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The Converse of Minlos' Theorem

Dedicated to Professor Tsuyoshi Ando on the occasion of his sixtieth birthday

By

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Abstract

Let $\mathscr{M}$ be the class of barrelled locally convex Hausdorff space E such that E'_b satisfies the property B in the sense of Pietsch. It is shown that if $E \in \mathscr{M}$ and if each continuous cylinder set measure on E' is $\sigma(E', E)$ -Radon, then E is nuclear. There exists an example of non-nuclear Fréchet space E such that each continuous Gaussian cylinder set measure on E' is $\sigma(E', E)$-Radon. Let q be $2 \leq q < \infty$. Suppose that $E \in \mathscr{M}$ and E is a projective limit of Banach space $\{E_\alpha\}$ such that the dual E'_α is of cotype q for every α. Suppose also that each continuous Gaussian cylinder set measure on E' is $\sigma(E', E)$-Radon. Then E is nuclear.

§1. Introduction

Let E be a nuclear locally convex Hausdorff space, then each continuous cylinder set measure on E' is $\sigma(E', E)$-Radon (Minlos' theorem, see Badrikian [2], Gelfand and Vilenkin [4], Minlos [11], Umemura [20] and Yamasaki [21]). We consider the converse problem. Let E be a locally convex Hausdorff space. If each continuous cylinder set measure on E' is $\sigma(E', E)$-Radon, then is E nuclear? The partial answers are known as follows.

(1) If E is a σ-Hilbert space or a Fréchet space, then the answer is affirmative (see Badrikian [2], Gelfand and Vilenkin [4], Minlos, [11], Mushtari [12], Umemura [20] and Yamasaki [21]).

(2) If E is barrelled and if E is a projective limit of L^0-embeddable Banach spaces, then the answer is affirmative (see Millington [10], Mushtari [12], Okazaki and Takahashi [14]).

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In this paper, we shall extend the case (1) for more general locally convex spaces. We introduce a class $\mathscr{M}$ in Section 4. $\mathscr{M}$ is the class of all barrelled locally convex Hausdorff space E such that the strong dual E'_b satisfies the property B in the sense of Pietsch (Pietsch [15] 1.5.5). The class $\mathscr{M}$ contain LF-spaces, barrelled DF-spaces and inductive limits of them. We prove the next theorem.

Theorem. *Let $E \in \mathscr{M}$. If each continuous discrete 1-stable cylinder set measure on E' is $\sigma(E',E)$-Radon, then E is nuclear.*

For the Gaussian cylinder set measures, the following result is well-known.

(3) Let E be a σ-Hilbert space. If each continuous Gaussian cylinder set measure on E' is $\sigma(E',E)$-Radon, then E is nuclear (see Gelfand and Vilenkin [4], Minlos [11], Umemura [20] and Yamasaki [21]).

In general, we can not conclude that E is nuclear even if each continuous Gaussian cylinder set measure on E' is $\sigma(E',E)$-Radon. We give a counter example. In this case, we prove the next result.

Theorem. *Let $2 \leq q < \infty$ be fixed and $E \in \mathscr{M}$. Suppose that E is a projective limit of Banach spaces $\{E_\alpha\}$ such that the dual E'_α is of cotype q for every α. Suppose also that each continuous Gaussian cylinder set measure on E' is $\sigma(E',E)$-Radon. Then E is nuclear.*

§2. Preliminaries

Let E be a locally convex Hausdorff space and E' be the topological dual of E. Denote by E'_b (resp. E'_s) the dual with the strong dual topology $\beta(E',E)$ (resp. weak $*$ topology $\sigma(E',E)$). The strong bidual of E is denoted by $(E'_b)'_b$. Let μ be a cylinder set measure on E'. Then we say that μ is a continuous cylinder set measure if the characteristic functional

$$\mu^\wedge(x) = \int_{E'} e^{i\langle x,a\rangle} d\mu(a), \quad x \in E,$$

is continuous on E.

The cylinder set measure μ on E' is called a continuous discrete p-stable cylinder set measure on E' if the characteristic functional $\mu^\wedge(x)$ is given by

$$\mu^\wedge(x) = \exp(-\|T(x)\|^p_{\ell_p}), \quad x \in E,$$

where $T: E \to \ell_p$ is a continuous linear operator and $0 < p \leq 2$. In the sequel, we consider only the cases $p = 1$ and 2. In the case where $p = 2, \mu$ is called a continuous Gaussian cylinder set measure. See Linde [7].

Let F, G be normed spaces and $0 < q, r < \infty$. A linear operator $S: F' \to G'$ is called (q,r)-summing if for every $\{a_n\} \subset F'$ with $\sum_{n=1}^{\infty} |\langle x, a_n\rangle|^r < \infty$ for every $x \in F$, it holds that $\sum_{n=1}^{\infty} \|S(a_n)\|_{G'}^q < \infty$. A linear operator $T: F \to G$ is called (q,r)-summing if for every $\{x_n\} \subset F$ with $\sum_{n=1}^{\infty} |\langle x_n, a\rangle|^r < \infty$ for every $a \in F'$, it holds that $\sum_{n=1}^{\infty} \|T(x_n)\|_G^q < \infty$. In the case where $r = q$, S and T are called r-summing, see Pietsch [15], Schwartz [18] and Tomczak-Jaegermann [19].

Let G be a Banach space and $2 \le q < \infty$. G is called of cotype q if there exists $K > 0$ such that for every n and every $z_1, z_2, \ldots, z_n \in G$, it holds that

$$\left[\sum_{i=1}^{n} \|z_i\|_G^q\right]^{1/q} \le K \int_{\Omega} \|\sum_{i=1}^{n} g_i(\omega) z_i\|_G \, dP(\omega),$$

where $\{g_i\}$ is a sequence of independent identically distributed Guassian random variables on a probability space (Ω, P) with the characteristic functional $e^{-|t|^2}$, see Linde [7], Maurey and Pisier [9], Tomczak-Jaegermann [19].

Let E be a locally convex Hausdorff space. For a closed absolutely convex neighborhood U of 0, we set $N(U) = \{x \in E: p_U(x) = 0\}$ where $p_U(x) = \inf\{t > 0: x \in tU\}$. Denote by $x(U)$ the equivalence class corresponding to $x \in E$ in the quotient space $E(U) = E/N(U)$. $E(U)$ is a normed space with norm $p[x(U)] = p_U(x)$ for $x \in E$.

For a closed absolutely convex bounded subset A of E, we set $E(A) = \{x \in E: x \in tA$ for some $t > 0\}$. $E(A)$ is a linear subspace of E. We put the norm on $E(A)$ by $p_A(x) = \inf\{t > 0: x \in tA\}$ for $x \in E(A)$.

For a neighborhood U of 0 in E, the polar $U^\circ = \{a \in E': |\langle x, a\rangle| \le 1$ *for* every $x \in U\}$ is weakly compact absolutely convex subset of E'_s. The normed space $E'(U^\circ)$ is a Banach space and $E(U)' = E'(U^\circ)$ by the duality $\langle x(U), a\rangle = \langle x, a\rangle$.

For two zero neighborhoods U, V with $V \subset U$, we define a canonical mapping $E(V,U): E(V) \to E(U)$ by associating $x(U)$ with $x(V)$.

For two closed absolutely convex bounded subsets A and B with $A \subset B$, it holds that $E(A) \subset E(B)$ and the canonical mapping $E(A,B): E(A) \to E(B)$ is defined by $E(A,B)(x) = x$ for $x \in E(A)$.

A locally convex Hausdorff space E is called nuclear if it contains a fundamental system $U_F(E)$ of zero neighborhoods which has the following equivalent properties (see Pietsch [15], 4.1.2):

(N_1) *For each* $U \in U_F(E)$ *there exists* $V \in U_F(E)$ *with* $V \subset U$ *such that the canonical mapping* $E(V,U): E(V) \to E(U)$ *is* 2*-summing.*
(N_2) *For each* $U \in U_F(E)$ *there exists* $V \in U_F(E)$ *with* $V \subset U$ *such that the canonical mapping* $E'(U^\circ, V^\circ): E'(U^\circ) \to E'(V^\circ)$ *is* 2*-summing.*

A locally convex Hausdorff space E is called dual nuclear if the strong dual E_b' is nuclear. For other basic notions of locally convex spaces, we refer to Schaefer [17].

§3. Summability and Dual Nuclearity

Let E be a locally convex Hausdorff space and $1 \leq p < \infty$. A sequence $(x_n) \subset E$ is called weakly p-summable if for every neighborhood U of 0, it holds that

$$\varepsilon_U^p((x_n)) = \sup\{(\sum_{n=1}^{\infty} |\langle x_n, a\rangle|^p)^{1/p} : a \in U^\circ\} < \infty$$

Denote by $l^p[E]$ the linear space of all weakly p-summable sequences. The topology of $l^p[E]$ given by the seminorms ε_U^p, $U \in U_F(E)$, is called the ε-topology where $U_F(E)$ is a fundamental system of zero neighborhoods of E.

A sequence $(x_n) \subset E$ is called absolutely p-summable if for every neighborhood U of 0, it holds that

$$\pi_U^p((x_n)) = (\sum_{n=1}^{\infty} p_U(x_n)^p)^{1/p} < \infty.$$

Denote by $l^p\{E\}$ the linear space of all absolutely p-summable sequences. The topology of $l^p\{E\}$ given by the seminorms π_U^p, $U \in U_F(E)$, is called the π-topology, where $U_F(E)$ is a fundamental system of zero neighborhoods of E. It holds that $(l^p\{E\}, \pi^p) \subset (l^p\{E\}, \varepsilon^p)$, where the inclusion is a continuous injection.

A sequence $(x_n) \subset E$ is called totally p-summable if there exists a closed absolutely convex bounded subset B such that $\sum_{n=1}^{\infty} p_B(x_n)^p < \infty$. Denote by $l^p\langle E\rangle$ the linear space of all totally p-summable sequences. It is clear that $l^p\langle E\rangle \subset l^p\{E\}$.

It is called that E has property B if for each bounded subset $\mathscr{B} \subset l^1\{E\}$ there exists a bounded set $B \subset E$ such that $\sum_{n=1}^{\infty} p_B(x_n) \leq 1$ for every $(x_n) \in \mathscr{B}$, see Pietsch [15], 1.5.5. If E has property B, then it holds that $l^1\{E\} \subset l^1\langle E\rangle$.

The nuclearity of the strong dual E_b' is characterized by the above summabilities as follows.

Lemma 1 (*Pietsch* [15] *Theorem* 4.2.11). *If E has property B and $l^1[E] = l^1\{E\}$, then E_b' is nuclear.*

It is known that the metrizable space or the dual metrizable space has property B (Pietsch [15] Theorem 1.5.8). We prove that the property B is retained by the projective or inductive limit operation.

Proposition 1. (1) *If each* E_n *has property B, then the projective limit* $\varprojlim E_n$ *has property B.*

(2) *Let* $E = \varinjlim E_n$ *be the strict inductive limit. Suppose that each* E_n *has property B and every bounded set B of E is contained and bounded in* E_k *for some k (k depends on B). Then E has property B.*

Proof. (1) Let $\mathscr{B}$ be bounded in $l^1\{E\}$. Let $\pi_n : E \to E_n$ be the canonical mapping. Then $\pi_n(\mathscr{B}) = \{(\pi_n x_i)_{i=1}^\infty : (x_i) \in \mathscr{B}\}$ is bounded in $l^1\{E_n\}$ for every n. By the property B of E_n, there exists a bounded set B_n in E_n such that $\sup\{\sum_{i=1}^\infty p_{B_n}(\pi_n x_i) : (x_i) \in \mathscr{B}\} \le 1$ for every n. We set $B = \{x \in E : \sum_{n=1}^\infty 2^{-n} p_{B_n} \cdot (\pi_n x) \le 1\}$. Then B is bounded in E and it holds that $p_B(x) = \sum_{n=1}^\infty 2^{-n} p_{B_n}(\pi_n x)$. So we have $\sum_{i=1}^\infty p_B(x_i) \le \sum_{i=1}^\infty 2^{-n} < \infty$ for every $(x_i) \in \mathscr{B}$.

(2) Let $\mathscr{B} \subset l^1\{E\}$ be bounded. Then the subset $C = \{x_i : i = 1, 2, \ldots, (x_j) \in \mathscr{B}\}$ is bounded in E. There exists k so that $C \subset E_k$ and C is bounded in E_k. Since E induces the topology on E_k, $\mathscr{B}$ is contained in $l^1\{E_k\}$ and bounded in $l^1\{E_k\}$. Hence there exists a bounded subset B in E_k such that $\sum_{i=1}^\infty p_B(x_i) \le 1$ for every $(x_i) \in \mathscr{B}$. This proves (2).

We investigate the property B of the strong dual E_b'.

Lemma 2. *Let* $E = \varinjlim E_n$ *be the inductive limit of locally convex spaces. If E is barrelled and if each* $(E_n)_b'$ *has property B, then* E_b' *has property B.*

Proof. Let $\mathscr{B} \subset l^1\{E_b'\}$ be bounded, that is, $\sup\{\sum_{i=1}^\infty p_{B^\circ}(a_i) : (a_i) \in \mathscr{B}\} < \infty$ for every bounded subset B in E. Let $\pi_n : E' \to E_n'$ be the canonical mapping. For every n, $\{(\pi_n(a_i)) : (a_i) \in \mathscr{B}\}$ is bounded in $l^1\{(E_n)_b'\}$ since each bounded set in E_n is also bounded in E. For every n, take a closed absolutely convex bounded set $K_n \subset (E_n)_b'$ such that $\sum_{i=1}^\infty p_{K_n}(\pi_n(a_i)) \le 1$ for every $(a_i) \in \mathscr{B}$. We set $K = \{a \in E' : \sum_{n=1}^\infty 2^{-n} p_{K_n}(a) \le 1\}$. Then K is bounded in E_b' since $\pi_n(K)$ is bounded in $(E_n)_b'$ for every n and E is barrelled (in fact, K° absorbs each point in E). We have $p_K(a) = \sum_{n=1}^\infty 2^{-n} p_{K_n}(a)$ for every $a \in E'(K)$. For each $(a_i) \in \mathscr{B}$, we obtain $\sum_{i=1}^\infty p_K(a_i) = \sum_{n=1}^\infty 2^{-n} (\sum_{i=1}^\infty p_{K_n}(\pi_n(a_i)) \le \sum_{n=1}^\infty 2^{-n} < \infty$. Thus E_b' has property B.

Proposition 2. *Let E be either*

(1) *metrizable,*

(2) *dual metrizable,*

(3) *LF-space,*

(4) *dual LF-space, or*

(5) $E=\varinjlim E_n$ *and E is barrelled, where* E_n *is one of* (1), (2), (3) *and* (4) *above. Then* E_b' *has property B.*

The next Lemma shall be used in Section 4, Theorem 2.

Lemma 3. *Let q be* $1\leq q<\infty$. *If E has property B, then for each bounded subset* $\mathscr{B}\subset l^q\{E\}$ *there exists a bounded set* $B\subset E$ *such that* $\sum_{n=1}^{\infty}p_B(x_n)^q\leq 1$ *for every* $(x_n)\in\mathscr{B}$.

Proof. Let s be $1/q+1/s=1$. Then the family $\mathscr{A}=\{(t_ix_i):(x_i)\in\mathscr{B}$ and $\|(t_i)\|_{\ell_s}\leq 1\}$ is bounded in $l^1\{E\}$ since it holds that for every zero neighborhood U $\sum_{i=1}^{\infty}p_U(t_ix_i)\leq(\sum_i|t_i|^s)^{1/s}(\sum_i p_U(x_i)^q)^{1/q}\leq(\sum_i p_U(x_i)^q)^{1/q}$ and since $\mathscr{B}$ is bounded in $\ell^q\{E\}$. By property B, there exists a bounded set B of E such that for every $(t_ix_i)\in\mathscr{A}$ it holds $\sum_{i=1}^{\infty}p_B(t_ix_i)=\sum_{i=1}^{\infty}|t_i|\,p_B(x_i)\leq 1$. Thus for every $(u_i)\in\ell_s$ with $\|u_i\|_{\ell_s}\leq 1$, we have $|\sum_i u_ip_B(x_i)|\leq\sum_i p_B(|u_i|x_i)\leq 1$. By the duality of ℓ_s and ℓ_q, it follows that $\sum_{i=1}^{\infty}p_B(x_i)^q\leq 1$ for every $(x_i)\in\mathscr{B}$, which shows the assertion.

§4. Converse of Minlos' Theorem

Lemma 4. *Let F, G be Banach spaces,* $\psi:G\to F$ *be a continuous linear mapping and* $\psi':F'\to G'$ *be the adjoint of* ψ. *Let* $(a_i)\subset F'$ *be* $\sum_{i=1}^{\infty}|\langle x,a_i\rangle|<\infty$ *for every* $x\in F$ *and* μ *be a continuous discrete* 1*-stable cylinder set measure on* F' *with* $\mu^\wedge(x)=\exp(-\sum_{i=1}^{\infty}|\langle x,a_i\rangle|)$. *Suppose that the image* $\psi'(\mu)$ *is* $\sigma(G',G)$*-Radon on* G_s'. *Then it holds that* $\sum_{i=1}^{\infty}\|\psi'(a_i)\|_{G'}<\infty$.

Proof. We follow Linde [7], Cor. 6.5.2 and Maurey [8], Prop.2b). For every N let λ_N, τ_N be the cylinder set measures on G' with

$$\lambda_N{}^\wedge(z)=\exp(-\sum_{n=1}^{N}|\langle z,\psi'(a_n)\rangle|)$$

$$\tau_N{}^\wedge(z)=\exp(-\sum_{n=N+1}^{\infty}|\langle z,\psi'(a_n)\rangle|),\ z\in G.$$

Then we have $\lambda_N*\tau_N=\psi'(\mu)$ as cylinder set measures, where $*$ denotes the convolution. Since $\psi'(\mu)$ is $\sigma(G',G)$-Radon, λ_N and τ_N are also $\sigma(G',G)$-Radon, see Okazaki [13], Lemma 1.

For $0<q<1$ it holds that

$$\int_{G'} \|a\|_{G'}^q d\lambda_N(a) = \int_{G'}\int_{G'} \|a\|_{G'}^q d\lambda_N(a) d\tau_N(b)$$

$$\leq 2^{-q}\int_{G'}\int_{G'} (\|a+b\|_{G'}^q + \|a-b\|_{G'}^q) d\lambda_N(a) d\tau_N(b)$$

$$\leq 2^{1-q}\int_{G'} \|a\|_{G'}^q d\psi'(\mu)(a),$$

since $\|2a\|_{G'}^q \leq \|a+b\|_{G'}^q + \|a-b\|_{G'}^q$ and τ_N is symmetric, see Hoffmann-Jørgensen [4], Theorem 2.6.

Let $\{f_n(\omega)\}$ be a sequence of independent identically distributed symmetric 1-stable random variables on a probability space (Ω, P) with the characteristic functional $e^{-|t|}$. Let q be fixed such that $0 < q < 1$. For every N, we set

$$S_N(\omega) = \sum_{n-1}^{N} \psi'(a_n) f_n(\omega).$$

S_N is a random variable which values in a finite-dimensional subspace of G' and the distribution of S_N is λ_N. If we set

$$H_N(\omega) = \operatorname*{Max}_{1\leq n\leq N} \| \psi'(a_n) f_n(\omega)\|_{G'}$$

then by Kwapien [6], Remark 1, it follows that

$$\int_\Omega H_N(\omega)^q d\mathrm{P}(\omega) \leq 8\int_\Omega \|S_N(\omega)\|_{G'}^q d\mathrm{P}(\omega)$$

$$= 8\int_{G'} \|a\|_{G'}^q d\lambda_N(a).$$

Consequently, we have

$$\int_\Omega H_N(\omega)^q d\mathrm{P}(\omega) \leq 8 \;\; 2^{1-q}\int_{G'} \|a\|_{G'}^q d\psi'(\mu)(a).$$

Since $\psi'(\mu)$ is a 1-stable $\sigma(G', G)$-Radon measure on G' and $0 < q < 1$, we have

$$L = \int_{G'} \|a\|_{G'}^q d\psi'(\mu)(a) < \infty,$$

see de Acosta [1], Linde [7] , Cor. 6.7.5. Thus we have

$$\int_\Omega \operatorname*{Max}_{1\leq n\leq N} \| \psi'(a_n) f_n(\omega)\|_{G'}^q d\mathrm{P}(\omega) \leq 8 \;\; 2^{1-q} L < \infty$$

for every $N = 1, 2, \cdots$. Letting $N \to \infty$, we have

$$\int_\Omega \sup_n \| \psi'(a_n) f_n(\omega)\|_{G'}^q dP(\omega) < \infty$$

Hence there exists $R > 0$ such that

$$P(\omega: \sup_n \| \psi'(a_n) f_n(\omega)\|_{G'} \leq R) = \prod_{n=1}^{\infty} \{1 - P(\omega : |f_n(\omega)| > R/\| \psi'(a_n)\|_{G'})\} > 0,$$

where we have used the independence of $\{f_n(\omega)\}$. This implies that

$$\sum_{n-1}^{\infty} P(\omega : |f_n(\omega)| > R/\| \psi'(a_n)\|_{G'} < \infty .$$

We remark that for every n.

$$\int_\Omega \| \psi'(a_n) f_n(\omega)\|_{G'}^q dP(\omega) = \| \psi'(a_n)\|_{G'}^q \int_\Omega |f_n(\omega)|^q dP(\omega) \leq 8\ 2^{1-q} L,$$

that is, $\sup_n \| \psi'(a_n)\|_{G'} < \infty$. Furthermore, it is known that $P(\omega: |f_n(\omega)| > t) \sim t^{-1}$ as $t \to \infty$, so we obtain for sufficiently large R.

$$P(\omega: |f_n(\omega)| > R / \| \psi'(a_n)\|_{G'}) \sim \| \psi'(a_n)\|_{G'} / R.$$

Hence it follows that $\sum_{n=1}^{\infty} \| \psi'(a_n)\|_{G'} < \infty$.

Remark 1. If $\psi'(\mu)$ is Radon with respect to the dual norm of G', then Lemma 4 is a direct consequence of the fact "every Banach space is of cotype 1-stable", see Linde [7], Cor. 6.5.2 and Maurey [8], Prop. 2 b).

Lemma 5. *Let E be a barrelled locally convex Hausdorff space. Suppose that each continuous discrete* 1*-stable cylinder set measure on* E' *is* $\sigma(E', E)$*-Radon. Then it holds that* $l^1[E'_s] = l^1\langle E'_b\rangle$.

Proof. Let $(a_i) \in l^1[E'_s]$, that is, $\sum_{i=1}^{\infty} |\langle x, a_i\rangle| < \infty$ for every $x \in E$. Since the semi-norm $|x| = \sum_{i=1}^{\infty} |\langle x, a_i\rangle|$ is lower semicontinuous on E, $|x|$ is continuous by the barrelledness. The continuous discrete 1-stable cylinder set measure μ on E' with $\mu^\wedge(x) = \exp(-\sum_{i=1}^{\infty} |\langle x, a_i\rangle|), x \in E$, is $\sigma(E', E)$-Radon. We can take a $\sigma(E', E)$-compact set $K \subset E'_s$ of the form $K = U^\circ, U \in U_F(E)$, satisfying that $\mu(K) > 0$ and $|\mu^\wedge(x) - 1| < 1/2$ for $x \in U$ by the barrelledness and the continuity of $\mu^\wedge(x)$. Consider the Banach space $E'(K) = \cup nK$ with the unit ball K. By the 0-1 law of a stable measure, it follows that $\mu(E'(K)) = 1$, see Dudley and Kanter [3]. Thus μ

is a $\sigma(E'(K), E(U))$-Radon measure. We claim that $(a_i) \subset E'(K)$. Take $\ell < \infty$ so that $|\exp(-|t|)-1| < 1/2$ implies $|t| < \ell$. Hence for every $x \in U$, it follows that $|\langle x, a_i \rangle| < \ell$ and $a_i \in \ell U^\circ = \ell K$, that is, $(a_i) \subset \ell K$. By Lemma 4, we obtain $\sum_{i=1}^{\infty} p_K(a_i) < \infty$, which shows $(a_i) \in l^1\langle E'_b \rangle$.

We introduce a class $\mathscr{M}$ of locally convex spaces as follows. $\mathscr{M}$ is the set of all barrelled locally convex Hausdorff space E such that the strong dual E'_b has property B. $\mathscr{M}$ contain LF-spaces and barrelled DF-spaces. $\mathscr{M}$ is closed under the operation taking a countable inductive limit (Proposition 2).

Theorem 1. *Let $E \in \mathscr{M}$ and suppose that every continuous discrete 1-stable cylinder set measure on E' is $\sigma(E', E)$-Radon. Then E is nuclear.*

Proof. By Lemma 5, we have $l^1[E'_b] = l^1\{E'_b\}$. By Lemma 1, it follows that $(E'_b)'_b$ is nuclear. Since E is barrelled, the topology of E is induced from $(E'_b)'_b$, which proves the Theorem.

In Theorem 1, we can not replace "1-stable" by "Gaussian" in general. We give an example later on.

Lemma 6. *Let $2 \le q < \infty$, E be a Banach space and $G = E'_b$ be the dual Banach space. Suppose that G is of cotype q. Let $(a_n) \subset G$ be $\sum_n |\langle x, a_n \rangle|^2 < \infty$ for every $x \in E$. If the continuous Gaussian cylinder set measure μ with $\mu^\wedge(x) = \exp(-\sum_n |\langle x, a_n \rangle|^2)$ is $\sigma(G, E)$-Radon, then it holds that $\sum_{n=1}^{\infty} \|a_n\|_G^q < \infty$.*

Proof. For every N, let λ_N, τ_N be the cylinder set measures on G with

$$\lambda_N{}^\wedge(x) = \exp(-\sum_{n=1}^{N} |\langle x, a_n \rangle|^2),$$

$$\tau_N{}^\wedge(x) = \exp(-\sum_{n=N+1}^{\infty} |\langle x, a_n \rangle|^2),\ x \in E.$$

Then we have $\lambda_N * \tau_N = \mu$. Since G is of cotype q, there exists $K > 0$ such that

$$\left[\sum_{n=1}^{N} \|a_n\|_G^q\right]^{1/q} \le K \int_Q \| \sum_{n=1}^{N} a_n g_n(\omega) \|_G \, dP(\omega)$$

$$= K \int_G \|a\|_G \, d\lambda_N(a)$$

$$\leq \int_G \|a\|_G \, d\mu(a)$$

for every N by the manner same to Lemma 4. Since μ is Gaussian, this last integral is finite, which implies the assertion.

Theorem 2. *Let q be $2 \leq q < \infty$. Let E be a locally convex Hausdorff space with a fundamental system $\{U_\alpha\}$ of zero neighborhoods such that the dual $E(U_\alpha)' = E'(U_\alpha^\circ)$ is of cotype q . Suppose that $E \in \mathscr{M}$ and each continuous Gaussian cylinder set measure on E' is $\sigma(E', E)$-Radon. Then E is nuclear.*

Proof. Firstly, we show that $l^2[E_b'] \subset l^q\langle E_b'\rangle \subset l^q\{E_b'\}$. Let $(a_i) \in l^2[E_b']$, that is, for every zero neighborhood W of E_b', $\sup\left\{\sum_{i=1}^{\infty}|\langle x, a_i\rangle|^2 : x \in W^\circ\right\} < \infty$. Then $h(x) = \left(\sum_{i=1}^{\infty}|\langle x, a_i\rangle|^2\right)^{1/2}$ is continuous on E since E is barrelled and $h(x)$ is lower semicontinuous. Hence $\exp(-h(x)^2)$ determines continuous Gaussian cylinder set measure μ on E' with $\mu^\wedge(x) = \exp(-h(x)^2)$ taking $T: E \to \ell_2$ be $T(x) = (\langle x, a_i\rangle)$. By the assumption, μ is $\sigma(E', E)$-Radon and so there exists α such that $\mu(E'(U_\alpha^\circ)) = 1$ by the 0-1 law of a Gaussian measure, see the proof of Lemma 5. Since $E'_{U_\alpha^\circ}$ is of cotype q it follows that $\sum_{i=1}^{\infty} p_{U_\alpha^\circ}(a_i)^q < \infty$ by Lemma 6.

Secondly, we show that each bounded set $\mathscr{B}$ in $l^2[E_b']$ is bounded also in $l^q[E_b']$. For every zero neighborhood W in E_b', there exists $M_w > 0$ such that

$$\sup_{(a_i) \in \mathscr{B}} \sup\left\{\sum_{i=1}^{\infty}|\langle x, a_i\rangle|^2 : x \in W^\circ\right\} < M_w.$$

Suppose that $\mathscr{B}$ is not bounded in $l^q\{E_b'\}$, that is, there is a zero neighborhood V in E_b' such that $\sup\left\{\sum_{i=1}^{\infty} p_v(a_i)^q : (a_i) \in \mathscr{B}\right\} = \infty$. For every n, take N_n and $(a_i^n) \in \mathscr{B}$ such that $\sum_{i=1}^{N_n} p_v(a_i^n)^q > 2^{nq}$. Remark that $\sup\left\{\sum_{i=1}^{\infty}|\langle x, a_i^n\rangle|^2 : x \in V^\circ, n = 1, 2, \ldots\right\} = C_v^2 < \infty$ since $\mathscr{B}$ is bounded in $l^2[E_b']$. Then we have for the sequence $\{2^{-n} a_i^n : 1 \leq i \leq N_n, n = 1, 2, \ldots\}$ and for every $x \in V^\circ, \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} |\langle x, 2^{-n} a_i^n\rangle|^2 \leq \sum_{n=1}^{\infty} 2^{-2n} C_v^2 < \infty$. On the other hand, $\sum_{n=1}^{\infty} \sum_{i=1}^{N_n} p_v(2^{-n} a_i^n)^q \geq \sum_{n=1}^{\infty} 2^{-nq} 2^{nq} < \infty$, which contradicts to $l^2[E_b'] \subset l^q\{E_b'\}$.

Thirdly, we prove that for every α, there exists β such that $E'(U_\alpha^\circ, U_\beta^\circ): E'(U_\alpha^\circ) \to E'(U_\beta^\circ)$ is $(q, 2)$-summing. For every $\alpha, A = U_\alpha^\circ$ is a bounded set in E_b'. We set $\mathscr{B} = \{(a_i) \in l^2[E_b'] : \varepsilon_A^2((a_i)) = \sup\{\sum_i |\langle x, a_i\rangle|^2 :$

$x \in A^\circ\} \le 1\}$. Since A is bounded in E'_b, $\mathscr{B}$ is bounded in $l^2[E'_b]$. Thus $\mathscr{B}$ is bounded in $l^2[E'_b]$ by the second step. By Lemma 3, there exists a bounded absolutely convex closed subset B in E'_b such that $\sum_{i=1}^{\infty} p_B(a_i)^q \le 1$ for every $(a_i) \in \mathscr{B}$. We can assume that $B = U_\beta^\circ$ for some β with $U_\beta \subset U_\alpha$ since E is barrelled. So we obtain $(\sum_n p_{U_\beta^\circ}(a_n)^q)^{1/q} \le \sup\{(\sum_n |\langle x, a_n\rangle|^2)^{1/2};\ p_{U_\alpha^\circ}(x) \le 1\}$, which shows the assertion.

Lastly, we show that for every α there exists β such that $E'(U_\alpha^\circ, U_\beta^\circ): E'(U_\alpha^\circ) \to E'(U_\beta^\circ)$ is 2-summing. Let α be arbitrarily fixed. By the third step, there exists α_1 such that the canonical injection $E'(U_\alpha^\circ, U_{\alpha_1}^\circ)$ is $(q,2)$-summing. Similarly we can find α_2 such that $E'(U_{\alpha_1}^\circ, U_{\alpha_2}^\circ)$ is $(q,2)$-summing. Repeatedly, we can find $\alpha_1, \alpha_2, \dots, \alpha_k$ such that $E'(U_{\alpha_i}^\circ, U_{\alpha_{i+1}}^\circ)$ is $(q,2)$-summing for every i. Let k be $k > q/2$. Then the k-composition $E'(U_\alpha^\circ, U_{\alpha_k}^\circ) = E'(U_{\alpha_{k-1}}^\circ, U_{\alpha_k}^\circ) \circ \cdots \circ E'(U_\alpha^\circ, U_{\alpha_1}^\circ)$ is 2-summing by Tomczak-Jaegermann [19], Theorem 22.5, since each $E'(U_{\alpha_k}^\circ)$ is of cotype q. This completes the proof.

Remark 2. In general, E is not necessarily nuclear even if each continuous cylinder set measure on E' is $\sigma(E',E)$-Radon. For example, let τ_s be the Sazonov topology on the infinite-dimensional Hilbert space H and consider $E = (H, \tau_s)$, see Sazonov [16]. Then E is not nuclear but each continuous cylinder set measure on E' is $\sigma(E',E)$-Radon, see Yamasaki [21], §20.

Counterexample. Let E be a Fréchet space. Suppose that each continuous Gaussian cylinder set measure on E' is σ-additive. Then can we conclude that E is nuclear? The answer is in general negative. We give a counterexample. The following result is well-known, see Schwartz [16].

Lemma 7. *Let G, F be Banach spaces and $\psi: G \to F$ be a continuous linear operator. Let ψ' be the adjoint of ψ and $0 < r < \infty$. Suppose that ψ' is r-summing. Then for every Gaussian cylinder set measure μ on F', the image $\psi'(\mu)$ is $\sigma(G',G)$-Radon.*

Example. Let $D_j = (n^{-(1/2)^{j+1}})_{n=1}^{\infty} : \ell_1 \to \ell_1$ be the diagonal operator given by

$$D_j((x_n)) = (n^{-(1/2)^{j+1}} x_n)_{n=1}^{\infty},\ (x_n) \in \ell_1 .$$

Let E be the projective limit of $\{\ell_1, D_j\}_{j=1}^{\infty}$. Explicitely, E is given by $E = \left\{(x_n) \in R^\infty : \sum_{n=1}^{\infty} n^{-(1/2)^{j+1}} |x_n| < \infty \text{ for each } j\right\}$. Let $E_j = \left\{(x_n) : |(x_n)|_j = \sum_n n^{-(1/2)^{j+1}} |x_n| < \infty\right\}$ with seminorm $|\ \ |_j$. Then we have $E = \cap_j E_j$ and

$$\begin{array}{ccccc} & E_{j-1} & \xleftarrow{\text{identity}} & E_j & \\ (n^{(1/2)^j}) \text{ isometry} & \uparrow & & \downarrow & (n^{-(1/2)^{j+1}}) \text{ isometry} \\ & \ell_1 & \xleftarrow[D_j]{} & \ell_1 & \end{array}$$

Then the dual E' is the inductive limit of $\{\ell_\infty, D_j\}$, where $D_j: \ell_\infty \to \ell_\infty$ be $D_j((x_n)) = (n^{-(1/2)^{j+1}} x_n)$. For every k, the composition $D_k \circ D_{k-1} \circ \ldots \circ D_1 : \ell_\infty \to \ell_\infty$ is the diagonal operator $(n^{-1/2+(1/2)^{k+1}})$, which is not 2-summing for every k. Remark that the diagonal operator $A = (a_n): \ell_\infty \to \ell_\infty$ (or into ℓ_r is r-summing if and only if $(a_n) \in \ell_r$. Thus E is not nuclear. We remark that each D_j is 2^{j+2}-summing since

$$\sum_{n=1}^{\infty} (n^{-(1/2)^{j+1}})^{2^{j+2}} = \sum_{n=1}^{\infty} n^{-2} < \infty .$$

By Lemma 7, for each continuous Gaussian cylinder set measure on E'_j is σ-additive on E'_{j+1} since the natural injection $\iota_{j+1,j}: E'_j \to E'_{j+1}$ is 2^{j+2}-summing. Hence each continuous Gaussian cylinder set measure on E' is also $\sigma(E', E)$-Radon.

Remark 3. In the above example, D_j is in fact defined on $\ell_{2^{j+1}}$ into $\ell_{2^{j+2}}$ which is also 2^{j+2}-summing. And the composition $D_k \circ D_{k-1} \circ \cdots \circ D_1 : \ell_4 \to \ell_{2^{k+2}}$ is not 2-summing. This shows that, in Theorem 2, we can not relax the condition "$E'(U_\alpha^\circ)$ is of cotyple q" by "$E'(U_\alpha^\circ)$ is of finite cotype". In Theorem 2, q must be uniform for every E_α.

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