$ and the following (2.7) :

$$e^{2\tau}\frac{\partial\tau}{\partial\sigma}-pe^{-\tau}\sin\theta+q\frac{\partial}{\partial\sigma}\Big(e^{\tau}\frac{\partial\theta}{\partial\sigma}\Big)=0 \quad \text{on} \quad \rho=1, \tag{2.7}$$

where $p=gL/(2\pi c^2)$, $q=2\pi T/(c^2Lm)$.

We next consider the case where the flow is finitely deep. In this case, ζ runs in $\eta<|\zeta|<1$, where $\eta=\exp(-2\pi a/cL)$. ω is an analytic function in the annulus $\{\eta<|\zeta|<1\}$. The condition (2.4′) is expressed as $\theta=0$ on $\rho=\eta$. Now (2.1–3, 4′) is rewritten as follows :

Find a function $\omega=\omega(\zeta)$ which is continuous on $\{\eta\le|\zeta|\le 1\}$, is analytic in $\{\eta<|\zeta|<1\}$, and satisfies the following (2.8) *and* (2.9).

$$e^{2\tau}\frac{\partial\tau}{\partial\sigma}-pe^{-\tau}\sin\theta+q\frac{\partial}{\partial\sigma}\Big(e^{\tau}\frac{\partial\theta}{\partial\sigma}\Big)=0 \quad \text{on} \quad \rho=1, \tag{2.8}$$

$$\theta=0 \quad \text{on} \quad \rho=\eta. \tag{2.9}$$

Both Levi-Civita [13] and Struik [22] considered the case where $q=0$, while [16] considered the general case. A compact derivation of (2.7, 8, 9) are given in [16, 25]. Once $\omega(\zeta)$ is obtained, we have the free boundary by (2.6). In fact, when $\zeta = \exp(i\sigma)$, (2.6) yields

$$\frac{dx}{d\sigma} = -\frac{L}{2\pi} e^{-\tau(e^{i\sigma})} \cos\theta(e^{i\sigma}),$$

$$\frac{dy}{d\sigma} = -\frac{L}{2\pi} e^{-\tau(e^{i\sigma})} \sin\theta(e^{i\sigma}).$$

After the integration with respect to σ, these formulae give us the parametric representation of the free boundary $\{(x(\sigma), y(\sigma))\ ;\ 0 \leq \sigma < 2\pi\}$. In this way the free boundary problem (2.1–4) and (2.1–3, 4′) have been transformed to the nonlinear boundary value problems of an analytic function satisfying (2.7) and (2.8, 9), respectively.

Since an analytic function is completely determined by its boundary value, a further reduction of the equation (2.7) is possible. In fact we can write (2.5) only by $\theta(1, \sigma)$ $(0 \leq \sigma < 2\pi)$. To this end, we make a definition :

Definition. For $\eta \in [0, 1)$ we define the following operator H_η :

$$H_\eta\left(\sum_{n=1}^{+\infty} (a_n \sin n\sigma + b_n \cos n\sigma)\right) = \sum_{n=1}^{\infty} \frac{1+\eta^{2n}}{1-\eta^{2n}}(-a_n \cos n\sigma + b_n \sin n\sigma)$$

When $\eta = 0$, we simply write H for H_0. We call this operator the Hilbert transform.

Consider the case of infinite depth. Then, we have $\tau(1, \sigma) = H(\theta^*)$, where $\theta^*(\sigma) = \theta(1, \sigma)$. Accordingly the problem is formulated as follows :

Find a 2π periodic function θ such that

$$e^{2H\theta}\frac{dH\theta}{d\sigma} - pe^{-H\theta}\sin\theta + q\frac{d}{d\sigma}\left(e^{H\theta}\frac{d\theta}{d\sigma}\right) = 0 \quad (0 \leq \sigma < 2\pi). \tag{2.10}$$

In this way, we have formulated the original free boundary problem in a second order equation for a scalar function of one variable. The difficulty of considering unknown boundary is replaced by the nonlinearity and the existence of the Hilbert transform H.

In the case of finite depth, the situation is almost the same. We note that $\tau(1, \sigma) = H_\eta(\theta(1, \sigma)) + \tau_0$, where τ_0 is a constant. If we replace p and q by $pe^{3\tau_0}$ and qe^{τ_0}, respectively, then we have

$$e^{2H_\eta\theta}\frac{dH_\eta\theta}{d\sigma} - pe^{-H_\eta\theta}\sin\theta + q\frac{d}{d\sigma}\left(e^{H_\eta\theta}\frac{d\theta}{d\sigma}\right) = 0 \qquad (0 \leq \sigma < 2\pi). \tag{2.11}$$

This form is very convenient in that both cases are treated in a unified manner. In fact, we obtain (2.10) if we put $\eta = 0$ in (2.11). Therefore the free boundary problem (2.1–4) or (2.1, 2, 3, 4′) is transformed to the single equation (2.11). It is therefore natural to define the following mapping

$$F(p, q, \eta\,;\theta)=\frac{d}{d\sigma}\Big(\frac{1}{2}e^{2H_\eta\theta}+qe^{H_\eta\theta}\frac{d\theta}{d\sigma}\Big)-pe^{-H_\eta\theta}\sin\theta,$$

and the following function spaces;

Definition.

$$X_n=H^n(S^1)/\mathbf{R}=\Big\{\sum_{k=1}^{\infty}(a_k\sin k\sigma+b_k\cos k\sigma)\,;\\ a_k\in\mathbf{R},\ b_k\in\mathbf{R},\ \sum_{k=1}^{\infty}(|a_k|^{2n}+|b_k|^{2n})<\infty\Big\}$$

Then we have the following theorem which is proved in [16]:

Theorem 2.1. *F is a C^∞-mapping from $\mathbf{R}^2\times[0, 1)\times X_2$ into X_0.*

By this theorem, we can apply any standard theory of static bifurcation theory like those in [3, 8, 9]. In the statement of Theorem 2.1, the smoothness is immediate. The point is that $F\in X_0=L^2(S^1)/\mathbf{R}$, i.e.,

$$\int_0^{2\pi}e^{-H_\eta\theta}\sin\theta(\sigma)d\sigma=0.$$

For the proof, see [16].

2.2. Explicit formula for the aspect ratio

We have to notice that the meaning of η is not intuitive. The constant η depends on the constant a. On the other hand, the mean depth h, which is defined as

$$h=\frac{1}{L}\int_0^L(y(x)-y_0)dx,$$

is a more intuitive parameter. We define the aspect ratio of the flow as $r=2\pi h/L$. We note that there is an implicit relation among η, r, and $\theta(\sigma)$. Consequently, r and η can not be taken independently. Then, in some sense, the name "parameter" seems to be more deserved by the triplet (p, q, r) rather than (p, q, η). In this view we should define the following mapping G.

$$G=(G_1, G_2)$$

$$G_1(p, q, r\,;\theta, \eta)=\frac{d}{d\sigma}\Big(\frac{1}{2}e^{2H_\eta\theta}+qe^{H_\eta\theta}\frac{d\theta}{d\sigma}\Big)-pe^{-H_\eta\theta}\sin\theta, \tag{2.12}$$

$$G_2(p, q, r\,;\theta, \eta)=\frac{2\pi}{L^2}\int_0^L(y(x)-y_0)dx-r. \tag{2.13}$$

Of course, to determine the set $\{(p, q, r; \theta, \eta); G(p, q, r; \theta, \eta)=(0, 0)\}$ is mathematically equivalent to the determination of the set $\{(p, q, \eta; \theta); F(p, q, \eta; \theta)=0, 0<\eta<1\}$. But η is a state variable in G, while it is regarded as a parameter in F. Since the mapping G may be better suited to the physical intuition, we here prove that G is a well-defined mapping and that we can apply a standard theorem of bifurcation to it. Namely we will set up an environment where we can use established bifurcation theorems.

To go further we need a representation of the right hand side of (2.13) by $\theta(\sigma)$. Suppose θ is given by the following Fourier series:

$$\theta=\sum_{k=1}^{\infty}(a_n\sin n\sigma+b_n\cos n\sigma).$$

Then, we define $\omega(\zeta)$ as

$$\omega(\zeta)=\sum_{n=1}^{\infty}\frac{1}{1-\eta^{2n}}\{(b_n-ia_n)\zeta^n-(b_n+ia_n)\eta^{2n}\zeta^{-n}\}\quad(\eta<|\zeta|<1). \tag{2.14}$$

This function satisfies $\mathrm{Re}[\omega(e^{i\sigma})]=\theta(\sigma)$ and $\mathrm{Re}[\omega(\eta e^{i\sigma})]\equiv 0$. We denote the real and the imaginary parts of ω by θ and τ. Thus θ is extended from the unit circle $|\zeta|=1$ to the annulus $\eta\leq|\zeta|\leq 1$ and satisfy $\theta(\eta e^{i\sigma})\equiv 0$. By the integration of (2.6), we have

$$z(\zeta)=z(1)-\frac{L}{2\pi i}\int_1^{\zeta}e^{i\omega(u)}\frac{du}{u},$$

which gives us

$$y(e^{i\sigma})=y(1)-\frac{L}{2\pi}\int_0^{\sigma}e^{-\tau(e^{i\sigma'})}\sin\theta(e^{i\sigma'})d\sigma',$$

and

$$y(\eta e^{i\sigma})=y(1)-\frac{L}{2\pi}\int_{\eta}^{1}e^{-\tau(t)}\cos\theta(t)\frac{dt}{t}-\frac{L}{2\pi}\int_0^{\sigma}e^{-\tau(\eta e^{i\sigma'})}\sin\theta(\eta e^{i\sigma'})d\sigma'.$$

Since $y(\eta e^{i\sigma})\equiv y_0$ and $\theta(\eta e^{i\sigma})\equiv 0$, we have

$$y_0=y(1)-\frac{L}{2\pi}\int_{\eta}^{1}e^{-\tau(t)}\cos\theta(t)\frac{dt}{t}.$$

We now have the following representation of G_2:

$$\begin{aligned}G_2(p, q, r; \theta, \eta)&=\frac{2\pi}{L^2}\int_0^{2\pi}[y(e^{i\sigma})-y_0]\left|\frac{dx}{d\sigma}\right|d\sigma-r\\&=\frac{1}{2\pi}\int_0^{2\pi}e^{-\tau(e^{i\sigma})}\cos\theta(e^{i\sigma})d\sigma\int_{\eta}^{1}e^{-\tau(t)}\cos\theta(t)\frac{dt}{t}\end{aligned}$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\sigma} e^{-\tau(e^{i\sigma'})}\sin\theta(e^{i\sigma'})d\sigma' e^{-\tau(e^{i\sigma})}\cos\theta(e^{i\sigma})d\sigma - r.$$

By the definition of ω, we have

$$\tau(t)=\sum_{n=1}^{\infty}\frac{-a_n}{1-\eta^{2n}}\left(t^n+\frac{\eta^{2n}}{t^n}\right) \tag{2.15}$$

and

$$\theta(t)=\sum_{n=1}^{\infty}\frac{b_n}{1-\eta^{2n}}\left(t^n-\frac{\eta^{2n}}{t^n}\right) \tag{2.16}$$

for $t\in[\eta, 1]$. Let $T_\eta\theta$ and $S_\eta\theta$ denote (2.15) and (2.16), respectively. The operators T_η and S_η are bounded operators from $H^2(S^1)/\mathbb{R}$ to $H^2((\eta,1))$. We finally get to the following representation of G_2:

$$G_2(p, q, r; \theta, \eta)=\frac{1}{2\pi}\int_0^{2\pi} e^{-H_\eta\theta(\sigma)}\cos\theta(\sigma)d\sigma\int_\eta^1 e^{-T_\eta\theta(t)}\cos S_\eta\theta(t)\frac{dt}{t} \tag{2.17}$$
$$-\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\sigma} e^{-H_\eta\theta(\sigma')}\sin\theta(\sigma')d\sigma' e^{-H_\eta\theta(\sigma)}\cos\theta(\sigma)d\sigma - \mathrm{r}$$

This is the formula of the aspect ratio. We now prove that G is a well-defined mapping.

Theorem 2.2. *G is a smooth mapping from $\mathbb{R}^2\times(0, +\infty)\times X_2\times[0, 1)$ to $X_0\times\mathbb{R}$.*

Proof. Since we have Theorem 2.1, we have only to show that G_2 is a smooth function. But the smoothness (actually it is analytic) is clear from the definition. ∎

We now see that the free boundary problem is transformed to obtain either $F^{-1}(0)$ or $G^{-1}(0, 0)$. The advantages and disadvantages of F and G are as follows:

(1) F contains infinitely deep waves as a special case that $\eta=0$. On the other hand, infinitely deep waves are not realized in G, since the waves correspond to the case $r\to\infty$.

(2) the aspect ratio r is a controllable parameter but η in F has less physical meaning.

The contents of the present paper have something in common with [18]. [18] considered F but not G. The Theorem 5.3 of § 5 below is given in [18] but other results of the present paper including numerical experiments are new.

§ 3. Bifurcation Equation

In this section we consider the bifurcation equation near the double bifurcation point of mode (m, n). This section is divided into two subsections: we consider F in the first half and G in the second half.

3.1. Bifurcation equation of F

Note first that $F(p, q, \eta; 0)\equiv 0$. We consider the bifurcation from the trivial solution $\theta=0$. Its Fréchet derivatives at $\theta=0$ is given by

$$D_\theta F(p, q, \eta; 0)w=\frac{dH_\eta w}{d\sigma}-pw+q\frac{d^2w}{d\sigma^2} \qquad (w\in X_2). \tag{3.1}$$

The proof of this formula is given in [**16**]. The following theorem follows immediately.

Theorem 3. 1. *$D_\theta F(p, q, \eta; 0): X_2\to X_0$ has a nontrivial null space if and only if (p, q, η) satisfies*

$$n\frac{1+\eta^{2n}}{1-\eta^{2n}}=p+n^2q \tag{3.2_n}$$

for some positive integer n. If $D_\theta F(p, q, \eta; 0)$ is regarded as an unbounded operator in X_0, then it is self-adjoint.

Definition. Consider (p, q, η) which satisfy (3.2_n) but not (3.2_m) with $m\neq n$. We call such a point a simple bifurcation point of mode n. A point which satisfies (3.2_m) and (3.2_n) with different m and n is called a double bifurcation point of mode (m, n).

It is proved in [**16**] that $(p, q, \eta)\in[0, \infty)^2\times[0, 1)$ can satisfy (3.2) for two different integers but not for three different integers. Thus it is sufficient to consider simple bifurcation points and double bifurcation points.

Let us define

$$B_n=\{(p, q, \eta)\in[0, \infty)\times(0, \infty)\times[0, 1)\ ;\ (3.2_n)\text{ holds true}\}.$$

Then the set of the double bifurcation points is equal to $B_m\cap B_n$, which is a curve (one dimensional). The following theorem is easy to prove.

Theorem 3. 2. *The null space of $D_\theta F(p, q, \eta; 0)$ is spanned by the two eigenfunctions*

$$\sin m\sigma, \quad \cos m\sigma$$

when $(p, q, \eta)\in B_m\setminus\cup_{k\neq m}B_k$. *When* $(p, q, \eta)\in B_m\cap B_n$, *the null space is spanned by the following four eigenfunctions*

$$\sin m\sigma, \quad \cos m\sigma, \quad \sin n\sigma, \quad \cos n\sigma.$$

We name double bifurcation point since $\cos m\sigma$ and $\cos n\sigma$ play essentially the same role as $\sin m\sigma$ and $\sin n\sigma$, respectively.

In a neighborhood of the simple bifurcation point, we can only prove the existence of waves whose profile has n troughs and n crests, which [1, 2] called regular n-waves. On the other hand, any neighborhood of the double bifurcation point has both regular n-waves and regular m-waves. The most significant characteristic of double bifurcation points is that there are solutions in which the wave profiles are of mixed nature. This occurs as a secondary bifurcation from the branches of regular waves. This fact is essentially known early in this century (Wilton [24]). The mathematical proof of the existence of the secondary branches is given in [12, 23]. A slightly different proof was given later in [16]. However, the global structure of the solution set has not been well understood until the numerical studies by [1, 2, 19, 20, 21].

We now define a bifurcation equation by the standard Lyapunov-Schmidt procedure: Let $(p_0, q_0, \eta_0; 0)$ be a double bifurcation point of mode (m, n) $(0<m<n)$. Let Q denote the L^2-projection from $L^2(S^1)$ onto the four dimensional subspace spanned by $\sin m\sigma$, $\cos m\sigma$, $\sin n\sigma$, and $\cos n\sigma$ (Theorem 3.2). Then, the equation

$$(I-Q)F(p, q, \eta; x \sin m\sigma+y \cos m\sigma+z \sin n\sigma+w \cos n\sigma + \phi(p, q, \eta; x, y, z, w))=0 \tag{3.3}$$

uniquely defines an $(I-Q)X_2$-valued mapping ϕ in some open set containing $(p_0, q_0, \eta_0; 0, 0, 0, 0)$. We define K by

$$K(p, q, \eta; x, y, z, w)=QF(*), \tag{3.4}$$

where the arguments of F, $*$, is the same as in (3.3). This mapping K is a bifurcation equation of F near (p_0, q_0, η_0).

3.2. Bifurcation equation of G

We first note that

$$G(p, q, r; 0, e^{-r})=(0, 0).$$

We consider the bifurcation from the trivial solution $(\theta, \eta)=(0, e^{-r})$. Computations go quite similarly to the previous case of F.

The Fréchet derivatives of G at $(\theta, \eta)=(0, e^{-r})$ are given by

$$D_\theta G_1(p, q, r; 0, e^{-r})w=\frac{dH_{e^{-r}}w}{d\sigma}-pw+q\frac{d^2w}{d\sigma^2} \qquad (w\in X_2) \tag{3.5}$$

$$\frac{d}{d\eta}G_1(p, q, r; 0, e^{-r})\equiv 0 \tag{3.6}$$

$$D_\theta G_2(p, q, r; 0, e^{-r})w\equiv 0 \qquad (w\in X_2) \tag{3.7}$$

$$\frac{d}{d\eta}G_2(p, q, r; 0, e^{-r})=-e^r \tag{3.8}$$

Proof. The formula (3.5) is the same as (3.1). The equalities (3.6) and (3.8) are clear from the definition. To show (3.7), we note that

$$D_\theta G_2(p, q, r; 0, e^{-r})w=-\int_\eta^1 (T_{e^{-r}}w)(t)\frac{dt}{t}-\frac{1}{2\pi}\int_0^{2\pi}\int_0^\sigma w(\sigma')d\sigma' d\sigma.$$

By making use of (2.15), we see that the right hand side vanishes for any $w\in X_2$. ∎

The following theorem follows immediately.

Theorem 3. 3. *$DG(p, q, r; 0, e^{-r})\equiv\begin{pmatrix} D_\theta G_1 & \frac{dG_1}{d\eta} \\ D_\theta G_2 & \frac{dG_2}{d\eta}\end{pmatrix}: X_2\times\mathbf{R}\to X_0\times\mathbf{R}$ has a non-trivial null space if and only if (p, q, r) satisfies*

$$n\frac{1+e^{-2nr}}{1-e^{-2nr}}=p+n^2q \tag{3.9$_n$}$$

for some positive integer n. If $DG(p, q, r; 0, e^{-r})$ is regarded as an unbounded operator in $X_0\times\mathbf{R}$, then it is self-adjoint.

Let us define

$$B_n=\{(p, q, r)\in[0, \infty)\times(0, \infty)\times(0, +\infty);\ (3.9_n) \text{ holds true}\}.$$

Then the set of the double bifurcation points is equal to $B_m\cap B_n$, which is a curve (one dimensional).

Theorem 3. 4. *The null space of $DG(p, q, r; 0, e^{-r})$ is spanned by the two vectors*

$$\textstyle\sum_1=(\sin m\sigma, 0), \quad \sum_2=(\cos m\sigma, 0),$$

when $(p, q, r)\in B_m\setminus\cup_{k\neq m}B_k$. *When* $(p, q, r)\in B_m\cap B_n$, *the null space is spanned by the following four vectors*

$$\Sigma_1=(\sin m\sigma,\ 0),\ \Sigma_2=(\cos m\sigma,\ 0),\ \Sigma_3=(\sin n\sigma,\ 0),\ \Sigma_4=(\cos n\sigma,\ 0).$$

We again define a bifurcation equation by the standard Lyapunov-Schmidt procedure : Let $(p_0, q_0, r_0\,;\, 0, e^{-r_0})$ be a double bifurcation point of mode (m, n) $(0<m<n)$. Let P denote the L^2-projection from $L^2(S^1)\times\mathbf{R}$ onto the four dimensional subspace spanned by Σ_1, Σ_2, Σ_3, and Σ_4 (Theorem 3.4). Then, the equation

$$(I-P)G(p, q, r\,;\,(0, e^{-r})+x\Sigma_1+y\Sigma_2+z\Sigma_3+w\Sigma_4 +\phi(p, q, r\,;\, x, y, z, w))=0. \tag{3.10}$$

uniquely defines an $(I-P)(X_2\times\mathbf{R})$-valued mapping ϕ in some open set containing $(p_0, q_0, r_0\,;\, 0, 0, 0, 0)$. We define K by

$$K(p, q, r\,;\, x, y, z, w)=PG(*), \tag{3.11}$$

where the arguments of G, $*$, is the same as in (3.10). This mapping K is a bifurcation equation of G.

§ 4. O(2)-Equivariance

In this section, we prove that F and G satisfy a certain property called O(2) -equivariance and that this property forces the bifurcation equation K to be of a special simple form ((4.5, 6) below). We first define an action of the orthogonal group O(2) on X_0 as follows: let us recall that O(2) is generated by rotations of angle $\alpha\in[0, 2\pi)$ and a reflection. Accordingly,

$$\gamma_\alpha\theta(\sigma)=\theta(\sigma-\alpha)\qquad(0\leq\alpha<2\pi)$$

$$\gamma_-\theta(\sigma)=-\theta(-\sigma)$$

defines an action of O(2) on X_0, where γ_α represents the element of O(2) representing the rotation of angle and γ_- the reflection. Then we have

Proposition 4. 1. *The mapping* $F:\mathbf{R}^2\times[0, 1)\times X_2\to X_0$ *is* O(2)*-equivariant, by which we mean*

$$F(p, q, \eta ; \gamma\theta) = \gamma F(p, q, \eta ; \theta) \qquad (\gamma \in \mathrm{O}(2)).$$

The mapping $G : \mathbf{R}^2 \times (0, \infty) \times X^2 \times [0, 1) \to X_0 \times \mathbf{R}$ *is* $\mathrm{O}(2)$*-equivariant, by which we mean*

$$G(p, q, r ; \gamma\theta, \eta) = (\gamma G_1(p, q, r ; \theta, \eta), G_2(p, q, r ; \theta, \eta)) \qquad (\gamma \in \mathrm{O}(2)).$$

Proof. By the definition, we have

$$H_\eta(\gamma_\alpha\theta) = \gamma_\alpha(H_\eta\theta), \quad \text{and} \quad H_\eta(\gamma_-\theta) = (H_\eta\theta)(-\sigma) = -\gamma_-(H_\eta\theta)$$

The proof of the $\mathrm{O}(2)$-equivariance of F is now easy to see. Since $G_1(p, q, r ; \gamma\theta, \eta) = \gamma G_1(p, q, r ; \theta, \eta)$ is the same as the $\mathrm{O}(2)$-equivariance of F, there remains to prove that $G_2(p, q, r ; \gamma\theta, \eta) = G_2(p, q, r ; \theta, \eta)$. To this end, we consider the following path integral :

$$\int_\Gamma \frac{e^{i\omega}}{\zeta} d\zeta, \tag{4.1}$$

where $\omega(\zeta)$ is defined by (2.14) and the path Γ is the closed path shown in Figure 2. Note that

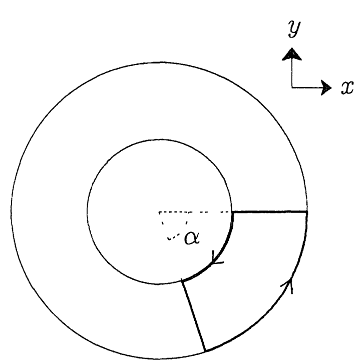

Figure 2. The path of integration.

$$\omega(te^{-i\alpha}) = S(\gamma_\alpha\theta) + iT(\gamma_\alpha\theta) \qquad (\eta < t < 1,\ 0 \leq \alpha < 2\pi)$$

The integral (4.1) is zero by Cauchy's theorem. Taking the real part of it, we obtain

$$-\int_\eta^1 e^{-T\theta} \cos S\theta \frac{dt}{t} + \int_{-\alpha}^0 e^{-H_\eta\theta} \sin\theta d\sigma + \int_\eta^1 e^{-T(\gamma_\alpha\theta)} \cos S(\gamma_\alpha\theta) \frac{dt}{t} = 0, \tag{4.2}$$

since $\mathrm{Re}[\omega(\eta e^{i\sigma})] \equiv 0$. The property $G_2(p, q, r ; \gamma_\alpha\theta, \eta) = G_2(p, q, r ; \theta, \eta)$ now follows from (4.2) and (2.17). On the other hand, $G_2(p, q, r ; \gamma_-\theta, \eta) = G_2(p, q,$

$r; \theta, \eta)$ is easy to see. Since $\{\gamma_\alpha\}$ and γ_- constitute a set of generators of $O(2)$, we are done. ∎

Proposition 4. 1 enables us to simplify the bifurcation equation. What we will write in the remaining part of this section holds equally to both the bifurcation equation of F and that of G. By Proposition 4. 1 and the fact that the bifurcation equation inherits the group equivariance from the basic differential equation ([8]), we see that K, too, has an $O(2)$-equivariance. To represent this more conveniently, we identify $(x, y, z, w) \in \mathbf{R}^4$ with $(\xi, \zeta) \in \mathbf{C}^2$ in the way that $\xi = x + iy$, $\zeta = z + iw$. Therefore, we can regard K as a mapping on (some open subset of) $\mathbf{R}^3 \times \mathbf{C}^2$. Similarly, we can regard that K takes its value in $\mathbf{C}^2$. Let (K_1, K_2) be the componentwise expression of K in $\mathbf{C}^2$. We then have

Proposition 4. 2. *The mapping K above is $O(2)$-equivariant in the sense that the following (4.3, 4) hold true.*

$$(4.3) \qquad K(p, q, r; e^{im\alpha}\xi, e^{in\alpha}\zeta) = (e^{im\alpha}K_1(p, q, r; \xi, \zeta), e^{in\alpha}K_2(p, q, r; \xi, \zeta)), \quad (\alpha \in [0, 2\pi))$$

$$(4.4) \qquad K(p, q, r; \bar{\xi}, \bar{\zeta}) = (\overline{K_1(p, q, r; \xi, \zeta)}, \overline{K_2(p, q, r; \xi, \zeta)}).$$

For the Proof, see [16]. Proposition 4. 2 forces the mapping K to be of a special form. Let us prepare some symbols.

Definition. We call a function $f: \mathbf{R}^3 \times \mathbf{C}^2 \to \mathbf{R}$ $O(2)$-invariant if

$$f(a; e^{im\alpha}\xi, e^{in\alpha}\zeta) \equiv f(a; \xi, \zeta) \qquad (\alpha \in [0, 2\pi))$$

and

$$f(a; \bar{\xi}, \bar{\zeta}) \equiv f(a; \xi, \zeta)$$

are satisfied. Here $a \in \mathbf{R}^3$, $\xi, \zeta \in \mathbf{C}$.

Remark. From now on, we write as $f: \mathbf{R} \to \mathbf{R}$, even when the defining domain of f is some small open set of $\mathbf{R}$. For instance, we consider mapping germs at the origin, although we write as if it were defined in the whole space.

The set of all germs (at the origin) of $O(2)$-invariant C^∞-functions is a commutative ring with a unit. Let $\mathscr{E}$ denote this ring. The set of all the mapping $K: \mathbf{R}^3 \times \mathbf{C}^2 \to \mathbf{C}^2$ satisfying (4.3, 4) is an $\mathscr{E}$-module. Let E denote this $\mathscr{E}$-module. In order to give a simple expression to $\mathscr{E}$ and E, we need to introduce two positive integers n' and m'. We define them as coprime positive integers satisfying $n/m = n'/m'$. We now have

Proposition 4. 3. *Any element* $f \in \mathscr{E}$ *is of the following form*

$$f(a\,;\xi,\zeta)=g(a\,;u,v,s)$$

where g *is a* C^{∞} *function of* $3+3$ *variables and* u, v, s *are defined by*

$$u=|\xi|^2,\qquad v=|\zeta|^2,\qquad s=Re[\bar{\xi}^{n'}\zeta^{m'}].$$

Proposition 4. 4. *The module* E *is generated over* $\mathscr{E}$ *by the following four elements*:

$$X_1=(\xi,0),\qquad X_2=(0,\bar{\xi}^{n'-1}\zeta^{m'}),\qquad X_3=(0,\zeta),\qquad X_4=(0,\xi^{n'}\bar{\zeta}^{m'-1}).$$

The proofs of Propositions 4.3, 4 can be found in [**9**, Chapter XX] and [**15**].

Corollary. *The mapping* K *at the bifurcation point of mode* (1, 2) *is of the following form*

$$K_1=f_1\xi+f_2\bar{\xi}\zeta, \tag{4.5}$$

$$K_2=f_3\zeta+f_4\xi^2, \tag{4.6}$$

where f_j *are of the following form*

$$f_j=f_j(p,q,r\,;|\xi|^2,|\zeta|^2,Re[\bar{\xi}^2\zeta]).\qquad (j=1,2,3,4)$$

Here the parameter r *should be replaced by* η *when we consider the bifurcation equation of* F.

§ 5. Bifurcation Equation of Mode (1, 2)

In this section we give a normal form of the bifurcation equation of mode (1, 2). Then we study the degeneracy.

We first consider the case of F. The bifurcation equation $K: \mathbf{R}^3\times\mathbf{C}^2\to\mathbf{C}^2$ is now written as (4.5, 6). We use the theory in [**8, 9**] in which mapping germs containing one parameter are considered. Our mapping, however, has three parameters. We thereby freeze (p, η) and use q as the bifurcation parameter. Note that the double bifurcation points of mode (m, n) are characterized by (3.2_m) and

(3.2_n). Therefore, for each $\eta\in[0, 1)$, there is one and only one (p, q) which satisfies the two conditions. We denote it by $(p(m, n; \eta), q(m, n; \eta))$. We then consider $K(p(m, n; \eta)+\mu, q(m, n; \eta)+\lambda, \eta; \xi, \zeta)$ and regard it as a mapping germ of $(\lambda, \mu, \eta; \xi, \zeta)$. Here $\lambda\in\mathbf{R}$ and $\mu\in\mathbf{R}$ run in a neighborhood of zero, $\xi\in\mathbf{C}$ and $\zeta\in\mathbf{C}$ run in a neighborhood of the origin, while η runs in $[0, 1)$. We then set $\mu=0$ and freeze η. We can now write as

$$K_1=f_1(\lambda; u, v, s)\xi+f_2(\lambda; u, v, s)\bar{\xi}\zeta, \tag{5.1}$$

$$K_2=f_3(\lambda; u, v, s)\zeta+f_4(\lambda; u, v, s)\xi^2. \tag{5.2}$$

We must, however, remember that K depends on η implicitly. If we have shown that this mapping is finitely determined and if we have computed universal unfoldings, then the equation for general (p, q, η) can be realized by one of the unfolded mappings ([8, 9]). Thus we are led to the analysis of (5.1, 2).

Since K is a bifurcation equation, all the derivatives of first order vanish at the origin. Accordingly $f_j(0; 0, 0, 0)=0$ $(j=1, 3)$. To be precise, $f_1(0; 0, 0, 0)=f_3(0; 0, 0, 0)=0$ for all $\eta\in[0, 1)$. In order to go further, we need to compute $f_2(0; 0, 0, 0)$ and $f_4(0; 0, 0, 0)$ as a functions of η. Generically, we can expect that neither of them vanishes. This is the generic bifurcation which we have mentioned. This generic case is analyzed in [9, 15]. We now recall the generic case. If $f_2(0; 0, 0, 0)\neq0$ and $f_4(0; 0, 0, 0)\neq0$, then the bifurcation equation (K_1, K_2) is O(2)-equivalent ([9, 15]) to

$$(K_1/f_2, K_4/f_4),$$

which may be written as

$$([\varepsilon\lambda+au+bv+cs+\phi_1]\xi+\bar{\xi}\zeta,\ [\delta\lambda+\hat{a}u+\hat{b}v+\hat{c}s+\phi_2]\zeta+\xi^2), \tag{5.3}$$

where $a, b, c, \hat{a}, \hat{b}, \hat{c}, \varepsilon$, and δ are real constants and ϕ_j are functions of u, v, and s of order ≥2. Concerning (5.3), we have the following theorem :

Theorem 5.1 ([15]). *Assume that $\varepsilon\delta\neq0$, $\hat{b}\neq0$, and $\varepsilon\hat{b}\neq\delta(b-\hat{a}/2)$. Then the bifurcation equation* (5.3) *is* O(2)*-equivalent to*

$$([\varepsilon'\lambda+b'v]\xi+\bar{\xi}\zeta,\ [\delta'\lambda+\hat{b}'v]\zeta+\xi^2), \tag{5.4}$$

where $\varepsilon'=\varepsilon/|\varepsilon|$, $b'=(b-\hat{a}/2)/|\hat{b}|$, $\delta'=\delta/|\delta|$, and $\hat{b}'=\hat{b}/|\hat{b}|$. A universal unfolding in the sense of [8] *of* (5.4) *is given by*

$$\tilde{K}(\alpha, \beta, \lambda; \xi, \zeta)=([\varepsilon'\lambda+\alpha+(b'+\beta)v]\xi+\bar{\xi}\zeta,\ [\delta'\lambda+\hat{b}'v]\zeta+\xi^2). \tag{5.5}$$

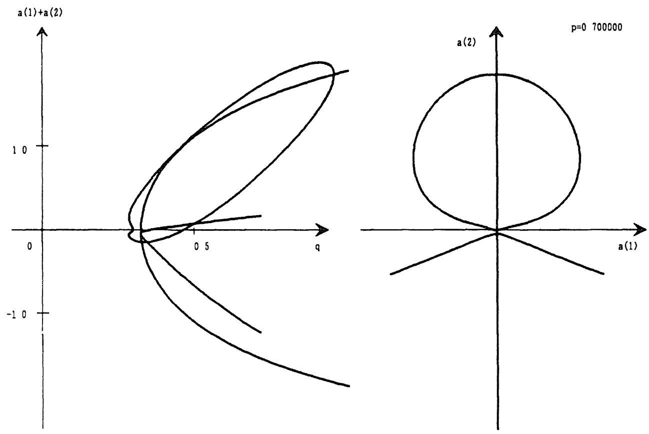

Figure 3. The bifurcation diagram. $p=0.70$. The bird's-eye view is on the left. The view from the q-axis is on the right. The mode 1 branch bifurcates at $q=0.3$ subcritically. The two branches of mode 1 have turning points and bend toward the branch of mode 2. They join with the branch of mode 2, forming a pitchfork. We see a contact point on the loop of mode 1 solutions but this is a spuriuos one caused by the projection onto the two dimensional plane. The mode 2 branch bifurcates supercritically. On the other side of the branch of mode 2, there is a supercritical pitchfork bifurcation. Thus, we have two primary pitchfork bifurcation points, two secondry pitchfork bifurcation points, and two turning points.

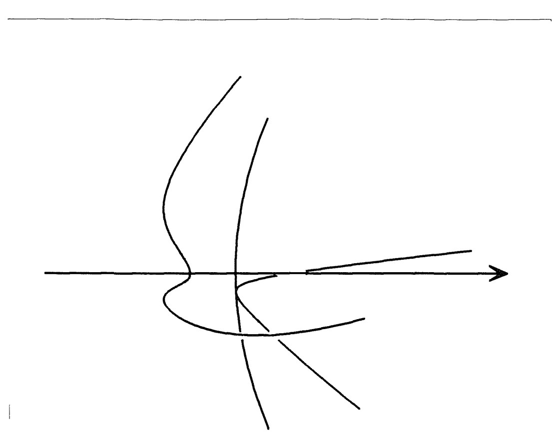

Figure 3-B. Blow-up of Figure 3 around the primary bifurcation.

where α and β are unfolding parameters.

We say that (5.4) is a normal form of (5.3). This theorem enables us to obtain a qualitative picture of solutions around a double bifurcation point of mode (1, 2). Since we consider only the symmetric waves, we may restrict ourselves to the

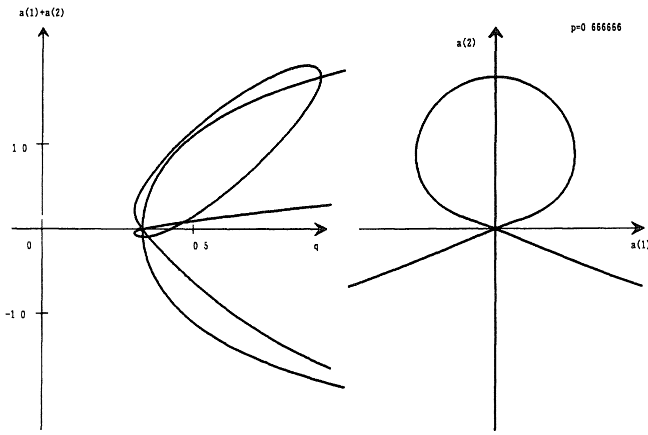

Figure 4. The bifurcation diagram. $p=2/3$. The point $q=1/3$ is the double bifurcation point of mode (1, 2).

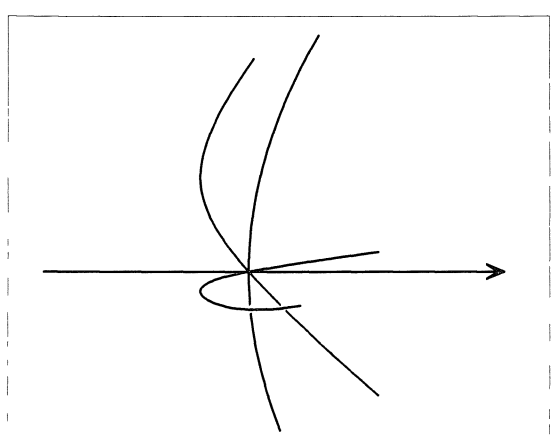

Figure 4-B. Blow-up of Figure 4 around the primary bifurcation.

solutions of $\tilde{K}(\alpha, \beta, \lambda, x, z)=0$, where x and z are real variables. After some computation, we note that the unfolding parameter β does not change the diagram $\tilde{K}^{-1}(0)$ qualitatively. Namely β is a modal parameter in the sense of [**8**].

We are now in a position to explain numerical solutions. Figures 3–7 show five numerical bifurcation diagrams, in each of which p is fixed and q is taken as a bifurcation parameter. Here $\eta=0$. We computed $(p, q, 0; \theta)$ such that $F(p, q, 0; \theta)=0$ for $p=0.47, 0.55, 0.61, 2/3, 0.7$. Note that the double bifurcation of mode (1, 2) takes place at $p=2/3$ (see (3.2)). In Figure 3-B, 4-B, and 5-B, we gave blow-ups of Figure 3, 4 and 5, respectively. Numerical continuation of the solution paths is carried out by H. B. Keller's method ([**10**]). Wave profiles of three solutions are shown in Figure 8. They are regular 1-wave, regular 2-wave, and the

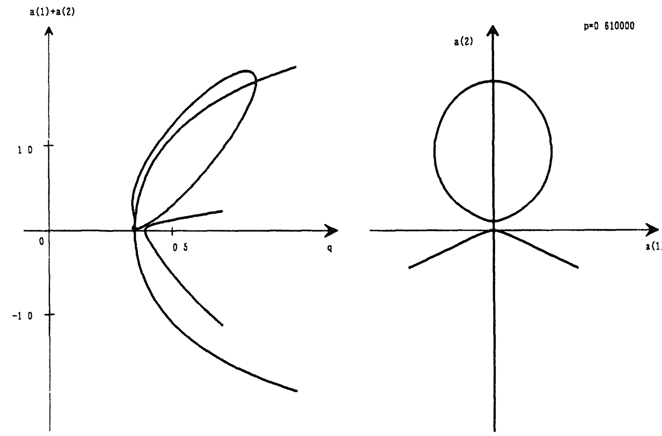

Figure 5. The bifurcation diagram. $p=0.61$. The branch, which appeared as a secondary one when $p=0.7$, now becomes the branch of mode 1. It occurs supercritically. The branch of mode 2 possesses a closed loop, which has two turning points on it. It is connected to the branch of mode 2, by two pitchforks.

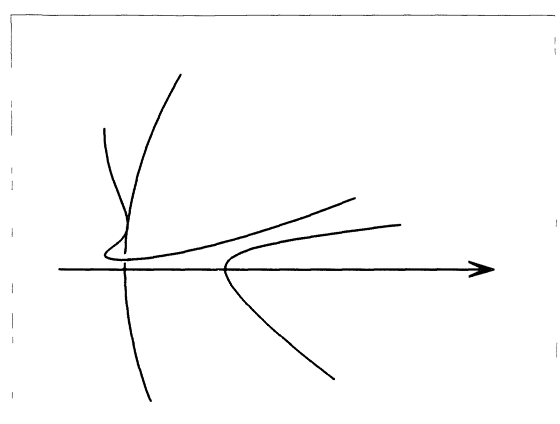

Figure 5-B. Blow-up of Figure 5 around the primary bifurcation.

mixed mode wave, respectively. We now compare Figure 4 with abstract bifurcation diagrams produced by (5.5). Since we consider real solutions only, (5.5) becomes as follows :

$$([\varepsilon' \lambda + \alpha + b' z^2]x + xz,\ [\delta' \lambda + \hat{b}' z^2]z + x^2), \tag{5.6}$$

where we put $\beta=0$, since it is a modal parameter. Figure 9 shows the case where $\varepsilon=\delta=1$, $b'=\hat{b}'=-1$. This is the same as (I2-h) of [**6**]. The figures faithfully reproduce a part of Figures 3–7 but they can not predict the existence of turning

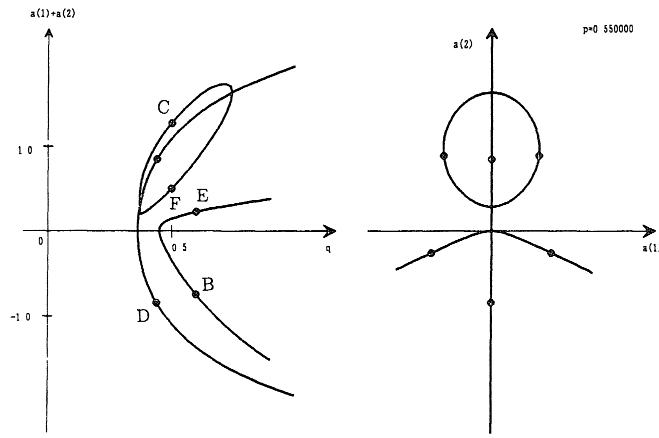

Figure 6. The bifurcation diagram. $p=0.55$. The difference between this figure and Figure 5 is that the loop does not have a turning point. The secondary bifurcation at the lower side of the loop, is almost vertical. The solid circles show the solutions whose profiles are drawn in Figure 8.

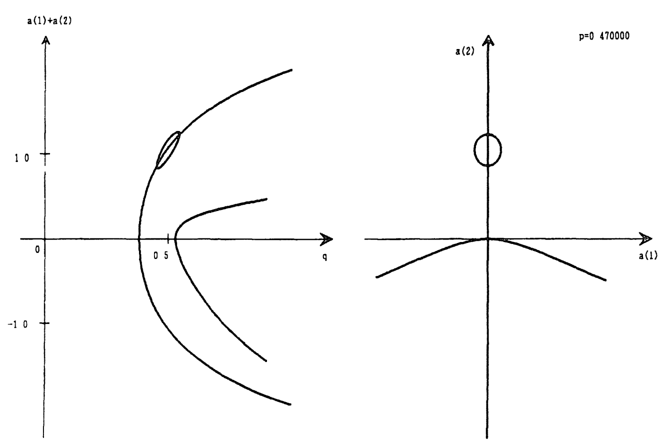

Figure 7. The bifurcation diagram. $p=0.47$. The loop has no turning point. It shrinks. When $p<0.45$, there is no secondary bifurcation.

(limit) points which are present in Figures 3 and 4. Here the following remark may be useful. The complete set of (5.6) may actually contain turning points. But this is away from the origin with finite distance. Since we consider the mapping germs, what matters is those which can appear in an arbitrary neighborhood of the origin as α varies about zero. In this sense, (5.6) can not reproduce the turning points. Therefore (5.5) is of limited use in the present problem.

We now suspect as follows : since $f_2(0\,;\,0,0,0)$ and $f_4(0\,;\,0,0,0)$ depend on the parameter η, our assumption $f_2 f_4 \neq 0$ may be violated at some η. If this violation happens, we surely have a different normal forms. If this is the case, we have two

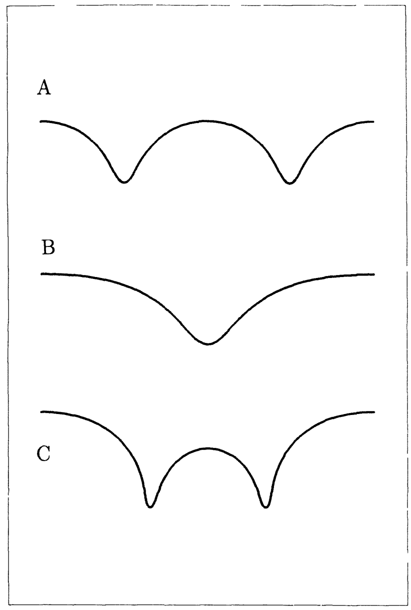

Figure 8. The wave profiles. $p=0.55$. (A) : regular 2 wave. (B) : regular 1 wave. (C) wave of mixed mode. The wave profile of the solution D is obtained by shifting (A) by half the wave length. The wave profile of the solution E is obtained from (B) by the shift. The wave profile of the solution F is obtained from (C) by the shift.

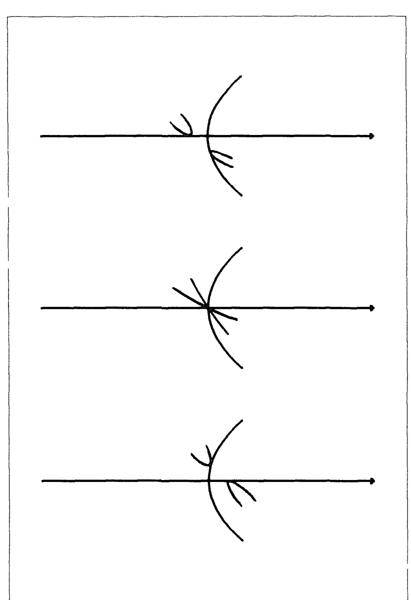

Figure 9. The bifurcation diagram of the mapping germ (5.6).

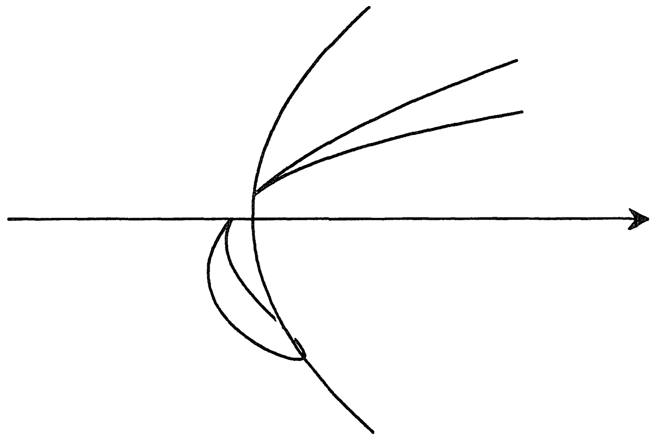

Figure 10.

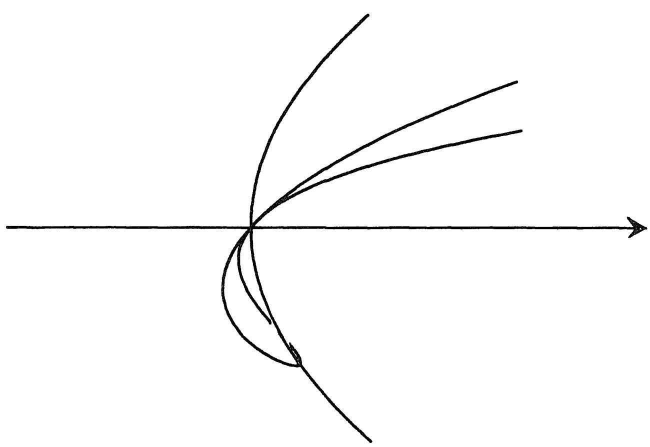

Figure 11.

possibilities : (1) $f_2(0\,;0,0,0)=0$ and $f_4(0\,;0,0,0)\neq 0$, (2) $f_2(0\,;0,0,0)\neq 0$ and $f_4(0\,;0,0,0)=0$. After some hand computations, we found that the case (1) fits our numerical results. So let us now assume that $f_2(0\,;0,0,0)=0$ and $f_4(0\,;0,0,0)\neq 0$. Under this assumption, the bifurcation equation may be written, after dividing K_2 by f_4, as follows :

$$(5.7)\qquad ((\varepsilon\lambda+au+bv+cs+\phi_1)\xi+(\eta\lambda+du+ev+kc+\phi_3)\bar{\xi}\zeta,\ (\delta\lambda+\hat{a}u+\hat{b}v+\hat{c}s+\phi_2)\zeta+\xi^2).$$

where $a,\cdots,e,k,\hat{a},\hat{b},\hat{c},\varepsilon,\delta,\eta$ are real constants. ϕ_j are functions of u, v, and s of order ≥ 2. We will show that this degeneracy assumption leads to Figure 3. As

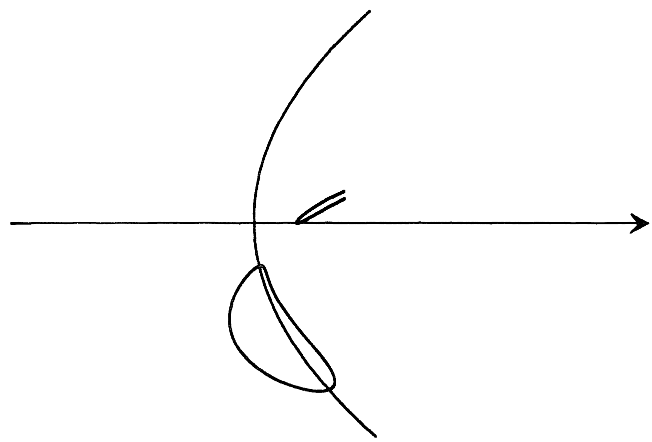

Figure 12.

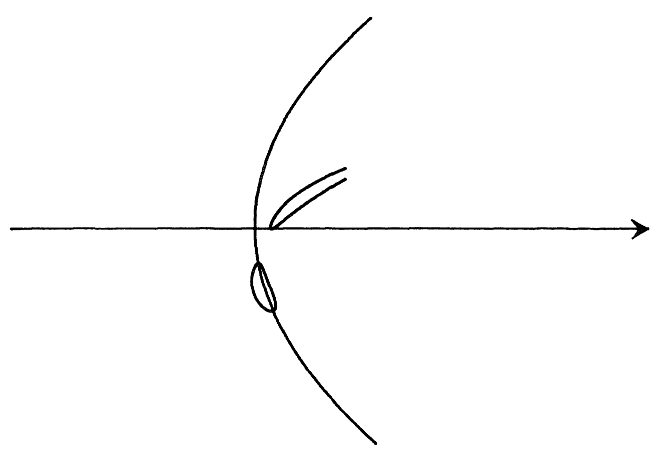

Figure 13.

in Theorem 5.1, we can have a normal form of (5.7) as follows

Theorem 5. 2 ([**15**]). *Under a certain generic assumption on the coefficients a, b, $\hat{a}$, $\hat{b}$, e, ε, δ,* (5.6) *is* O(2)*-equivalent to*

$$((\varepsilon\lambda + au + bv + cs)\xi + ev\bar{\xi}\zeta,\ (\delta\lambda + \hat{a}u + \hat{b}v)\zeta + \xi^2).$$

A universal unfolding of this is given by

$$((\varepsilon\lambda + \alpha + (a+\gamma_1)u + (b+\gamma_2)v + (c+\gamma_3)s)\xi + (\beta + (e+\gamma_4)v)\bar{\xi}\zeta, \quad (\delta\lambda + \hat{a}u + \hat{b}v)\zeta + \xi^2). \tag{5.8}$$

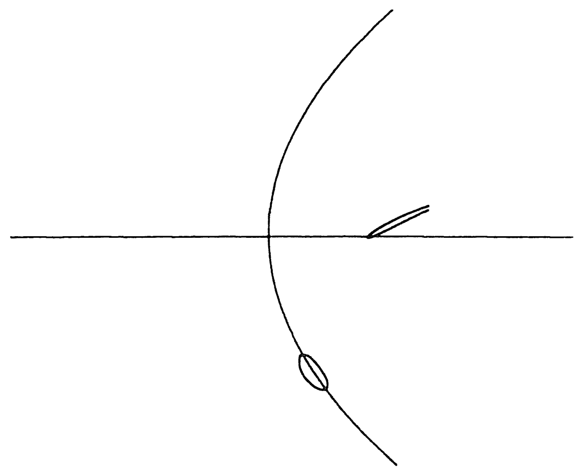

Figure 14.

where α, β, and $\gamma_j (j=1, 2, 3, 4)$ are unfolding parameters among which α and β are essential and γ_j are modal parameters.

We now use this theorem to draw bifurcation diagrams. Some drawings are done in [**7**, **17**]. In fact Figure 19 through 24 in [**17**] are obtained in this way (namely under the degeneracy assumption $f_2(0; 0, 0, 0)=0$), by choosing the constants a, b, etc. Figures 10–14 are taken from [**17**]. Here applies a remark similar to the one we presented to Figure 9. When we say that Figures 10–14 are obtained, it means that these figures do appear in the mapping germs as we vary α and β. We see that Figures 3–7 and the diagrams in Figure 10–14 are qualitatively the same, if we make the change of variables $(x, z) \longmapsto (-x, -z)$. Thus a universal unfolding of the degenerate bifurcation equation can explain the phenomena well. Therefore it would be quite reasonable to expect that there is a degenerate bifurcation point (p, q, η) for some $\eta \in [0, 1)$.

We now examine the existence of the hypothetical degeneracy. Namely we wish to know if $f_2(0; 0, 0, 0)=0$ for some η in our mapping K. We hereafter check if this is the case or not.

Theorem 5.3. *It holds that*

$$f_2(0; 0, 0, 0) = -\frac{1+4\eta^2+\eta^4}{2(1-\eta^2)^2}, \quad f_4(0; 0, 0, 0) = -\frac{2(1+4\eta^2+\eta^4)}{(1-\eta^2)^2}.$$

In particular, neither of them vanishes for any $\eta \in [0, 1)$.

Proof. Actually this theorem is proved in [**18**]. We reproduce the proof here, since the volume (called Kokyu-roku) containing [**18**] is of a private nature. We have

(5.9) $$4q(1, 2; \eta)+p(1, 2; \eta)=2\frac{1+\eta^4}{1-\eta^4}, \qquad q(1, 2; \eta)+p(1, 2; \eta)=\frac{1+\eta^2}{1-\eta^2}$$

since $m=1$ and $n=2$. We note that

$$f_2(0; 0, 0, 0)=\frac{\partial^2 K_1}{\partial x \partial z}(0; 0, 0).$$

It holds that

(5.10) $$\frac{\partial K}{\partial x}(\lambda; \xi, \zeta)=QD_\theta F(p(1, 2; \eta), q(1, 2; \eta)+\lambda, \eta; \#)(\sin\sigma+\phi_x),$$

where # denotes $x\sin\sigma+y\cos\sigma+z\sin 2\sigma+w\cos 2\sigma+\phi$. Differentiating (5.10) in z, we obtain

$$\frac{\partial^2 K}{\partial x \partial z}(0; 0, 0)=QD_\theta F^0(\phi^0_{xz})+QD^2_\theta F^0(\sin\sigma, \sin 2\sigma)$$

where ^{0} implies that the functions are evaluated at $(p(1, 2; \eta), q(1, 2; \eta), \eta; 0)$. Since ϕ is $(I-Q)X_2$-valued and since $D_\theta F^0$ commutes with Q, the first term of the right hand side vanishes. Hence $f_2(0; 0, 0)$ is the coefficient of $\sin\sigma$ in $D^2_\theta F^0$ $(\sin\sigma, \sin 2\sigma)$. We now compute the second order Fréchet derivative of F. We have ([**16**])

$$D^2_\theta F(p, q, \eta; 0)(f, g)=2dd\sigma(HfHg)+p(fHg+gHf)+q\frac{d}{d\sigma}\Big(Hf\frac{dg}{d\sigma}+Hg\frac{df}{d\sigma}\Big)$$

for all $f, g\in X_2$, where we write H instead of H_η. This formula yields

$$\begin{aligned} D^2_\theta F(p, q, \eta; 0)(\sin\sigma, \sin 2\sigma)&=2\frac{1+\eta^2}{1-\eta^2}\frac{1+\eta^4}{1-\eta^4}\frac{d}{d\sigma}(\cos\sigma\cos 2\sigma)\\ &-p\Big(\frac{1+\eta^4}{1-\eta^4}\cos 2\sigma\sin\sigma+\frac{1+\eta^2}{1-\eta^2}\cos\sigma\sin 2\sigma\Big)\\ &+q\frac{d}{d\sigma}\Big(-\frac{1+\eta^2}{1-\eta^2}\cos\sigma\cdot 2\cos 2\sigma-\frac{1+\eta^4}{1-\eta^4}\cos 2\sigma\cos\sigma\Big). \end{aligned}$$

Consequently we obtain

$$\begin{aligned} QD^2_\theta F(p, q, \eta; 0)(\sin\sigma, \sin 2\sigma)&=\frac{1+\eta^2}{1-\eta^2}\frac{1+\eta^4}{1-\eta^4}(-\sin\sigma)\\ &+p\Big(\frac{1}{2}\frac{1+\eta^4}{1-\eta^4}-\frac{1}{2}\frac{1+\eta^2}{1-\eta^2}\Big)\sin\sigma+q\Big(\frac{1+\eta^2}{1-\eta^2}+\frac{1}{2}\frac{1+\eta^4}{1-\eta^4}\Big)\sin\sigma. \end{aligned}$$

By this equality and (5. 9) we have

$$f_2(0\,;0,0,0)=-\frac{3p(1,2\,;\eta)(p(1,2\,;\eta)+q(1,2\,;\eta))}{4}=-\frac{1+4\eta^2+\eta^4}{2(1-\eta^2)^2}.$$

In a similar way we obtain

$$f_4(0\,;0,0,0)=QD_\theta^2G_1(p,q,\eta\,;0)(\sin\sigma,\sin\sigma)=-\frac{2(1+4\eta^2+\eta^4)}{(1-\eta^2)^2}.$$

This completes the proof. ∎

We now consider the bifurcation equation of G. The double bifurcation points are now written as $(p(1,2\,;r),q(1,2\,;r))$ for $r\in(0,\infty)$. We again get to (5.1, 2) depending implicitly on $r\in(0,\infty)$.

Theorem 5. 4. *It holds that*

$$f_2(0\,;0,0,0)=-\frac{1+4e^{-2r}+e^{-4r}}{2(1-e^{-2r})^2},\quad f_4(0\,;0,0,0)=-\frac{2(1+4e^{-2r}+e^{-4r})}{(1-e^{-2r})^2}.$$

Proof. We have

$$\text{(5.11)}\quad 4q(1,2\,;r)+p(1,2\,;r)=2\frac{1+e^{-4r}}{1-e^{-4r}},\quad q(1,2\,;r)+p(1,2\,;r)=\frac{1+e^{-2r}}{1-e^{-2r}}$$

since $m=1$ and $n=2$. We note that

$$f_2(0\,;0,0,0)=\frac{\partial^2K_1}{\partial x\partial z}(0\,;0,0).$$

It holds that

$$\text{(5.12)}\qquad \frac{\partial K}{\partial x}(\lambda\,;\xi,\zeta)=PDG(p(1,2\,;r),q(1,2\,;r)+\lambda,r\,;\#)(\Sigma_1+\phi_x),$$

where # denotes $(0,e^{-r})+x\Sigma_1+y\Sigma_2+z\Sigma_3+w\Sigma_4+\phi$. Differentiating (5.12) in z, we obtain

$$\frac{\partial^2K}{\partial x\partial z}(0\,;0,0)=PDG^0(\phi_{xz}^0)+PD^2G^0(\Sigma_1,\Sigma_3)$$

where ^{0} implies that the function is evaluated at $(p(1,2\,;r),q(1,2\,;r),r\,;0,e^{-r})$. Since ϕ is $(I-P)X_2$-valued and since DG^0 commutes with P, the first term of the right hand side vanishes. On the other hand, since the second components of Σ_j are zero, it follows that $PD^2G^0(\Sigma_1,\Sigma_3)=QD_\theta^2G_1^0(\sin\sigma,\sin 2\sigma)$. Hence $f_2(0\,;0,0)$ is the coefficient of $\sin\sigma$ in $D_\theta^2G_1^0(\sin\sigma,\sin 2\sigma)$. Since G_1 is the same as F, we get the same conclusion as that of Theorem 5.3.

§ 6. Conclusion

The conclusion of § 5 is that neither of F nor G contain the degenerate bifurcation point. On the other hand, the hypothesis of the existence of the degeneracy reproduces the numerical diagrams by an abstract way. Therefore we must look for a new parameter, say χ, and a new mapping $\tilde{F}(p, q, \chi; \bullet)$ which satisfies the following two conditions :

(1) $\tilde{F}(p, q, \chi; \bullet)$ is an extended mapping of the mapping $F(p, q, 0; \theta)$ which is defined in § 2, in the following sense ; At a point χ_0, the mapping $\tilde{F}(p, q, \chi_0; \bullet)$ is equal to $F(p, q, 0; \theta)$.

(2) The mapping $\tilde{F}(p, q, \chi; \bullet)$ has a degenerate bifurcation point. In other words, there exists a χ_d at which $f_2(0; 0, 0, 0)=0$

The depth of the flow (or the aspect ratio) is not the parameter by the Theorems 5.3 and 5.4. We must look for another parameter. This is our conclusion.

If we consider the two phase flow (i.e., we consider both the flows above and beneath the free boundary), then we may have a candidate of the new parameter. The parameter is the ratio of mass densities of the two fluids. The present problem is realized as a special case of zero density ratio. The two phase flow problem is studied by Kotchine [**11**]. We will examine the degeneracy in this new context in the forthcoming paper.

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