Publ. RIMS, Kyoto Univ.
30 (1994), 671–688

On the Irregularity of Special Non-Canonical Surfaces

By

Kazuhiro KONNO *

Abstract

We consider minimal surfaces of general type whose canonical map is "special" meaning that it is composed of a pencil or its degree is high. We characterize, to some extent, Beauville's examples of irregularity 2 in the pencil case, and show that the irregularity is at most 12 when the canonical degree is 5.

Introduction

Let S be a minimal surface of general type defined over C, and let $K=K_S$ denote a canonical divisor. If $p_g>1$, we can consider the rational map associated with $|K|$, the canonical map $\Phi_K : S \to P^{p_g-1}$. We put $\Sigma=\Phi_K(S)$ and let $\phi_K : S \to \Sigma$ be the induced rational map. When ϕ_K is not birational, some important results were obtained by Beauville and Xiao :

(1) Suppose that Σ is a curve, that is, $|K|$ is composed of a pencil. We get a relatively minimal fibration $f: X \to B$ after blowing up the base points and taking the Stein factorization if necessary. Put $b=g(B)$ and let g be the genus of a general fibre of f. Beauville [1] showed that $g \leq 5$ when p_g is large. Later, Xiao [12] showed that either $b=q=1$ or $b=0$, $q \leq 2$.

(2) Suppose that Σ is a surface. It is well-known that Σ is a ruled surface when its degree is small (cf. [1], [14] or [10]). Hence, if $d_{can}:=\deg\phi_K$ is large, Miyaoka-Yau's inequality implies that Σ is ruled and, as in the previous case, S has a pencil of curves of genus g induced by the ruling of Σ. Beauville [1] showed that $d_{can} \leq 9$ when p_g is large enough. Xiao showed that $d_{can}=9$ is actually impossible for $p_g>132$ ([14]), and that $q \leq 3$ when $d_{can} \geq 7$, $p_g>115$ ([14] and [16]). He also

Communicated by S. Mori, July 21, 1993.
Supported in part by a Grant under The Monbusho International Scientific Research Program : 04044081
1991 Mathematics Subject Classification : 14J29
* Department of Mathematics, Faculty of Science, Osaka University, Toyonaka, Osaka 560, Japan.

proved that there is a bound on q, g when $d_{can}=5$, 6. After that, Sun [11] has shown $q\leq 5$ when $d_{can}=6$ and $p_g>55$, along an analogous line.

The purpose of this article is to give a slight refinement of the above results. Our main interest is in the cases $q=2$ in (1) and $d_{can}=5$ in (2). We show that a surface with $q=2$ whose canonical map is composed of a pencil is essentially an example of Beauville [1, 2.5] when the Albanese map is not surjective (Theorem 3.6), and that $q\leq 12$ if the canonical map is of degree 5 onto the image (Theorem 4.5). As one may learn from (1) and (2), we are naturally led to studying fibred surfaces $f: X\to B$. We use the powerful methods due to Xiao in order to analize $f_*\omega_X$. Hence the paper should be regarded as an appendix to his remarkable papers, especially to [14].

§ 1. Irregularity of Fibred Surfaces

In this and the next sections, we recast Xiao's method in [14] and prepare some results for the later use. See also [12], [15], [16], [1], [3] and [9].

1.1. Let $\mathscr{E}$ be a locally free sheaf on a non-singular projective curve B. We put $\mathscr{E}^*=\mathit{Hom}(\mathscr{E}, \omega_B)$ and $\mu(\mathscr{E})=\deg(\mathscr{E})/\mathrm{rk}(\mathscr{E})$. According to [4], we have a filtration of $\mathscr{E}$ by locally free subsheaves $\mathscr{E}_i$:

$$0=\mathscr{E}_0\subset\mathscr{E}_1\subset\cdots\subset\mathscr{E}_l=\mathscr{E}$$

which satisfies

(i) $\mathscr{E}_i/\mathscr{E}_{i-1}$ is semi-stable,

(ii) $\mu_i(\mathscr{E})>\mu_{i+1}(\mathscr{E})$, where $\mu_i(\mathscr{E}):=\mu(\mathscr{E}_i/\mathscr{E}_{i-1})$.

As usual, we call such a filtration the Harder-Narashimhan filtration of $\mathscr{E}$. Note that we have

$$(\mathrm{rk}(\mathscr{E})-1)\mu_1(\mathscr{E})+\mu_l(\mathscr{E})\geq\deg(\mathscr{E}). \tag{1.1}$$

Let $\pi: P(\mathscr{E})\to B$ be the associated projective bundle. We denote by $H(\mathscr{E})$ and F a (relatively ample) tautological divisor and a fibre of π, respectively. The locally free sheaf $\mathscr{E}$ is called nef if and only if $H(\mathscr{E})$ is nef. By [9], the Q-divisors $H(\mathscr{E})-\mu_l(\mathscr{E})F$ and $H(\mathscr{E})-\mu_1(\mathscr{E})F$ are respectively nef and pseudo-effective.

1.2. Let $f: X\to B$ be a relatively minimal fibration of non-singular projective surface X onto a non-singular projective curve B of genus b. We assume that X is of general type and $p_g>0$. We let g denote the genus of a general fibre D of f. Then $g\geq 2$. By Arakelov's theorem [2], the relative dualizing sheaf $\omega_{X/B}$ is nef. By [3], $f_*\omega_X$ is a direct sum of a locally free sheaf and $q-b$ copies of ω_B, and $f_*\omega_{X/B}=$

$f_*\omega_X \otimes \omega_B^{-1}$ is nef.

From now on, we let $\mathscr{E}$ be the locally free subsheaf of $f_*\omega_X$ generically generated by elements in $H^0(f_*\omega_X)$; the quotient $\mathscr{E}' = f_*\omega_X/\mathscr{E}$ is also locally free. Put $r = \mathrm{rk}(\mathscr{E})$ and let $0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_l = \mathscr{E}$ be the Harder-Narashimhan filtration for $\mathscr{E}$.

For each i, the natural sheaf homomorphism $f^*\mathscr{E}_i \to f^*f_*\omega_X \to \omega_X$ induces a rational map $\phi_i : X \to \boldsymbol{P}(\mathscr{E}_i)$. Let $\rho_i : X_i \to X$ be the elimination of the indeterminacy of ϕ_i, and let $\tilde{\phi}_i : X_i \to \boldsymbol{P}(\mathscr{E}_i)$ be the induced holomorphic map. We denote by $\tilde{M}_i$ the pull-back to X_i of $H(\mathscr{E}_i)$ via $\tilde{\phi}_i$. Then $\tilde{M}_i - \mu_i(\mathscr{E})\rho_i^* D$ is nef, since so is $H(\mathscr{E}_i) - \mu_i(\mathscr{E})F$. Put $M(\mathscr{E}_i) = (\rho_i)_* \tilde{M}_i$. Then $M(\mathscr{E}_i) - \mu_i(\mathscr{E})D$ is nef. Note that $K_X \equiv M(\mathscr{E}_i) + Z(\mathscr{E}_i)$ with an effective divisor $Z(\mathscr{E}_i)$, where $\equiv$ denotes the numerical equivalence.

Let $\mathscr{F}'$ be the locally free subsheaf of $\mathscr{E}^*$ generated by $H^0(\mathscr{E}^*)$, and put $\mathscr{F} = (\mathscr{F}')^*$. Then $\mathscr{F}$ and $\mathscr{F}^*$ are both nef. We have $p_g = h^0(f_*\omega_X) = h^0(\mathscr{E})$ and $h^1(\mathscr{E}) = h^1(\mathscr{F}) = h^0(\mathscr{F}^*)$ by the choice of $\mathscr{E}$ and $\mathscr{F}$.

Proposition 1.3. *With the above notation,*

$$q(X) \leq b + \mathrm{rk}(\mathscr{F}) - (b-1)(g-r).$$

If the equality holds here, then $\deg(\mathscr{E}') = 2(b-1)(g-r)$ *and* $\mathscr{F}$ *is a direct sum of* $\mathrm{rk}(\mathscr{F})$ *copies of* ω_B.

Proof. The following inequalities were shown in [14, p. 477] :

$$h^1(\mathscr{E}) = h^1(\mathscr{F}) \leq b\,\mathrm{rk}(\mathscr{F}) - \frac{1}{2}\deg(\mathscr{F}), \tag{1.2}$$

$$\deg(\mathscr{E}') \geq 2(b-1)(g-r). \tag{1.3}$$

$$\deg(\mathscr{F}) + \deg(\mathscr{E}') \geq 2(b-1)(g-r+\mathrm{rk}(\mathscr{F})), \tag{1.4}$$

We have an exact sequence

$$0 \to \mathscr{E} \to f_*\omega_X \to \mathscr{E}' \to 0$$

and $h^0(\mathscr{E}) = h^0(f_*\omega_X) = p_g$. Since we have $h^1(f_*\omega_X) = q(X) - b$ by [3, Theorem (3.1)], we get $q - b = h^1(\mathscr{E}) - \chi(\mathscr{E}')$. By the Riemann-Roch theorem and (1.3), we have $\chi(\mathscr{E}') = \deg(\mathscr{E}') - (b-1)(g-r) \geq \deg(\mathscr{E}')/2$. Applying (1.2), we get $q - b \leq b\,\mathrm{rk}(\mathscr{F}) - (\deg(\mathscr{F}) + \deg(\mathscr{E}'))/2$. Hence the inequality follows from (1.4). If the equality holds there, then the equalities hold in (1.2), (1.3) and (1.4). Hence we have $\deg(\mathscr{E}') = 2(b-1)(g-r)$, $\deg(\mathscr{F}) = 2(b-1)\mathrm{rk}(\mathscr{F})$ and $h^1(\mathscr{F}) = \mathrm{rk}(\mathscr{F})$. Since $\mathscr{F}$ and $\mathscr{F}^*$ are both nef, we see that $\mathscr{F}^*$ is semi-stable of degree 0 and $h^0(\mathscr{F}^*) = \mathrm{rk}(\mathscr{F})$. Since $\mathscr{F}^*$ is generated by its global sections, it is a direct sum of $\mathcal{O}_B$. Q. E. D.

Corollary 1.4. *Assume that* $\mathrm{rk}(\mathcal{F})=r$. *Then* $p_g \leq br$. *Furthermore,* $f: X \to B$ *is locally trivial if (and only if) the equality holds in* (1.4). *In particular, when* $q=b+r-(b-1)(g-r)$, *f is locally trivial and* $\mathcal{E} \simeq \omega_B^{\oplus r}$.

Proof. Since $\mathrm{rk}(\mathcal{F})=r$, we have $\mathcal{F}=\mathcal{E}$. Hence $\mathcal{E}^*$ is also nef and $\deg(\mathcal{E}^*)=-\deg(\mathcal{E})+2(b-1)r \geq 0$. By Clifford's theorem, $p_g=h^0(\mathcal{E}) \leq \deg(\mathcal{E})/2+r \leq br$. By (1.4), we have

$$\deg(f_*\omega_X)=\deg(\mathcal{E})+\deg(\mathcal{E}') \geq 2(b-1)g.$$

If $\deg(f_*\omega_X)=2(b-1)g$, then we have $\deg(f_*\omega_{X/B})=\deg(f_*\omega_X)-2(b-1)g=0$. Hence f is locally trivial. The rest may be clear from Proposition 1.3. Q. E. D.

Lemma 1.5. *If* $b=0$, *then* $q(X) \leq r+1$.

Proof. Assume that $q \geq r+2$. Let S be the minimal model of X and let $\alpha: S \to \mathrm{Alb}(S)$ be the Albanese map. Since $q > r+1$, it follows from [16, Theorem 2] that $\alpha(S)$ cannot be a surface. Hence $C=\alpha(S)$ is a non-singular irreducible curve of genus q. Let $\beta: X \to C$ be the fibration induced by α. We denote by h the genus of a general fibre D_1 of β. Since $b=0$, we have $\deg(\mathcal{E})=p_g-r$ and $\mu_1(\mathcal{E}) \geq p_g/r-1$. Hence $K_X-(p_g/r-1)D$ is pseudo-effective, and we have $(K_X-(p_g/r-1)D)D_1 \geq 0$. Since X is non-ruled, we have $DD_1 \geq 2$. It follows that $2h-2 \geq DD_1(p_g/r-1) \geq 2(p_g/r-1)$. On the other hand, we have $\deg \beta_*\omega_{X/C}=\chi(\mathcal{O}_X)-(h-1)(q-1) \geq 0$. Since $q \geq r+2$, we get

$$\chi(\mathcal{O}_X) \geq (h-1)(q-1) \geq \frac{q-1}{r}(p_g-r) \geq \frac{r+1}{r}(\chi+1)$$

which is impossible. Q. E. D.

Now we can show the following :

Theorem 1.6. *Let* $f: X \to B$ *be a relatively minimal fibration of genus* $g \geq 2$, $b=g(B)$, *and assume that X is of general type. Assume that the global sections of* $f_*\omega_X$ *generically generate a locally free subsheaf of rank r.*

(1) *If* $b=0$, *then* $q(X) \leq \min\{g-r, r+1\}$.

(2) *If* $b=1$, *then* $q(X) \leq r$.

(3) *If* $b>1$ *and* $g>r$, *then* $q(X) \leq r$ *and* $g \leq r+r/(b-1)$.

(4) *If* $b>1$ *and* $g=r$, *then* $q(X) \leq b+g-1$ *unless X is a product of B and a curve of genus g.*

Proof. Assume that $b=0$. Then $\mathscr{F}=0$ and we have $q\leq g-r$ by Proposition 1.3. Hence we get (1) by Lemma 1.5.

Assume that $b>0$ and $g>r$. Then the inequality in Proposition 1.3 says that

$$q\leq b+\mathrm{rk}(\mathscr{F})-(b-1)(g-r)\leq b+\mathrm{rk}(\mathscr{F})-(b-1)\leq \mathrm{rk}(\mathscr{F})+1\leq r+1.$$

We assume that $q(X)=r+1$, and show that this eventually leads us to a contradiction. We have $\mathrm{rk}(\mathscr{F})=r$, $g=r+1$ (or $b=1$). Corollary 1.4 shows that f is locally trivial and $\mathscr{E}\simeq\omega_B^{\oplus r}$. This cannot happen for $\mathrm{b}=1$, since X is of general type. Hence we can assume that $b>1$. Since f is locally trivial, we have an exact sequence

$$0\to f^*\omega_B\to\Omega_X^1\to\omega_{X/B}\to 0.$$

Then, as in [12, § 1], one can see that this sequence splits. Recall that $\mathscr{E}\otimes\omega_B^{-1}\simeq\mathcal{O}_B^{\oplus r}$ is a subsheaf of $f_*\omega_{X/B}$. Hence, we have

$$\begin{aligned}h^0(\Omega_X^1)&=h^0(f^*\omega_B)+h^0(f_*\omega_{X/B})\\&\geq b+h^0(\mathscr{E}\otimes\omega_B^{-1})\\&=b+r,\end{aligned}$$

which is impossible, since $r+1=q=h^0(\Omega_X^1)$ and $b>1$. Therefore, $q(X)\leq r$. By $q\geq b$, we have $(b-1)(g-r)\leq\mathrm{rk}(\mathscr{F})\leq r$. Hence $g\leq r+r/(b-1)$ when $b>1$ and $g>r$.

Assume that $b>1$ and $g=r$. Then Proposition 1.3 gives us $q(X)\leq b+\mathrm{rk}(\mathscr{F})\leq b+g$. If $q=b+g$, then f is globally trivial as is well-known. Q. E. D.

We close the section with the following :

Lemma 1.7. *Assume that $b=0$, $q=r+1$ and that the Albanese image is a curve C. Let S be the minimal model of X. Then the Albanese pencil of S is a locally trivial hyperelliptic fibration of genus p_g/r, $K_S^2=8\chi(\mathcal{O}_S)$, and $\mathscr{E}$ is the direct sum of r copies of $\mathcal{O}(p_g/r-1)$.*

If $g=2r+1$, then $X=S$ and S is a double covering of $P=\boldsymbol{P}^1\times C$ with branch locus $2(p_g/r+1)$ distinct fibers of $p_1:P\to\boldsymbol{P}^1$. If $g>2r+1$, then $m=2r(g-2r-1)/(p_g+r)$ is an integer greater than 1 and $K_X^2\leq K_S^2-2m$.

Proof. We use the same notation as in the proof of Lemma 1.5. Then the same argument there easily gives us $\mu_1(\mathscr{E})=p_g/r-1$, $DD_1=2$, $h=p_g/r$ and $\chi=(h-1)(q-1)$. The last equality shows that $\alpha:S\to C$ is locally trivial. Since $DD_1=2$, D_1 is a double cover of $\boldsymbol{P}^1$. Hence it is a hyperelliptic curve. Since $\mu(\mathscr{E})=\mu_1(\mathscr{E})$ and B

$=\mathbf{P}^1$, we see that $\mathscr{E}\simeq\mathcal{O}_{\mathbf{P}^1}(p_g/r-1)^{\oplus r}$.

We have a holomorphic map $\phi: X\to P=\mathbf{P}^1\times C$ putting $\phi=f\times\beta$. Since $DD_1=2$, ϕ is of degree 2. Put $g=2r+k$. It follows from Theorem 1.6 that k is a positive integer. By the Riemann-Hurwitz formula, we see that the branch locus B_0 of ϕ is linearly equivalent to 2ξ, where $\xi=p_1^*\mathcal{O}_{\mathbf{P}^1}(h+1)+p_2^*(\eta)$ and η is a divisor of degree $k-1$ on C. Furthermore, X is birationally equivalent to a double covering X_0 of P constructed in the total space of $[\xi]$ with branch locus B_0. Note that B_0 is free from multiple components. The dualizing sheaf of X_0 is induced by $K_P+\xi$. Hence $\chi(\mathcal{O}_{X_0})=\chi+(k-1)h$ and $\omega_{X_0}^2=8\chi+4(k-1)(h-1)$, where $\chi=\chi(\mathcal{O}_S)$.

If $k=1$, then $\chi(\mathcal{O}_{X_0})=\chi$. Furthermore, B_0 consists of fibers of p_1 and $2\eta=0$. In particular, since B_0 is smooth, X_0 is isomorphic to X. Note that $\eta\neq 0$, since, otherwise, X is a product of C and a curve of genus h contradicting $q(X)=r+1=g(C)$. Therefore, η is a 2-torsion element. Note further that $X=S$ in this case. Conversely, if we take a 2-torsion element $\eta\in\mathrm{Pic}^0(C)$ and construct a double covering X_0 of P in $[p_1^*\mathcal{O}(p_g/r+1)+p_2^*\eta]$ with branch locus consisting of $2(p_g/r+1)$ distinct fibres of p_1, then an easy calculation shows that X_0 satisfies our requirements.

Assume that $k>1$. We take the canonical resolution X^* of X_0 (see, [5]). Let m_i denote the multiplicity of the singular point of B_0 appearing in the process of the canonical resolution. The difference of the invariants of X_0 and X^* can be measured by the formula in [5]. Since $\chi(\mathcal{O}_{X^*})=\chi$, we have

$$\sum_i\left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right]-1\right)=2(k-1)h. \tag{1.5}$$

Since $K_S^2=8\chi$ and $K_{X^*}^2\leq K_X^2\leq 8\chi$, we have

$$\sum_i\left(\left[\frac{m_i}{2}\right]-1\right)^2\geq 2(k-1)(h-1). \tag{1.6}$$

Since $k>1$, we can assume that $[m_1/2]>1$. It follows from (1.5) and (1.6) that $2(k-1)\geq\Sigma([m_i/2]-1)$. Then, from (1.5), we get $\Sigma([[m_i/2]-h)([m_i/2]-1)\geq 0$. This allows us to assume $[m_1/2]\geq h$. Then the fibre Γ_1 of $p_2: P\to C$ passing through this singular point induces on X^* a rational curve. Since $\alpha: S\to C$ is locally trivial, this implies $X^*\neq S$ and, hence, the equality does not hold in (1.6). Then, as above, we see that $[m_1/2]=h+1$. Since every fibre of α is non-singular, the singular point must be a $2(h+1)$-ple point which becomes an ordinary $2(h+1)$-ple point after, say, k_1-times of blowing-ups ($k_1\geq 0$), and Γ_1 is not a component of B_0. Hence, on X^*, the inverse image of Γ_1 consists of a non-singular curve of genus h, two (-1)-curves coming from the proper transform of Γ_1 and $2k_1$ (-2)-curves which are "infinitely near" (-1)-curves. These (-1)-curves must remain on X, since we have the holomorphic map $f: X\to\mathbf{P}^1$. Hence $K_X^2\leq K_S^2-2(k_1+1)=$

$8\chi-2(k_1+1)$, $m_1=\cdots=m_{k_1+1}=2h+2$.

As in (1.5), (1.6), we get

$$\sum_{i\geq k_1+2}\left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right]-1\right)=\{2(k-1)-(k_1+1)(h+1)\}h.$$

and

$$\sum_{i\geq k_1+2}\left(\left[\frac{m_i}{2}\right]-1\right)^2\geq\{2(k-1)-(k_1+1)(h+1)\}(h-1).$$

If $2(k-1)>(k_1+1)(h+1)$, then similarly as above, one can show that there is a singular point of B_0 of multiplicity $2(h+1)$ which becomes an ordinary $2(h+1)$-ple point after, say, k_2-times of blowing-ups. Let Γ_2 be the fibre of p_2 passing through this singular point. Then it creates two (-1)-curves and $2k_2$ infinitely near (-1)-curves on X. Hence $K_X^2\leq 8\chi-2(k_1+1)-2(k_2+1)$.

We can repeat such a procedure unless $2(k-1)$ is some multiple of $h+1$. Hence $m=2(k-1)/(h+1)$ is a positive integer and $K_X^2\leq K^2-2m$. If $m=1$, one can easily see that the fibre of $p_1: P\to P^1$ passing through the singular point of multiplicity $2h+2$ of B_0 is a multiple component of B_0, which is impossible.

Q. E. D.

§ 2. Inequalities

In this section, we give some inequalities generalizing one in [14, Lemma 3] along an analogous line there. We freely use the notation in the previous section. In particular, let $0\subset\mathscr{E}_1\subset\cdots\subset\mathscr{E}_l=\mathscr{E}$ be the Harder-Narashimhan filtration of $\mathscr{E}$. Put $d_i=M(\mathscr{E}_i)D$ and $a_i=2g-2-d_i$ for $1\leq i\leq l$. We put $d=d_l$, $a=a_l$, $M=M(\mathscr{E})$ and $Z=Z(\mathscr{E})$ for the sake of simplicity. If there are no danger of confusion, we also put $r_i=\mathrm{rk}(\mathscr{E}_i)$, $\mu_i=\mu_i(\mathscr{E})$, $M_i=M(\mathscr{E}_i)$ and $Z_i=Z(\mathscr{E}_i)$.

Lemma 2.1. *With the above notation, the following hold.*

(1) $2r_i-2\leq d_i\leq 2g-2$.

(2) *Let* $Z_i=\Sigma m_jG_j$ *be the irreducible decomposition and put*

$$\alpha_i=\max_j\{m_j\mid DG_j>0\}.$$

Then $tK_{X/B}+M_i-\mu_iD+Z_i$ *is nef for any* $t\geq\alpha_i$. *In particular,* $(a_i+1)K_X-(\mu_i+2a_i(b-1))D$ *is nef.*

Proof. (1): We clearly have $d_i\leq 2g-2$. Since d_i equals to the degree of the linear system $|M_i||_D$ which is of dimension r_i-1, Clifford's theorem shows $d_i\geq$

$2r_i-2$.

(2) : Recall that $K_{X/B}$ and $M_i-\mu_i D$ are nef. Let C be any irreducible curve on X. If C is not a component of Z_i, then $Z_iC\geq 0$ and $(\alpha_i K_{X/B}+M_i-\mu_i D+Z_i)C\geq 0$. Assume that $C=G_j$ for some j. If $DG_j=0$, then $(\alpha_i K_{X/B}+M_i-\mu_i D+Z_i)G_j=(\alpha_i+1)K_{X/B}G_j\geq 0$. If $DG_j>0$, then $(\alpha_i K_{X/B}+M_i-\mu_i D+Z_i)G_j=(\alpha_i-m_j)K_{X/B}G_j+m_j(K_{X/B}+G_j)G_j+(Z_i-m_jG_j)G_j\geq 0$. Hence $\alpha_i K_{X/B}+M_i-\mu_i D+Z_i=(\alpha_i+1)K_X-(\mu_i+2\alpha_i(b-1))D$ is nef. Since $a_i=DZ_i$, we always have $a_i\geq\alpha_i$. Therefore, $(a_i+1)K_X-(\mu_i+2a_i(b-1))D$ is nef. Q. E. D.

Lemma 2.2. *If* $\mathrm{rk}(\mathscr{F})\leq r-1$, *then* $\deg(\mathscr{E})\geq p_g-r+b(r-\mathrm{rk}(\mathscr{F}))$. *If* $\mathrm{rk}(\mathscr{F})=r$, *then* $\deg(\mathscr{E})\geq 2(p_g-r)$.

Proof. By the Riemann-Roch theorem and $p_g=h^0(\mathscr{E})$, we get $\deg(\mathscr{E})=p_g+r(b-1)-h^1(\mathscr{E})$. Since $\mathscr{F}$ is nef, we have $\deg(\mathscr{F})\geq 0$. Hence, by (1.2), we have $h^1(\mathscr{E})\leq b\,\mathrm{rk}(\mathscr{F})$. If $\mathrm{rk}(\mathscr{F})=r$, then $\mathscr{F}=\mathscr{E}$ and Clifford's theorem shows $p_g=h^0(\mathscr{E})\leq\deg(\mathscr{E})/2+r$. Hence $\deg(\mathscr{E})\geq 2(p_g-r)$. Q. E. D.

Corollary 2.3. *If* $r>1$ *and* $p_g\geq\min\{(3r-2)b+r+1,\ 2(g-1)b+g+q+1\}$, *then the canonical map of X separates fibers of f.*

Proof. Let L be a line bundle of degree $2b+1$ on B. Then it is very ample.

Assume that $p_g\geq(3r-2)b+r+1$. Since $p_g>br$, we have $\mathrm{rk}(\mathscr{F})<r$ by Corollary 1.4, and Lemma 2.2 shows that $\deg(\mathscr{E})\geq p_g-r+b$. We have $\deg(\mathscr{E}(-L))\geq p_g-r+b-r(2b+1)\geq(r-1)(b-1)$ by assumption. Hence, by [8, Corollary], L can be chosen so that $H^0(\mathscr{E}(-L))\neq 0$. Since $|f^*L|+(K_X-f^*L)$ is a subsystem of $|K_X|$, the canonical map separates fibers of f.

Assume that $p_g\geq 2(g-1)b+g+q+1$. Since $\deg(f_*\omega_X)=\chi(\mathscr{O}_X)+(g+1)(b-1)$, we have $\deg(f_*\omega_X(-L))\geq(g-1)(b-1)$. Hence, as above, the canonical map can separate fibers also in this case. Q. E. D.

Lemma 2.4.

$$K_X^2\geq\frac{4g(g-1)-d_1^2}{2g-d_1-1}\mu_1(\mathscr{E})+\frac{2(2g-2-d_1)^2}{2g-d_1-1}(b-1).$$

In particular,

$$K_X^2\geq\frac{4g(g-1)-d_1^2}{2g-d_1-1}\mu(\mathscr{E})+\frac{2(2g-2-d_1)^2}{2g-d_1-1}(b-1).$$

Proof. For each i, we have

$$\begin{aligned}(2.1)\qquad K_X^2 &= K_X(M_i+Z_i)\\ &= (K_X-\mu_1 D)(M_i-\mu_i D)+(2g-2-a_i)\mu_1+2(g-1)\mu_i+K_XZ_i\\ &\geq (2g-2-a_i)\mu_1+2(g-1)\mu_i+K_XZ_i\end{aligned}$$

Since $((a_i+1)K_X-(\mu_i+2a(b-1))D)Z_i\geq 0$, we have

$$(2.2)\qquad K_XZ_i\geq\frac{a_i}{a_i+1}(\mu_i+2a_i(b-1)).$$

Now, put $i=1$. It follows from (2.1) and (2.2) that

$$(2.3)\qquad K_X^2\geq 4(g-1)\mu_1-\frac{a_1^2}{a_1+1}(\mu_1-2(b-1)).$$

Hence we get the inequalities, if we note $\mu_1\geq\mu(\mathscr{E})$. Q. E. D.

Corollary 2.5. *If* $\deg(\mathscr{E})\geq 2r(b-1)$, *then*

$$K_X^2\geq\frac{4g(g-1)}{2g-1}\Big(\mu(\mathscr{E})+2\Big(1-\frac{1}{g}\Big)(b-1)\Big).$$

Proof. Since $a_1=2g-2-d_1\leq 2g-2r_1\leq 2g-2$, we have $a_1^2/(a_1+1)\leq 4(g-1)^2/(2g-1)$. Since $\mu_1\geq\mu(\mathscr{E})\geq 2(b-1)$, (2.3) gives the inequality. Q. E. D.

When d is small enough, we can give a better bound.

Lemma 2.6. *Assume that* $0<d\leq\min\{2g-r, 2g-3\}$ *and* $\deg(\mathscr{E})\geq 2(b-1)d/(2g-1)$. *Then*

$$K_X^2\geq\frac{4g(g-1)}{(2g-1)r-d}\Big(\deg(\mathscr{E})+2(b-1)\Big(r-\frac{d+r}{g}\Big)\Big).$$

Proof. $(a+1)K_X-(\mu_l+2a(b-1))D$ is nef by Lemma 2.1. Since $K_X-\mu_1 D$ is pseudo-effective, we have $(K_X-\mu_1 D)((a+1)K_X-(\mu_l+2a(b-1))D)\geq 0$. If follows from this and (1.1) that

$$\begin{aligned}(2.4)\qquad (a+1)K_X^2 &\geq 2(g-1)((a+1)\mu_1+\mu_l+2a(b-1))\\ &\geq 2(g-1)((a-r+2)\mu_1+\deg(\mathscr{E})+2a(b-1)).\end{aligned}$$

On the other hand, (2.1) and (2.2) for $i=l$ give us

$$K_X^2 \geq (2g-2-a)\mu_1 + 2(g-1)\mu_l + K_X Z$$
$$\geq (2g-2-a)\mu_1 + 2(g-1)\mu_l + \frac{a}{a+1}(\mu_l + 2a(b-1)).$$

Hence it follows from (1.1) that

$$(2.5) \qquad (a+1)K_X^2 \geq -(a((r-1)(2g-1)+a+1) - 2(g-1)(a-r+2))\mu_1 + ((2g-1)a+2g-2)\deg(\mathscr{E}) + 2a^2(b-1).$$

Note that we have $2(g-1)(a-r+2) \leq ((r-1)(2g-1)+a+1)a$.

Since $a>0$, the desired inequality follows from (2.4) when $((r-1)(2g-1)+a+1))\mu_1 \geq (2g-1)\deg(\mathscr{E}) - 2(b-1)(2g-2-a)$ and, otherwise, it follows from (2.5). Q. E. D.

By using the same method, one can also get a slight improvement of [15, Corollary 3].

Lemma 2.7. *Let $f: X \to B$ be a relatively minimal fibration of genus $g \geq 2$, $b = g(B)$, and put $h = q(X) - b$. If $g - h > 0$, then*

$$K_{X/B}^2 \geq \frac{4g(g-1)}{(2g-1)(g-h)} \deg(f_* \omega_{X/B}).$$

When f is not locally trivial, the equality holds only if $g - h = 1$.

Proof. By [3, Theorem 3.1], $f_*\omega_{X/B} = \mathscr{H} \oplus \mathscr{O}_B^{\oplus h}$. Hence $\deg(\mathscr{H}) = \deg f_*\omega_{X/B}$ and $\mathrm{rk}(\mathscr{H}) = g - h$. Since $\mathscr{H}$ is a direct factor of $f_*\omega_{X/B}$, it is nef.

Let $0 \subset \mathscr{H}_1 \subset \cdots \subset \mathscr{H}_k = \mathscr{H}$ be the Harder-Narashimhan filtration for $\mathscr{H}$. The natural sheaf homomorphism $f^*\mathscr{H}_1 \to f^* f_*\omega_{X/B} \to \omega_{X/B}$ induces a rational map $\phi: X \to P(\mathscr{H}_1)$. Let M be the pull-back of a tautological divisor by ϕ. Then $K_{X/B} \equiv M + Z$ with an effective divisor Z, and $M - \mu_1(\mathscr{H})D$ is nef, where D denotes a general fibre of f. Put $a = DZ$. Since $\mu_1(\mathscr{H}) \geq \mu(\mathscr{H}) = \deg f_*\omega_{X/B}/(g-h)$, it is sufficient to show

$$(2.6) \qquad K_{X/B}^2 \geq \frac{4g(g-1)}{2g-1}\mu_1(\mathscr{H}).$$

Similarly as in Lemma 2.1, one can show that $(a+1)K_{X/B} - \mu_1(\mathscr{H})D$ is nef. Hence $K_{X/B} Z \geq a\mu_1(\mathscr{H})/(a+1)$ and we get

$$K_{X/B}^2 \geq ((a+1)(4g-4-a)+a)\mu_1(\mathscr{H})/(a+1)$$

similarly as in (2.3). Since $a \leq 2g-2$, we get (2.6) with equality holding only if $a = 2g-2$ (hence $\mathrm{rk}(\mathscr{H}_1)=1$ since $2g-2-a \geq 2\mathrm{rk}(\mathscr{H}_1)-2$ by Clifford's theorem).
Q. E. D.

Proposition 2.8. *If $f\colon X \to B$ is a relatively minimal fibration of genus $g \geq 2$ which is not locally trivial. Then*

$$q(X)-b \leq \frac{g(5g-2)}{3(2g-1)} < \frac{5g+1}{6}.$$

When f is of hyperelliptic type,

$$q(X)-b \leq \begin{cases} \dfrac{(5g^2+g-1)g}{(2g-1)(3g+1)}, & \text{if } g \text{ is even,} \\ \dfrac{(5g^3-6g^2+5g-1)g}{(2g-1)(3g^2-2g+2)}, & \text{if } g \text{ is odd.} \end{cases}$$

Proof. If f is not locally trivial, its slope $\lambda(f)=K_{X/B}^2/\deg(f_*\omega_{X/B})$ is well-defined and satisfies $\lambda(f) \leq 12$ by [15, Theorem 2]. If f is a hyperelliptic fibration, then [7, Theorem 4.0.4] shows

$$\lambda(f) \leq \begin{cases} \dfrac{4(g-1)(3g+1)}{g^2}, & \text{if } g \text{ is even,} \\ \dfrac{4(3g^2-2g+2)}{g^2+1}, & \text{if } g \text{ is odd.} \end{cases}$$

Since we have $\lambda(f) \geq 4g(g-1)/(2g-1)(g-h)$ by Lemma 2.7, an easy calculation shows the assertions.
Q. E. D.

Corollary 2.9. *Let the situation be as in Theorem 1.6, and assume that $b>0$, $g = r \geq 2$. If $q(X)=b+g-1$, then one of the following holds:*

(1) *$p_g = gb-1$, $g \leq 3$, f is locally trivial and $K_X^2 = 8\chi(\mathcal{O}_X)$.*

(2) *$p_g \geq gb$, $g \leq 6$, and*

$$K_X^2 \geq \begin{cases} \dfrac{4(g-1)}{2g-1}(g\chi(\mathcal{O}_X)-(g^2-5g+2)(b-1)) & \text{if } g \geq 3, \\ 4p_g-4 & \text{if } g=2. \end{cases}$$

Proof. We have $\deg(f_*\omega_{X/B})=p_g-gb+1$. Since it is a non-negative integer, we get $p_g \geq gb-1$.

Assume that $p_g = gb-1$. Then f is locally trivial, and we get $q-b \leq (g+1)/2$

by the proof of [15, Corollary 3]. Since $q-b=g-1$, we get $g\leq 3$.

Assume that $p_g\geq gb$. Since f is not locally trivial and $q-b=g-1$, it follows from Lemma 2.7 and Proposition 2.8 that $K^2_{X/B}\geq (4g(g-1)/(2g-1))\deg f_*\omega_{X/B}$ and $g\leq 6$, respectively. We also have $K^2_{X/B}\geq 4\deg f_*\omega_{X/B}$ by [15, Theorem 1]. Hence we get (2). Q. E. D.

§ 3. Surfaces whose Canonical Map Is a Pencil

From now on, we let S be a minimal surface of general type with $p_g\geq 2$. In this section, we assume that the canonical image is a curve Σ. Let $\sigma: X\to S$ be the elimination of the base points of the variable part of $|K|$. Then taking the Stein factorization, we get a relatively minimal fibration $f: X\to B$ of genus g, $b=g(B)$. In this case, $\mathscr{E}$ is a line bundle and $M(\mathscr{E})\equiv\deg(\mathscr{E})D$. Hence $d=M(\mathscr{E})D=0$.

Theorem 3.1. *Assume that the canonical map of S is composed of a pencil. Then $b=q=1$ or $b=0$, $q\leq 2$. If $q=2$, then $g\geq 3$. Furthermore,*

$$K^2\geq K_X^2\geq \frac{4g(g-1)}{2g-1}\left(p_g+(b-1)\left(3-\frac{2}{g}\right)\right). \tag{3.1}$$

Proof. The statement for b, q follows from Theorem 1.6. Then, since $b\leq 1$ and since $\mathscr{E}$ is a line bundle with $h^0(\mathscr{E})=p_g>1$, we have $\deg(\mathscr{E})=p_g-1+b$. Hence we get (3.1) by Lemma 2.4 putting $d=d_1=0$, $r=1$. Q. E. D.

Remark 3.2. The statement for b, q in Theorem 3.1 already can be found in [12]. Unfortunately, (3.1) may not be sharp : When $g=2$ and $p_g\geq 3$, we can find the following bound in [13] :

$$K^2\geq\begin{cases}4p_g-6, & \text{if } (b,q)=(0,0)\\ 4p_g-4, & \text{if } (b,q)=(0,1)\\ 4p_g, & \text{if } (b,q)=(1,1).\end{cases}$$

When $b=0$, we can write $|K|=|(p_g-1)D_0|+Z_0$, where $D_0=\sigma_*D$ and $Z_0=\sigma_*Z$.

Lemma 3.3. *Let the notation be as above and assume that $b=0$.*

(1) *If $q=1$, then $K^2\geq 4p_g-4$ with equality holding only if the Albanese pencil is hyperelliptic.*

(2) *If $D_0^2=0$, then $K^2\geq 2(g-1)(p_g-1)$.*

(3) *If $D_0^2>0$, then $K^2\geq\max\{D_0^2(p_g-1)^2, (2g-2-D_0^2)(p_g-1)\}$. In particu-*

lar, $K^2 \geq 2(g-1)(1-1/p_g)(p_g-1)$.

Proof. (1) : Let $\alpha : S \to \mathrm{Alb}(S)$ be the Albanese map, and let D_1 be a general fibre of α. Since $K-(p_g-1)D_0$ is pseudo-effective, we have $0 \leq (K-(p_g-1)D_0)D_1 = 2h-2-D_0D_1(p_g-1) \leq 2h-2-2(p_g-1)$, where $h=g(D_1)$. Hence $h \geq p_g$ with equality holding only if D_1 is a hyperelliptic curve. On the other hand, we have $K^2 \geq (4-4/h)\chi$ by [15, Theorem 2]. Hence $K^2 \geq (4-4/p_g)p_g = 4p_g-4$.

(2) : Since K is nef, we have $K^2 = (p_g-1)KD_0 + KZ_0 = 2(g-1)(p_g-1) + KZ_0 \geq 2(g-1)(p_g-1)$.

(3) : We have $Z = \sigma^* Z_0 + \sum((p_g-1)m_i+1)E_i$, where m_i denotes the multiplicity of a base point of $|D_0|$ appearing in σ, and E_i is the inverse image of the base point. Hence $2g-2-\sum m_i = KD_0 = (p_g-1)D_0^2 + D_0Z_0$. $K^2 = (2g-2-\sum m_i)(p_g-1) + KZ_0 = (p_g-1)^2D_0^2 + (K+(p_g-1)D_0)Z_0 \geq (p_g-1)^2D_0^2$. We also note that $D_0^2 \geq \sum m_i$. Hence $K^2 \geq (2g-2-D_0^2)(p_g-1)$. Q. E. D.

Corollary 3.4. *Let S be a minimal surface of general type whose canonical map is composed of a pencil. Then* $K^2 \geq 4p_g-7$.

Proof. By Remark 3.2, we can assume that $g \geq 3$. By Lemma 3.3, we only have to consider the case that $b=0$, $D_0^2>0$ and $p_g \leq 4$. If $p_g=4$, then Lemma 3.3, (3) implies that $K^2 \geq 3(p_g-1) = 4p_g-7$. Assume that $p_g=3$. If $D_0^2 \geq 2$, then we are done. If $D_0^2=1$, then $KD_0 = 2+D_0Z_0$. Since $KD_0+D_0^2$ is even, D_0Z_0 is a positive odd integer. It follows $K^2 \geq (3-1)^2+(3-1) = 6 = 4p_g-6$. Assume that $p_g=2$. Then $K^2 \geq 1 = 4p_g-7$. Q. E. D.

Corollary 3.5. *Let the notation and assumption be as above. Assume that the variable part of* $|K|$ *is free from base points, when* $b=0$. *Then the following hold.*

(1) *If* $b=q=1$, *then* $g \leq 5$.

(2) *If* $b=0$ *and* $p_g \geq 20-9q$, *then* $g \leq 5$.

Proof. By Miyaoka-Yau's inequality, we have $K^2 \leq 9\chi$. Hence (1) and (2) follow from (3.1) and Lemma 3.3. Q. E. D.

When $q=2$, we can say more :

Theorem 3.6. *Let S be a minimal surface of general type with* $q=2$ *whose canonical map is composed of a pencil of genus g. Assume that the Albanese map is not surjective. Then* $K^2=8\chi$ *and the Albanese pencil is a locally trivial hyperelliptic fibration of genus* p_g. *Furthermore,* $g=3$ *and S is an example of Beauville* [1, 2.5] *except possibly when* $(p_g, g) = (2, 6)$, $(2, 9)$ or $(3, 7)$.

Proof. Except for the last sentence, this is clear from Lemma 1.7. Assume that

$g>3$ and put $m=2(g-3)/(p_g+1)$. Then $D_0^2\geq 2m$ as we saw in the proof of Lemma 1.7. Since $K^2=8\chi=8(p_g-1)$, Lemma 3.3 gives us $8\geq D_0^2(p_g-1)\geq 2m(p_g-1)$. Since $m\geq 2$, we have $2(p_g+1)\geq (g-3)(p_g-1)\geq (p_g+1)(p_g-1)$. Since m is an integer, we obtain the list of the exceptions. Q. E. D.

§ 4. Surfaces with High Canonical Degree

In this section, we assume that the canonical map of S induces a rational map $\phi_K : S\to\Sigma\subset P^{p_g-1}$ of degree $d_{can}>1$ onto the image Σ.

The following lemma due to Xiao [14, Lemma 1] guarantees that Σ is ruled by rational curves of small degree when d_{can} is large. See also [10].

Lemma 4.1. *If there exists a positive integer δ such that*

$$\deg\Sigma<\frac{2(\delta+1)}{\delta+2}\Big(p_g-1-\frac{9}{8}(\delta+1)\Big),$$

then Σ has a pencil of rational curves of degree $\leq\delta$. Furthermore, when $\delta=1$, the above inequality can be weakened to

$$\deg\Sigma<\frac{4}{3}(p_g-3)$$

except if $p_g=10$ and $(\Sigma, \mathcal{O}(1))\simeq(P^2, \mathcal{O}(3))$.

Assume that Σ is ruled by rational curves of degree δ. Let Λ be a pencil of curves on S induced by the ruling of Σ via ϕ_K. Let $\sigma : X\to S$ be the composite of blowing-ups which eliminates $\mathrm{Bs}\Lambda$. Then, taking the Stein factorization if necessary, we get a relatively minimal fibration $f: X\to B$. As before, we denote by g the genus of a general fibre D of f and put $b=g(B)$.

Let $\mathcal{E}$ be the locally free subsheaf of $f_*\omega_X$ generically generated by its global sections. Since D is mapped onto a rational curves of degree δ, the restriction map $H^0(K_X)\to H^0(K_D)$ is of rank $\leq\delta+1$. Hence $r=\mathrm{rk}(\mathcal{E})\leq\delta+1$. Put $d=M(\mathcal{E})D$ as before. Let $\phi : X\to P(\mathcal{E})$ be, as in 1.2, the rational map associated with $f^*\mathcal{E}\to\omega_X$. Then, by the choice of $\mathcal{E}$, the canonical map Φ_{K_X} is a composite of ϕ and the rational map of $P(\mathcal{E})$ induced by $H(\mathcal{E})$ which we denote by Φ_H.

Lemma 4.2. *Assume that the canonical image is ruled by rational curves of degree δ.*

(1) *d_{can} is a multiple of d/δ. If Φ_H separates fibers of $P(\mathcal{E})\to B$, then $d=d_{can}\delta$. If d_{can} is a prime number, then $d=d_{can}\delta$.*

(2) *If $g=r$, then f is of hyperelliptic type, $d=2\delta$ and d_{can} is even.*

Proof. (1) : Since the image of D under the canonical map is a rational curve of degree δ, d is a multiple of δ, and d/δ equals the degree of $\Phi_{K_X}\mid_D$, hence, ϕ is of degree d/δ onto its image.

(2) : Since $\mathrm{rk}(\mathscr{E})=g$, the restriction map $H^0(K_X)\to H^0(K_D)$ is surjective. By the assumption, it follows that D is mapped onto a rational curve via its canonical map. Hence D is a hyperelliptic curve. By what we saw above, ϕ is of degree 2 onto the image. Hence d_{can} must be even. Q. E. D.

Note that S has no pencil of hyperelliptic curves if d_{can} is odd. Hence Theorem 1.6, Lemma 1.7 and Lemma 4.2 give us the following generalization of [16, Theorem 3].

Theorem 4.3. *Assume that Σ is ruled by rational curves of degree δ. Assume further that $g>\delta+1$ or d_{can} is odd. Then $q\leq\delta+2$. If $q=\delta+2$, then $b=0$ and $g\geq 2\delta+3$. If d_{can} is odd and $q=\delta+2$, the Albanese image of S is a surface.*

Lemma 4.4. *Suppose that $b>1$ and $g=\delta+1$.*

(1) *Assume that $\delta=1$. Then d_{can} is an even integer not exceeding 10. If $d_{can}=10$, then $b=q=2$, $p_g=3$. If $d_{can}=8$, then $(b, q, p_g)=(2, 2, 3)$, $(2, 3, 3)$ or $(3, 3, 4)$. If $d_{can}=6$, then $(b, q, p_g)=(2, 2, 3)$, $(2, 2, 4)$, $(2, 3, 3)$, $(3, 3, 4)$, $(3, 3, 5)$ or $(4, 4, 6)$.*

(2) *If $\delta=2$ and $d_{can}=6$, then $(b, q, p_g)=(2, 2, 4)$, $(2, 2, 6)$, $(3, 3, 6)$ or $(4, 4, 9)$.*

Proof. We can assume that $\mathscr{E}=f_*\omega_X$. Put $H=H(f_*\omega_X)$. Since $\deg f_*\omega_{X/B}\geq 0$, we have

$$p_g\geq q+\delta(b-1)-1 \tag{4.1}$$

(1) : Though this is essentially contained in [13, p. 74], we give a proof for the sake of completeness. Put $d_{can}=2m$. Then Φ_H is a map of degree m onto the image Σ. Hence $H^2\geq m\deg\Sigma$. Since $H^2=\deg f_*\omega_X=\chi+3(b-1)$ and $\deg\Sigma\geq p_g-2$, we get

$$(m-1)p_g\leq 3b-q+2m-2. \tag{4.2}$$

From (4.1) and (4.2), we get $mq+(m-3)b\leq 4m-4$. If $q\geq 3$, then we have $m\leq 4$, since $b\geq 2$. Assume that $q=b=2$. Since $p_g\geq 3$, it follows from (4.2) that $4=3b-q\geq m-1$. Hence we get $m\leq 5$. The rest follow from an easy calculation.

(2) : Let V be the image of $\phi: X\to P(f_*\omega_X)$. Then V is numerically equivalent to $2H-\nu F$ with an integer ν. Since V is a relative hyperquadric of rank 3, one can

easily show $3v \leq 2\deg(f_*\omega_X)$ (see, e. g., [6]). Since H induces a map of degree 3, we have $H^2(2H - vF) \geq 3\deg\Sigma$, that is, $2\deg(f_*\omega_X) - v \geq 3\deg\Sigma$. Hence $\deg(f_*\omega_X) \geq (9/4)\deg\Sigma$. On the other hand, since Σ is not ruled by straight lines, Lemma 4.1 gives us $\deg\Sigma \geq (4/3)(p_g - 1 - 9/4)$. Therefore, $\deg(f_*\omega_X) \geq 3p_g - 9$. Since $\deg(f_*\omega_X) = \chi + 4(b-1)$, we have

$$2p_g \leq 4b - q + 6. \tag{4.3}$$

It follows from (4.1) and (4.3) that $q \leq 4$. Furthermore, since $p_g \geq 4$, we get

$$\begin{aligned}
(b, q) &= (2, 2) : 4 \leq p_g \leq 6 \\
(b, q) &= (2, 3) : p_g = 4, 5 \\
(b, q) &= (2, 4) : p_g = 5 \\
(b, q) &= (3, 3) : p_g = 6, 7 \\
(b, q) &= (3, 4) : p_g = 7 \\
(b, q) &= (4, 4) : p_g = 9.
\end{aligned}$$

It is known that surfaces with degree $p_g - 2$ in $P^{p_g - 1}$ is ruled by straight lines unless it is the Veronese surface, $p_g = 6$. Hence, if $p_g \neq 6$, we can assume that $\deg\Sigma \geq p_g - 1$. Since $\deg f_*\omega_X \geq (9/4)(p_g - 1)$, we have

$$\deg f_*\omega_X \geq \begin{cases} 7, & \text{if } p_g = 4, \\ 9, & \text{if } p_g = 5, \\ 14, & \text{if } p_g = 7. \end{cases}$$

Hence we can exclude several cases and get (2). Q. E. D.

In [14, Theorem 5], it is shown that there is a bound of q, g when $d_{can} \geq 5$. Now we can give a bound on q.

Theorem 4.5. *Let S be a surface of general type whose canonical map is a rational map of degree $d_{can} > 4$ onto its image.*

(1) *If $d_{can} \geq 7$, then $q \leq 3$ except possibly when $d_{can} = 7$, $p_g = 10$, $q = 4$, $K^2 = 63$ and Σ is P^2 embedded into P^9 by $|\mathcal{O}(3)|$.*

(2) *If $d_{can} = 6$, then $q \leq 5$.*

(3) *If $d_{can} = 5$, then $q \leq 12$, and $q \neq 12$ when $p_g > 136$.*

Proof. (1) : Assume that $q \geq 4$. Miyaoka-Yau's inequality gives us

$$\deg\Sigma \leq K^2/d_{can} \leq 9\chi/d_{can} \leq (9/d_{can})(p_g - 3).$$

Hence Lemma 4.1 implies that Σ is ruled by lines unless we are in the case excepted in (1). But then, Theorem 4.3 and Lemma 4.4 give us $q \leq 3$, a contradiction.

(2) : Assume that $q \geq 6$. By the same reasoning as above, Lemma 4.1 implies that Σ is ruled by rational curves of degree $\delta \leq 2$. In this case, however, Theorem 4.3 and Lemma 4.4 give us $q \leq 4$, a contradiction.

(3) : Assume that $q \geq 13$. By the same reasoning as above, Lemma 4.1 implies that Σ is ruled by rational curves of degree $\delta \leq 8$. But, Theorem 4.3 shows $q \leq 10$ contradicting our initial assumption. Quite similarly, assuming $q=12$ and $p_g > 136$, we can show that Σ is ruled by rational curves of degree $\delta \leq 9$. But Theorem 4.3 tells us $q \leq 11$. Q. E. D.

Remark 4.6. In the above theorem, (1) and (2) respectively can weaken the assumption on p_g in [16, p. 602, Corollary] and [11, Theorem 3].

As for g, we can show, for example, the following :

Proposition 4.7. *Let the notation and assumption be as above.*

(1) *If $d_{can}=6$ and $p_g > 190$, then $g \leq 16$.*

(2) *If $d_{can}=5$ and $p_g > 1324$, then $g \leq 44$.*

Proof. We show only (2), because (1) can be treated similarly if we note that $d=6\delta$ holds when p_g is large enough by Corollary 2.3 and Lemma 4.2.

If $p_g > 1324$, then

$$\deg \Sigma \leq \frac{9}{5}(p_g+1) < \frac{2(9+1)}{9+2}\left(p_g - 1 - \frac{9}{8}(9+1)\right).$$

Hence, by Lemma 4.1, Σ is ruled by rational curves of degree $\delta \leq 9$. We assume $g \geq 45$ and show that this leads us to a contradiction. By Theorem 1.6, we can suppose $b \leq 1$. By Lemma 4.2, we have $d=5\delta$. Since $5\delta \leq 45 < 2g-10 \leq 2g-\delta-1$, it follows from Lemma 2.6 (and Lemma 2.4 when $\mathcal{E}$ is semi-stable) that $K^2 \geq (1584/169)(p_g-28)$. However, since $p_g > 728$, this contradicts Miyaoka-Yau's inequality $K^2 \leq 9(p_g+1)$. Hence $g \leq 44$. Q. E. D.

References

[1] Beauville, A., L'application canonique pour les surfaces de type général, *Invent. Math.*, **55** (1979), 121–140.

[2] ————, L'inegalite $p_g \geq 2q-4$ pour les surfaces de type général, *Bull. Soc. Math. France*, **110** (1982), 343–346.

[3] Fujita, T., On Kähler fiber spaces over curves, *J. Math. Soc. Japan*, **30** (1978), 779–794.

[4] Harder, G. and Narashimhan, M. S., On the cohomology groups of moduli spaces of vector bundles on curves, *Math. Ann.*, **212** (1975), 215–248.

[5] Horikawa, E., On deformations of quintic surfaces, *Invent. Math.*, **31** (1975), 43–85.

[6] Konno, K., Non-hyperelliptic fibrations of small genus and certain irregular canonical surfaces, *Ann. Sc. Norm. Sup. Pisa. Ser. N,* **XX** (1993), 575–595.

[7] Matsusaka, S., Some numerical invariants of hyperelliptic fibrations, *J. Math. Kyoto Univ.,* **30** (1990), 33–57.

[8] Mukai, S. and Sakai, F., Maximal subbundles of vector bundles on a curve, *Manuscripta Math.,* **52** (1985), 251–256.

[9] Nakayama, N., Zariski-decomposition problem for pseudo-effective divisors, In : *Proceedings of the Meeting and the workshop "Algebraic Geometry and Hodge Theory",* vol. I, Hokkaido Univ. Technical report series in Math. no. 16 (1990), 189–217.

[10] Reid, M., Surfaces of small degree, *Math. Ann.,* **275** (1986), 71–80.

[11] Sun, X. T., Algebraic surfaces whose canonical image has a pencil of rational curves of degree two, *Math. Z.,* **209** (1992), 67–74.

[12] Xiao, G., L'irrégularité des surfaces de type général dont le systéme canonique est composé d'un pinceau, *Compositio Math.,* **56** (1985), 251–257.

[13] ————, Surfaces fibrées en courbes de genre deux, *Lec. Notes in Math.* **1137**, Berlin, Heidelberg, New York, Springer, 1985.

[14] ————, Algebraic surfaces with high canonical degree, *Math. Ann.,* **274** (1986), 473–483.

[15] ————, Fibred algebraic surfaces with low slope, *Math. Ann.,* **276** (1987), 449–466.

[16] ————, Irregularity of surfaces with a linear pencil, *Duke Math. J.,* **55** (1987), 597–602.