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An Estimate of the Depth from an Intermediate Subfactor

By

Jerzy WIERZBICKI*

Abstract

We show that for a triple $K\subset N\subset M$ of type II_1 factors the depth of the inclusion "$K\subset M$" is not greater than the maximum of depths of the inclusions "$K\subset N$" and "$N\subset M$", provided there is such a factor P, that the diagram $\begin{matrix} P & \subset & M \\ \cup & & \cup \\ K & \subset & N \end{matrix}$ is commuting and co-commuting square (or a non-degenerate commuting square) of type II_1 factors.

In [B] D. Bisch proved, that if depth of inclusion $K\subset M$ of two type II_1 factors is finite then for any intermediate subfactor N, the depths of "$K\subset N$" and of "$N\subset M$" are finite too. In this note we give a partial converse to the assertion. Similar result was obtained recently in the case of depth two irreducible inclusions in [S] by T. Sano, who used a different method. After the work had been completed, the author learned about another, much shorter proof of Theorem 6 below, based on the bimodule technique ([K]).

§1. Preliminaries

We recall here shortly the basic notions we need. We will follow [GHJ], [P1], [SW] and [PP2]. Let $N\subset M$ be an inclusion of type II_1 factors with $[M:N]<\infty$. We have the corresponding Jones' tower

$$N\subset M\subset^{e_0}M_1\subset^{e_1}M_2\cdots$$

with Jones' projections $e_i\in M_{i+1}$. Consider also the tower of relative commutants $\{Y_i\}_{i\geq -1}$, $Y_{-1}=N'\cap N$, $Y_0=N'\cap M$, $Y_i=N'\cap M_i$. If "$N\subset M$" is

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* Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060, Japan

of finite depth then for some $i>1$, Y_i becomes the basic construction for "$Y_{i-2}\subset Y_{i-1}$".

Definition 1. The minimal integer i with the above property is called the depth of the inclusion "$N\subset M$". Let us denote it by $d(N\subset M)$.

In [WW] we used notion of co-commuting square of type II_1 factors.

Definition 2. A diagram $\begin{matrix} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{matrix}$ of finite factors is a co-commuting square, if their commutants $\begin{matrix} M' & \subset & K' \\ \cup & & \cup \\ L' & \subset & N' \end{matrix}$ form a commuting square.

From [SW] we see, that the co-commuting square, which is also a commuting square of type II_1 factors coincides with the notion of non-degenerate commuting square of II_1 factors, which was introduced by S. Popa in [P1].

We need also the algebraic basic construction as introduced in [PP2].

Definition 3. Suppose that N is a subfactor of a type II_1 factor M with $[M:N]<\infty$. Let M_1 be a von Neumann algebra with a finite, faithful and normal trace τ_1. Assume that e is a projection in M_1. If there is a trace preserving *-isomorphism $\phi:\langle M,e_N^M\rangle \rightarrow M_1$ of the basic construction of "$N\subset M$" onto M_1 such that $\forall x\in M$, $\phi(x)=x$ and $\phi(e_N^M)=e$ then we call M_1 algebraic basic construction for the inclusion "$N\subset M$". For convenience we will write $M_1=\langle M,N,e\rangle$.

§2. The Result

We construct a system of type II_1 factors from a given commuting and co-commuting square of II_1 factors: $\begin{matrix} Q_{1,0} & \subset & Q_{1,1} \\ \cup & & \cup \\ Q & \subset & Q_{0,1} \end{matrix}$, where the Jones index $[Q_{1,1}:Q]$ is finite. In the first step we define $Q_{2,1}=\langle Q_{1,1},e_{Q_{0,1}}^{Q_{1,1}}\rangle$ as the basic construction for the pair $Q_{0,1}\subset Q_{1,1}$ and also $Q_{1,2}=\langle Q_{1,1},e_{Q_{1,0}}^{Q_{1,1}}\rangle$ and $Q_{2,2}=\langle Q_{1,1},e_{Q}^{Q_{1,1}}\rangle$. This way we obtained the following "bigger" diagram $\begin{matrix} Q_{2,1} & \subset & Q_{2,2} \\ \cup & & \cup \\ Q_{1,1} & \subset & Q_{1,2} \end{matrix}$, which by

[SW] is a commuting and co-commuting square too. Similarly as in the first step, we put $Q_{n+1,n}=\langle Q_{n,n}, e^{Q_{n,n}}_{Q_{n-1,n}}\rangle$, $Q_{n,n+1}=\langle Q_{n,n}, e^{Q_{n,n}}_{Q_{n,n-1}}\rangle$ and $Q_{n+1,n+1}=\langle Q_{n,n}, e^{Q_{n,n}}_{Q_{n-1,n-1}}\rangle$ for $n=2,3,4\cdots$. Let us use the following notation: $e_n=e^{Q_{n,n}}_{Q_{n,n-1}}$, $f_n=e^{Q_{n,n}}_{Q_{n-1,n}}$, $g_n=e^{Q_{n,n}}_{Q_{n-1,n-1}}$.

The above construction of increasing commuting and co-commuting squares can be found in [SW] and we know that for $i,j\geq 1$, $e_if_j=f_je_i$ and $g_i=e_if_i$. We complete the above system by defining von Neumann subalgebras $Q_{i,j}$ with the following iterative formulas:

$$Q_{i+1,j}=Q_{i,j}\vee\{f_i\} \text{ and } Q_{i,j+1}=Q_{i,j}\vee\{e_j\},$$

where "$\vee$" denotes generation of a von Neumann algebra.

Lemma 4. *With the above notation we have*:

(i) $\forall i,j\geq 1, Q_{i,j}$ *are type* II_1 *factors.*

(ii) $\forall i,j\geq 0$, $Q_{i+2,j}=\langle Q_{i+1,j}, Q_{i,j}, f_{i+1}\rangle$ *and* $Q_{i,j+2}=\langle Q_{i,j+1}, Q_{i,j}, e_{j+1}\rangle$.

Since the above result is implicit in [SW] and since it is known to other specialists (e.g. [K], [P1]) we omit the proof.

The next lemma can in fact be read off from the proof of [P3] Proposition 2.1. Let us state it explicitly. If a von Neumann algebra A contains a projection e then we will write $C(e,A)$ for the central support of e in A. For a subalgebra B we define the following projection:

$$V(e,B)=\bigvee\{ueu^*| u \text{ is a unitary in } B\}.$$

Lemma 5. *Let $N\subset M$ be type II_1 factors and $[M:N]<\infty$. B and A are such von Neumann subalgebras that the diagram* $\begin{matrix} N & \subset & M \\ \cup & & \cup \\ B & \subset & A \end{matrix}$ *is a commuting square. Suppose that a projection $e\in A$ satisfies $E^M_N(e)=[M:N]^{-1}$. Then $C(e,A)=V(e,B)$.*

With the above preparation the proof of our main result becomes easy.

Theorem 6. *Let $K\subset M\subset L$ be type* II_1 *factors with* $[L:K]<\infty$. *Suppose that* $\mathrm{d}(K\subset M)<\infty$ *and* $\mathrm{d}(M\subset L)<\infty$. *If there is a type* II_1 *factor* N, *such that* $K\subset N\subset L$ *and such that the diagram* $\begin{matrix} N & \subset & L \\ \cup & & \cup \\ K & \subset & M \end{matrix}$ *is a nondegenerate commuting square, then*

$$\mathrm{d}(K\subset L)\leq\max(\mathrm{d}(K\subset M),\mathrm{d}(M\subset L)).$$

Proof. Let us construct a system of type II_1 factors $\{Q_{i,j}\}$ as in Lemma 4 with $L=Q_{1,1}, N=Q_{1,0}, M=Q_{0,1}$ and $K=Q_{0,0}=Q$. Denote $g_i=e_if_i$ and $q=\max(\mathrm{d}(K\subset M),\mathrm{d}(M\subset L))$. From Lemmas 4, 5 and [P3] 3.1 we obtain:

$$\begin{aligned} & V(g_q,Q'\cap Q_{q,q})\geq V(g_q,Q'\cap Q_{q,q}\cap\{f_q\}') \\ (*)\qquad & =\bigvee\{ue_qu^*f_q|u\in U(Q'\cap Q_{q-1,q})\}\geq V(e_q,Q'\cap Q_{0,q})\,f_q \\ & =C(e_q,Q'\cap Q_{0,q+1})\,f_q=f_q, \end{aligned}$$

so that

$$\begin{aligned} & C(g_q,Q'\cap Q_{q+1,q+1})=V(g_q,Q'\cap Q_{q,q})=V(V(g_q,Q'\cap Q_{q,q}),Q'\cap Q_{q,q}) \\ (**)\qquad & \geq V(f_q,Q'\cap Q_{q,q})\geq V(f_q,(Q_{0,1}\vee\{e_2,e_3\cdots e_q\})'\cap Q_{q,q}) \\ & =V(f_q,Q'_{0,1}\cap Q_{q,1})=C(f_q,Q'_{0,1}\cap Q_{q+1,1})=1. \end{aligned}$$

Q.E.D.

Remark. The example $\begin{matrix} L\otimes K & \subset & L\otimes M \\ \cup & & \cup \\ N\otimes K & \subset & N\otimes M \end{matrix}$, where $K\subset M$ and $N\subset L$ are finite index inclusions of type II_1 factors, shows that the above estimate is sharp. However in some cases equality does not hold. Let α be an outer action of the symmetric group S_3 on a type II_1 factor K. Consider the following commuting and co-commuting square:

$$\begin{matrix} K\rtimes_\alpha\langle(1,2,3)\rangle & \subset & K\rtimes_\alpha S_3 \\ \cup & & \cup \\ K & \subset & K\rtimes_\alpha S_2. \end{matrix}$$

We see that

$$\mathrm{d}(K\subset K\rtimes_\alpha S_3)=2<\max(\mathrm{d}(K\subset K\rtimes_\alpha S_2),\mathrm{d}(K\rtimes_\alpha S_2\subset K\rtimes_\alpha S_3))=\max(2,4).$$

Using the above method we can show a little more.

Corollary 7. *Let the diagram* $\begin{matrix} N & \subset & L \\ \cup & & \cup \\ K & \subset & M \end{matrix}$ *be as in Theorem* 6. *We denote* $a=\mathrm{d}(N\subset L)$, $b=\mathrm{d}(K\subset M)$, $c=\mathrm{d}(K\subset N)$ *and* $d=\mathrm{d}(M\subset L)$. *Then*

$$\mathrm{d}(K\subset L)\leq \max(\min(a,b),\ \min(c,d))$$

$$=\min(\max(a,c),\ \max(a,d),\ \max(b,c),\ \max(b,d)).$$

Proof. If q is the right-hand side of the above inequality then the inequality $(*)$ in the proof of Theorem 6 may be replaced by the following one:

$$(*)' \qquad \bigvee\{ue_q u^* f_q \mid u\in U(Q'\cap Q_{q-1,q})\}\geq V(e_q, Q'_{1,0}\cap Q_{1,q})f_q,$$

except in the case $q=1$ which we consider later. Also the inequality $(**)$ above may be replaced by:

$$(**)' \qquad V(f_q, Q'\cap Q_{q,q})\geq V(f_q, Q'\cap Q_{q,0}).$$

By symmetry this ends the proof in all cases except when $a=d=1$ and $c,d>1$. This can be obtained from Theorem 6 and the known fact that an inclusion of type II_1 factors is of depth 1, iff it is isomorphic to the inclusion "$N\subset N\otimes M_n(\boldsymbol{C})$". Q.E.D.

For a given pair $Q\subset N$ of II_1 factors with finite index A. Ocneanu ([O]) and also Bisch ([B]) considered a set of projections corresponding to intermediate subfactors between Q and N denoted in [B] by $IS(Q,N)$. If $N_1=\langle N, e_Q^N\rangle$ is the basic construction for $Q\subset N$ then

$$IS(Q,N)=\{q\in P(Q'\cap N_1)\mid qe_Q^N=e_Q^N, E_N^{N_1}(q)\in \boldsymbol{C} \text{ and } qNq\subset Nq\}.$$

Remark. Let $Q\subset K\subset N$ be a triple of type II_1 factors with $[N:Q]<\infty$ and $Q'\cap N=\boldsymbol{C}$. Then there exists a type II_1 factor B such that $\begin{matrix} B & \subset & N \\ \cup & & \cup \\ Q & \subset & K \end{matrix}$ is a nondegenerate commuting square, if and only if there is in $IS(Q,N)$ a Jones' projection e corresponding to the inclusion $Q\subset K$ i.e. such that $\forall x\in K, exe=E_Q(x)e$ and $E_K^{N_1}(e)=[K:Q]^{-1}$.

Indeed, if $e\in IS(Q,N)$ and $B=\{e\}'\cap N$ then by [PP2] 1.2, [B] 4.2 and [SW] Theorem 7.1 the diagram $\begin{matrix} B & \subset & N \\ \cup & & \cup \\ Q & \subset & K \end{matrix}$ is a commuting and co-commuting square.

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