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On Unbounded Positive *-Representations on Fréchet-Domains

By

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Abstract

Let D be a Fréchet-domain from Op*-algebra, abbreviated F-domain. The present paper is concerned with the study of positive *-representations of $L^+(D)$, of the Calkin representation of $L^+(D)$ and of bounded sets in ultrapower $D_{\mathfrak{U}}$. For this the density property plays an important role. It was introduced by S. Heinrich for locally convex spaces in [2].

In the paper [3] we gave several characterizations of the density property of an F-domain D. In this work we give a characterizations of continuity of positive *-representations and Calkin representation of $L^+(D)$ by the density property of D. This generalizes the well-known result due to K. Schmüdgen, see [12]. Further we describe bounded subsets in ultrapower $D_{\mathfrak{U}}$. If D has the density property, then every bounded set $M \subset D_{\mathfrak{U}}$ has a simple structure: For each bounded set $M \subset D_{\mathfrak{U}}$ there exists a bounded set $N \subset D$ with $M \subset N_{\mathfrak{U}}$. S. Heinrich proved an analogous result for bounded ultrapowers on locally convex spaces.

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§1. Preliminaries

Throughout the paper, D denotes a dense linear subspace of a Hilbert space H. We denote the norm, unit ball, and the scalar product of H by $\|\cdot\|$, U_H and $\langle\cdot,\cdot\rangle$, respectively. For a closable linear operator T on H, let $\bar{T}$, $D(T)$ and $\|T\|$ denote the closure, domain, and the norm of T (provided the later exists), respectively. The set of linear operators

$$L^+(D) := \{T \in \mathrm{End}(D) : D \subset D(T^*) \text{ and } T^*(D) \subset D\}$$

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is the maximal Op*-algebra on the domain D with the involution $T^+ := T^*|_D$. The domain D will be endowed with the weakest locally convex topology such that $D \ni \varphi \mapsto \|T\varphi\|$ are continuous seminorms for all $T \in L^+(D)$. This topology is called the **graph topology** t. Throughout this paper, we assume that D is a Fréchet space. In this case we say that D is an F-domain. These assumptions imply that there exists a sequence (A_k) in $L^+(D)$ such that the following conditions are satisfied, see [5]:

1. The topology of D is generated by the sequence of seminorms $(\|A_k \cdot\|)$, i.e. for each $T \in L^+(D)$ there exists $k \in N$ such that $\|T\varphi\| \leq \|A_k\varphi\|$ for all $\varphi \in D$.
2. $A_1 = I_D$, $A_k = A_k^+$, $\langle \varphi, A_k^2 \varphi \rangle \leq \langle \varphi, A_{k+1}\varphi \rangle$ and $\|A_k^2\varphi\| \leq \|A_{k+1}\varphi\|$ for all $\varphi \in D$.

Throughout this paper, we fix a sequence $(A_k) \subset L^+(D)$ for each F-domain D such that conditions 1. and 2. are satisfied.

Let us now define a sequence of scalar products of D by

$$\langle \varphi, \psi \rangle_l := \langle A_l\varphi, A_l\psi \rangle \qquad \text{for all } \varphi, \psi \in D; l \in N$$

and let D_l denote the unitary space $(D, \langle \cdot, \cdot \rangle_l)$. The Hilbert norm of D_l is $\|\varphi\|_l := \|A_l\varphi\|$ and the completion of D_l is the Hilbert space $H_l := D(\bar{A}_l)$. Remark that $D_1 = D$ and $H_1 = H$ are valid.

Let us consider the locally convex topology on D_l generated by the sequence of seminorms $(\|A_k \cdot\|_l)$. Since for all $k \in N$ there exists $l_k \in N$ such that

$$\|A_k\varphi\|_l = \|A_lA_k\varphi\| \leq \|A_{l_k}\varphi\| \qquad \text{for all } \varphi \in D,$$

it follows that this topology coincides with the graph topology t, i.e. D and D_l coincide as locally convex spaces. In general, $A_k \notin L^+(D_l)$, however (A_k) is an operator family in the sense of [13] (this is used in Proposition 1.1). An operator $T \in L^+(D_l)$ is called **l-hermitian** if $\langle \varphi, T\psi \rangle_l = \langle T\varphi, \psi \rangle_l$ for all $\varphi, \psi \in D$ and an l-hermitian operator T is called **positive** if $\langle \varphi, T\varphi \rangle_l \geq 0$. In this case we write $T \geq_l 0$.

If E and F are locally convex spaces, we denote by $\mathscr{L}(E, F)$ the linear space of all continuous linear operators mapping E into F. Let $l \in N$. We define

$$\mathscr{C}(H_l, D) := \{T \in \mathscr{L}(D, D) : \text{There is } S \in \mathscr{L}(H_l, D) \text{ such that}$$
$$T\varphi = S\varphi \text{ for all } \varphi \in D\}.$$

The following proposition is valid, see [13], Theorem 2.4.1.

Proposition 1.1. *Let $l \in N$. If $M \subset D$ is a bounded set, then there exists $B \in \mathscr{C}(H_l, D_l)$ such that $B \geq_l 0$ and $M \subset \bar{B}(U_{H_l})$.*

The algebra $L^+(D_l)$ will be endowed with the topology τ_b of uniform convergence on bounded sets. This topology is generated by the system of seminorms

$$q_B(T) = \|BTB\|, \quad B \in \mathscr{C}(H, D) \qquad \text{with } 0 \leq B$$

or for an arbitrary fixed $l \in N$ by the system of seminorms

$$q_{B,l}(T) = \|BTB\|_l, \quad B \in \mathscr{C}(H_l, D) \qquad \text{with } 0 \leq_l B.$$

Given $B \in \mathscr{C}(H_l, D)$ with $0 \leq_l B$, we can define the positive operator $T := (\bar{B}^2 + I)^{-1}|_D$. We set $\psi := T\varphi \in H_l$ for a $\varphi \in D$, this implies $\varphi = \bar{B}^2\psi + \psi$. Since $\varphi \in D$ and $\bar{B}^2\psi \in D$, we get $\psi \in D$ and $T \in L^+(D_l)$. It follows from $BTB \leq_l I$ that consequently $q_{B,l}(T) \leq 1$.

Proposition 1.2. *Suppose that D is an F-domain and $l \in N$. Then*

$$\rho_l : L^+(D_l) \ni T \mapsto A_l^2 T \in L^+(D)$$

is a continuous mapping.

Proof. Let $S \in L^+(D_l)$ be an l-hermitian operator. We have

$$\langle \varphi, A_l^2 S\psi \rangle = \langle \varphi, S\psi \rangle_l = \langle S\varphi, \psi \rangle_l = \langle A_l^2 S\varphi, \psi \rangle$$

for all $\varphi, \psi \in D$. This implies $(A_l^2 S)^+ = A_l^2 S$ and $A_l^2 S \in L^+(D)$. Since each element $T \in L^+(D_l)$ can be expressed through the form $T = S_1 + iS_2$ with $S_1, S_2 \in L^+(D_l)$ and l-hermitian, it follows that the above mapping makes sense. It is well-known that $B \in \mathscr{C}(H, D)$ and $0 \leq B$ implies $BA_l^2 \in \mathscr{C}(H, D)$. It follows that ρ_l is a continuous mapping. ◇

§2. The Density Property of an F-Domain D

The density property, abbreviation (DP), was introduced by S. Heinrich in [2]. A lot of topological properties of the algebra $L^+(D)$ are characterized by the (DP) of the domain D. These relationships were established in [3] and we will repeat here some results. We start with the definition of the (DP) for metrizable locally convex spaces.

Definition 2.1. Let E denote a metrizable locally convex space, $(U_k)_{k\in N}$ a countable base of closed absolutely convex 0-neighbourhoods in E, and $\mathscr{B}$ the system of all bounded subsets of E. Then E has the **density property** if following holds:

Given a positive sequence (λ_k) and an $n\in N$, there exist $n_0\in N$ and $M\in\mathscr{B}$ such that

$$\bigcap_{k=1}^{n_0} \lambda_k U_k \subset U_n + M.$$

Now we give a characterization of the (DP) for F-domains D by partial order properties of the Op*-algebra $L^+(D)$.

Theorem 2.2. ([3]). *For an F-domain D, the following assertions are equivalent*:

1. *D has the (DP).*
2. *Given a positive sequence (λ_k) and an $n\in N$, there exist $n_0\in N$ and $B\in\mathscr{C}(H_n, D)$ with $B\geq_n 0$ such that*

$$A_n^2(I+\bar{B})^{-1}\leq \sum_{k=1}^{n_0}\lambda_k^{-1}A_k.$$

3. *Given a positive sequence (λ_k) and an $n\in N$, there exist $n_0\in N$ and $P\in\mathscr{C}(H_n, D)$ such that $\bar{P}$ is an orthogonal projection in the Hilbert space H_n and*

$$A_n^2(I-P)\leq \sum_{k=1}^{n_0}\lambda_k^{-1}A_k.$$

We denote by τ_n the finest locally convex topology on $L^+(D)$ for which the positive cone $L^+(D)_+ := \{T \in L^+(D) : T \geq 0\}$ is normal. The topology τ_n is called **normal topology**. Since $L^+(D)_+$ is τ_b-normal cone, we have $\tau_b \subseteq \tau_n$.

Theorem 2.3. ([3]). *For an F-domain D, the following assertions are equivalent*:

1. *D has the (DP).*
2. *$L^+(D)$ has the normal topology, i.e. $\tau_b = \tau_n$.*

Commutatively dominated F-domains are of the form

$$D := \bigcap_{k=1}^{\infty} D(h_k(T)),$$

where T is a self-adjoint operator on a Hilbert space H and (h_k) is a sequence of real measurable functions on the spectrum $\sigma(T)$ of T such that

$$1 = h_1(t) \qquad \text{and } h_k(t)^2 \leq h_{k+1}(t) \qquad \text{a.e.}$$

for each $k \in \boldsymbol{N}$, see [8] Proposition 3.2.

Definition 2.4. We say that the functions (h_k) fulfill the **condition (*)**, if for each positive sequence (λ_k) there is an $n \in \boldsymbol{N}$ such that all functions (h_k) are essentially bounded on

$$M_n := \{t \in \sigma(T) : h_1(t) \leq \lambda_1, \cdots, h_n(t) \leq \lambda_n\}.$$

Proposition 2.5. ([3]). *Let D be a commutatively dominated F-domain. Then we have the assertion*:

D has the (DP) if and only if (h_k) fulfill the condition (*).

The domain $S(\boldsymbol{R}^n)$ of tempered test functions has the (DP). One can find an example which does not fulfill the condition (*) (and has not the (DP)) in [11]. We will give a new example which does not fulfill the condition (*). The example was constructed by K.-D. Kürsten in [6] for the realization of the Heisenberg algebra for systems in infinitely many degrees of freedom.

Example. Let $\Lambda := \{(n_j)_{j=1}^{\infty} : n_j \in \boldsymbol{N} \cup \{0\}\}$ be an uncountable index set and let $\chi : \Lambda \to [0,1]$ be a bijection. Furthermore let H be a (non-separable) Hilbert space with an orthonormal basis $\{\varphi_t : t \in [0,1]\}$. We define by $T\varphi_t := t\varphi_t$ a continuous, self-adjoint operator on H with spectrum $[0,1]$. Using real measurable functions

$$h_k(t) := \left(1 + \sum_{j=1}^{k} n_j\right)^k \qquad \text{with } (n_j) := \chi^{-1}(t)$$

we obtain the F-domain $D := \bigcap_{k=1}^{\infty} D(h_k(T))$. Suppose the functions (h_k) fulfill the condition (*). We set $\lambda_k := 1$ for all $k \in N$. By assumption there exists an $n \in N$ such that all functions (h_k) are bounded on

$$N := \{t \in [0,1] : h_1(t) \leq 1, \cdots, h_n(t) \leq 1\}.$$

We set

$$t_l := \chi((0, \cdots, 0, \underset{\uparrow (n+1)}{l}, 0, \cdots))$$

with $l \in N$. If $k \leq n$, then we have $h_k(t_l) = 1$ for all $l \in N$, i.e. $t_l \in N$ for all $l \in N$. Since $h_{n+1}(t_l) = (1+l)^{n+1}$, it follows that h_{n+1} is unbounded on N. This contradiction implies that (h_k) does not fulfill the condition (*).

§3. Positive *-Representations of $L^+(D)$

If D has (DP), then $\tau_b = \tau_n$ is valid on $L^+(D)$ and each positive *-representation on $L^+(D)$ is continuous. In this section we will prove that if $\tau_b \neq \tau_n$ on $L^+(D)$, then there is a non-continuous positive *-representation of $L^+(D)$. A similar assertion is true for a faithful *-representation of the generalized Calkin algebra of $L^+(D)$.

Let D, D_0 be domains. By a ***-representation** ω of $L^+(D)$ on D_0 we mean a *-homomorphism of $L^+(D)$ in $L^+(D_0)$ satisfying $\omega(Id_D) = Id_{D_0}$. The domain D_0 will be endowed with the graph topology of the Op*-algebra $\omega(L^+(D))$, i.e. the weakest locally convex topology such that $D_0 \ni \varphi \mapsto \|T\varphi\|$ are continuous seminorms for all $T \in \omega(L^+(D))$. The algebra $\omega(L^+(D))$ will also be endowed with the topology of uniform convergence on bounded sets of D_0. The representation ω is called **weakly continuous** if for each $\varphi \in D_0$ the linear functional $\langle \omega(\cdot)\varphi, \varphi \rangle$ is continuous on $L^+(D)$. We say ω is **continuous**, if ω is a continuous mapping of $L^+(D)$ onto $\omega(L^+(D))$. The representation ω is **positive**, if $0 \leq T$ implies $0 \leq \omega(T)$.

In order to define *-representations of the Calkin algebra of $L^+(D)$, we consider an F-domain D, a free ultrafilter $\mathfrak{U}$ on $\boldsymbol{N}$ and the following linear spaces:

$$\tilde{D}_\infty := \{(\varphi_i) \in D^{\boldsymbol{N}} : (\varphi_i) \quad \text{is bounded}\},$$

$$D_\infty := \{(\varphi_i) \in D^{\boldsymbol{N}} : (\varphi_i) \quad \text{is } \sigma(D, D')\text{-0-sequence}\},$$

$$K_{\mathfrak{U}} := \{(\varphi_i) \in \tilde{D}_\infty : \lim_{\mathfrak{U}} \|\varphi_i\| = 0\},$$

$$\tilde{D}_{\mathfrak{U}} := \tilde{D}_\infty / K_{\mathfrak{U}}, \qquad D_{\mathfrak{U}} := D_\infty / (D_\infty \cap K_{\mathfrak{U}}).$$

The elements from $\tilde{D}_{\mathfrak{U}}$ will be denoted by $(\tilde{\varphi}_i)_{\mathfrak{U}}$ or $\tilde{f}$ and the elements from $D_{\mathfrak{U}}$ will be denoted by $(\varphi_i)_{\mathfrak{U}}$ or f. The domains $\tilde{D}_{\mathfrak{U}}$ and $D_{\mathfrak{U}}$ will be endowed with the topologies which are generated by the seminorms

$$\tilde{p}_k((\tilde{\varphi}_i)_{\mathfrak{U}}) := \lim_{\mathfrak{U}} \|A_k \varphi_i\| \qquad \text{and } p_k((\varphi_i)_{\mathfrak{U}}) := \lim_{\mathfrak{U}} \|A_k \varphi_i\|,$$

respectively. The space $D_{\mathfrak{U}}$ is called **(ordinary) ultrapower** of D. On $\tilde{D}_{\mathfrak{U}}$ and $D_{\mathfrak{U}}$ we can define scalar products by

$$\langle (\tilde{\varphi}_i)_{\mathfrak{U}}, (\tilde{\psi}_i)_{\mathfrak{U}} \rangle := \lim_{\mathfrak{U}} \langle \varphi_i, \psi_i \rangle \qquad \text{and}$$

$$\langle (\varphi_i)_{\mathfrak{U}}, (\psi_i)_{\mathfrak{U}} \rangle := \lim_{\mathfrak{U}} \langle \varphi_i, \psi_i \rangle,$$

respectively. It is well-known that $\tilde{D}_{\mathfrak{U}}$ and $D_{\mathfrak{U}}$ are F-domains and the graph topologies t are generated by $\tilde{p}_k$ and p_k, respectively. See [4], Satz 3.3.1. or [9], Proposition 3.7. The formula

$$\pi(T)(\varphi_i)_{\mathfrak{U}} := (T\varphi_i)_{\mathfrak{U}}, \qquad T \in L^+(D), \qquad (\varphi_i)_{\mathfrak{U}} \in D_{\mathfrak{U}}$$

defines a positive *-representation π of $L^+(D)$ on $D_{\mathfrak{U}}$. The *-representation π will be termed **(unbounded) Calkin representation**. For more details see [10], [12] or [7]. The kernel ker π is the closed ideal $\mathscr{V}$ of all operators in $L^+(D)$ which map each bounded subset of D into a relatively compact subset of D. The quotient algebra $\mathscr{A}_c := L^+(D)/\mathscr{V}$ is called the **Calkin algebra** of $L^+(D)$. Let $\boldsymbol{\iota}$ denote the quotient map of algebra $L^+(D)$ onto $\mathscr{A}_c$. Then $\pi = \sigma \circ \boldsymbol{\iota}$ defines a faithful *-representation σ of the *-algebra $\mathscr{A}_c$. We endow $\mathscr{A}_c$ with the quotient topology of $L^+(D)$, which is generated by the seminorms

$$q_B(a) := \|\pi(B)\sigma(a)\pi(B)\| \qquad a \in \mathscr{A}_c, \qquad B \in \mathscr{C}(H, D) \text{ with } 0 \leq B,$$

see [12], Theorem 2.1 or [4], Satz 3.3.5.

Let us now prove some preliminary lemmas.

Lemma 3.1. *Suppose that D is an F-domain, $\varphi_i, \varphi \in D$ and (φ_i) weakly converges to φ. Then $\tilde{g} := (\tilde{\varphi}_i)_{\mathfrak{U}} \in \tilde{D}_{\mathfrak{U}}$, $g := (\varphi_i - \varphi)_{\mathfrak{U}} \in D_{\mathfrak{U}}$ and the equation*

$$\langle g, \pi(T)g \rangle = \langle \tilde{g}, \tilde{\pi}(T)\tilde{g} \rangle - \langle \varphi, T\varphi \rangle$$

is true for all $T \in L^+(D)$.

Proof. By definition of $D_{\mathfrak{U}}$ and $\tilde{D}_{\mathfrak{U}}$ we get immediately $\tilde{g} \in \tilde{D}_{\mathfrak{U}}$ and $g \in D_{\mathfrak{U}}$. Choose $T \in L^+(D)$. Since

$$\lim_{\mathfrak{U}} \langle \varphi_i, T\varphi \rangle = \langle \varphi, T\varphi \rangle = \lim_{\mathfrak{U}} \langle \varphi, T\varphi_i \rangle,$$

it follows that

$$\langle g, \pi(T)g \rangle = \lim_{\mathfrak{U}} \langle (\varphi_i - \varphi), T(\varphi_i - \varphi) \rangle$$

$$= \lim_{\mathfrak{U}} \langle \varphi_i, T\varphi_i \rangle - \lim_{\mathfrak{U}} \langle \varphi, T\varphi_i \rangle - \lim_{\mathfrak{U}} \langle \varphi_i, T\varphi \rangle + \langle \varphi, T\varphi \rangle$$

$$= \langle \tilde{g}, \tilde{\pi}(T)\tilde{g} \rangle - \langle \varphi, T\varphi \rangle.$$

◇

Lemma 3.2. *Suppose that D is an F-domain. Given a positive sequence (λ_k) and an $m \in N$, there exists $B \in \mathscr{C}(H_m, D)$ with $0 \leq_m B$ such that*

$$\langle \varphi, (I + \bar{B}^2)^{-1} \varphi \rangle_m < \frac{1}{2} \sum_{k=1}^{\infty} 2^{-k} \lambda_k^{-1} \langle \varphi, A_k \varphi \rangle$$

for all $\varphi \in D$ (the value ∞ on the right hand side is possible).

Proof. Given an arbitrary $\varphi \in D$ with $\varphi \neq 0$. We set

$$\varphi_0 := \left(\sum_{k=1}^{\infty} 2^{-(k+2)} \lambda_k^{-1} \langle \varphi, A_k \varphi \rangle \right)^{-\frac{1}{2}} \varphi$$

(if the denominator is ∞, then we set $\varphi_0:=0$). Then

$$\begin{aligned}\|A_n\varphi_0\|^2 &= \left(\sum_{k=1}^{\infty} 2^{-(k+2)}\lambda_k^{-1}\langle\varphi, A_k\varphi\rangle\right)^{-1}\|A_n\varphi\|^2 \\ &\leq \left(\sum_{k=1}^{\infty} 2^{-(k+2)}\lambda_k^{-1}\langle\varphi, A_k\varphi\rangle\right)^{-1}\langle\varphi, A_{n+1}\varphi\rangle \\ &\leq 2^{n+3}\lambda_{n+1}\end{aligned}$$

for all $n\in N$. It follows that there exists a fixed bounded set $M\subset D$ with the property that for all $\varphi\in D$ the corresponding φ_0 belongs to M. By Proposition 1.1 there exists a $B\in\mathscr{C}(H_m, D)$ with $0\leq_m B$ such that $M\subset\bar{B}(U_{H_m})$. Using

$$\sup_{\psi\in M}|\langle\psi, T\psi\rangle_m|\leq\|BTB\|_m \qquad \text{for all } T\in L^+(D)$$

and taking $T:=(I+\bar{B}^2)^{-1}$, we get the inequality

$$|\langle\varphi_0,(I+\bar{B}^2)^{-1}\varphi_0\rangle_m|\leq 1.$$

We remove the normalization for φ_0 and obtain

$$\langle\varphi,(I+\bar{B}^2)^{-1}\varphi\rangle_m\leq\frac{1}{4}\sum_{k=1}^{\infty}2^{-k}\lambda_k^{-1}\langle\varphi, A_k\varphi\rangle \qquad \text{for all } \varphi\in D,$$

hence

$$\langle\varphi,(I+\bar{B}^2)^{-1}\varphi\rangle_m<\frac{1}{2}\sum_{k=1}^{\infty}2^{-k}\lambda_k^{-1}\langle\varphi, A_k\varphi\rangle \qquad \text{for all } \varphi\in D.$$

◇

Lemma 3.3. *Let D be an F-domain. Suppose that for each positive sequence (λ_k) and an $m\in N$ there exists always a $B\in\mathscr{C}(H_m, D)$ with $0\leq_m B$ such that*

$$\langle g, \pi\circ\rho_m((I+\bar{B}^2)^{-1})g\rangle<\frac{1}{2}\sum_{k=1}^{\infty}2^{-k}\lambda_k^{-1}\langle g, \pi(A_k)g\rangle$$

for all $g\in D_{\mathfrak{u}}$ (the value ∞ on the right hand side is possible). Then D has the (DP).

Proof. Suppose that D does not satisfy the (DP). By Theorem 2.2 there exist an $m \in N$ and a positive sequence (λ_k) such that for each $n \in N$ and $B \in \mathscr{C}(H_m, D)$ with $0 \le_m B$, we can find a $\varphi_{n,B} \in D$ with

$$1 = \langle \varphi_{n,B}, (I+\bar{B})^{-1}\varphi_{n,B}\rangle_m > \sum_{k=1}^{n} \lambda_k^{-1}\langle \varphi_{n,B}, A_k\varphi_{n,B}\rangle. \tag{1}$$

By assumption, there exist an $m \in N$ and a $B_1 \in \mathscr{C}(H_m, D)$ with $0 \le_m B_1$ such that

$$\langle g, \pi \circ \rho_m((I+\bar{B}_1^2)^{-1})g\rangle < \frac{1}{2}\sum_{k=1}^{\infty} 2^{-k}\lambda_k^{-1}\langle g, \pi(A_k)g\rangle \tag{2}$$

for all $g \in D_{\mathfrak{U}}$. By Lemma 3.2 there is a $B_2 \in \mathscr{C}(H_m, D)$ with $0 \le_m B_2$ such that

$$\langle \varphi, A_m^2(I+\bar{B}_2^2)^{-1}\varphi\rangle < \frac{1}{2}\sum_{k=1}^{\infty} 2^{-k}\lambda_k^{-1}\langle \varphi, A_k\varphi\rangle \tag{3}$$

for all $\varphi \in D$. We set $B_0 := B_1^2 + B_2^2 \in \mathscr{C}(H_m, D)$ and replace $\bar{B}_1^2$ and $\bar{B}_2^2$ in (2) and (3) by $\bar{B}_0$. Remark that the inequalities are true with $\bar{B}_0$. Using (1), we get

$$\|A_k\varphi_{n,B_0}\|^2 \le \langle \varphi_{n,B_0}, A_{k+1}\varphi_{n,B_0}\rangle \le \lambda_{k+1}$$

for all $n \in N$ with $k+1 \le n$. Therefore $(\varphi_{n,B_0})_{n=1}^{\infty}$ is a bounded sequence in D. Since D is semireflexive, we obtain that the set $\{\varphi_{n,B_0} : n \in N\}$ is relatively weakly compact in D. Thus, there exist a $\psi_0 \in D$ and a subsequence $\psi_i := \varphi_{n_i,B_0}$ which weakly converges to ψ_0. Let $g_0 := (\psi_i - \psi_0)_{\mathfrak{U}} \in D_{\mathfrak{U}}$, $\tilde{g}_0 := (\tilde{\psi}_i)_{\mathfrak{U}} \in \tilde{D}_{\mathfrak{U}}$, $r \in \{1, m\}$ and $T \in L^+(D_r)$. By Lemma 3.1 we get

$$\langle g_0, \pi(A_r^2T)g_0\rangle = \langle \tilde{g}_0, \tilde{\pi}(A_r^2T)\tilde{g}_0\rangle - \langle \psi_0, A_r^2\psi_0\rangle. \tag{4}$$

Choose now $n_0 \in N$ such that (see (2) and (3))

$$\langle g_0, \pi \circ \rho_m((I+\bar{B}_0)^{-1})g_0\rangle < \frac{1}{2}\sum_{k=1}^{n_0} 2^{-k}\lambda_k^{-1}\langle g_0, \pi(A_k)g_0\rangle, \tag{5}$$

$$\langle \psi_0, A_m^2(I+\bar{B}_0)^{-1}\psi_0\rangle < \frac{1}{2}\sum_{k=1}^{n_0} 2^{-k}\lambda_k^{-1}\langle \psi_0, A_k\psi_0\rangle. \tag{6}$$

Using the equation (4) and the inequality (5) we obtain

$$\langle\tilde{g}_0,\tilde{\pi}(A_m^2(I+\bar{B}_0)^{-1})\tilde{g}_0\rangle-\langle\psi_0,A_m^2(I+\bar{B}_0)^{-1}\psi_0\rangle$$

$$\leq\frac{1}{2}\sum_{k=1}^{n_0}2^{-k}\lambda_k^{-1}\langle\tilde{g}_0,\tilde{\pi}(A_k)\tilde{g}_0\rangle-\frac{1}{2}\sum_{k=1}^{n_0}2^{-k}\lambda_k^{-1}\langle\psi_0,\pi(A_k)\psi_0\rangle. \tag{7}$$

We add the inequality (6) and get

$$\langle\tilde{g}_0,\tilde{\pi}(A_m^2(I+\bar{B}_0)^{-1})\tilde{g}_0\rangle<\frac{1}{2}\sum_{k=1}^{\infty}2^{-k}\lambda_k^{-1}\langle\tilde{g}_0,\tilde{\pi}(A_k)\tilde{g}_0\rangle. \tag{8}$$

Now let us construct a contradiction. Using (1) we get

$$\langle\tilde{g}_0,\tilde{\pi}(A_m^2(I+\bar{B}_0)^{-1})\tilde{g}_0\rangle=\lim_{\mathfrak{U}}\langle\psi_i,A_m^2(I+\bar{B}_0)^{-1}\psi_i\rangle$$

$$=\lim_{\mathfrak{U}}\langle\varphi_{n_i,B_0},A_m^2(I+\bar{B}_0)^{-1}\varphi_{n_i,B_0}\rangle=1. \tag{9}$$

On the other hand, by (1) we have

$$\langle\tilde{g}_0,\tilde{\pi}(A_k)\tilde{g}_0\rangle=\lim_{\mathfrak{U}}\langle\psi_i,A_k\psi_i\rangle=\lim_{\mathfrak{U}}\langle\varphi_{n_i,B_0},A_k\varphi_{n_i,B_0}\rangle\leq\lambda_k,$$

too. This implies

$$\frac{1}{2}\sum_{k=1}^{\infty}2^{-k}\lambda_k^{-1}\langle\tilde{g}_0,\tilde{\pi}(A_k)\tilde{g}_0\rangle\leq\frac{1}{2}<1=\langle\tilde{g}_0,\tilde{\pi}(A_m^2(I+\bar{B}_0)^{-1})\tilde{g}_0\rangle$$

and we have a contradiction with the inequality (8). Thus D has the (DP). ◇

We can now prove the main result in this section. The following theorem generalizes the result due to K. Schmüdgen to the case of an arbitrary F-domains, see [12].

Theorem 3.4. *Suppose that D is an F-domain. $\mathfrak{U}$ is a free ultrafilter on N and $\mathscr{A}_c$ the Calkin algebra of $L^+(D)$. Then the following assertions are equivalent*:

1. *D has the (DP).*

2. *Each positive *-representation ω of $L^+(D)$ is continuous.*
3. *The faithful *-representation $\sigma:\mathscr{A}_c \to L^+(D_{\mathfrak{u}})$ is continuous.*
4. *Each weakly continuous *-representation ω of $L^+(D)$ is continuous.*

Proof. (1)$\Rightarrow$(2). Using Theorem 2.3, we have $\tau_b = \tau_n$ on $L^+(D)$, i.e. $L^+(D)$ has the normal topology. Let $\omega: L^+(D) \to L^+(D_0)$ be a positive *-representation. The uniform topology on $L^+(D_0)$ is generated by the family of seminorms

$$p_M(S) := \sup_{\psi \in M} |\langle \psi, S\psi \rangle| \qquad M \subset D_0 \text{ is bounded}, \quad S \in L^+(D_0).$$

Since the set

$$U_M := \{T \in L^+(D) : p_M(\omega(T)) \leq 1\}$$

is absolutely convex and $L^+(D)_+$-saturated, it follows that U_M is a 0-neighbourhood in $L^+(D)$. This proves the continuity of ω.

(2)$\Rightarrow$(3). Note that $\pi: L^+(D) \to L^+(D_{\mathfrak{u}})$ is a positive *-representation. Since the quotient map $\iota: L^+(D) \to \mathscr{A}_c$ is continuous and $\pi = \sigma \circ \iota$, it follows that σ is continuous.

(3)$\Rightarrow$(1). Given an arbitrary $g \in D_{\mathfrak{u}}$, $g \neq 0$. We set

$$g_0 := \left(\sum_{k=1}^{\infty} 2^{-(k+1)} \lambda_k^{-1} \langle g, \sigma \circ \iota(A_k) g \rangle \right)^{-\frac{1}{2}} g.$$

The inequality

$$\|\pi(A_k) g_0\|^2 = \left(\sum_{k=1}^{\infty} 2^{-(k+1)} \lambda_k^{-1} \langle g, \pi(A_k) g \rangle \right)^{-1} \|\pi(A_k) g\|^2$$

$$\leq \left(\sum_{k=1}^{\infty} 2^{-(k+1)} \lambda_k^{-1} \langle g, \pi(A_k) g \rangle \right)^{-1} \langle g, \pi(A_{k+1}) g \rangle \leq 2^{k+2} \lambda_{k+1}$$

implies that there exists a fixed bounded set $M \subset D_{\mathfrak{u}}$ with $g_0 \in M$ (for all $g \in D_{\mathfrak{u}}$). By assumption, σ is continuous and by Lemma 1.2,

$$\sigma \circ \iota \circ \rho_m : L^+(D_m) \mapsto L^+(D_{\mathfrak{u}})$$

is continuous for all $m \in N$, too. For each $m \in N$ there exists a $B \in \mathscr{C}(H_m, D)$ with $0 \leq_m B$ such that

$$\sup_{h\in M}|\langle h,\sigma\circ \iota\circ\rho_m(T)h\rangle|<\|BTB\|_m$$

is valid for all $T\in L^+(D_m)$. We set $h:=g_0$ and $T:=(I+\bar{B}^2)^{-1}$ and get

$$|\langle g_0,\sigma\circ\iota\circ\rho_m(T)g_0\rangle|<1,$$

$$\langle g,\pi\circ\rho_m((I+\bar{B}^2)^{-1})g\rangle<\frac{1}{2}\sum_{k=1}^{\infty}2^{-k}\lambda_k^{-1}\langle g,\pi(A_k)g\rangle$$

for all $g\in D_{\mathfrak{U}}$. By Lemma 3.3, D has the (DP).

(2)$\Rightarrow$(4). According to [12], Lemma 1.4, each weakly continuous *-representation ω of $L^+(D)$ is a positive *-representation. By assumption, ω is continuous.

(4)$\Rightarrow$(3). Since $\pi=\sigma\circ\iota$ is weakly continuous, it follows that π and σ are continuous. ◇

§4. Bounded Sets in the Ultrapower of *D*

The aim of this section is to describe bounded sets in ultrapower $D_{\mathfrak{U}}$. If D has the (DP), then every bounded subset $M\subset D_{\mathfrak{U}}$ has a simple structure. Namely, we can find a bounded subset $N\subset D$ such that $M\subset N_{\mathfrak{U}}$, i.e. for each $f\in M$ there exist $\varphi_i\in N$ such that $f=(\varphi_i)_{\mathfrak{U}}$. Remark, that (φ_i) is not a weak 0-sequence in general case. We shall show converse, too. S. Heinrich proved an analogous result for bounded ultrapowers of locally convex spaces, see [2]. We start with the definition due to S. Heinrich.

Let $\mathfrak{U}$ be a free ultrafilter on N and let D be an F-domain. We denote the elements of the set-theoretical ultrapower of D with $[\varphi_i]_{\mathfrak{U}}$ and we consider the following linear spaces:

$$\hat{D}_{\infty,\mathfrak{U}}:=\{[\varphi_i]_{\mathfrak{U}}:\text{there exists } F\in\mathfrak{U} \text{ such that}$$

$$\sup_{i\in F}\|A_k\varphi_i\|<\infty \qquad \text{for all } k\in N\},$$

$$\hat{K}_{\mathfrak{U}}:=\{[\varphi_i]_{\mathfrak{U}}\in\hat{D}_{\infty,\mathfrak{U}}:\lim_{\mathfrak{U}}\|A_k\varphi_i\|=0 \qquad \text{for all } k\in N\},$$

$$\hat{D}_{\mathfrak{U}}:=\hat{D}_{\infty,\mathfrak{U}}/\hat{K}_{\mathfrak{U}}.$$

The elements of $\hat{D}_{\mathfrak{U}}$ will be denoted by $(\hat{\varphi_i})_{\mathfrak{U}}$ or $\hat{f}$. The space $\hat{D}_{\mathfrak{U}}$ will be endowed with the topology which is generated by the seminorms

$$p_k((\hat{\varphi_i})_{\mathfrak{U}}):=\lim_{\mathfrak{U}}\|A_k\varphi_i\|.$$

The locally convex space $\hat{D}_{\mathfrak{U}}$ is called the **bounded ultrapower** of D.

Lemma 4.1. *The locally convex spaces $\tilde{D}_{\mathfrak{U}}$ and $\hat{D}_{\mathfrak{U}}$ are topologically isomorphic. The isomorphism is*

$$J:\tilde{D}_{\mathfrak{U}}\ni(\tilde{\varphi_i})_{\mathfrak{U}}\mapsto(\hat{\varphi_i})_{\mathfrak{U}}\in\hat{D}_{\mathfrak{U}}.$$

Proof. Taking $(\tilde{\varphi_i})_{\mathfrak{U}}$, $(\tilde{\psi_i})_{\mathfrak{U}}\in\tilde{D}_{\mathfrak{U}}$ with $(\tilde{\varphi_i})_{\mathfrak{U}} =(\tilde{\psi_i})_{\mathfrak{U}}$, i.e. $\lim_{\mathfrak{U}}\|\varphi_i-\psi_i\|=0$. Since $(\varphi_i-\psi_i)$ is bounded in D, we get

$$\lim_{\mathfrak{U}}\|A_k(\varphi_i-\psi_i)\|^2=\lim_{\mathfrak{U}}\langle A_k^2(\varphi_i-\psi_i),(\varphi_i-\psi_i)\rangle$$

$$\leq\sup_i\|A_k^2(\varphi_i-\psi_i)\|\lim_{\mathfrak{U}}\|\varphi_i-\psi_i\|=0$$

for all $k\in N$. This implies $[(\varphi_i-\psi_i)]_{\mathfrak{U}}\in\hat{K}_{\mathfrak{U}}$, i.e. $(\hat{\varphi_i})_{\mathfrak{U}}=(\hat{\psi_i})_{\mathfrak{U}}$. Therefore J defines a linear mapping. It is clear, that J is a one-to-one mapping. Taking an arbitrary $(\hat{\psi_i})_{\mathfrak{U}}\in\hat{D}_{\mathfrak{U}}$, there exists an $F\in\mathfrak{U}$ such that $\{\psi_i : i\in F\}$ is a bounded set in D. Set $\varphi_i:=\psi_i$ for all $i\in F$ and $\varphi_i:=0$ otherwise. We obtain $(\varphi_i)\in\tilde{D}_\infty$ and $J((\tilde{\varphi_i})_{\mathfrak{U}})=(\hat{\psi_i})$. This implies that J is a mapping onto $\hat{D}_{\mathfrak{U}}$. According to the definitions of the corresponding topologies, J is a homeomorphism. ◇

Theorem 4.2. *Suppose that D is an F-domain and $\mathfrak{U}$ is a free ultrafilter on N. The following assertions are equivalent:*

1. *D has the (DP).*
2. *For each bounded subset $M\subset\tilde{D}_{\mathfrak{U}}$ there exists a bounded subset $N\subset D$ with $M\subset\tilde{N}_{\mathfrak{U}}$.*
3. *For each bounded subset $M\subset D_{\mathfrak{U}}$ there exists a bounded subset $N\subset D$ with $M\subset N_{\mathfrak{U}}$.*

Proof. (1)$\Rightarrow$(2). According to [2], Theorem 1.4 and Lemma 4.1, the assertion (2) is true.

(2)$\Rightarrow$(3). This implication is clear, because $D_{\mathfrak{U}}$ is a topological subspace of $\tilde{D}_{\mathfrak{U}}$.

(3) $\Rightarrow$ (1). The proof is shown in similar to the proof of (3) $\Rightarrow$ (1) in Theorem 3.4. Given an arbitrary $g \in D_{\mathfrak{U}}$, $g \neq 0$. We set

$$g_0 := \left(\sum_{k=1}^{\infty} 2^{-(k+2)} \lambda_k^{-1} \langle g, \sigma \circ \iota(A_k) g \rangle \right)^{-\frac{1}{2}} g,$$

too. There exists a fixed bounded set $M \subset D_{\mathfrak{U}}$ with $g_0 \in M$ for all $g \in D_{\mathfrak{U}}$. We choose an $m \in N$. By assumption, there exists a $B \in \mathscr{C}(H_m, D)$ with $0 \leq_m B$ such that $M \subset (\bar{B}(U_{H_m}))_{\mathfrak{U}}$, i.e. there exists a sequence (φ_i) with $\varphi_i \in U_{H_m}$ and $g_0 = (\bar{B}\varphi_i)_{\mathfrak{U}}$. We have

$$|\langle g_0, \pi \circ \rho_m(T) g_0 \rangle| = |\langle (\bar{B}\varphi_i)_{\mathfrak{U}}, (A_m^2 T \bar{B} \varphi_i)_{\mathfrak{U}} \rangle|$$

$$= \lim_{\mathfrak{U}} |\langle \bar{B}\varphi_i, T\bar{B}\varphi_i \rangle_m| \leq \|BTB\|_m$$

for all $T \in L^+(D_m)$. We set $T := (I + \bar{B}^2)^{-1}$ and obtain

$$\langle g, \pi \circ \rho_m((I + \bar{B}^2)^{-1}) g \rangle < \frac{1}{2} \sum_{k=1}^{\infty} 2^{-k} \lambda_k^{-1} \langle g, \pi(A_k) g \rangle$$

for all $g \in D_{\mathfrak{U}}$. According to Lemma 3.3, D has the (DP). ◇

Proposition 4.3. *Suppose that D is an F-domain and $\mathfrak{U}$ is a free ultrafilter on N. If D has the (DP), then $D_{\mathfrak{U}}$ also has the (DP).*

Proof. Since D has the (DP), it follows by Theorem 2.2 that the following assertion is true: Given a positive sequence (λ_k) and an $n \in N$, there exist $n_0 \in N$ and $P \in \mathscr{C}(H_n, D)$ such that $\bar{P}$ is an orthogonal projection in the Hilbert space H_n and

$$A_n^2(I - P) \leq \sum_{k=1}^{n_0} \lambda_k^{-1} A_k.$$

The *-representation π is positive. Hence we have

$$\pi(A_n)^2 - \pi(A_n^2 P) \leq \sum_{k=1}^{n_0} \lambda_k^{-1} \pi(A_k).$$

Remark that $(\pi(A_k)) \subset L^+(D_{\mathrm{U}})$ is a sequence which satisfies the conditions 1 and 2 in section 1. Using Lemma 1.2, we get $A_n^2 P \in L^+(D)$. Let us consider the map

$$\tilde{\pi}(T) := (\bar{T}\varphi_i)_{\mathrm{U}} \qquad T \in \mathscr{C}(H_n, D), \qquad (\tilde{\varphi}_i)_{\mathrm{U}} \in (H_n)_{\mathrm{U}}$$

which is an extension of π. It is easy to see that $\tilde{\pi}$ is an element of $\mathscr{C}((H_n)_{\mathrm{U}}, D_{\mathrm{U}})$, where $(H_n)_{\mathrm{U}}$ is a Hilbert space. We obtain

$$\pi(A_n)^2 - \pi(A_n)^2 \tilde{\pi}(P) \leq \sum_{k=1}^{n_0} \lambda_k^{-1} \pi(A_k)$$

and the assertion follows from Theorem 2.2. ◇

Problem 4.4. *Is the assertion "If D_{U} has the (DP), then D has the (DP)" true?*

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