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Presentations of *AF* Algebras Associated to *T*-Graphs

By

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§1. Introduction

An *AF* algebra is an inductive limit of finite dimensional C^*-algebras F_n, and an embedding of F_n in F_{n+1} is represented by a graph whose edges are the (multiplicity of the) embeddings of the simple factors of F_n in those of F_{n+1} (see [1, 3]). Thus from a graph Γ, with distinguished vertex $*$, we can build up a unital *AF* algebra $A(\Gamma)$, by iteration of embeddings represented by Γ, but starting with the complex number $\mathbf{C}$ at $*$. The space of semi-infinite paths $\hat{\Gamma}$ in Γ beginning at $*$ will be the graph of a Bratteli diagram for $A(\Gamma)$.

Suppose Γ is locally finite, and let $\Gamma^{(0)}$ and $\Gamma^{(1)}$ denote the vertices and edges respectively of Γ, and Δ the incidence matrix of Γ. We write $\|\Gamma\| = \|\Delta\|$. A Markov trace Tr on $A(\Gamma)$ is given by a solution $(\phi_v : v \in \Gamma^{(0)}) > 0,\ y > 0$ to

$$y\phi_v = \sum_{w \in \Gamma^{(0)}} \Delta(v,w)\phi_w . \tag{1.1}$$

Using a solution to (1.1) projections $\{e_i : i \in \mathbf{N}\}$ in $A(\Gamma)$ can be defined which satisfy the relations:

$$e_n e_{n\pm1} e_n = \tau e_n,\ e_n e_m = e_m e_n,\ |m-n| > 1 \tag{1.2}$$

$$\mathrm{Tr}(ae_m) = \tau \mathrm{Tr}(a),\ a \in C^*(1, e_1, \ldots, e_{m-1}) \tag{1.3}$$

where $\tau = y^{-2}$, [15,12,11,13]. Let $A(\tau)$ be the C^*-algebra generated by projections $\{e_i : i \in \mathbf{N}\}$ satisfying relations (1.2) and (1.3) for some $\tau \in \mathbf{R}$, and trace Tr on $A(\tau)$. Then we have a pair of *AF* algebras $A(\tau) \subseteq A(\Gamma)$. Moreover we know by [10] that

$$1/\tau \in \{4\cos^2(\pi/l) : l = 3,4,\ldots\} \cup [4,\infty) \tag{1.4}$$

$$A(\tau) \cong A(A_{l-1}) \text{ if } 1/\tau = 4\cos^2(\pi/l),\quad l = 3,4,5,\ldots \tag{1.5}$$

$$A(\tau) \cong A(A_\infty) \text{ if } 1/\tau \geq 4, \tag{1.6}$$

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where A_m, $3 \leq m \leq \infty$ denote the usual Dynkin diagrams (see Figure 4).

In this paper we give an algebraic characterisation of $A(T_{p,2,r})$ for $1 \leq r \leq \infty$ (see Figure 1, 2, 3 for these graphs). Let $e_{\bar{p}}, e_1, e_2, \ldots$ be a sequence of projections satisfying relations (1.2) and additionally:

$$e_{\bar{p}} e_n = e_n e_{\bar{p}} \quad n = 1, 2, \ldots, p-1, p+1, p+2, \ldots \quad (\text{and } e_{\bar{p}} e_1 = 0, \text{ if } p = 2) \tag{1.7}$$

$$e_{\bar{p}} e_p e_{\bar{p}} = \tau e_{\bar{p}} \tag{1.8}$$

$$e_p e_{\bar{p}} e_p = \tau(1 - e_1 \vee \ldots \vee e_{p-2}) e_p . \tag{1.9}$$

Then we show in Theorem 3.1 that $A(\tau, p) = C^*(1, e_{\bar{p}}, e_1, e_2, \ldots)$ is non-trival only when

$$\beta = 1/\sqrt{\tau} \in \{\|T_{p,2,r}\| : r \geq 1\} \cup [\|T_{p,2,\infty}\|, \infty). \tag{1.10}$$

In which case there exists a surjective *-homomorphism

$$A(T_{p,2,r}) \oplus \mathbf{C}(1 - e_1 \vee \ldots \vee e_{p+r-2} \vee e_{\bar{p}}) \to A(\tau, p), \quad \text{when } \beta = \|T_{p,2,r}\|, \ \text{r} < \infty. \tag{1.11}$$

$$A(T_{p,2,\infty}) \to A(\tau, p), \quad \text{when } \beta \geq \|T_{p,2,\infty}\|. \tag{1.12}$$

If $r < \infty$, i.e. $\beta < \|T_{p,2,\infty}\|$, then this map (1.11) is automatically an isomorphism as $A(T_{p,2,r})$ is simple. If there exists a Markov trace on $A(\tau, p)$ (cf. (1.3), or see §2 and the statement of Theorem 3.1 for a precise definition) then in all cases (1.11) and (1.12) we have an isomorphism between $A(T_{p,2,r})$, $1 \leq r \leq \infty$ and $A(\tau, p)$; moreover in the case $r < \infty$, $\beta = \|T_{p,2,r}\|$, we have

$$1 = e_1 \vee \ldots \vee e_{p+r-2} \vee e_{\bar{p}} . \tag{1.13}$$

We give a constructive proof of the existence of the above homomorphisms (1.11)–(1.12) constructing matrix units in $A(\tau, p)$ labelled by paths in the graph $\hat{T}_{p,2,r}$. Thus even in the case of $p = 1$, our proof does not reduce to that of Jones for the A_n-series. Indeed we prove a stronger result in that the existence of the homomorphism in (1.11) and (1.12) does not depend on the existance of a Markov trace. Moreover we show that the homomorphism in (1.11) is an isomorphism even without the assumption of a Markov trace. It is also striking to note that by throwing in the extra relations (1.7)–(1.9) to those of Temperley-Lieb and Jones (1.2), we find a rigidity *above* index four. Note also that our construction of matrix units is different from that proposed by [14] in the A_n-case. This result was announced in [4].

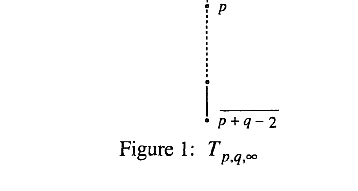

Figure 1: $T_{p,q,\infty}$

Thus $T_{2,1,\infty} = A_\infty$, $T_{2,2,\infty} = D_\infty$, $T_{3,2,\infty} = E_\infty$, as in Figure 2.

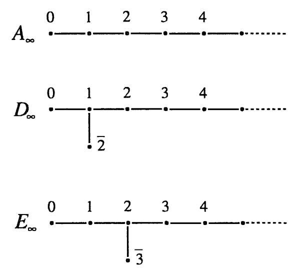

Figure 2

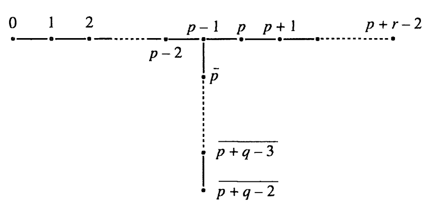

Figure 3: $T_{p,q,r}$

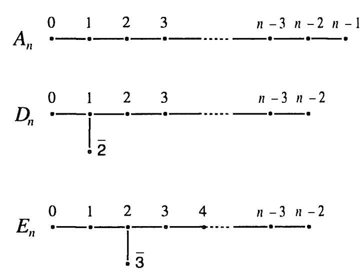

Figure 4

§2. Preliminaries

Let Γ be a graph with distinguished vertex $*$. We assume throughout that Γ is unoriented, connected and locally finite i.e. the number of edges adjacent to a vertex is finite. We say that $\gamma \in \Gamma^{(0)}$ is even (respectively odd) if it can be joined to $*$ by an even (respectively odd) number of vertices. On the Cantor set $\hat{\Gamma}$ of sequences $(x_\upsilon)_{\upsilon=0}^{\infty}$, with

$$(x_\upsilon, x_{\upsilon+1}) \in \Gamma^{(1)} \tag{2.1}$$

$$x_0 = * \tag{2.2}$$

consider the equivalence relation $\sim$ with countable orbits given by $(x_\upsilon) \sim (y_\upsilon)$ if and only if $x_\upsilon = y_\upsilon$ except for finitely many υ. Let $A = A^\Gamma = A(\Gamma)$ be the corresponding

C^*-algebra, with Bratteli diagram identified with $\hat{\Gamma}$. For each finite subset Λ of $\mathbf{N}$, let $A(\Lambda) = A^{\Gamma}(\Lambda)$ be the C^*-algebra [2,6,7] generated by the following partial isometries $f_{\gamma,\gamma'}$: Both γ and γ' are elements of

$$\mathcal{G}^{\Lambda} = \{\gamma \in (\Gamma^{(0)})^{\Lambda'} : (\gamma_i, \gamma_{i+1}) \in \Gamma^{(1)}\}, \tag{2.3}$$

where $\Lambda' = \{i : d(i,\Lambda) \le 1\}$, and $\gamma(j) = \gamma'(j)$ if $j \notin \Lambda$. The partial isometry $f_{\gamma,\gamma'}$ has as initial domain the cylinder set $Z_\gamma = \{(x_\upsilon) : x_\upsilon = \gamma_\upsilon,\ \upsilon \in \Lambda'\}$, and it replaces any such (x_υ) in Z_γ, by (y_υ) in the cylinder set $Z_{\gamma'}$, where $y_\upsilon = x_\upsilon$, $\upsilon \notin \Lambda$, $y_\upsilon = \gamma'_\upsilon$, $\upsilon \in \Lambda$. Then we let $A(\Gamma)_n = A^{\Gamma}[0,n]$, denote the algebra at the n$^{\text{th}}$ level of the Bratteli diagram $\hat{\Gamma}$. Let Δ be the incidence matrix of the graph. Note that Δ is symmetric. If $v \in \Gamma^{(0)}$, let $t(v) = \{w \in \Gamma^{(0)} : (w,v) \in \Gamma^{(1)}\}$. Let $(\phi_v : v \in \Gamma^{(0)}) > 0$ be a solution to

$$y\phi_v = \sum_{w \in \Gamma^{(0)}} \Delta(v,w)\phi_w, \tag{2.4}$$

for some positive number y. Then $X(v) = (\sqrt{(\phi_w / \phi_v)}) : w \in t(v))$ defines a unit vector in $\ell^2(t(v))$. If $k \le l \in \mathbf{N}$, $s,t \in \Gamma^{(0)}$, let $\mathcal{G}^{[k,l]}_{s,t} = \{\gamma \in \mathcal{G}^{[k,l]} :\ \gamma_{k-1} = s,\ \gamma_{l+1} = t\}$. Then for each $n \in \mathbf{N}$, let

$$e_n = \sum_v X(v)(\gamma_n) X(v)(\gamma'_n) f_{\gamma,\gamma'} \tag{2.5}$$

where the summation is over all $v \in \Gamma^{(0)}$, and $\gamma, \gamma' \in \mathcal{G}^{\{n\}}_{v,v}$. Then e_n is a projection, being identified with a sum of the rank one projections on $X(v)$ in $\mathrm{End}(\ell^2(t(v)))$. The family $\{e_n : n = 0,1,2,\ldots\}$ satisfy the relations

$$e_n e_{n\pm1} e_n = \tau e_n,\ \ e_n e_m = e_m e_n,\ \ |m-n| \ge 2 \tag{2.6}$$

where $\tau = y^{-2}$ [11,13].

We define a trace Tr, called a Markov trace, on a $A(\Gamma)$ to be the unique state on $A(\Gamma)$ such that

$$\mathrm{Tr}\, f_{\gamma,\gamma'} = 0 \ \text{ if } \ \gamma \ne \gamma' \tag{2.7}$$

$$\mathrm{Tr}\, f_{\gamma,\gamma} = y^{-(l-k+2)} v_\delta v_\beta \ \text{ if } \ \gamma \in \mathcal{G}^{[k,l]}_{s,t}. \tag{2.8}$$

Then

$$\mathrm{Tr}\,(ae_m) = y^{-2}\mathrm{Tr}(a) \quad a \in A[0,m-1] \tag{2.9}$$

$$\mathrm{Tr}(e_m) = y^{-2}, \quad \mathrm{Tr}(1) = 1. \tag{2.10}$$

Note that if the graph Γ is finite and connected, then by the Perron Frobenius theory there is an unique normalised strictly positive solution to (2.5) and $y = \|\Delta\|$.

If Γ contains no cycle of odd length then there is a partition $\Gamma = \Gamma^{(0)}_+ \cup \Gamma^{(0)}_-$, with $\Gamma^{(0)}_+ \cap \Gamma^{(0)}_- = \phi$, such that there are no edges between two vertices in $\Gamma^{(0)}_+$ (respectively $\Gamma^{(0)}_-$). Such a graph is called bipartite. Then it is more convenient to describe $\hat{\Gamma}$ as follows [11]. There is a distance function $d: \Gamma^{(o)} \to \mathbf{N}$, where $d(v)$ is the number of edges in a minimal path from $*$ to v. Then we can identify

$$\hat{\Gamma}^{(0)} = \{(v, d(v) + 2k): v \in \Gamma^{(0)},\ k = 1, 2, \ldots\}$$

with distinguished vertex $(*,0)$, and where there are p edges between vertices (v, n) and (w,m) in $\hat{\Gamma}^{(0)}$ if and only if $|n - m| = 1$ and there are p edges between v and w in $\Gamma^{(0)}$. We identify Γ with the subgraph of $\hat{\Gamma}$, called the underlying graph, having vertices $\{(v, d(v)): v \in \Gamma^{(0)}\}$ and whose edges are those in $\hat{\Gamma}^{(0)}$ connecting these vertices. The distance function d on Γ extends to a distance function on $\hat{\Gamma}^{(0)}$ also denoted by d, where $d(v, m) = m$.

To construct matrix units in $A(\tau, p)$ we will need a certain family of rational functions associated with the graphs $T_{p,q,\infty}$. Here we give some properties of these functions that will be needed later (see [5] for more details).

If Δ is the incidence matrix of Γ, we will aim to find a family $\{\phi_v: v \in \Gamma^{(0)}\}$ of rational functions in an indeterminate x satisfying

$$x\phi_v = \sum_{w \in \Gamma^{(0)}} \Delta(v, w)\phi_w \tag{2.11}$$

$$\phi_* = 1. \tag{2.12}$$

Consider the graph $\Gamma = T_{p,q,\infty}$ with $p \geq q \geq 1$ and $* = 0$ (see Figure 1). Then functions $\{\phi_v\}$ satisfying (3.1) and (3.2) exist and are unique. They are

$$\begin{aligned}
&\phi_r = S_r, \quad 0 \leq r \leq p - 1 \\
&\phi_p = S_{p+q-1} / S_{q-1} \\
&\phi_{\bar{r}} = S_{p+q-2-r} S_{p-1} / S_{q-1}, \quad \text{if} \quad q \geq 2, \quad p \leq r \leq p + q - 2 \\
&\phi_r = x\phi_{r-1} - \phi_{r-2}, \quad r \geq p + 1,
\end{aligned} \tag{2.13}$$

where $S_n \in \mathbf{Z}[x]$ are Chebyshev polynomials of the second kind satisfying

$$S_r = xS_{r-1} - S_{r-2}, \quad S_0 = 1, \quad S_{-1} = 0. \tag{2.14}$$

Let $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ be the sequence of subgraphs of $T_{p,2,\infty}$ given by: $\Gamma_1 = A_{p+1}$, consisting of vertices $0, 1, 2, \ldots, p - 1, \bar{p}$, and all edges of $T_{p,2,\infty}$ joining these vertices. For $r \geq 2$, $\Gamma_r = T_{p,2,r}$, consisting of vertices $0, 1, 2, \ldots, p - 1, \bar{p}, p, \ldots, p + r - 2$ and all edges of $T_{p,2,\infty}$ joining these vertices (see Figures 3, 4).

Proposition 2.1 [5]. *Let $\{\phi_v\}$ be the family of rational functions associated to the graph $T_{p,2,\infty}$, $p \geq 2$ given by* (3.3). *The roots of ϕ_v are real, and if β_r, γ_r denote the largest, and second largest respectively, roots of ϕ_{p+r-1} for $r \geq 1$ then:*

(a) $\beta_r = \|\Gamma_r\|$,

(b) *the sequence $\{\beta_r\}$ is strictly increasing and converges to $\|T_{p,2,\infty}\|$,*

(c) $\gamma_{r+1} < \beta_r < \beta_{r+1}$, *for all* $r \geq 1$,

(d) *if* $\beta_r < \beta < \beta_{r+1}$, *then* $\phi_{p+r-1}(\beta) / \beta\phi_{p+r-2}(\beta) < 0$, *for all* $r \geq 1$,

(e) $\phi_v(\beta_r) > 0$ *for all* $v \in \Gamma_r^{(0)}$, $r \geq 1$.

Let $\{\phi_v\}$ be a family of rational functions associated to a graph Γ, satisfying (2.11) and (2.12). Then we define, for $v \in \Gamma^{(0)}$:

$$Q_v(t) = x^{-d(v)}\phi_v(x) \tag{2.15}$$

where $t = x^{-2}$. Then for $\Gamma = T_{p,q,\infty}$, $p \geq q \geq 2$

$$\begin{aligned} Q_r &= P_r \quad 0 \leq r \leq p-1 \\ Q_p &= P_p - tP_{p-1}P_{q-2} / P_{q-1} = P_{p+q-1} / P_{q-1} \\ Q_{\bar{r}} &= t^{r+1-p}P_{p-1}P_{p+q-2-r} / P_{q-1}, \ (\text{if } q \geq 2) \ \ p \leq r \leq p+q-2 \\ Q_r &= Q_{r-1} - tQ_{r-2}, \ r \geq p+1 \end{aligned} \tag{2.16}$$

where $P_r \in \mathbf{Z}[t]$, $r = 0, 1, 2, \ldots\ldots$ are defined by

$$P_r(t) = x^{-r}S_r(x) \tag{2.17}$$

$t = x^{-2}$, and are the Jones' polynomials [10]:

$$P_r = P_{r-1} - tP_{r-2}, \ P_0 = 1, P_{-1} = 0. \tag{2.18}$$

Note that $Q_v(t) \in \mathbf{Z}[t]$ for all $v \in T_{p,q,\infty}^{(0)}$, if and only if $q = 2$, or $q \mid p$.

The vertices of the graph $\hat{T}_{p,q,\infty}$, are labelled as in Figure 5.

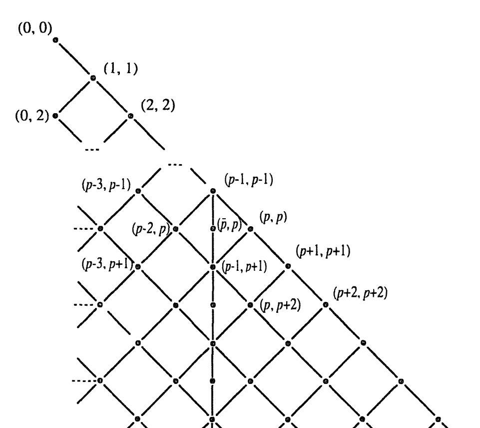

Figure 5: $\hat{T}_{p,2,\infty}$

We associate to each vertex (v,n) of $\hat{T}_{p,q,\infty}$, the polynomial

$$Q_{(v,n)}(t) = t^{(n-d(v))/2}Q_v(t) \tag{2.19}$$

where d is the distance function on $T_{p,q,\infty}$. Thus our notation is consistent with the embedding of $T_{p,q,\infty}$ in $\hat{T}_{p,q,\infty}$.

§3. An Algebraic Presentation and Matrix Units for $A(T_{p,2,r})$

Consider the graph $T_{p,2,r}$ as in Figures 1 and 3 where $2 \le p < \infty$, $2 \le r \le \infty$. We have already noted in Section 2 that $A(\tau) \subset A(T_{p,2,r})$ where $1/\tau = \|T_{p,2,r}\|^2$ if $r < \infty$, and $1/\tau \ge \|T_{p,2,\infty}\|^2$ otherwise. In the path algebra $A(T_{p,2,r})$, the projection e_n may be described as follows. In the notation of Section 2,

$$A[n-1,n+1] \supseteq \bigoplus_{v} \mathrm{End}\ell^2(t(v)) \tag{3.1}$$

where the summation is over all even (respectively odd) vertices $v \in T^{(0)}_{p,2,r}$ with $(v,n-1) \in \hat{T}^{(0)}_{p,2,r}$ when n is odd (respectively even). Three situation arise:

$$\mathrm{End}\ell^2(t(v)) = \mathbf{C} \text{ if } v=0, \text{ or } v=p+r-2, \text{ when } r<\infty, \text{ or } v=\overline{p}. \tag{3.2}$$

$$\mathrm{End}\ell^2(t(v)) = M_3(\mathbf{C}), \text{ if } v=p-1. \tag{3.3}$$

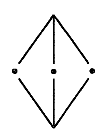

Figure 6

$$\mathrm{End}\ell^2(t(v)) = M_2(\mathbf{C}) \text{ othrewise.} \tag{3.4}$$

In the case (3.3) the matrix algebras $\mathrm{End}\ell^2(t(v))$ live on those portions of the Bratteli diagram shown in Figure 6. In the identification of (3.3) and (3.4), we will order paths with initial vertex (v,n-1), and final vertex (v,n+1) from left to right. In the first case (3.2), e_n will be 1 on these components and in the second and third cases will be the rank one projections in these components given by

$$\frac{1}{\beta\phi_{p-1}}\begin{bmatrix} \phi_{p-2} & (\phi_{p-2}\phi_{\overline{p}})^{\frac{1}{2}} & (\phi_{p-2}\phi_p)^{\frac{1}{2}} \\ (\phi_{\overline{p}}\phi_{p-2})^{\frac{1}{2}} & \phi_{\overline{p}} & (\phi_{\overline{p}}\phi_p)^{\frac{1}{2}} \\ (\phi_p\phi_{p-2})^{\frac{1}{2}} & (\phi_p\phi_{\overline{p}})^{\frac{1}{2}} & \phi_p \end{bmatrix} \tag{3.5}$$

$$\frac{1}{\beta\phi_v}\begin{bmatrix} \phi_{v-1} & (\phi_{v-1}\phi_{v+1})^{\frac{1}{2}} \\ (\phi_{v-1}\phi_{v+1})^{\frac{1}{2}} & \phi_{v+1} \end{bmatrix} \tag{3.6}$$

respectively. We now introduce a new projection, $e_{\overline{p}} \in A[p-1,p+1]$, which lives in $\mathrm{End}\ell^2(t(p-1))$ and is given by the rank one operator corresponding to the middle path, namely $(p-1,\overline{p},p-1)$ in $T_{p,2,r}$ or $((p-1,p-1),(\overline{p},p),(p-1,p+1))$ in $\hat{T}_{p,2,r}$:

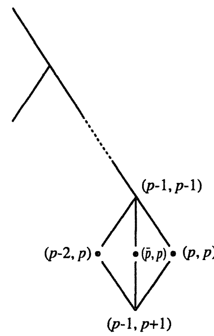

Figure 7

i.e. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in $\mathrm{End}(\ell^2(p-1))$. We observe the following facts:

(3.7) The projection $e_1e_3e_5 \ldots e_{2n+1}$ (a projection by (1.2)) corresponds to the projection $f_{\delta,\delta}$ given by the extreme left hand path δ as shown in Figure 8.

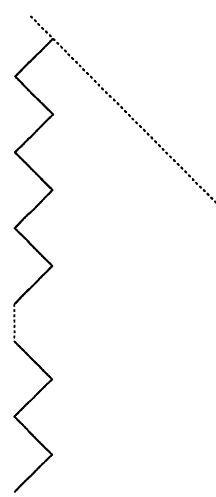

Figure 8

(3.8) The projection $f_n = 1 - e_1 \vee \ldots \vee e_{n-1}$ corresponds to the projection $f_{\eta,\eta}$ given by the extreme right hand path η as shown in Figure 9, for $n \leq p-1$.

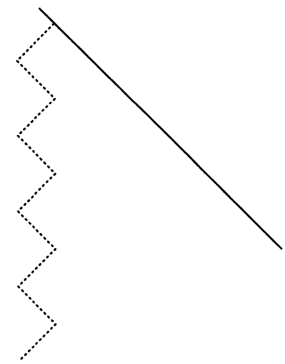

Figure 9

We know by Wenzl [16] see also [8] that if $e_1,\ldots,e_N$ is a sequence of projections satisfying (1.2) and if $s=(N+4)/2$, then either

$$\beta = 4\cos^2(\pi / q) \tag{3.9}$$

for some integer q with $3 \le q \le s$, or

$$\beta \ge 4\cos^2(\pi / s). \tag{3.10}$$

In which case

$$f_0 = f_1 = 1; \tag{3.11}$$

$$f_{\iota+1} = f_\iota - (\beta S_{\iota-1} / S_\iota) f_\iota e_\iota f_\iota \tag{3.12}$$

where $S_i = S_\iota(\beta)$, for $i = 1, 2, \ldots, N-2$, for details see below (3.30)–(3.31). We can then easily verify (3.8) from (3.11)–(3.12). Moreover, we can then deduce the following relations:

$$e_n e_{\bar{p}} = 0, \quad n = 1, 2, \ldots, p-1 \tag{3.13}$$

$$e_n e_{\bar{p}} = e_{\bar{p}} e_n, \; n = p+1, p+2, \ldots.. \tag{3.14}$$

$$e_{\bar{p}} e_p e_{\bar{p}} = \tau e_{\bar{p}} \tag{3.15}$$

$$e_p e_{\bar{p}} e_p = \tau(1 - e_1 \vee \ldots \vee e_{p-2}) e_p. \tag{3.16}$$

Conditions (3.13–3.16) together with the Temperley Lieb relation for $e_1, e_2, \ldots$ in the presence of a Markov trace serve to characterise $A(T_{p,2,r})$ $r = 2, 3, \ldots, \infty$.

Thoerem 3.1. *Let* $p \ge 2$, $\tau > 0$, *and let* $e_1, e_2, \ldots, e_{\bar{p}}$ *be a sequence of projections satisfying*

$$e_n e_m = e_m e_n, \quad m, n = 1, 2, \ldots, |m-n| \ge 2 \tag{3.17}$$

$$e_n e_{\bar{p}} = e_{\bar{p}} e_n, \; n \ne p \;\; (\textit{and} \;\; e_1 e_{\bar{p}} = 0 \;\; \textit{if} \;\; p = 2) \tag{3.18}$$

$$e_n e_{n\pm1} e_n = \tau e_n \tag{3.19}$$

$$e_{\bar{p}} e_p e_{\bar{p}} = \tau e_{\bar{p}} \tag{3.20}$$

$$e_p e_{\bar{p}} e_p = \tau(1 - e_1 \vee \ldots \vee e_{p-2}) e_p. \tag{3.21}$$

Let

$$A(\tau, p) = C^*(1, e_1, e_2, \ldots, e_{\bar{p}}). \tag{3.22}$$

Then $A(\tau, p)$ *is non-trivial if and only if*

$$\beta = 1/\sqrt{\tau} \in \{\|T_{p,2,r}\|: r \geq 1\} \cup [\|T_{p,2,\infty}\|, \infty) \tag{3.23a}$$

where $T_{p,2,1} = A_{p+1}$. *Moreover there exists surjective* $*$*-homomorphisms*

$$A(T_{p,2,r}) \oplus \mathbf{C}(1 - e_1 \vee \ldots \vee e_{p+r-2} \vee e_{\bar{p}}) \to A(\tau, p) \text{ when } \beta = \|T_{p,2,r}\|, \tag{3.23b}$$

and

$$A(T_{p,2,\infty}) \to A(\tau, p) \text{ when } \beta \geq \|T_{p,2,r}\|. \tag{3.23c}$$

If $r < \infty$, *i.e.* $\beta = \|T_{p,2,r}\| < \|T_{p,2,\infty}\|$, *then* (3.23b) *is automatically an isomorphism.*

Suppose there exists a trace tr *on* $A(\tau, p)$ *such that*

$$\operatorname{tr}(xe_n) = \tau \operatorname{tr} x, \quad x \in A(\tau, p)_n \tag{3.24}$$

where

$$A(\tau, p)_n = \begin{cases} C^*(1, e_1, e_2, \ldots, e_{n-1}) & n < p \\ C^*(1, e_1, e_2, \ldots, e_{n-1}, e_{\bar{p}}) & n \geq p. \end{cases} \tag{3.25}$$

Then

$$1 = e_1 \vee \ldots \vee e_{p+r-2} \vee e_{\bar{p}} \tag{3.26a}$$

and

$$A(\tau, p) \simeq \begin{cases} A(T_{p,2,r}), & \beta = \|T_{p,2,r}\| \quad 1 \leq r < \infty \\ A(T_{p,2,\infty}), & \beta \geq \|T_{p,2,\infty}\|. \end{cases} \tag{3.26b}$$

We will give a constructive proof of (3.26), obtaining expressions for matrix units in $A(\tau, p)_n$ under conditions (3.17)–(3.21). This yields a $*$-homomorphism from $A(T_{p,2,r})$ into $A(\tau, p)$ for appropriate $r \leq \infty$, depending on τ. This will be a $*$-isomorphism under the assumption of a Markov trace on $A(\tau, p)$ (3.24).

To describe the matrix units in $A(T_{p,2,r})$, it is convenient to label paths in the Bratteli diagram $\hat{T}_{p,2,\infty}$ by certain sequences of half-integers as follows. In the first place, if $\alpha, \beta \in \hat{T}^{(0)}_{p,2,\infty}$, are on level m, respectively n, where $m \leq n$, let Path (v,w) denote the paths of length $n - m$ from v to w in $\hat{T}_{p,2,\infty}$. For $\alpha = (v, m)$ labelled as in Figure 5, put $n = (m - d(v))/2$. Then if

$$I = \{0, 1, 2, 3, \ldots, p-2, \varepsilon, p-1, p, \ldots\} \tag{3.27}$$

where $\varepsilon = p - 2 + 1/2$ define

$$I_\alpha = \{i = (i_1, \ldots, i_n) \in I^n : i_n \leq d(v) - \delta_{v,\bar{p}}, i_{n-1} \leq i_n + 1, \ldots, i_1 \leq i_2 + 1\}. \quad (3.28)$$

Then we may identify the sets Path $(*, \alpha)$ and I_α as illustrated in Figure 10.

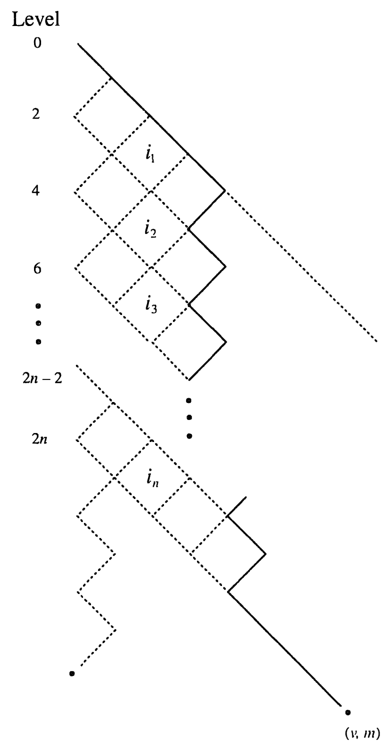

Figure 10

The numbers $i_1, i_2, \ldots, i_n$ correspond to the number of diamonds in the diagonal strip where:

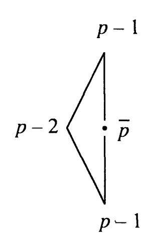

Figure 11

counts as half a diamond. For example when $p = 3$, $\alpha = (4, 10)$, $i = (3/2, 2, 3/2)$ in I_α corresponds to the path in Figure 12:

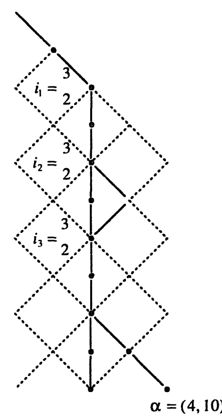

Figure 12

On $A(T_{p,2,r})$ we have an endomorphism obtained, essentially by shifting each vertex of a path down two levels, and then rejoining this path to $(*,0)$ via $(**,1)$. If $\alpha=(v,m)\in\hat{T}^{(0)}_{p,2,r}$, and $i\in \text{Path}(*,\alpha)$, i.e. $i=(i_1,i_2,\dots,i_n)$, where $n=(m-d(v))/2$, then put $i'=(0,i_1,\dots,i_n)\in \text{Path}(*,(v,m,+2))$, as in Figure 13.

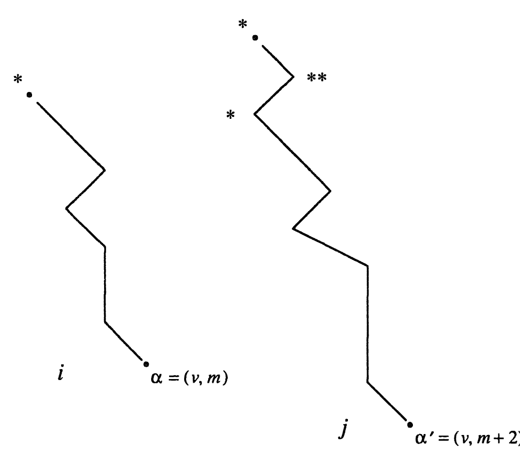

Figure 13

Then there exists an induced $*$-endomorphism of $A(T_{p,2,r})$ such that

$$\gamma(f_{i,j})=f_{i',j'}. \tag{3.29}$$

One can obtain a formula, inductively, for the projection g_v corresponding to the extreme right hand path in terms of $1, e_1, e_2, \dots, e_{\bar{p}}$. First take $g_0=1$, then suppose we have g_v for $1\le v\le p-1$. On level $v+1$ of $\hat{T}_{p,2,r}$, g_v splits into two paths, i.e. we have $g_v=g_{v+1}+i$, as shown in Figure 14.

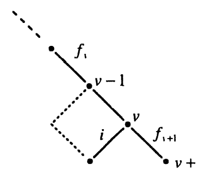

Figure 14

But the path i clearly corresponds to the projection $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in End $\ell^2(t(v-1))$, and by (3.4) and (3.8) we see that, since $g_{v+1}e_v = 0$, we have

$$g_v e_v g_v = (\phi_v / \beta\phi_{v-1})i. \tag{3.30}$$

It then follows that

$$g_{v+1} = g_v - (\beta\phi_{v-1} / \phi_v)\, g_v e_v g_v. \tag{3.31}$$

For $v = p-1$, note that the path g_{p-1} splits as a sum of three paths on level p, as shown in Figure 15, i.e. $g_{p-1} = g_p + g_{\bar{p}} + i$

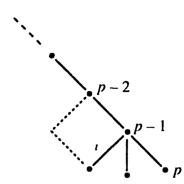

Figure 15

where $g_{\bar{p}} = e_{\bar{p}}$. Again it is clear that i corresponds to the projection $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in End $(\ell^2(t(p-2)))$. Hence

$$i = (\beta\phi_{p-2} / \phi_{p-1})\, g_{p-1}e_{p-1}g_{p-1},$$

and so

$$g_p = g_{p-1} - (\beta\phi_{p-2} / \phi_{p-1})g_{p-1}e_{p-1}g_{p-1} - g_{\bar{p}}. \tag{3.32}$$

The situation for $v \geq p$ is similar to that for $v < p-1$.

Consider, for $v \neq 0, p-1$, the operator $e_{v+1}g_{v+1}$ (where $g_v = f_v$, for $v = 0, ..., p-1$) contained in $A[v, v+2]$. This is given by

$$\frac{1}{\beta\phi_v}\begin{pmatrix} 0 & (\phi_{v-1}\phi_{v+1})^{\frac{1}{2}} \\ 0 & \phi_{v+1} \end{pmatrix} \tag{3.33}$$

on End $(\ell^2(t(v)))$, and is zero on the other components in the decomposition (8.1). Defining

$$u_v = (\beta\phi_v / \sqrt{(\phi_{v-1}\phi_{v+1})})\, e_{v+1}g_{v+1} \tag{3.34}$$

we see from (3.33) that $u_v' \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, i.e. u_v' flips the left hand path of End $(\ell^2(t(v)))$ to the right hand path as shown in Figure 16.

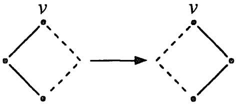

Figure 16

When $v = p-1$, the operator $e_p g_{\bar{p}}, e_p g_p$ are both elements of $A[p-1, p+1]$, and are only non-zero on the component End $(\ell^2(t(p-1)))$. It is clear from (3.5) that, if we define

$$u_\varepsilon = (\beta\phi_{p-1} / \sqrt{(\phi_{p-2}\phi_{\bar{p}})})e_p g_{\bar{p}} \tag{3.35}$$

$$u_{p-1} = (\beta\phi_{p-1} / \sqrt{(\phi_{p-2}\phi_p)})\, e_p g_p \tag{3.36}$$

where $\varepsilon = p-2+1/2$, then $u_\varepsilon' \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and $u_{p-1}' \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. This is illustrated in Figure 17.

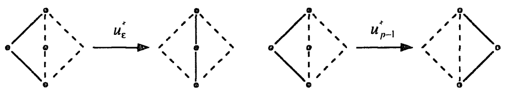

Figure 17

Note also, that for $v \neq \varepsilon$, $\sqrt{(\phi_{v-1} / \beta\phi_v)}u_v$ is a partial isometry with final projection, $e_{v+1}f_v$, and initial projection, $(\beta\phi_v / \phi_{v+1})f_{v+1}e_{v+1}f_{v+1}$. Also $\sqrt{(\phi_{p-2} / \beta\phi_{p-1})}u_\varepsilon$ is a partial isometry with final projection, $e_p g_{p-1}$, and initial projection, $e_{\bar{p}}$.

Matrix units for $A(T_{p,2,r})$ are constructed as follows. Let $\alpha = (v, m) \in \hat{T}^{(0)}_{p,2,r}$, then if $n = (m - d(v))/2$, put $G_\alpha = \gamma^n(g_v)$, then by considering Figure 13, we see that G_α corresponds to the path shown in Figure 18. To obtain an expression for the diagonal matrix unit G_α' in the component labelled by α corresponding to the path

$i=(i_1, i_2, \ldots, i_n)$ shown in Figure 10, one conjugates by the operator $\gamma^{n-1}(\Delta_{i_n})\gamma^{n-2}(\Delta_{i_{n-1}})\ldots\gamma(\Delta_{i_2})\Delta_{i_1}$, where $\Delta_k = u_1u_2\ldots u_k$.
Thus, $\gamma^{n-1}(\Delta_{i_n})' G_\alpha \gamma^{n-1}(\Delta_{i_n})$ corresponds to the path obtained from that in Figure 18 by flipping i_n diamonds in the n^{th} diagonal strip shown in Figure 10. Conjugating the new path by $\gamma^{n-2}(\Delta_{i_{n-1}})$, flips i_{n-1} diamonds in the $(n-1)^{\text{th}}$ strip etc. Off-diagonal matrix units corresponding to pairs of distinct paths are constructed in a similar way.

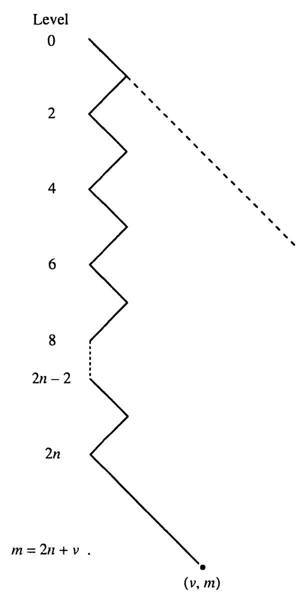

Figure 18

Proof of Theorem 3.1.

Lemma 3.2. $e_m e_{\bar{p}} = 0, \quad m = 1, 2, \ldots, p-1$.

Proof. We see from (3.21) that $e_m e_p e_{\bar{p}} e_p = 0, \quad m = 1, 2, \ldots, p-2$. Thus $(e_m e_p e_{\bar{p}})(e_{\bar{p}} e_p e_m) = 0$, shows that $e_m e_p e_{\bar{p}} = 0$. Then using $[e_m, e_{\bar{p}}] = 0$, and $e_{\bar{p}} e_p e_{\bar{p}} = \tau e_{\bar{p}} g$ we see that $e_m e_{\bar{p}} = 0$ for $m = 1, 2, \ldots, p-2$. In particular $e_{p-2} e_{\bar{p}} = 0$. Consequently $\tau e_{p-1} e_{\bar{p}} = e_{p-1} e_{p-2} e_{p-1} e_{\bar{p}} = 0$, as $[e_{p-1}, e_{\bar{p}}] = 0$.

Lemma 3.3. *Let* $p \geq 2$, $\tau > 0$, *such that* $e_{\bar{p}}, e_1, e_2, \ldots$ *is a sequence of projections satisfying* (3.17)–(3.21). *Define*

$$\gamma_n(x) = \tau^{-(n-1)} e_1 e_2 \ldots e_n x e_n \ldots e_2 e_1, \quad x \in A(\tau, p). \tag{3.37}$$

*Then there exists an unique *-endomorphism* γ *of* $A(\tau, p)$ *such that*

$$\gamma(x) = \lim_{n\to\infty} \gamma_n(x), \quad x \in A(\tau, p) \tag{3.38}$$
$$\gamma(x) = \gamma_n(x), \quad x \in A(\tau, p)_{n-1} \tag{3.39}$$
$$\gamma(1) = e_1 \tag{3.40}$$
$$\gamma(e_m) = e_1 e_{m+2} \tag{3.41}$$
$$\gamma(e_{\bar{p}}) = \tau^{-p} e_1 e_2 \dots e_{p-1} e_p e_{\bar{p}} e_{p+1} e_p e_{p-1} \dots e_2 e_1 . \tag{3.42}$$

Proof. Let A_0 denote the set of $y \in A(\tau, p)$ such that $\lim_{n\to\infty} \gamma_n(y)$ exists. For $y \in A_0$, let $\gamma(x) = \lim_{n\to\infty} \gamma_n(y)$. Then elementary computations show that $1, e_{\bar{p}}, e_1, e_2, \dots \in A_0$, (3.40) – (3.42) hold, and indeed

$$\gamma_n(1) = e_1, \qquad n \geq 1 \tag{3.43}$$
$$\gamma_n(e_m) = e_1 e_{m+2}, \qquad n > m+1 \tag{3.44}$$
$$\gamma_n(e_{\bar{p}}) = \tau^{-p} e_1 e_2 \dots e_{p-1} e_p e_{p+1} e_{\bar{p}} e_p e_{p-1} \dots e_2 e_1, \quad n > p. \tag{3.45}$$

Then if $x, y \in A_0$,

$$e_n \dots e_2 e_1 e_2 \dots e_n = \tau^n e_n , \tag{3.46}$$

cf. (3.43), and so:

$$\gamma_n(x)\gamma_n(y) = \tau^{-n} e_1 e_2 \dots e_n x\, e_n y\, e_n \dots e_1 . \tag{3.47}$$

But $[e_n, x] \to 0$ as $n \to \infty$, for any $x \in A(\tau, p)$, as $[e_n, e_\nu] = 0$ for n large, $\nu \in T^{(0)}_{p,2,\infty}$. Thus $xy \in A_0$ and $\gamma(xy) = \gamma(x)\,\gamma(y)$. Thus A_0 is a dense $*$-subalgebra of $A(\tau, p)$ and (3.39) holds. Now

$$\begin{aligned} &\|\tau^{-n} e_1 \dots e_n\, x\, e_n \dots e_1\| \leq \tau^{-n} \|e_1 \dots e_n\|^2 \|x\| \\ &= \tau^{-n} \|e_1 \dots e_n e_n \dots e_1\| \, \|x\| = \|e_1\| \, \|x\| \qquad \text{by (3.46).} \end{aligned}$$

Hence γ_n is a contraction, and so A_o is closed. Thus $A_o = A(\tau, p)$ and the Lemma follows.

Definition. Suppose $e_{\bar{p}}, e_1, e_2, \dots,$ is a sequence of projections satisfying (3.17)–(3.21) where $1/\sqrt{\tau} = \beta$ is such that $\phi_\nu(\beta) \neq 0$ for all $\nu \in T^{(0)}_{p,2,r-1}$, and some $r \geq 2$, where $\{\phi_\nu(x) \colon \nu \in T^{(0)}_{p,2,\infty}\}$ is the family of rational functions associated with the graph $T_{p,2,\infty}$ as in Section 2. Then we can define a sequence of operators $g_\nu \in A(\tau, p)_{d(\nu)}$ for $\nu \in T^{(0)}_{p,2,r}$ by

$$g_0 = 1, \tag{3.48}$$
$$g_{\nu+1} = g_\nu - (\beta\phi_{\nu-1} / \phi_\nu) g_\nu e_\nu g_\nu, \quad \nu = 0, 1, \dots, p-2 \tag{3.49}$$
$$g_{\bar{p}} = e_{\bar{p}} \tag{3.50}$$
$$g_p = g_{p-1} - (\beta\phi_{p-2} / \phi_{p-1}) g_{p-1} e_{p-1} g_{p-1} - g_{\bar{p}} \tag{3.51}$$

$$g_{v+1} = g_v - (\beta\phi_{v-1} / \phi_v) g_v e_v g_v \qquad p \le v \le p+r-3 \tag{3.52}$$

where $\phi_j = \phi_j(\beta)$.

Lemma 3.4. *Under the preceeding conditions, the family* $\{e_k, g_v, k, v \in T^{(0)}_{p,2,r-1}, k \ne \overline{p}\}$ *satisfy*

(a) $e_k g_v = g_v e_k \qquad v = 0, 1, \dots, p+r-2, \quad k \ge v+1$

(b) (i) $e_p g_{\overline{p}} e_p = (\phi_{\overline{p}} / \beta\phi_{p-1}) e_p g_{p-1}$

(ii) $e_v g_v e_v = (\phi_v / \beta\phi_{v-1}) e_v g_{v-1} \qquad v = 1, 2, \dots, p+r-2$

(c) $e_k g_v = 0 \qquad v = 2, 3, \dots, p+r-2, \quad k = 1, 2, \dots, v-1$

(d) $g_v^2 = g_v = g_v^* \qquad v \in T^{(0)}_{p,2,r-1} = \{0, 1, \dots, p+r-2, \overline{p}\}$

(e) (i) $g_{\overline{p}} g_v = g_{\overline{p}} \qquad v = 0, 1, \dots, p-1$

(ii) $g_{\overline{p}} g_v = 0 \qquad v = p, p+1, \dots, p+r-2$

(iii) $g_k g_v = g_v \qquad v = 0, 1, \dots, p+r-2, \quad k = 1, 2, \dots, v.$

(f) $g_v = \begin{cases} 1 - e_1 \vee \dots \vee e_{v-1} & v = 2, 3, \dots, p-1 \\ 1 - e_1 \vee \dots \vee e_{v-1} \vee e_{\overline{p}} & v = p, p+1, \dots . \end{cases}$

Proof. For $v = 0, 1, 2, \dots, p-1$, the relevant parts of the lemma are clear. Next note that $\phi_{\overline{p}} = \beta^{-1}\phi_{p-1}$, and so (b)(i) follows immediately from (3.21). To see (e)(i), note that $g_v = 1 - e_1 \vee \dots \vee e_{v-1}$, for $v = 2, 3, \dots, p-1$, and so $g_{\overline{p}} g_v = e_{\overline{p}}(1 - e_1 \vee \dots \vee e_{v-1}) = e_{\overline{p}} = g_{\overline{p}}$ by (8.5). Moreover, to show (e)(ii):

$$\begin{aligned} g_{\overline{p}} g_p &= g_{\overline{p}}(g_{p-1} - (\beta\phi_{p-2} / \phi_{p-1}) g_{p-1} e_{p-1} g_{p-1} - g_{\overline{p}}) \\ &= g_{\overline{p}} - (\beta\phi_{p-2} / \phi_{p-1}) g_{\overline{p}} e_{p-1} g_{p-1} - g_{\overline{p}} = 0 \end{aligned}$$

since $g_{\overline{p}} g_{p-1} = g_{\overline{p}}$, and $g_{\overline{p}} e_{p-1} = e_{\overline{p}} e_{p-1} = 0$ by Lemma 3.2. It follows inductively on $v = p, p+1, \dots, p+r-3$ using (3.15) that $g_{\overline{p}} g_v = 0$ for such v, i.e.(e)(ii) holds.

We now prove the properties listed for g_p. It is clear from (3.48–50) that g_p is in the algebra generated by $1, e_1, e_2, \dots, e_{p-1}$ and $e_{\overline{p}}$. Thus (a) holds for $v = p$. Next, since e_p and g_{p-1} commute, we have

$$\begin{aligned} e_p g_p e_p &= e_p[g_{p-1} - (\beta\phi_{p-2} / \phi_{p-1}) g_{p-1} e_{p-1} g_{p-1} - g_{\overline{p}}] e_p \\ &= e_p g_{p-1} - (\beta\phi_{p-2} / \phi_{p-1}) g_{p-1} e_p e_{p-1} e_p g_{p-1} - e_p g_{\overline{p}} e_p \\ &= e_p g_{p-1} - (\phi_{p-2} / \beta\phi_{p-1}) g_{p-1} e_p g_{p-1} - (\phi_{\overline{p}} / \beta\phi_{p-1}) e_p g_{p-1} \qquad \text{using (b)(i)} \\ &= 1 - (\phi_{p-2} / \beta\phi_{p-1}) - (\phi_{\overline{p}} / \beta\phi_{p-1}) e_p g_{p-1}. \end{aligned}$$

But $\phi_p = \beta\phi_{p-1} - \phi_{p-2} - \phi_{\overline{p}}$, and so we obtain (b)(ii) for $v = p$.

We know that (c) holds for $v \le p-1$ by definition of g_i (3.12), and $e_k g_{\overline{p}} = 0$ by Lemma 3.2 for $k = 1, 2, \dots, p-1$. Thus $e_k g_p = 0$ by Lemma 3.4(c), for $k = 1, 2, \dots, p-2$. Also we have, using (b)(ii) for $v = p-1$, and noting that $e_{p-1} g_{\overline{p}} = 0$ that

$$e_{p-1} g_p = e_{p-1} g_{p-1} - (\beta\phi_{p-2} / \phi_{p-1}) e_{p-1} g_{p-1} e_{p-1} g_{p-1} - e_{p-1} g_{\overline{p}}$$

$$= e_{p-1}g_{p-1} - e_{p-1}g_{p-2}g_{p-1}.$$

But $g_{p-2}g_{p-1} = g_{p-1}$ by (e)(iii) for $v = p-1$, $k = p-2$, and so $e_{p-1}g_p = 0$. Thus (c) holds for $v = p$.

Next note that by (b)(ii) for $v = p-1$, one easily shows that $g_{p-1} - (\beta\phi_{p-2} / \phi_{p-1})g_{p-1}e_{p-1}g_{p-1}$ is a projection. Consequently,

$$\begin{aligned} g_p^2 &= g_{p-1} - (\beta\phi_{p-2} / \phi_{p-1})g_{p-1}e_{p-1}g_{p-1} - 2g_{\bar{p}}[g_{p-1} - (\beta\phi_{p-2} / \phi_{p-1})g_{p-1}e_{p-1}g_{p-1}] + g_{\bar{p}} \\ &= g_{p-1} - (\beta\phi_{p-2} / \phi_{p-1})g_{p-1}e_{p-1}g_{p-1} - 2g_{\bar{p}} + g_{\bar{p}} = g_p. \end{aligned}$$

Here we have used Lemma 3.2 and the fact that $g_{p-1}g_{\bar{p}} = g_{\bar{p}}$ by (e)(i). This gives (d) for $v = p$, and (e) for $v = p$ is clear.

Now suppose, for some v, $p < v < p + r - 2$, that g_v has the properties listed. Then we show that g_{v+1} also satisfies these properties. In the first place, (a) follows from the definition of g_{v+1}. Then since $\phi_{v+1} = \beta\phi_v - \phi_{v-1}$, for $v > p$, we have

$$\begin{aligned} e_{v+1}g_{v+1}e_{v+1} &= e_{v+1}[g_v - (\beta\phi_{v-1} / \phi_v)g_ve_vg_v]e_{v+1} \\ &= g_ve_{v+1} - (\beta\phi_{v-1} / \phi_v)g_ve_{v+1}e_ve_{v+1}g_v \\ &= g_ve_{v+1} - (\beta\phi_{v-1} / \phi_v)\beta^{-2}g_ve_{v+1}g_v \\ &= (1 - (\phi_{v-1} / \beta\phi_v))g_ve_{v+1} = (\phi_{v+1} / \beta\phi_v)g_ve_{v+1}. \end{aligned}$$

Next, by the inductive hypothesis, we have $e_jg_v = 0$ for $1 \le j \le v-1$, and so $e_jg_{v+1} = 0$ for $1 \le j \le v-1$. Moreover, by b(ii), and e(iii) for v,

$$e_vg_{v+1} = e_vg_v - (\beta\phi_{v-1} / \phi_v)e_vg_ve_vg_v = e_vg_v - e_vg_{v-1}g_v = e_vg_v - e_vg_v = 0.$$

Thus (c) holds for $v+1$. For (d), one has, using $g_v^2 = g_v$ and (b)(ii) for v that

$$\begin{aligned} g_{v+1}^2 &= g_v - (2\beta\phi_{v-1} / \phi_v)g_ve_vg_v + (\beta\phi_{v-1} / \phi_v)^2 g_ve_vg_ve_vg_v \\ &= g_v - (2\beta\phi_{v-1} / \phi_v)g_ve_vg_v + (\beta\phi_{v-1} / \phi_v)g_ve_vg_{v-1}g_v = g_{v+1}. \end{aligned}$$

Finally (e) for $v + 1$ is clear.

It follows from (c) and e(ii) that $1 - g_v$ is an upper bound for $e_1, e_2, \ldots, e_{v-1}, e_{\bar{p}}$. To show that it is the least upper bound note that $1 - g_v$ is a linear combination of monomials in $e_1, e_2, \ldots, e_{v-1}, e_{\bar{p}}$.

Definition. Let $p, r \ge 2$ be fixed, $\beta > 0$ with $\phi_v(\beta) > 0$ for $v \in T^{(0)}_{p,2,r}$. Put $\varepsilon = p - 2 + 1/2$. Then we define operators $u_1, u_2, \ldots, u_{p+r-3}, u_\varepsilon, \overline{u_\varepsilon}$ as follows:

$$u_k = \beta\phi_k(1 / \sqrt{(\phi_{k-1}\phi_{k+1})})e_{k+1}g_{k+1}, \quad k = 1, 2, \ldots, p + r - 3 \tag{3.53}$$

$$u_\varepsilon = \beta\phi_{p-1}(1 / \sqrt{(\phi_{p-2}\phi_{\bar{p}})})e_pg_{\bar{p}} \tag{3.54}$$

$$\overline{u_\varepsilon} = \beta\phi_{p-1}(1 / \sqrt{(\phi_{\bar{p}}\phi_p})e_pg_p. \tag{3.55}$$

Note that $u_n \in A(\tau, p)_{k+2}, u_\varepsilon \in A(\tau, p)_{p+1}$ and

$$u_\varepsilon\overline{u_\varepsilon} = u_{p-1} \tag{3.56}$$

for Lemma 3.1 (b)(i) and (e)(iii). For $k = 1, 2, ..., p + r - 3$ put

$$\Delta_k = u_1 u_2 \dots u_k, \tag{3.57}$$

$$\Delta_\varepsilon = u_1 u_2 \dots u_{p-2} u_\varepsilon. \tag{3.58}$$

Lemma 3.5. *Let $p, r \geq 2$ be fixed, and $e_1, e_2, \dots, e_{\bar{p}}$, a sequence of projections satisfying* (3.17)–(3.21) *where $\tau = \beta^{-2} > 0$, and $\phi_v(\beta) > 0$ for all $v \in T^{(0)}_{p,2,r}$. Then we have*:

(a) $\Delta_k \Delta_v^* = 0$ *for* $k, v \in \{1, 2, \dots, p+r-3, \varepsilon\}$, *and* $k \neq v$.

(b) $\Delta_k g_v \Delta_k^* = \Delta_k \Delta_k^*, \quad v \neq \bar{p}, \quad 1 \leq v \leq k+1$

$\Delta_\varepsilon g_{\bar{p}} \Delta_\varepsilon^* = \Delta_\varepsilon \Delta_\varepsilon^*, \quad k = \varepsilon,$

(c) $e_1 \Delta_k \Delta_k^* e_1 = \gamma(g_k), \quad k = 1, \dots, p+r-3$

$e_1 \Delta_\varepsilon \Delta_\varepsilon^* e_1 = \gamma(g_{p-1})$

(d) $u_k \gamma(x) = 0, \quad x \in A(\tau, p), \quad k \geq 1$.

(e) $u_i g_v = 0, \quad v \in T^{(0)}_{p,2,r}, \quad d(v) > i+1, \quad i \neq \varepsilon$.

$u_\varepsilon g_v = 0, \quad v = p, p+1, \dots$

(f) $\Delta_v^* \gamma(g_v) \Delta_v = (\beta\phi_v / \phi_{v+1}) f_{v+1} e_{v+1} f_{v+1}, \quad v \in T^{(0)}_{p,2,r}, \quad v \neq 0, \bar{p}, p+r-2 \quad$ (*if* $r < \infty$).

(g) $\Delta_\varepsilon^* \gamma(g_{p-1}) \Delta_\varepsilon = g_{\bar{p}}$.

Proof. (a) For $v \neq \varepsilon$,

$$u_\varepsilon u_v^* = \beta\phi_{p-1}(1/\sqrt{(\phi_{p-2}\phi_{\bar{p}})})\beta\phi_v(1/\sqrt{(\phi_{v-1}\phi_{v+1})})e_p g_{\bar{p}} g_{v+1} e_{v+1} = 0$$

since if $v \leq p-2$, $g_{\bar{p}} g_{v+1} = g_{\bar{p}}$ and $g_{\bar{p}} e_{v+1} = 0$, whereas if $v \geq p-1$, we have $g_{\bar{p}} g_{v+1} = 0$. Similarly for $v, k \neq \varepsilon$, we have, assuming that $k < v$:

$$u_k u_v^* = \beta\phi_k(1/\sqrt{(\phi_{k-1}\phi_{k+1})})\beta\phi_v(1/\sqrt{(\phi_{v-1}\phi_{v+1})})e_{k+1} g_{k+1} g_{v+1} e_{v+1} = 0$$

since $g_{k+1} g_{v+1} = g_{v+1}$, and $e_{k+1} g_{v+1} = 0$.

(b) For $v \neq \bar{p}$, one has $u_k g_v u_k^* = u_k u_k^*$ for $k \leq v+1$, since $g_k g_{v+1} = g_{v+1}$ for $1 \leq v \leq k+1$. For $1 \leq k \leq p-1$, one has $u_k g_{\bar{p}} u_k^* = 0$, since $g_{k+1} g_{\bar{p}} = g_{\bar{p}}$, and $g_{\bar{p}} e_{k+1} = 0$ for $1 \leq k \leq p-2$, and $g_{k+1} g_{\bar{p}} = 0$ for $k = p-1$. Moreover $u_\varepsilon g_{\bar{p}} u_\varepsilon^* = u_\varepsilon u_\varepsilon^*$ as $g_{\bar{p}}^2 = g_{\bar{p}}$.

(c) For $k = 1, 2, \dots, p+r-3$:

$$\begin{aligned}
\Delta_k \Delta_k^* &= u_1 u_2 \dots u_k u_k^* \dots u_2^* u_1^* \\
&= \eta_k e_2 g_2 e_3 g_3 \cdots e_{k+1} g_{k+1} e_{k+1} \cdots g_3 e_3 g_2 e_2 \\
&= \eta_k e_2 e_3 \dots e_{k+1} g_2 g_3 \cdots g_k g_{k+1} g_k \cdots g_3 g_2 e_{k+1} e_3 e_2 \\
&= \eta_k e_2 e_3 \dots e_{k+1} g_{k+1} e_{k+1} \cdots e_2
\end{aligned} \tag{3.59}$$

where we have used Lemma 3.1(a) and (e), and

$$\eta_k = (\beta\phi_1)^2 \dots (\beta\phi_k)^2 / (\phi_0 \phi_2 \dots \phi_{k-1} \phi_{k+1}). \tag{3.60}$$

Then from (3.23) and Lemma 3.1(b)(ii) we obtain

$$\Delta_k \Delta_k^* = \eta_k (\phi_{k+1} / \beta\phi_k) e_2 e_3 \dots e_k e_{k+1} g_k e_k \dots e_2$$

and so by Lemma 3.2 we have

$$e_1\Delta_k\Delta_k^* e_1 = \eta_k(\phi_k / \beta\phi_{k-1})e_1e_2e_3 \dots e_ke_{k+1}f_ke_{k+1}e_k \dots e_2 = \eta_k(\phi_{k+1} / \beta\phi_k)\beta^{-2k}\gamma(f_k).$$

But

$$(\eta_k)(\phi_k / \beta\phi_{k-1})(1/\beta^{2k}) =$$
$$(\beta^{2k})((\phi_1^2\phi_2^2 \dots \phi_k^2)/(\phi_0\phi_2\phi_1\phi_3 \dots \phi_{k-1}\phi_{k+1}))(\phi_{k+1}/\beta\phi_k)(1/\beta^{2k}) = 1, \qquad (3.61)$$

which establishes (c) for $k \neq \varepsilon$.

Similarly, one uses $e_pg_{\bar{p}}e_p = (\phi_{\bar{p}} / \beta\phi_{p-1})e_pg_{p-1}$ and Lemma 3.2 to show that $e_1\Delta_\varepsilon\Delta_\varepsilon^* e_1 = \gamma(g_{p-1})$.

(d) This follows because $\gamma(x) = e_1\gamma(x)$ by (3.16), and $g_{k+1}e_1 = 0 = g_{\bar{p}}e_1$ for $k = 1, 2, \dots, p+r-3$.

(e) This follows immediately from Lemma 3.4(c), and (e).

(f) We have by Lemma 3.2, and Lemma 3.1(a) and (e) that for v as stated:

$$\begin{aligned}\Delta_v^*\gamma(g_v)\Delta_v &= \beta^{2v}\Delta_v^* e_1e_2 \dots e_{v+1}g_ve_{v+1} \dots e_2e_1\Delta_v \\ &= \beta^{2v}\eta_v g_{v+1}e_{v+1} \dots g_2e_2e_1e_2 \dots e_{v+1}g_ve_{v+1} \dots e_2e_1e_2g_2 \dots e_{v+1}g_{v+1} \\ &= \beta^{2v}\eta_v g_{v+1}g_v \dots g_2e_{v+1} \dots e_2e_1e_2 \dots e_{v+1}g_ve_{v+1} \dots e_2e_1e_2 \dots e_{v+1}g_2 \dots g_{v+1} \\ &= \beta^{2v}\eta_v g_{v+1}e_{v+1} \dots e_2e_1e_2 \dots e_{v+1}g_ve_{v+1} \dots e_2e_1e_2 \dots e_{v+1}g_2 \dots g_{v+1}\end{aligned}$$

where $\eta_v = \beta^{2v}(\beta\phi_v / \phi_{v+1})$ by (3.61). But using (3.46) and Lemma 3.4(a) and (e)(iii) we have

$$\begin{aligned}\Delta_v^*\gamma(g_v)\Delta_v &= \beta^{-2v}\eta_v g_{v+1}e_{v+1}g_ve_{v+1}g_{v+1} \\ &= (\beta\phi_v / \phi_{v+1})g_{v+1}g_ve_{v+1}g_{v+1} = (\beta\phi_v / \phi_{v+1})g_{v+1}e_{v+1}g_{v+1}\end{aligned}$$

(g) Similarly, we have

$$\begin{aligned}&\Delta_\varepsilon\gamma(g_{p-1})\Delta_\varepsilon^* = \beta^{2(p-1)}\Delta_\varepsilon e_1e_2 \dots e_pg_{p-1}e_p \dots e_2e_1\Delta_\varepsilon^* \\ &= \beta^{2(p-1)}\beta^{2(p-1)}(\phi_1\phi_{p-1} / \phi_0\phi_{\bar{p}})g_{\bar{p}}e_pg_{p-1}e_{p-1} \dots g_2e_2e_1e_2 \dots e_pg_{p-1}e_p \dots e_1e_2g_2 \dots e_pg_{\bar{p}} \\ &= \beta^{2(p-1)}\beta^{2p-1}(\phi_{p-1} / \phi_{\bar{p}})g_{\bar{p}}g_{p-1} \dots g_2e_pe_{p-1} \dots e_2e_1e_2 \dots e_pg_{p-1}e_p \dots e_2e_1e_2 \dots e_pg_2 \dots g_{p-1}g_{\bar{p}} \\ &= \beta^{-2(p-1)}\beta^{2p-1}(\phi_{p-1} / \phi_{\bar{p}})g_{\bar{p}}e_pg_{p-1}e_pg_{\bar{p}} \\ &= \beta^2(\phi_{p-1} / \beta\phi_{\bar{p}})g_{\bar{p}}e_pg_{p-1}e_pg_{\bar{p}}.\end{aligned}$$

But $\phi_{p-1} = \beta\phi_{\bar{p}}$, and by Lemma 3.4 we have $g_{\bar{p}}e_pg_{p-1}e_pg_{\bar{p}} = g_{\bar{p}}e_pg_{\bar{p}} = \beta^{-2}g_{\bar{p}}$ and the result follows.

Assumption. Let p, r, and $\tau = \beta^{-2}$ be fixed, where $p \geq 2$, $2 \leq r \leq \infty$, and suppose that

$$\phi_v(\beta) > 0 \text{ for all } v \in T_{p,2,r}^{(0)}. \qquad (3.62)$$

Recall from Proposition 2.1, that if $r < \infty$, and $\beta = \|T_{p,2,r}\|$, or if $r = \infty$, and $\beta \geq \|T_{p,2,\infty}\|$, then (3.62) is true.

Definition. Let $\alpha = (v, m) \in \hat{T}^{(0)}_{p,2,r}$, and put $n = (m - d(v))/2$. Then define

$$G_\alpha = \gamma^n(g_v) \tag{3.63}$$

and for $i, j \in I_\alpha = \text{Path}(*, \alpha)$ define

$$G^{ij}_\alpha = \Delta^*_{i_1}\gamma(\Delta_{i_2})^* \ldots \gamma^{n-1}(\Delta_{i_n})^* \gamma^n(g_v)\gamma^{n-1}(\Delta_{j_n}) \ldots \gamma(\Delta_{j_2})\Delta_{j_1}. \tag{3.64}$$

Note that $G^{00}_\alpha = G_\alpha$, and $(G^{ij}_\alpha)^* = G^{ji}_\alpha$. Put $G^i_\alpha = G^{ii}_\alpha$, and $v_{\alpha,i} = G_\alpha \gamma^{n-1}(\Delta_{i_n}) \ldots \gamma(\Delta_{i_2})\Delta_{i_1}$.

Lemma 3.6. *For* $\alpha = (v, m)$, $\beta = (w, m) \in \hat{T}^{(0)}_{p,2,r}$, $i, j \in \text{Path}(*, \alpha)$, $k, l \in \text{Path}(*, \beta)$ *we have*

$$G^{ij}_\alpha G^{kl}_\beta = \delta_{\alpha\beta}\delta_{jk}G^{il}_\alpha. \tag{3.65}$$

Proof. (a) $G^{ij}_\alpha G^{jl}_\alpha = G^{il}_\alpha$.
It is clear that $G^2_\alpha = G_\alpha$. Now suppose that $n \geq 1$, and $i \neq 0$. We prove by induction on *n* that

$$e_1 e_3 \ldots e_{2n-1}\gamma^{n-1}(\Delta_{i_n}) \ldots \Delta_{i_1}\Delta^*_{i_1} \ldots \gamma^{n-1}(\Delta_{i_n})^* e_{2n-1} \ldots e_3 e_1 = \gamma^n(g_z) \tag{3.66}$$

where $z = [i_n]$. If $n = 1$, this follows immediately from Lemma 3.5(c) since $i_1 \neq 0$. Now suppose that (3.66) is true for $n > 1$. Then by the induction hypothesis, and Lemma 3.5(c) we have

$$\begin{aligned}
&e_1 e_3 \ldots e_{2n+1}\gamma^n(\Delta_{i_{n+1}}) \ldots \Delta_{i_1}\Delta^*_{i_1} \ldots \gamma^n(\Delta_{i_{n+1}})^* e_{2n+1} \ldots e_3 e_1 \\
&= e_{2n+1}\gamma^n(\Delta_{i_{n+1}})[e_1 e_3 \ldots e_{2n-1}\gamma^{n-1}(\Delta_{i_n}) \ldots \Delta_{i_1}\Delta^*_{i_1} \ldots \gamma^{n-1}(\Delta_{i_n})^* e_{2n-1} \ldots e_3 e_1]\gamma^n(\Delta_{i_{n+1}})^* e_{2n+1} \\
&= e_{2n+1}\gamma^n(\Delta_{i_{n+1}})\gamma^n(g_z)\gamma^n(\Delta^*_{i_{n+1}})e_{2n+1} = \gamma^n(e_1\Delta_{i_{n+1}}g_z\Delta^*_{i_{n+1}}e_1).
\end{aligned}$$

But $z \leq i_{n+1} + 1$, and so by Lemma 3.5(b) and (c) we have

$$e_1\Delta_{i_{n+1}}g_z\Delta^*_{i_{n+1}}e_1 = e_1\Delta_{i_{n+1}}\Delta^*_{i_{n+1}}e_1 = \gamma(g_{z'}),$$

where $z' = [i_{n+1}]$. Hence (3.66) is true for all *n*. Now it follows from (3.66), noting that $G_\alpha = \gamma^n(g_v) = e_1 e_3 \ldots e_{2n-1}\gamma^n(g_v)$, that

$$v_{\alpha,i}v^*_{\alpha,i} = G_\alpha\gamma^n(g_z)G_\alpha = \gamma^n(g_v g_z g_v) = \gamma^n(g_v) = G_\alpha$$

since $z \leq d(v)$. This gives (a).

(b) $G^{ij}_\alpha G^{kl}_\alpha = 0$, for $j \neq k$.

We may assume that $n \geq 1$. We show that for $i \neq j$, $v_{\alpha,i}v^*_{\alpha,j} = 0$. If $i \neq j$, then there exists a $k \leq n$ such that $i_1 = j_1, \ldots, i_k = j_k$, and $i_{k+1} \neq j_{k+1}$. Then by (3.66) we have

$$v_{\alpha,i}v^*_{\alpha,j} = G_\alpha\gamma^{n-1}(\Delta_{i_n}) \ldots \gamma^k(\Delta_{i_{k+1}})\gamma^k(g_z)\gamma^k(\Delta_{j_{k+1}})^* \ldots \gamma^{n-1}(\Delta_{j_n})^* G_\alpha.$$

But $z \le i_{k+1}+1$, and so $\Delta_{i_{k+1}} g_z = \Delta_{i_{k+1}}$. It follows from Lemma 3.5(a) that

$$\gamma^k(\Delta_{i_{k+1}})\gamma^k(g_z)\gamma^k(\Delta_{j_{k+1}})^* = \gamma^k(\Delta_{i_{k+1}} g_z \Delta^*_{j_{k+1}}) = \gamma^k(\Delta_{i_{k+1}}\Delta^*_{j_{k+1}}) = 0$$

since $i_{k+1} \ne j_{k+1}$.

(c) $G^{ij}_\alpha G^{i'j'}_\beta = 0$, for $v \ne w$.

Put $k = (m - d(w))/2$. Suppose that $n = k$, then $v = p$, $w = \overline{p}$, or $v = \overline{p}$, and $w = p$. We must show that

$$v_{\alpha,i} v^*_{\beta,j} = G_\alpha \gamma^{n-1}(\Delta_{i_n}) \dots \Delta_{i_1}\Delta^*_{j_1} \dots \gamma^{n-1}(\Delta_{j_n})^* G_\beta$$

vanishes. First suppose that there exists $t < n$ such that $i_1 = j_1, \dots, i_t = j_t$, and $i_{t+1} \ne j_{t+1}$. Then as in the proof of (a) we have

$$v_{\alpha,i} v^*_{\beta,j} = G_\alpha \gamma^{n-1}(\Delta_{i_n}) \dots \gamma^t(\Delta_{i_{t+1}})\gamma^t(g_{i_t})\gamma^t(\Delta_{j_{t+1}})^* \dots \gamma^{n-1}(\Delta_{j_n})^* G_\beta$$

if $i_t \ne \varepsilon$, otherwise we replace $\gamma^t(g_{i_t})$ by $\gamma^t(g_{p-1})$. Then since $i_t \le i_{t+1}+1$, we see that $\Delta_{i_{t+1}} g_{i_t} = \Delta_{i_{t+1}}$, and so

$$v_{\alpha,i} v^*_{\beta,j} = G_\alpha \gamma^n(\Delta_{i_n}) \dots \gamma^t(\Delta_{i_{t+1}}\Delta^*_{j_{t+1}}) \dots \gamma^n(\Delta_{j_n})^* G_\beta.$$

But $i_{t+1} \ne j_{t+1}$, and so $\Delta_{i_{t+1}}\Delta^*_{j_{t+1}} = 0$ by Lemma 3.5(a). Similarly, if $i_t = \varepsilon$ then $\Delta_{i_{t+1}} g_{p-1} = \Delta_{i_{t+1}}$. If no such *t* exists, then $i_1 = j_1, \dots, i_n = j_n$ and so

$$v_{\alpha,i} v^*_{\beta,j} = G_\alpha \gamma^n(g_z) G_\beta = \gamma^n(g_v)\gamma^n(g_z)\gamma^n(g_w) = \gamma^n(g_v g_z g_w)$$

where $z = i_n$ if $i_n \ne \varepsilon$, and $z = p-1$, if $i_n = \varepsilon$. But $i_n \le d(v)$, and so $g_v g_z = g_v$, and by Lemma 3.4(e) $g_v g_w = 0$.

Now suppose that $n \ne k$, with $n > k$. Note that $d(w) = d(v) + 2(n-k)$. If there is a $t < k$, such that $i_1 = j_1, \dots, i_t = j_t$, and $i_{t+1} \ne j_{t+1}$ then as before we have

$$v_{\alpha,i} v^*_{\beta,j} = G_\alpha \gamma^{n-1}(\Delta_{i_n}) \dots \gamma^t(\Delta_{i_{t+1}})\gamma^t(g_z)\gamma^t(\Delta_{j_{t+1}})^* \dots \gamma^{k-1}(\Delta_{j_k})^* G_\beta$$

where $z = i_t$ if $i_t \ne \varepsilon$, $z = p-1$ otherwise. But $z \le i_{t+1}+1$, and so $\Delta_{i_{t+1}} g_z = \Delta_{i_{t+1}}$, then by Lemma 3.5(a) $v_{\alpha,i} v^*_{\beta,j} = 0$, since $i_{t+1} \ne j_{t+1}$. Finally if $i_1 = j_1, \dots, i_k = j_k$, then we have

$$\begin{aligned} v_{\alpha,i} v^*_{\beta,j} &= G_\alpha \gamma^{n-1}(\Delta_{i_n}) \dots \gamma^k(\Delta_{i_{k+1}})\gamma^k(g_z) G_\beta \\ &= \gamma^k[\gamma^{n-k}(g_v)\gamma^{n-k-1}(\Delta_{i_n}) \dots \Delta_{i_{k+1}} g_z g_w] \end{aligned}$$

where $z = [i_t]$. But $z \le d(w)$, and so $g_z g_w = g_w$. Note also that by definition of $i = (i_1, \dots, i_n)$, we have $i_{k+1} \le d(v) + n - k - 1$, and so

$$i_{k+1} + 1 \le d(v) + n - k < d(v) + 2(n-k) = d(w).$$

But this implies that $u_{i_{k+1}} g_w = 0$ by Lemma 3.5(e). Hence $\Delta_{i_{k+1}} g_w = 0$, and so $v_{\alpha,i} v^*_{\beta,j} = 0$.

Lemma 3.7. (a) $G^{\iota}_{(0,m)} = G^{\iota}_{(1,m+1)}$ $i \in I_{(0,m)} \subset I_{(1,m+1)}$.

(b) $G^{\iota}_{(\nu,m)} = G^{(\iota,\nu-1)}_{(\nu-1,m+1)} + G^{\iota}_{(\nu+1,m+1)}$, $i \in I_{(\nu,m)}$, $\nu \in \hat{T}^{(0)}_{p,2,r-1}$. $\nu \neq 0, p-1, \bar{p}$.

(c) $G^{\iota}_{(\bar{p},m)} = G^{(\iota,\varepsilon)}_{(p-1,m+1)}$, $i \in I_{(\bar{p},m)}$.

(d) $G^{\iota}_{(p-1,m)} = G^{(\iota,\nu-1)}_{(p-2,m+1)} + G^{\iota}_{(\bar{p},m+1)} + G^{\iota}_{(p,m+1)}$.

(e) *When* $\beta = \|T_{p,2,r}\|$, *and* $r < \infty$, $G_{(p+r-2,p+r-2)} = G^{(p+r-3)}_{(p+r-3,p+r-1)} + g_{p+r-1}$, *and for* $m > p+r-2$, $G^{\iota}_{(p+\nu-2,m)} = G^{(\iota,p+r-3)}_{(p+r-3,m+1)}$, $i \in I_{(p+r-2,m)}$. *If there exists a faithful trace satisfying* (3.24), *then* $g_{p+r-1} = 0$.

Proof. For (a) note that $g_0 = g_1$, and if $\alpha = (0,m) \in \hat{T}^{(0)}_{p,2,r}$ then m is even. Thus

$$G_{(0,m)} = \gamma^{m/2}(g_0) = \gamma^{m/2}(g_1)G_{(1,m+1)}$$

and so (a) follows.

For (b) note that when $\nu \neq 0, p-1, \bar{p}, (p+r-2 \text{ if } r < \infty)$, then by Lemma 3.5(f)

$$g_\nu = g_{\nu+1} + (\beta\phi_{\nu-1}/\phi_\nu)g_\nu e_\nu g_\nu = g_{\nu+1} + \Delta^*_{\nu-1}\gamma(g_{\nu-1})\Delta_{\nu-1}.$$

Then since $G_{(\nu,m)} = \gamma^n(g_\nu)$ where $n = (m-\nu)/2$, we have

$$G_{(\nu,m)} = \gamma^n(g_{\nu+1}) + \gamma^n(\Delta_{\nu-1})^*\gamma^{n+1}(g_{\nu-1})\gamma^n(\Delta_{\nu-1}),$$

and so

$$\begin{aligned} G^{\iota}_{(\nu,m)} = {} & \Delta^*_{\iota_1}\dots\gamma^{n-1}(\Delta_{\iota_n})^*\gamma^n(g_{\nu+1})\gamma^{n-1}(\Delta_{\iota_n})\dots\Delta_{\iota_1} \\ & +\Delta^*_{\iota_1}\dots\gamma^{n-1}(\Delta_{\iota_n})^*\gamma^n(\Delta_{\nu-1})^*\gamma^{n+1}(g_{\nu-1})\gamma^n(\Delta_{\nu-1})\gamma^{n-1}(\Delta_{\nu-1})\dots\Delta_{\iota_1}. \end{aligned}$$

But $\gamma^n(g_{\nu+1}) = G_{(\nu+1,m+1)}$, and $\gamma^{n+1}(g_{\nu-1}) = G_{(\nu-1,m+1)}$, and so (b) follows.

By Lemma 3.5(g), we have $g_{\bar{p}} = \Delta^*_\varepsilon\gamma(g_{p-1})\Delta_\varepsilon$, and so if $n = (m-p)/2$, we have

$$G_{(\bar{p},m)} = \gamma^n(g_{\bar{p}}) = \gamma^n(\Delta_\varepsilon)^*\gamma^{n+1}(g_{p-1})\gamma^n(\Delta_\varepsilon),$$

but $\gamma^{n+1}(g_{p-1}) = G_{(p-1,m+1)}$, and so (c) follows.

For (d) we use Lemma 3.5(f) to obtain

$$g_{p-1} = g_p + (\beta\phi_{p-2}/\phi_{p-1})g_{p-1}e_{p-1}g_{p-1} + g_{\bar{p}} = g_p + \Delta^*_{p-2}\gamma(g_{p-2})\Delta_{p-2} + g_{\bar{p}}.$$

Hence if $n = (m-(p-1))/2$, we have

$$\begin{aligned} G_{(p-1,m)} &= \gamma^n(g_{p-1}) = \gamma^n(g_p) + \gamma^n(\Delta_{p-2})^*\gamma^{n+1}(g_{p-2})\gamma^n(\Delta_{p-2}) + \gamma^n(g_{\bar{p}}) \\ &= G_{(p,m+1)} + \gamma^n(\Delta_{p-2})^* G_{(p-2,m+1)}\gamma^n(\Delta_{p-2})^* + G_{(\bar{p},m+1)} \end{aligned}$$

and (d) follows.

(e) If $\beta = \|T_{p,2,r}\|$, then $\phi_{p+r-1}(\beta) = 0$. Then putting $t = p+r-1$, it follows by Lemma 3.4(b) that $e_t g_t (e_t g_t)^* = e_t g_t e_t = (\phi_t/\beta\phi_{t-1})e_t g_{t-1} = 0$, and so $e_t g_t = 0$. But then we have

$$e_{t+1}g_t = \beta^2 e_{t+1}e_t e_{t+1}g_t = \beta^2 e_{t+1}(e_t g_t)e_{t+1} = 0$$

and by induction $e_k g_t = 0$ for all $k \geq t$, and thus for all k.

It follows that $\gamma^n(g_t) = 0$ for all $n \geq 1$. Then from (3.16) and Lemma 3.5 we have $g_{t-1} = g_t + \Delta^*_{t-2}\gamma(g_{t-2})\Delta_{t-2}$, from which we obtain (e) by applying γ.

If a faithful trace, tr satisfying (3.24) exists, then since $\phi_t(\beta) = 0$,

$$\begin{aligned}\operatorname{tr}(g_t) &= \operatorname{tr}(g_{t-1}) - (\beta\phi_{t-2}/\phi_{t-1})\operatorname{tr}(g_{t-1}e_{t-1}) \\ &= (1-(\phi_{t-2}/\beta\phi_{t-1}))\operatorname{tr}(g_{t-1}) = (\phi_t/\beta\phi_{t-1})\operatorname{tr}(g_{t-1}) = 0.\end{aligned}$$

Hence $g_{p+r-1} = 0$.

Lemma 3.8. *Suppose that* (3.62) *holds for* $r = \infty$, *then we have for each* $m \geq 0$

$$1 = \sum_{v} G^i_{(v,\,m)} \tag{3.67}$$

where the summation is over all vertices (v, m) *on level m of* $\hat{T}^{(0)}_{p,2,r}$ *and all* $i \in I_{(v,m)}$. *If* (3.62) *holds for some* $r < \infty$, *then* (3.67) *is true for* $m \leq p+r-2$, *and for each* $m > p+r-2$ *we have*

$$1 = \sum G^i_{(v,m)} + g_{p+r-1}. \tag{3.68}$$

Proof. We use the splitting rules for $G^i_{(v,\,m)}$ in Lemma 3.7.

Definition. Let $m \geq 1$, $p \geq 2$, and $r \in \{2,3,\ldots,\infty\}$. Let $\alpha = (v, m+1) \in \hat{T}^{(0)}_{p,2,r}$, with $d(v) < m+1$. Put $n = (m+1-d(v))/2$. Note that for such α, we have $\alpha' = (v, m-1) \in \hat{T}^{(0)}_{p,2,r}$. Let $\tilde{I}_\alpha = \{i \in I_\alpha; (i_1, i_2, \ldots, i_{n-1}) \in I_{\alpha'}\}$. For example, if $v \neq 0, p-1, \bar{p}$, and if $r < \infty, v \neq p+r-2$, then $\tilde{I}_\alpha$ consists of all $i \in I_\alpha$ with $i_n = v-1$, or $i_n = v$ and $i_{n-1} \leq v$.

Lemma 3.9. (a) *Let* $t \in \{1, 2, \ldots, p-2, \varepsilon, p-1, \ldots\}$, *and* $s \geq 1$, *then if* $m \geq t+2s+3$ *we have* $\gamma^s(\Delta_t)e_m = e_m\gamma^s(\Delta_t)$.

(b) *Let* $i = (i_1, \ldots, i_n) \in I_\alpha$, *then we have* $i_k \leq i_{n-t} + (n-k-t)$ *for* $t = 0, 1, \ldots, n-1$, *and* $k = 1, 2, \ldots, n-t$.

(c) *For* $v \neq 0, \bar{p}$, *and if* $r < \infty$, $v \neq p+r-2$, *we have* $\gamma(g_v)\Delta_v e_{v+1} = \sqrt{(\phi_{v+1}/\beta\phi_v)}$ $\cdot\beta^v e_1 e_2 \ldots e_{v+1} g_v$.

(d) *For* $v \neq 0, 1, \bar{p}$, *we have* $\gamma(g_v)\Delta_{v-1}e_{v+1} = \sqrt{(\phi_{v-1}/\beta\phi_v)}\beta^v e_1 e_2 \ldots e_{v+1} g_v$, *and when* $v=1$ *we have* $\gamma(g_1)e_2 = \sqrt{(\phi_0/\beta\phi_1)}\beta e_1 e_2 g_1$.

(e) *For* $v \neq 0$, *and if* $r < \infty$, $v \neq p+r-2$, *we have* $\gamma(\Delta_v)\Delta_{v+1}e_{v+3} = 0$.

(f) $\gamma(g_{\bar{p}})\Delta_\varepsilon e_{p+1} = \beta^p e_1 e_2 \ldots e_{p+1} g_{\bar{p}}$.

(g) $\gamma(g_{p-1})\Delta_\varepsilon e_p = \beta^{p-1}\sqrt{(\phi_{\bar{p}}/\beta\phi_{p-1})}e_1 e_2 \ldots e_p g_{p-1}$.

(h) $G_\alpha e_m = 0$.

Proof. (a) First note that $e_m\Delta_r = \Delta_r e_m$ for $m \geq r+3$, and $r \neq \varepsilon$. Also $e_m\Delta_\varepsilon = \Delta_\varepsilon e_m$, for $m \geq p+2$, i.e. $m \geq \varepsilon + 3$. Then since $m = k+2s$, with $k \geq t+3$, we have

$$e_m\gamma^S(\Delta_t) = \gamma^S(e_k)\gamma^S(\Delta_t) = \gamma^S(e_k\Delta_t) = \gamma^S(\Delta_t e_k) = \gamma^S(\Delta_t)e_m.$$

(b) This is clear from the definition of I_α.

(c) By Lemma 3.4(a), (b) we have

$$\begin{aligned}\Delta_\nu e_{\nu+1} &= (\beta\phi_\nu/\sqrt{(\phi_{\nu-1}\phi_{\nu+1})})e_2g_2e_3g_3\ldots e_{\nu+1}g_{\nu+1}e_{\nu+1}\\ &= \beta^\nu\sqrt{(\phi_1\phi_\nu/\phi_0\phi_{\nu+1})}e_2g_2\ldots e_\nu g_\nu.(\phi_{\nu+1}/\beta\phi_\nu)e_{\nu+1}g_\nu)\\ &= \beta^\nu\sqrt{(\phi_1\phi_\nu/\phi_0\phi_{\nu+1})}(\phi_{\nu+1}/\beta\phi_\nu)e_2e_3\ldots e_\nu e_{\nu+1}g_\nu = \beta^\nu\sqrt{(\phi_{\nu+1}/\beta\phi_\nu)}e_2e_3\ldots e_{\nu+1}g_\nu.\end{aligned}$$

Hence by Lemma 3.3 we have

$$\begin{aligned}\gamma(g_\nu)\Delta_\nu e_{\nu+1} &= \beta^{2\nu}e_1e_2\ldots e_{\nu+1}g_\nu e_{\nu+1}\ldots e_1\beta^\nu\sqrt{(\phi_{\nu+1}/\beta\phi_\nu)}e_2e_3\ldots e_{\nu+1}g_\nu\\ &= \beta^\nu\sqrt{(\phi_{\nu+1}/\beta\phi_\nu)}e_1e_2\ldots e_{\nu+1}g_\nu(\beta^{2\nu}e_{\nu+1}\ldots e_2e_1e_2\ldots e_{\nu+1})g_\nu\\ &= \beta^\nu\sqrt{(\phi_{\nu+1}/\beta\phi_\nu)}e_1e_2\ldots e_{\nu+1}g_\nu e_{\nu+1}g_\nu\end{aligned}$$

and so (c) follows using Lemma 3.4.

(d) Using Lemma 3.3, and Lemma 3.4(a), (e) and (3.46) we have

$$\begin{aligned}\gamma(g_\nu)\Delta_\nu e_{\nu+1} &= \beta^{2\nu}\beta^{\nu-1}\sqrt{(\phi_1\phi_{\nu-1}/\phi_0\phi_\nu)}e_1e_2\ldots e_{\nu+1}g_\nu e_{\nu+1}\ldots e_1e_2g_2\ldots e_\nu g_\nu e_{\nu+1}\\ &= \beta^{\nu-1}\sqrt{(\beta\phi_{\nu-1}/\phi_\nu)}e_1e_2\ldots e_{\nu+1}g_\nu(\beta^{2\nu}e_{\nu+1}\ldots e_1e_2e_1\ldots e_{\nu+1})g_\nu\\ &= \beta^{\nu-1}\sqrt{(\beta\phi_{\nu-1}/\phi_\nu)}e_1e_2\ldots e_{\nu+1}g_\nu e_{\nu+1}g_\nu.\end{aligned}$$

But $\beta^{\nu-1}\sqrt{(\beta\phi_{\nu-1}/\phi_\nu)} = \beta\sqrt{(\phi_{\nu-1}/\beta\phi_\nu)}$, and $e_{\nu+1}g_\nu e_{\nu+1}g_\nu = e_{\nu+1}g_\nu$, and so we have the first part of (d). Also

$$\gamma(g_1)e_2 = \beta^2e_1e_2g_1e_2e_1e_2 = e_1e_2g_1 = \beta\sqrt{(\phi_0/\beta\phi_1)}e_1e_2g_1.$$

(e) First note that $\gamma(\Delta_\nu) = \beta^{2(\nu+2)}e_1e_2\ldots e_{\nu+3}\Delta_\nu e_{\nu+3}\ldots e_2e_1$, and so, using Lemma 3.4(a), (e) and (3.46) we have

$$\begin{aligned}\gamma(\Delta_\nu)\Delta_{\nu+1}e_{\nu+3} &= \delta e_1\ldots e_{\nu+3}\Delta_\nu e_{\nu+3}\ldots e_1e_2g_2e_3g_3\ldots e_{\nu+2}g_{\nu+2}e_{\nu+3}\\ &= \delta e_1\ldots e_{\nu+3}\Delta_\nu e_{\nu+3}\ldots e_2e_1e_2\ldots e_{\nu+3}g_{\nu+2}\\ &= \delta' e_1\ldots e_{\nu+3}\Delta_\nu e_{\nu+3}g_{\nu+2}\end{aligned}$$

where δ, δ' are scalars. But $\Delta_\nu = u_1u_2\ldots u_\nu$, and

$$u_\nu e_{\nu+3}g_{\nu+2} = \lambda e_{\nu+1}g_{\nu+1}e_{\nu+3}g_{\nu+2} = \lambda e_{\nu+1}e_{\nu+3}g_{\nu+2} = \lambda e_{\nu+3}e_{\nu+1}g_{\nu+2} = 0$$

where λ is a scalar, by Lemma 3.4(a), (c) and (e), and so $\gamma(\Delta_\nu)\Delta_{\nu+1}e_{\nu+3} = 0$.

(f) As in (d), we have

$$\gamma(g_{\bar{p}})\Delta_\varepsilon e_{p+1}$$

$$= \beta^{2p}\beta^{p-1}\sqrt{(\phi_1\phi_{p-1}/\phi_0\phi_{\bar p})}e_1e_2\dots e_{p+1}g_{\bar p}e_{p+1}\dots e_1e_2g_2e_3g_3\dots e_pg_{\bar p}e_{p+1}$$
$$= \beta^{p}\sqrt{(\phi_{p-1}/\beta\phi_{\bar p})}e_1e_2\dots e_{p+1}g_{\bar p}(\beta^{2p}e_{p+1}\dots e_2e_1e_2\dots e_{p+1})g_{\bar p}$$
$$= \beta^{p}\sqrt{(\phi_{p-1}/\beta\phi_{\bar p})}e_1e_2\dots e_{p+1}g_{\bar p}e_{p+1}g_p.$$

But $\phi_{p-1}/\beta\phi_{\bar p}$, and so (f) follows.

(g) Since $e_pg_{\bar p}e_p = (\phi_{\bar p}/\beta\phi_{p-1})e_pg_{p-1}$, we have

$$\gamma(g_{p-1})\Delta_\varepsilon e_p$$
$$= \beta^{2(p-1)}e_1e_2\dots e_pg_{p-1}e_p\dots e_1\beta^{p-1}\sqrt{(\phi_1\phi_{p-1}/\phi_0\phi_{\bar p})}e_2g_2\dots e_{p-1}g_{p-1}e_pg_{\bar p}e_p$$
$$= \beta^{p-1}\sqrt{(\beta\,\phi_{p-1}/\phi_{\bar p})}e_1e_2\dots e_pg_{p-1}(\beta^{2(p-1)}e_p\dots e_2e_1e_2\dots e_p)\sqrt{(\phi_{\bar p}/\beta\phi_{p-1})}g_{p-1}$$
$$= \beta^{p-1}\sqrt{(\beta\,\phi_{p-1}/\phi_{\bar p})}e_1\dots e_pg_{p-1}e_p\sqrt{(\phi_{\bar p}/\beta)}\phi_{p-1}g_{p-1}$$
$$= \beta^{p-1}\sqrt{(\phi_{\bar p}/\beta\phi_{p-1})}e_1e_2\dots e_pe_{p-1}.$$

(h) Since $m = 2n + v - 1$, we have by Lemma 3.4(c) that

$$G_\alpha e_m = \gamma^n(g_v)\gamma^n(e_{v-1}) = \gamma^n(g_ve_{v-1}) = 0.$$

Proposition 3.10. *If* $\alpha = (v, m+1) \in \hat T^{(0)}_{p,2,r}$, $i, j \in I_\alpha$; *then* $G^i_\alpha e_m G^j_\alpha = \gamma^{ij}_\alpha G^{ij}_\alpha$, *if* $i, j \in \tilde I_\alpha$ *we have*

$$\gamma^{ij}_\alpha = \begin{cases} \dfrac{\sqrt{\phi_{\omega(i)}\phi_{\omega(j)}}}{\beta\phi_v}\delta_{i_1,j_1}\dots\delta_{i_{n-1},j_{n-1}}, & v \neq 0, p-1, \bar p, \text{ and if } r<\infty, v \neq p+r-2 \\ \dfrac{\sqrt{\phi_{\zeta(i)}\phi_{\zeta(j)}}}{\beta\phi_{p-1}}\delta_{i_1,j_1}\dots\delta_{i_{n-1},j_{n-1}}, & v = p-1 \\ 1\delta_{i_1,j_1}\dots\delta_{i_{n-1},j_{n-1}}, & v = 0, \bar p, \text{ and if } r<\infty, v = p+r-2 \end{cases}$$

and if i, or $j \notin \tilde I_\alpha$, *then* $\gamma^{ij}_\alpha = 0$. *Here we have*

$$\omega(i) = \begin{cases} v-1 \text{ if } i_n = v-1 \\ v+1 \text{ if } i_n = v \end{cases}$$

and

$$\zeta(i) = \begin{cases} p-2 & \text{if } i_n = p-2 \\ \bar p & \text{if } i_n = \varepsilon = p-2+1/2 \\ p & \text{if } i_n = p-1. \end{cases}$$

Moreover if $\beta = (w, m+1) \in \hat T^{(0)}_{p,2,r}$, *with* $w \neq v$, *then* $G^i_\alpha e_m G^j_\beta = 0$ *for all* $i \in I_\alpha, j \in I_\beta$. *Finally we have* $G_{(m+1,m+1)}e_m = 0$.

Proof. First suppose that $v \neq 0, \overline{p}, p-1$, or if $r < \infty$, $v \neq p+r-2$, . Let $i \in I_\alpha$, then $i_{n-1} \leq v$, and so by Lemma 3.9(b) we have $i_k \leq v+n+k-1$, for $k = 1, 2, \ldots, n-1$. Then since $i_k + 2(k-1) + 3 \leq v + 2n - 1 = m$, for $k = 1, 2, \ldots, n-1$, it follows from Lemma 3.9(a) that

$$v_{\alpha,i} e_m = G_\alpha \gamma^{n-1}(\Delta_{i_n}) \ldots \Delta_{i_1} e_m = G_\alpha \gamma^{n-1}(\Delta_{i_n}) e_m \gamma^{n-2}(\Delta_{i_{n-1}}) \ldots \Delta_{i_1}.$$

But $e_m = \gamma^{n-1}(e_{v+1})$, and so by Lemma 3.9(c) or (d) we have

$$G_\alpha \gamma^{n-1}(\Delta_{i_n})\, e_m = \gamma^{n-1}(\gamma(g_v)\Delta_{i_n} e_{v+1}) = \sqrt{(\phi_{\omega(i_n)}/\beta\phi_v)}\beta^v \gamma^{n-1}(e_1 e_2 \ldots e_{v+1} g_v).$$

Now, it follows from the proof of Lemma 3.6(b) that for $j \in I_\alpha$

$$v_{\alpha,i} v^*_{\alpha,i} = G_\alpha \gamma^{n-1}(\Delta_{i_n}) e_m \gamma^{n-2}(\Delta_{i_{n-1}}) \ldots \Delta_{i_1} \Delta'_{j_n} \ldots \gamma^{n-2}(\Delta_{j_{n-1}})' e_m \gamma^{n-1}(\Delta_{j_n})' G_\alpha$$

vanishes if $(i_1, i_2, \ldots, i_{n-1}) \neq (j_1, j_2, \ldots, j_{n-1})$, otherwise using (3.66) we have

$$\begin{aligned} v_{\alpha,i} v'_{\alpha,j} &= \gamma^{n-1}(\gamma(g_v)\Delta_{i_n} e_{v+1})\gamma^{n-1}(g_{[i_{n-1}]})\gamma^{n-1}(e_{v+1}\Delta'_{j_n}\gamma(g_v)) \\ &= (\sqrt{(\phi_{\omega(i_n)}\phi_{\omega(j_n)})}/\beta\phi_v)\gamma^{n-1}(\beta^{2v} e_1 e_2 \ldots e_{v+1} g_v g_{[i_{n-1}]} g_v e_{v+1} \ldots e_1). \end{aligned}$$

But $g_{[i_{n-1}]} g_v = g_v$, and so by Lemma 3.3

$$v_{\alpha,i} v^*_{\alpha,j} = (\sqrt{(\phi_{\omega(i_n)}\phi_{\omega(j_n)})}/\beta\phi_v)\gamma^{n-1}(\gamma(g_v)) = \gamma^{ij}_\alpha \gamma^n(g_v) = \gamma^{ij}_\alpha G_\alpha.$$

Hence $G^i_\alpha e_m G^j_\alpha = \gamma^{ij}_\alpha G^{ij}_\alpha$.

Now suppose that $i \notin I_\alpha$, then either $i_n < v-1$, or $i_{n-1} = v+1$. If $i_n < v-1$, then by Lemma 3.9(b), $i_k \leq v+n-k-2$, for $k = 1, 2, \ldots, n$. Then since $i_k + 2(k-1) + 3 \leq v + 2n - 1 = m$, for $k = 1, \ldots, n$ we have

$$v_{\alpha,i} e_m = G_\alpha e_m \gamma^{n-1}(\Delta_{i_n}) \ldots \Delta_{i_1}$$

but this vanishes by Lemma 3.9(h). If $i_{n-1} = v+1$, then $i_k + 2(k-1) + 3 \leq m$ for $k = 1, 2, \ldots, n-2$, and so

$$v_{\alpha,i} e_m = G_\alpha \gamma^{n-1}(\Delta_{i_n})\gamma^{n-2}(\Delta_{i_{n-1}}) e_m \gamma^{n-3}(\Delta_{i_{n-2}}) \ldots \Delta_{i_1}.$$

Now if $i_{n-1} = v+1$, then $i_n = v$, thus since $e_m = \gamma^{n-2}(e_{v+3})$ we have

$$G_\alpha \gamma^{n-1}(\Delta_{i_n})\gamma^{n-2}(\Delta_{i_{n-1}}) = \gamma^{n-2}(\gamma^2(g_v)\gamma(\Delta_v)\Delta_{v+1} e_{v+3})$$

and so it follows from Lemma 3.9(e) that $v_{\alpha,i} e_m = 0$.

The proof that $G^i_\alpha e_m G^j_\alpha = \gamma^{ij}_\alpha G^{ij}_\alpha$, for $\alpha = (v, m+1)$, with $v = 0, p-1, \overline{p}$, or $p+r-2$ when $r < \infty$, is similar, using Lemma 3.9 (f) and (g) for example.

The proof that $G_{\alpha}^{i} e_m G_{\beta}^{j} = 0$ for $\alpha \neq \beta$, is essentially a combination of the above proof, and that of Lemma 3.6(c). Thus if $i \notin I_\alpha$, then $G_{\alpha}^{i} e_m = 0$, otherwise we proceed as above to get

$$\begin{aligned} v_{\alpha,i} e_m v_{\beta,j}^{*} &= G_\alpha \gamma^{n-1}(\Delta_{i_n}) \ldots \gamma^{k-1}(\Delta_{i_k}) e_m \gamma^{k-2}(\Delta_{i_{k-1}}) \ldots \Delta_{i_1} . \Delta_{j_1}^{*} \ldots \gamma^{t-2}(\Delta_{j_{t-1}})^{*} e_m \gamma^{t-1}(\Delta_{i_t})^{*} \\ &\qquad \ldots \gamma^{s-1}(\Delta_{i_s})^{*} G_\beta , \end{aligned}$$

and then either $\gamma^{k-2}(\Delta_{i_{k-1}}) \ldots \Delta_{i_1} . \Delta_{j_1}^{*} \ldots \gamma^{t-2}(\Delta_{j_{t-1}})^{*} = 0$, or more detailed arguments are necessary as in the proof of Lemma 3.6(c).

Finally we have $G_{(m+1,m+1)} e_m = g_{m+1} e_m = 0$.

Lemma 3.11. *For $m \geq 1$, we have $e_m = \sum \gamma_{\alpha}^{ij} G_{\alpha}^{ij}$ where the summation is over all vertices α on level $m+1$ of $\hat{T}_{p,2,r}$, and all $i, j \in I_\alpha$, and the coefficients $\gamma_{ij} \in \mathbf{C}$ are given in Proposition* 3.10.

Proof. By Lemma 3.8 we have $1 = \sum G_{\alpha}^{i} + u$, where we can take $u = 0$, if $r = \infty$, or if $m \leq p + r - 3$, otherwise note that $u e_k = 0$ for all k, and the summation is over all vertices α on level $m+1$ of $\hat{T}_{p,2,r}$, and $i \in I_\alpha$. It follows using Proposition 3.10 that

$$e_m = 1 e_m 1 = (\sum G_{\alpha}^{i} + u) e_m (\sum G_{\alpha}^{j} + u) = \sum G_{\alpha}^{i} e_m G_{\alpha}^{j} = \sum \gamma_{\alpha}^{ij} G_{\alpha}^{ij}.$$

Remark 3.12. It follows immediately that G_{α}^{i} is a minimal idempotent in $A(\tau, p)_{m+1}$ for each $\alpha = (v, m+1)$, on level $m+1$ of $\hat{T}_{p,2,r}$, and $i \in I_\alpha$.

Lemma 3.13. *Let $p \geq 2$, $\tau > 0$, and $e_1, e_2, \ldots, e_{\bar{p}}$ a sequence of projections satisfying the relations* (3.17)–(3.21). *If $\tau = \|A_{p+1}\|^{-2}$, then $A(\tau, p) \cong A(\tau)$, the Jones algebra with parameter τ, otherwise $A(\tau, p)$ is trivial unless*

$$\beta = 1/\sqrt{\tau} \in \{\|T_{p,2,r}\|; r \geq 2\} \cup [\|T_{p,2,\infty}\|, \infty).$$

Proof. We can clearly assume that if $\beta < 2$, then $\beta = 2\cos(\pi/m)$ for some $m \geq 3$. We first show that $\beta \leq \beta_1$ is not allowed. Suppose that $\beta = \beta_1$, then using Proposition 2.1(c) we have $(g_p e_p)^{*}(g_p e_p) = e_p g_p e_p = (\phi_p / B\phi_{p-1}) e_p g_{p-1} = 0$. Hence $g_p e_p = 0$. Next, from (3.12) and (3.50) we have $f_p = g_p + e_{\bar{p}}$, where $f_p = 1 - e_1 \vee \ldots \vee e_{p-1}$. Then since $\beta = \|A_{p+1}\|$, and $S_{p+1}(\beta) = 0$, we have $\beta S_{p-1}(\beta) / S_p(\beta) = \beta^2$. Thus

$$\begin{aligned} f_{p+1} = f_p - (\beta S_{p-1} / S_p) f_p e_p f_p &= f_p - \beta^2 f_p e_p f_p = g_p + e_{\bar{p}} - \beta^2 (g_p + e_{\bar{p}}) e_p (g_p + e_{\bar{p}}) \\ &= g_p + e_{\bar{p}} - \beta^2 e_{\bar{p}} e_p e_{\bar{p}} = g_p . \end{aligned}$$

It follows that $e_{\bar{p}} = f_p - f_{p+1}$ is in the C^*-algebra generated by $1, e_1, e_2, \ldots, e_p$.

The only other cases we need to consider for $\beta < \beta_1$, are when $\beta = \|A_K\| = 2\cos(\pi/(k+1)), k = 3,\ldots,p$. Then, if $f_k = 1 - e_1 \vee \ldots \vee e_{k-1}$, since $S_k(\beta) = 0$, we have $e_k f_k e_k = (S_k / \beta S_{k-1}) f_{k-1} e_k = 0$, and hence $f_k e_l = 0 = e_k f_k$. Then we have

$$0 = e_{k+1} f_k e_k e_{k+1} = f_k e_{k+1} e_k e_{k+1} = \beta^{-2} f_k \beta_{k+1}$$

and by induction $f_k e_k = 0$ for all $l \geq k$. It then follows that

$$1 - f_{k+1} = (1 - f_k) v e_k = e_k + (1 - f_k) - e_k(1 - f_k) = 1 - f_k,$$

and by induction that $1 - f_l = 1 - f_k$ for all $l \geq k$. In particular $f_p = f_{p+1}$. But $e_{\bar{p}} f_p = e_{\bar{p}}$, and so $e_{\bar{p}} \leq f_p = f_{p+1}$. It then follows that $(e_{\bar{p}} e_p)^* e_{\bar{p}} e_p = e_p e_{\bar{p}} e_p \leq e_p f_{p+1} e_p = 0$ and so $e_{\bar{p}} e_p = 0$. Thus $e_{\bar{p}} = \beta^2 e_{\bar{p}} e_p e_{\bar{p}} = 0$.

Now suppose that $\beta_r < \beta < \beta_{r+1}$, for $r \geq 1$, then since $\phi_{p+r-1} / \beta \phi_{p+r-2} < 0$, by Proposition 2.1(d) we have, putting $t = p + r - 1$,

$$0 < (g_t e_t g_t)^2 = (\phi_t / \beta \phi_{t-1}) g_t e_t g_t = (\phi_t / \beta \phi_{t-1})(e_t g_t)^* (e_t g_t) \leq 0,$$

and so $e_t g_t = 0$. Then using Proposition 2.1(d) again gives $0 = e_t g_t e_t = (\phi_t / \beta \phi_{t-1}) e_t g_{t-1}$, and so $e_t g_{t-1} = 0$.

If $r = 1$, then we have by (3.51)

$$0 = e_p g_p = e_p (g_{p-1} - (\beta \phi_{p-2} / \phi_{p-1}) g_{p-1} e_{p-1} g_{p-1} - g_{\bar{p}}) = -e_p g_{\bar{p}} = -e_p e_{\bar{p}}$$

and so $e_{\bar{p}} = \beta^2 e_p - e_p e_{\bar{p}} = 0$.

For $r \geq 2$, note first that $e_k g_t = 0$ for all k (see the proof of Proposition 3.7(e)), and it is clear also that when $m + 1 \geq t$, we can write the identity as $1 = \sum G_\alpha^\iota + g_t$, where α runs over all vertices on level $m + 1$ of $\hat{T}_{p,2,r}$ and $i \in I_\alpha$. We now show that if $m = 2t$, then $G_\alpha^\iota = 0$ for all α on level $m + 1$ of $\hat{T}_{p,2,r}$, and all $i \in I_\alpha$. Note that for $\alpha = (v, 2t+1) \in \hat{T}_{p,2,r}$, $d(v)$ is odd, and if $n = (2t + 1 - d(v))/2$, then we can assume $n \geq 2$. If $n = 2$, then $d(v) = 2t - 3 = 2(p + r - 1) - 3 \geq p + r - 1$, since $p, r \geq 2$, and so we can take $i = (i_1, i_2) \in I_\alpha$, with $i_1 = 0$, and $i_2 = t - 2$. Next we show that if $n > 2$, then we can choose $i \in I_\alpha$, with $i_1 = 0$, $i_2 = t - 2$. First note that by Lemma 3.9(b), $i_2 \leq i_n + (n - 2)$, and that

$$i_n + (n-2) = i_n + (2t + 1 - d(v))/2 - 2 = i_n - d(v)/2 - 1/2 + (t - 2).$$

Now if $v \neq \bar{p}$, $d(v) = v$ is odd, and so if we consider paths $i \in I_\alpha$, with $i_n = v$, then $i_n - d(v)/2 - 1/2 = v/2 - 1/2 \geq 0$, i.e. $i_n + (n - 2) \geq t - 2$. It follows that a path with $i_2 = t - 2$ is allowed. If $v = \bar{p}$, then $d(\bar{p}) = p$ is odd, and so $p \geq 3$. Thus taking $i \in I_\alpha$, with $i_n = p - 1$, we have $i_n - d(\bar{p})/2 - 1/2 = p - 1 - p/2 - 1/2 = (p - 3)/2 \geq 0$. Then since $i_n + (n - 2) \geq t - 1$, we can choose $i \in I_\alpha$, with $i_2 = t - 2$.

Next note that $\gamma(\Delta_{t-2})\gamma(g_{t-1}) = \gamma(\Delta_{t-2} g_{t-1}) = \gamma(\Delta_{t-2})$. But $g_t e_t = 0$, and so $\gamma(g_{t-1}) = 0$, which means $\gamma(\Delta_{t-2}) = 0$. But if $i \in I_\alpha$, is chosen as above with $i_2 = t - 2$, then it follows that $G_\alpha^\iota = 0$, and finally since G_α^j is equivalent to G_α^ι for all $j \in I_\alpha$, that $G_\alpha^j = 0$ for all $j \in I_\alpha$. It follows that $1 = g_t$, and so $e_m = 0$ for all $m \geq 1$.

Lemma 3.14. *Let* $\beta = 1/\sqrt{\tau} = \|T_{p,2,r}\|$, *for some* r, $1 \leq r \leq \infty$. *Suppose that there exists a faithful trace* tr, *satisfying* (3.24). *Then we have*

(a) $\mathrm{tr}\,(\gamma(x)) = \tau\,\mathrm{tr}(x)$

(b) $\mathrm{tr}\,(g_v) = Q_V(\tau)$, *for* $v \in T_{p,2,r}^{(0)}$

(c) $\mathrm{tr}\,(G_\alpha) = Q_\alpha(\tau)$, *for* $a \in \hat{T}_{p,2,r}^{(0)}$

where Q_V, Q_α *are as defined in* (2.15) *and* (2.19).

Proof. (a) For $x \in A(\tau, p)$, we have by Lemma 3.3 that

$$\gamma(x) = \tau^{-n} e_1 e_2 \dots e_{n+1} x e_{n+1} \dots e_2 e_1,$$

and so by (3.46), (3.24)

$$\begin{aligned}\mathrm{tr}\,(\gamma(x)) &= \tau^{-n}\ \mathrm{tr}(e_1 e_2 \dots e_{n+1} x e_{n+1} \dots e_2 e_1)\\ &= \tau^{-n}\ \mathrm{tr}(e_{n+1} \dots e_2 e_1 e_2 \dots e_{n+1} x)\\ &= \tau^{-n}\ \mathrm{tr}(\tau^n e_{n+1} x) = \mathrm{tr}(e_{n+1} x) = \tau\,\mathrm{tr}(x).\end{aligned}$$

(b) Now $g_0 = g_1 = 1$, and $Q_0 = Q_1 = 1$, and so (b) is true for $v = 0, 1$. For $v = 2, \dots, p-1$, we have

$$g_v = g_{v-1} - (\beta\phi_{v-2} / \phi_{v-1}) g_{v-1} e_{v-1} g_{v-1}$$

and so by (3.24), and noting that $\phi_v = \beta\phi_{v-1} - \phi_{v-2}$ we see

$$\begin{aligned}\mathrm{tr}\,(g_v) &= \mathrm{tr}\,(g_{v-1}) - (\beta\phi_{v-2} / \phi_{v-1})\,\mathrm{tr}\,(e_{v-1} g_{v-1})\\ &= (1 - (\phi_{v-2} / \beta\phi_{v-1}))\,\mathrm{tr}\,(g_{v-1})\\ &= (\phi_v / \beta\phi_{v-1})\,\mathrm{tr}\,(g_{v-1}).\end{aligned}$$

It follows that for $v = 2, \dots, p-1$,

$$\mathrm{tr}\,(g_v) = (\phi_v / \beta\phi_{v-1})(\phi_{v-1} / \beta\phi_{v-2}) \dots (\phi_1 / \beta\phi_0)\,\mathrm{tr}\,(g_0) = \phi_v / \beta^v = Q_v(\tau).$$

Next, by (3.20), (3.21) and (3.24), we have

$$\begin{aligned}\mathrm{tr}(e_{\bar{p}}) = \mathrm{tr}(g_{\bar{p}}) &= \tau^{-1}\mathrm{tr}(e_{\bar{p}} e_p e_{\bar{p}}) = \tau^{-1}\mathrm{tr}(e_p e_{\bar{p}} e_p)\\ = \tau^{-1}\mathrm{tr}(\tau e_p g_{p-1}) &= \mathrm{tr}(e_p g_{p-1}) = \tau\,\mathrm{tr}(g_{p-1}) = \tau Q_{p-1}(\tau) = Q_{\bar{p}}(\tau).\end{aligned}$$

Then by (3.14), (3.24), and the facts that $\phi_{\bar{p}} / \beta\phi_{p-1} = \beta^{-2}$, and $\beta\phi_{p-1} - \phi_{p-2} - \phi_{\bar{p}} = \phi_p$, we have

$$\begin{aligned}\mathrm{tr}(g_p) &= \mathrm{tr}(g_{p-1}) - (\beta\phi_{p-2} / \phi_{p-1})\,\mathrm{tr}\,(g_{p-1} e_{p-1}) - \mathrm{tr}\,(e_{\bar{p}})\\ &= (1 - (\phi_{p-2} / \beta\phi_{p-1}) - (1/\beta^2))\,\mathrm{tr}\,(g_{p-1}) = (\phi_p / \beta\phi_{p-1})(\phi_{p-1} / \beta^{p-1}) = Q_p(\tau).\end{aligned}$$

For $v > p$, one shows that $\operatorname{tr}(g_v) = (\phi_v / \beta\phi_{v-1}) \operatorname{tr}(g_{v-1})$, using (3.15), and (b) follows.
(c) Let $\alpha = (v, m)$, and $n = (m - d(v))/2$, then $G_\alpha = \gamma^n(g_v)$ and by (a) we have

$$\operatorname{tr}(\gamma^n(g_v)) = \tau^n \operatorname{tr}(g_v) = \tau^n Q_v(\tau) = Q_\alpha(\tau).$$

Proof of Theorem 3.1 *continued*. For τ as in (3.23) choose the corresponding r, $2 \le r \le \infty$, and define a map $\Psi: A(T_{p,2,r}) \oplus \mathbf{C}(1 - e_1 \vee \ldots \vee e_{p+r-2} \vee e_{\bar{p}}) \to A(\tau, p)$ as follows. Put $q = 1 - e_1 \vee \ldots \vee e_{p+r-2} \vee e_{\bar{p}}$. For $\alpha \in \hat{T}^0_{p,2,r}$, and $i, j \in I_\alpha = \text{Path}(*, \alpha)$

$$\Psi(f_{ij}) = G^{ij}_\alpha, \Psi(q) = q.$$

It is clear from Lemma 3.7 that this map is well defined. From Lemma 3.6 we see that it defines a $*$-homomorphism and by Lemmas 3.8, and 3.11, it is surjective. It remains only to show that the map is injective under the stated conditions.

When $r < \infty, A(T_{p,2,r})$ is simple, and so Ψ is injective. Suppose there exists a Markov trace. To show that Ψ is injective in this case, it is enough to show that $\operatorname{tr}(G_\alpha) > 0$ for each $\alpha \in \hat{T}^0_{p,2,r}$. But by Lemma 3.14(c) we have $\operatorname{tr}(G_\alpha) = Q_\alpha(\tau)$, which we know is positive if $1/\tau \ge \|T_{p,2,\infty}\|$ (see Proposition 2.1).

Remark 3.15. The method employed in the proof of Theorem 3.1 should also work for infinite graph Γ of the type indicated by Figure 19. Here Γ is a tree, with an infinite branch which has attached to it a finite number of branches of length one, and a distinguished vertex $*$.

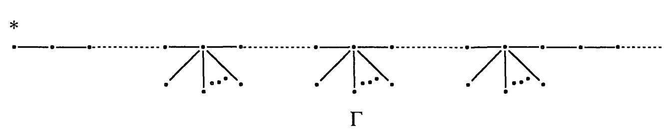

Figure 19

In these cases a presentation of $A(\Gamma)$ would be as follows. Let $\{e_v; v \in \Gamma^{(0)}\}$ be a set of projections indexed by the vertices of Γ, such that the following relations are satisfied:

$$e_v e_w = e_w e_v,\ d(v,w) \ge 2 \tag{3.69}$$

$$e_v e_w e_v = \tau e_v,\ d(v,w) = 1,\ v, w \notin \partial\Gamma / \{*\}, \text{ or } v \in \partial\Gamma / \{*\} \text{ and } w \notin \partial\Gamma / \{*\} \tag{3.70}$$

$$e_v e_w e_v = \tau f_v e_v,\ d(v,w) = 1,\ v, w \notin \partial\Gamma / \{*\},\ w \notin \partial\Gamma / \{*\} \tag{3.71}$$

$$e_w e_v = 0,\quad v, w \in \partial\Gamma / \{*\} \tag{3.72}$$

where $f_w = 1 - \bigvee_u e_u$, and the join is over all $u \in \Gamma^{(0)}$ such that $d(*, u) \le d(*, w) - 2$, and $\partial\Gamma$ denotes the boundary of Γ. This would include certain star shaped graphs considered in [17].

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