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Rings of Fractions of $B(H)$

By

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§ 1. Introduction

In this paper we discuss the following question : What are rings of fractions of $B(H)$, the algebra of all bounded linear operators on a separable, infinite dimensional, Hilbert space H ? We recall the definition of a ring of fractions of a (generally non-commutative) ring according to [4].

Definition. A subset S of a ring A with a unit 1 is called a *(right) denominator set* if S satisfies the following conditions :

(S0) If $s, t \in S$, then $st \in S$, and $1 \in S$.
(S1) If $s \in S$ and $a \in A$, then there exist $t \in S$ and $b \in A$ such that $sb = at$.
(S2) If $sa = 0$ with $s \in S$, then $at = 0$ for some $t \in S$.
(S3) S does not contain 0. (to avoid triviality).

Definition. The *ring* $A[S^{-1}]$ *of fractions* of a ring A with respect to a (right) denominator set S is defined by $A[S^{-1}] = (A \times S)/\sim$, where $\sim$ is the equivalence relation on $A \times S$ defined as $(a, s) \sim (b, t)$ if there exist $c, d \in A$ such that $ac = bd$ and $sc = td \in S$. We define addition and multiplication of $(a, s)^\sim$, $(b, t)^\sim \in (A \times S)/\sim$ in the obvious way :

$$(a, s)^\sim + (b, t)^\sim = (ac + bd, u)^\sim \text{ for some } c \in A,\ u \text{ and } d \in S \text{ with } u = sc = td,$$

$$(a, s)^\sim \cdot (b, t)^\sim = (ac, tu)^\sim \text{ for some } c \in A \text{ and } u \in S \text{ with } sc = bu.$$

Moreover if A has a scalar (complex number) multiple, then also does $A[S^{-1}]$. Then $\varphi(a) = (a, 1)^\sim$ defines a homomorphism $\varphi : A \to (A \times S)/\sim = A[S^{-1}]$.

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Our main theorem asserts that any ring of fractions $B(H)[S^{-1}]$ is isomorphic to $B(H)$ or the quotient ring $B(H)/J$ of $B(H)$ by the ideal J of finite rank operators. The next problem is the existence of such a denominator set S. It clear that $B(H)[S^{-1}]=B(H)$ if we take $S=\{1\}$. We shall show that there exist at least countably infinite many different denominator sets S such that $B(H)[S^{-1}]$ are isomorphic to $B(H)/J$.

§ 2. Main Theorem

An operator $x\in B(H)$ is a Fredholm operator if ran x is closed, dim ker x is finite and dim ker x^* is finite, where ran x is the range of x and ker x is the kernel of x. The collection of Fredholm operators is denoted by F. The ind is the function from F to the integers Z defined by ind $x=\dim\ker x-\dim\ker x^*$. This function enjoys the following property : For $x, y\in F$, ind $xy=$ ind $x+$ ind y, ind $x^*=-$ ind x, ind $1=0$. Put $F_0=\{x\in F \mid \text{ind } x=0\}$. Then F and F_0 satisfy (S0). Moreover F and F_0 are invariant under compact perturbations ([1]). If x and $y\in B(H)$ satisfy $xyx=x$, $yxy=y$, $(xy)^*=xy$ and $(yx)^*=yx$, then y is called a Moore-Penrose inverse of x and y is denoted by $x^\dagger$. A Moore-Penrose inverse $x^\dagger$ does not always exist but it is unique if it exists. It is known that $x^\dagger$ exists if and only if ran x is closed ([3]). In particular if x is in F, then x has $x^\dagger$.

We need the following Theorem in [2 ; Theorem 3. 6] :

Theorem F-W. *Let S be in $B(H)$. If* ran s *is not closed, then there exists a unitary $u\in B(H)$ such that* ran $s\cap$ ran $us=\{0\}$.

We shall show that a denominator is automatically a Fredholm operator.

Theorem 1. *If a subset $S\subset B(H)$ is a denominator set of $B(H)$, then S is contained in the set F of Fredholm operators.*

Proof. Let $s\in S$. Assume that ran s is not closed. Then by Theorem F-W, there exists a unitary u such that ran $s\cap$ ran $us=\{0\}$. The condition (S1) implies that there exist $t\in S$ and $b\in B(H)$ such that $sb=(us)t$. Then

$$\text{ran } ust=\text{ran } sb=\text{ran } sb\cap\text{ran } ust\subset\text{ran } s\cap\text{ran } us=\{0\}.$$

Therefore $ust=0$. Then S contains $st=0$. This contradicts to (S3). Hence ran s is closed. Next assume that dim ker $s^*=+\infty$. Then there exists a unitary u such that ran $u\cap$ ran $us=\{0\}$, since dim (ran $s)^\perp=$ dim ker $s^*=+\infty$. By the same argument of the proceeding paragraph, S contains 0. This is a contradiction. Therefore dim ker $s^*<+\infty$. Next we shall show that dim ker $s<+\infty$. Since ran s

is closed, $s^\dagger$ exists. Put $a=1-s^\dagger s$, then $sa=0$. By (S2) there exists $t\in S$ such that $at=0$. Since $a=a^*$, $t^*a=0$, that is, ran $a\subset$ ker t^*. Then dim ran $a\leqq$ dim ker $t^*<+\infty$, because $t\in S$. Thus dim ker $s=$ dim ran $a<+\infty$. Therefore $s\in S$ is a Fredholm operator.

Consider the canonical homomorphism $\varphi: B(H)\to B(H)[S^{-1}]$ defined by $\varphi(x)=(x, 1)^\sim$.

Lemma 2. *The canonical map* $\varphi: B(H)\to B(H)[S^{-1}]$ *is onto.*

Proof. Take $(a, s)^\sim\in B(H)[S^{-1}]$. Then $s^\dagger$ exists by Theorem 1. Put $z=1-s^\dagger s$. Since $sz=0$, there exists $c\in S$ such that $zc=0$ by (S2). Then $c=s^\dagger sc$. Put $x=as^\dagger$ and $d=sc$. Then

$$ac=as^\dagger sc=as^\dagger d\in B(H) \quad \text{and} \quad sc=ld\in S.$$

This shows that $(a, s)\sim(as^\dagger, 1)$. Then $\varphi(x)=(as^\dagger, 1)^\sim=(a, s)^\sim$. Thus φ is onto.

The following main theorem gives the possible rings of fractions of $B(H)$ completely :

Theorem 3. *Let S be a denominator set of* $B(H)$. *If S contains a non-invertible operator, then the ring* $B(H)[S^{-1}]$ *of fractions is isomorphic to the quotient ring* $B(H)/J$ *of* $B(H)$ *by the ideal J of finite rank operators. If S does not, then* $B(H)[S^{-1}]$ *is isomorphic to* $B(H)$.

Proof. By Lemma 2, $B(H)[S^{-1}]$ is isomorphic to $B(H)/\ker\varphi$. We note that

$$(*) \qquad \ker\varphi=\{x\in B(H) \mid xc=0 \text{ for some } c\in S\}.$$

If S does not contain non-invertible elements, then $\ker\varphi=\{0\}$, so $B(H)[S^{-1}]$ is isomorphic to $B(H)$. Now suppose that S contains a non-invertible operator s. Then $s^\dagger s\neq 1$ or $ss^\dagger\neq 1$. If $ss^\dagger\neq 1$, then $x=1-ss^\dagger\neq 0$ and $x\in\ker\varphi$, because $xs=s-ss^\dagger s=0$ and $s\in S$. If $s^\dagger s\neq 1$, put $x=1-s^\dagger s$. Since $sx=0$, $xt=0$ for some $t\in S$ by (S2). Thus $x\neq 0$ and $x\in\ker\varphi$. In any case we have that $\ker\varphi\neq\{0\}$. Next we shall show that $\ker\varphi\subset J$. Let $x\in\ker\varphi$. By $(*)$ there exists $c\in S$ such that $xc=0$. Since $c^*x^*=0$, ran $x^*\subset$ ker c^*. By Theorem 1, c is a Fredholm operator and dim ker $c^*<+\infty$. Hence x^* is a finite rank operator, so $x\in J$. Since J is a non-trivial minimal two-sided ideal of $B(H)$, $\ker\varphi=J$. Therefore if S contains a non-invertible element, then $B(H)[S^{-1}]$ is isomorphic to $B(H)/J$.

§ 3. Examples of Denominator Sets

In this section we shall give some examples of a denominator set S such that $B(H)[S^{-1}]$ is ismorphic to $B(H)/J$. In fact there exist at least countably infinite many denominator sets with this property, although we have not yet determined all of them.

Theorem 4. *If S is a semigroup such that $F_0 \subset S \subset F$, then S is a denominator set. In particular F_0 and F are denominator sets.*

Proof. It is clear that S satisfies (S0) and (S3). We shall show that S satisfies (S1). Take $s \in S$ and $a \in B(H)$. Since $s \in F$, $s^{\dagger}$ exists. Then $1-ss^{\dagger} \in J$, because dim ran $(1-ss^{\dagger}) = \dim \ker s^* < +\infty$. Put $c=(1-ss^{\dagger})a$. Then c is also in J, so ran c is closed and $c^{\dagger}$ exists. Then $c^{\dagger}c$ is in J. Put $t=1-c^{\dagger}c$. Since t is a compact perturbation of 1, $t \in F_0 \subset S$. Put $b=s^{\dagger}at$. Then

$$at-sb=(1-ss^{\dagger})at=(1-ss^{\dagger})a(1-c^{\dagger}c)=c(1-c^{\dagger}c)=0.$$

So $sb=at$. Thus S satisfies (S1). Next we shall show that S satisfies (S2). Take $s \in S$ and $s \in B(H)$ such that $sa=0$. Since ran $a \subset \ker s$, a is in J. Consider a polar decomposition $a=u|a|$. We may assume that u is a unitary. Put $t=u^*s^*s$. Then ind $t=$ ind u^*- ind $s+$ ind $s=0$. Hence $t \in F_0 \subset S$. And $at=u|a|u^*s^*s=ua^*s^*s=u(sa)^*s=0$. Thus S satisfies (S2).

Finally we shall give two kinds of examples of denominator sets of $B(H)$ which do not contain F_0. Let K be a separable, infinite dimensional, Hilbert space and n be a positive integer. Put $H=K \oplus \cdots \oplus K$ (n times). Then $B(H)$ can be identified with the set $M_n(B(K))$ of $n \times n$ matrices whose entries are in $B(K)$. Let S be a denominator set of $B(K)$. Define S_n and $S^n \subset B(H)$ by

$$S_n=\left\{\begin{pmatrix} s & & 0 \\ & s & \\ & & \ddots \\ 0 & & & s\end{pmatrix} \in B(H) \;\middle|\; s \in S\right\}$$

$$S^n=\left\{\begin{pmatrix} s_1 & & 0 \\ & s_2 & \\ & & \ddots \\ 0 & & & s_n\end{pmatrix} \in B(H) \;\middle|\; s_1, \cdots, s_n \in S\right\}.$$

By [4; page 61, Exercises 4], S_n is a denominator set of $B(H)$. Similarly we can show that S^n is also a denominator set of $B(H)$. Therefore we get the following :

Theorem 5. *There exist countably infinite many denominator sets S of $B(H)$ such that $B(H)[S^{-1}]$ are isomorphic to $B(H)/J$.*

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