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Representations of Unitary Groups and Free Convolution

By

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To each finite dimensional representation of a unitary group $U(n)$ is associated a probability measure on the set of integers, depending on the highest weights which occur in this representation. We show that asymptotically for large n and large irreducible representations of $U(n)$ the measure associated to the tensor product of two representations, or to the restriction of a representation to a subgroup $U(m)$ with m comparable to n, can be expressed in terms of the measures associated to the first representations by means of the notion of free convolution (namely additive free convolution for the tensor product problem and multiplicative free convolution for the restriction problem).

Introduction

Let $U(n)$ be the group of complex $n \times n$ unitary matrices. In the representation theory of this group two basic questions are the following.

i) Given an irreducible representation R of $U(n)$ and $m \leq n$, decompose the restriction of R to $U(m)$ (embedded as a subgroup of $U(n)$) into irreducible components.

ii) Given two irreducible representations R_1 and R_2 of $U(n)$ decompose the (Kronecker) tensor product representation $R_1 \otimes R_2$ into irreducible components.

The first problem can be dealt with by using recursively Weyl's branching rule, which says that given an irreducible representation of $U(n)$ with highest weight $(\nu_1, \dots, \nu_n)$ (where ν_j for $1 \leq j \leq n$ are integers and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$), then in the restriction of R to $U(n-1)$ only the irreducible representations with highest weights $(\lambda_1, \dots, \lambda_{n-1})$ satisfying $\nu_1 \geq \lambda_1 \geq \nu_2 \geq \cdots \geq \lambda_{n-1} \geq \nu_n$ appear, each with multiplicity 1. From this one can determine the multiplicity $a_{l,\lambda}^{n,\nu}$ of the dominant weight λ in the restriction of the representation of $U(n)$ with highest weight ν to $U(l)$.

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For the second problem, Weyl's character formula for representations of semi-simple Lie groups can be used to give an explicit formula for the multiplicity $a^{\lambda}_{\mu\nu}$ with which the dominant weight λ occurs in a tensor product of two irreducible representation with highest weights μ and ν (see [6]). In the case of unitary groups the Littlewood-Richardson rule provides another approach to this problem (see [4]).

In this paper we shall be interested in asymptotic results on these problems when both the dimension of the group and the representations become large (where for an irreducible representation to be large we mean that the components of its highest weight become large). In this case the explicit formulas for the multiplicities in both problems become quickly intractable and one has to find other means of obtaining information on them. One way to do this is to try to estimate the asymptotic behaviour of some linear combinations of the coefficients. This is what we do in this paper using the notion of free convolution of measures, introduced by D. Voiculescu in the context of operator algebras (see [1]).

We will associate to each finite dimensional representation of a unitary group a measure on the set of integers $\mathbb{Z}$ in the following way: if R is an irreducible representation of $U(n)$ then the measure associated to R is $\mathcal{M}(R) = \frac{1}{n}\sum_{k=1}^{n} \delta_{\mu_k}$ where δ_x is the Dirac measure at x and $\mu_k \in \mathbb{Z}$ are the components of the highest weight of R. If $R = R_1 \oplus R_2$ decomposes into a direct sum of representations then $\mathcal{M}(R)$ is the convex combination $\mathcal{M}(R) = \frac{\dim(R_1)}{\dim(R)}\mathcal{M}(R_1) + \frac{\dim(R_2)}{\dim(R)}\mathcal{M}(R_2)$.

When the representation R is irreducible the measure $\mathcal{M}(R)$ determines completely its isomorphism class. When R is not irreducible, although this measure does not in general determine the decomposition of R into irreducible components, it gives some partial information on the multiplicities of the irreducible representations which occur in R.

We will show that asymptotically the map $\mathcal{M}$ converts the tensor product of representations into the additive free convolution of measures, and that the operation of restriction to a subgroup can be expressed through $\mathcal{M}$ with the help of the multiplicative free convolution by a measure.

Informally the results that we prove can be expressed as follows: denote by $\mathcal{M}^{\varepsilon}(R)$ the image of the measure $\mathcal{M}(R)$ by the transformation $x \mapsto \varepsilon x$ on $\mathbb{R}$ for a small $\varepsilon \in \mathbb{R}_{+}$, and let R_1 and R_2 be two irreducible finite dimensional representations of $U(n)$. If n is large, such that $\mathcal{M}^{\varepsilon}(R_1)$ and $\mathcal{M}^{\varepsilon}(R_2)$ are close to some probability measures α_1 and α_2 with compact support on $\mathbb{R}$, then $\mathcal{M}^{\varepsilon}(R_1 \otimes R_2)$ is close to the measure $\alpha_1 \boxplus \alpha_2$ which is the free convolution of α_1 and α_2. In the same vein we will have that if m is an integer close to

pn for $p \in]0, 1]$ then $\mathscr{M}^{\varepsilon}(R_{1|U(m)})$ (where $(R_{1|U(m)}$ is the restriction of R_1 to $U(m))$ is close to the measure $\frac{1}{p}(\alpha_1 \otimes (p\delta_1 + (1-p)\delta_0) - (1-p)\delta_0)$ where $\otimes$ denotes the multiplicative free convolution of measures, and δ_x the Dirac measure on $x \in \mathbb{R}$.

The proof of these results will consist in a two-step approximation argument. In the first step one shows that, for fixed n, and for an irreducible representation R of $U(n)$ with a large highest weight $(\mu_1, \ldots, \mu_n)$, the operators $dR(E_{kl})$ (where dR is the corresponding representation of the complexified Lie algebra $gl_n(\mathbb{C})$ and E_{kl} is the canonical basis of $gl_n(\mathbb{C})$), considered as non-commutative random variables, can be approximated in distribution by the components of a random matrix with spectrum close to $\{\mu_1, \ldots, \mu_n\}$. This is just a simple consequence of Kirillov's formula for the character of an irreducible representation. In fact the result that we prove could be extended to arbitrary compact semi-simple Lie groups.

The second step consists in considering a matrix canonically associated to any representation of $U(n)$ by the formula $C(R) = \sum_{kl} E_{kl} \otimes dR(E_{kl})$. Using the result of the first part of the paper, as well as results of D. Voiculescu and R. Speicher on asymptotic freeness of large independent matrices, we show that the matrices $C(R)$ corresponding to independent representations (i.e. acting on different components of a tensor product space) are asymptotically free random variable with distribution close to $\mathscr{M}(R)$, and are also asymptotically free with diagonal matrices. This asymptotic freenes allows us to link the decomposition problems for representations with free convolution.

In fact our results show how to construct in a natural way families of *non-random* matrices which are asymptotically free random variables with prescribed distributions.

This paper is organized as follows. In the first part, we recall some definitions and results and establish the notations. In the second part we use Kirillov's character formula to give an estimate which relates traces of some operators in an irreducible representation of $U(n)$ with the integrals of products of coordinate functions on the orbit of the highest weight in the coadjoint representation. In the third part, we introduce the probability measure and the C-matrix associated to a representation of $U(n)$ and give some of their properties, then we prove the main result of the paper, and we deduce from this the results concerning the tensor product and the restriction problem. Finally in the end we treat an explicit example.

§ 1

We recall here several definitions and notations.

1.1. First we deal with free families and free convolution (see [1] for more details).

A *non-commutative probability space* is a couple $(\mathscr{A}, \phi)$ where $\mathscr{A}$ is a unital $*$-algebra and ϕ a positive linear functional on $\mathscr{A}$ such that $\phi(1) = 1$. The elements of $\mathscr{A}$ are called non-commutative random variables. Let $\mathbb{C}\langle y_\iota, y_\iota^*\rangle_{\iota \in I}$ be the free algebra generated by the symbols y_ι, y_ι^*, $\iota \in I$, with the involution $y_\iota \mapsto y_\iota^*$.

The *law* (or *distribution*) of a family $(Y_\iota)_{\iota \in I}$ of non-commutative random variables is the linear map

$$D_Y\colon \mathbb{C}\langle y_\iota, y_\iota^*\rangle_{\iota \in I} \to \mathbb{C}$$
$$P \mapsto \phi(P(Y_\iota, Y_\iota^*))$$

where elements of $\mathbb{C}\langle y_\iota, y_\iota^*\rangle_{\iota \in I}$ are considered as non-commutative polynomials. If $(Y_\iota^{(n)})_{\iota \in I}$ for $n \in \mathbb{N}$ and $(Y_\iota^{\infty})_{\iota \in I}$ are families of non-commutative random variables, one says that the law of $(Y_\iota^{(n)})_{\iota \in I}$ converges to that of $(Y_\iota^{\infty})_{\iota \in I}$ as $n \to \infty$ if for every $P \in \mathbb{C}\langle y_\iota, y_\iota^*\rangle_{\iota \in I}$ one has $D_{Y^{(n)}}(P) \to D_{Y^{\infty}}(P)$.

A family $(\mathscr{A}_\iota)_{\iota \in I}$ of subalgebras of $\mathscr{A}$ is said to be *free* if for any $k \in \mathbb{N}$ and any sequence $a_1 \in \mathscr{A}_{\iota_1}, \dots a_k \in \mathscr{A}_{\iota_k}$ one has $\phi(a_1 \dots a_k) = 0$ whenever $\iota_j \neq \iota_{j+1}$ for $j = 1, \dots, k-1$ and $\phi(a_j) = 0$ for all $j = 1, \dots, k$. A family of sets of non-commutative random variables $(E_\iota)_{\iota \in I}$ is said to be free if the family of algebras $(\mathscr{A}_\iota)_{\iota \in I}$ is free where for each ι, $\mathscr{A}_\iota$ is the algebra generated by the set E_ι.

The law of a free family $(Y_\iota)_{\iota \in I}$ is determined by the law of each variable Y_ι.

A sequence of families of sets of non-commutative random variables $(E_\iota^{(n)})_{\iota \in I}$ with $E_\iota^{(n)} = \{Y_\sigma^{(n)} | \sigma \in J_\iota\}$ is said to be asymptotically free if the law of the family random variables $(Y_\sigma^{(n)}, \sigma \in J_\iota, \iota \in I)$ converges towards the law of some family of random variables $(Y_\sigma^{\infty}, \sigma \in J_\iota, \iota \in I)$ such that the sets $E_\iota^{\infty} = \{Y_\sigma^{\infty} | \sigma \in J_\iota\}$ for $\iota \in I$ form a free family of sets of random variables.

If (X, Y) is a free family then the distribution of $X + Y$ depends only on the distributions of X and of Y, it is called the additive free convolution $D_{X+Y} = D_X \oplus D_Y$. In the same way, the distribution of XY depends only on D_X and D_Y and is called the multiplicative free convolution denoted by $D_X \otimes D_Y = D_{XY}$. If $\mathscr{A}$ is a C^*-algebra, X and Y are positive elements then $D_X \otimes D_Y$ is also the distribution of $X^{1/2} Y X^{1/2}$ so that the multiplicative free convolution of two probability measures with compact supports on $\mathbb{R}_+$ is again a probability measure with compact support on $\mathbb{R}_+$.

1.2. Let $U(n)$ be the group of $n \times n$ complex unitary matrices and $\mathfrak{u}(n)$ the corresponding Lie algebra of $n \times n$ anti-hermitian matrices. Let $\mathfrak{d}_n$ be the Lie subalgebra of diagonal matrices. The complexification of $\mathfrak{u}(n)$ is $gl_n(\mathbb{C}) = \mathfrak{u}(n) + i\mathfrak{u}(n)$, the Lie algebra of the complex group $GL_n(\mathbb{C})$ and $d_n = \mathfrak{d}_n + i\mathfrak{d}_n$ is the complexification of $\mathfrak{d}(n)$. We call $(E_{kl})_{1 \leq k, l \leq n}$ the canonical basis of $gl_n(\mathbb{C}) \sim M_n(\mathbb{C})$.

Let $\mathfrak{u}(n)^*$ be the dual space of $\mathfrak{u}(n)$ and $\mathfrak{d}(n)^*$ the dual space of $\mathfrak{d}(n)^*$. Let P_+ be the set of dominant weights which are the elements μ of $\mathfrak{d}(n)^*$ such that $\mu(iE_{kk}) \in \mathbb{Z}$ and $\mu(iE_{11}) \geq \cdots \geq \mu(iE_{nn})$. Every irreducible representation of $U(n)$ (or equivalently finite dimensional irreducible representation of $GL_n(\mathbb{C})$) has a highest weight with respect to the ordered basis $(iE_{kk})_{1\leq k\leq n}$ of $\mathfrak{d}_n$, which is a dominant weight and it characterizes the representation up to isomorphism. There is an inclusion $\mathfrak{d}_n^* \subset \mathfrak{u}_n^*$ dual to the canonical projection $\mathfrak{u}_n \to \mathfrak{d}_n$, and elements of $\mathfrak{u}_n^*$ can be extended to complex linear functionals on $gl_n(\mathbb{C})$.

For a dominant weight μ we put $|\mu| = \sup_k |\mu(iE_{kk})|$. Let η_k be the dual basis of $(iE_{kk})_{1\leq k\leq n}$ in $\mathfrak{d}(n)^*$ and ρ be half the sum of positive roots $\rho = \sum_{k=1}^{n} (n-k+1)\eta_k$. For $\nu \in \mathfrak{u}(n)^*$ let $\mathcal{O}_\nu^*$ be the orbit of ν under the coadjoint action of $U(n)$.

Let R_μ be a unitary irreducible representation with highest weight μ on some Hilbert space V. In the non-commutative probability space $(\mathcal{L}(V), tr)$ (where tr is the normalized trace on $\mathcal{L}(V)$) let us consider the random variables $\xi_{kl}^\mu = dR_\mu(E_{kl})$ (where dR_μ is the corresponding complex representation of $gl_n(\mathbb{C})$).

The Kirillov character formula for R_μ (see [2]) can be stated as

$$tr\left(\exp\left(i\sum_{kl}\alpha_{kl}\xi_{kl}^\mu\right)\right) = \left(\prod_{k<l}\frac{\alpha_{ll}-\alpha_{kk}}{sh(\alpha_{ll}-\alpha_{kk})}\right)\int_{\mathcal{O}_{\mu+\rho}^*}\exp\left(i\sum\alpha_{kl}\omega(E_{kl})\right)d\omega \quad (1)$$

for complex numbers α_{kl} such that $\alpha_{kl} = \bar{\alpha}_{lk}$, and where $d\omega$ denotes the normalized invariant measure on $\mathcal{O}_{\mu+\rho}^*$.

1.3. A *random matrix* is a random variable of some non-commutative probability space of the form $(M_n(\mathbb{C}) \otimes L^\infty(\Omega, \mathcal{F}, P), tr \otimes P)$ where $(\Omega, \mathcal{F}, P)$ is a probability space and tr is the normalized trace on $M_n(\mathbb{C})$.

Let $\mathcal{O}_\lambda$ denote the orbit of the matrix $\sum_{k=1}^{n} \lambda_k E_{kk}$ (where $\lambda_k \in \mathbb{R}$ for $1 \leq k \leq n$) under the adjoint action of $U(n)$ on $i\mathfrak{u}(n)$, with its invariant probability measure. Let Y_{kl}^λ denote the kl coordinate function on $\mathcal{O}_\lambda$. A uniform random matrix with spectrum $\{\lambda_1, \ldots, \lambda_n\}$ is a random matrix $\sum_{kl} E_{kl}H_{kl}$ where the joint law of the random variables H_{kl}, $1 \leq k$, $l \leq n$ is the same as the joint law of the random variables Y_{kl}^λ, $1 \leq k$, $l \leq n$.

Let $\nu \in \mathfrak{u}^*(n)$, then ν is in the orbit of some element of $\mathfrak{d}^*(n)$ of the form $\sum_k \nu_k\eta_k$. Define the measurable maps $X_{kl}^\nu: (\mathcal{O}_\nu^*, d\omega) \to \mathbb{C}$ as $X_{kl}^\nu(\omega) = \omega(E_{kl})$. It is easy to see that in the non-commutative probability space $(M_n(\mathbb{C}) \otimes L^\infty(\mathcal{O}_\nu^*, d\omega), tr \otimes d\omega)$ the random variable defined as $\sum_{kl} E_{kl} \otimes X_{kl}^\nu$ is a uniform random matrix with spectrum $\{\nu_1, \ldots, \nu_n\}$.

For a family $\nu(s)_{s\in S}$ (where $\nu(s) = \nu_1(s)\eta_1 + \cdots \nu_n(s)\eta_n$) of elements of $\mathfrak{u}_n$, in the noncommutative probability space $M_n(\mathbb{C}) \otimes L^\infty(\prod_{s\in S}\mathcal{O}_{\nu(s)}^*, \prod_{s\in S} d\omega)$ the elements $(\sum E_{kl} \otimes X_{kl}^{\nu(t)})_{t\in S}$ where $X_{kl}^{\nu(t)}(\prod_{s\in S}\omega_s) = \omega_t(E_{kl})$ form a family of uniform random matrices with respective spectra $\{\nu_1(s)\ldots\nu_n(s)\}$, and with independent coordinates.

1.4. Following results of D. Voiculescu (see [7]), R. Speicher proved that the trace of the spectral measure of the sum of two large symmetric matrices is given asymptotically by the free convolution of their spectral measures, for almost all choices of the matrices with given spectrum (see [5] part 3 for the precise statement of the theorem). If one looks carefully at his arguments one can see that they imply the following result. Let n_m be a sequence of positive integers such that $n_m \to \infty$ as $m \to \infty$, and for each m let $X(m, s)_{s \in S}$ be a family of uniform random $n_m \times n_m$ matrices with independent entries and with respective spectra $\{\lambda_1(m, s) \dots \lambda_n(m, s)\}$. Suppose that the measures $\frac{1}{n_m} \sum_{k=1}^{n_m} \delta_{\lambda_k(m,s)}$ converge weakly to some measure $\alpha(s)$ with compact support on $\mathbb{R}$. Let $D(t, m)_{t \in T}$ be families of $n_m \times n_m$ diagonal matrices which have a limit distribution as $m \to \infty$, and such that $\sup_m \|D(t, m)\| < \infty$ for all $t \in T$. One considers the matrices $D(t, m)$ as random matrices with constant entries. Then $(X(m, s)_{s \in S}, \{D(t, m)_{t \in T}\})_{m \geq 0}$ is an asymptotically free family of sets of random variables.

§2

2.1. Let $\mu \in P_+$ and R_μ be as in 1.2. In this section we will prove a technical estimate which relates the moments of the random variables ξ^μ_{kl} and $X^{\mu+\rho}_{kl}$ defined in 1.2 and 1.3. We denote by E the expectation on the (commutative) probability space $L^\infty(\mathcal{O}^*_{\mu+\rho}, d\omega)$.

Proposition 1. *There exist constants C_r depending only on r such that for any sequence $(k_1, l_1), \dots (k_r, l_r)$ in $[1, n]^2$ one has*

$$|tr(\xi^\mu_{k_1 l_1} \dots \xi^\mu_{k_r l_r}) - E[X^{\mu+\rho}_{k_1 l_1} \dots X^{\mu+\rho}_{k_r l_r}]| \leq 6nC_r(|\mu| + 6n)^{r-1} .$$

Proof. We first give a series of lemma.

2.2. Lemma 1. *For every sequence $(k_1, l_1), \dots (k_r, l_r)$ in $[1, n]^2$ one has*

$$|tr(\xi^\mu_{k_1 l_1} \dots \xi^\mu_{k_r l_r})| \leq |\mu|^r$$

and

$$|E[X^{\mu+\rho}_{k_1 l_1} \dots X^{\mu+\rho}_{k_r l_r}]| \leq (|\mu| + n)^r .$$

Proof. The norm of $dR_\mu(E_{kl})$ on V is less than $|\mu|$ so that the first inequality follows. For the second, one has $|X^{\mu+\rho}_{kl}| \leq |\mu + \rho| \leq |\mu| + n$ on $\mathcal{O}^*_{\mu+\rho}$ hence the second inequality.

Let us notice that the function $\prod_{k<l} \frac{\alpha_{ll} - \alpha_{kk}}{sh(\alpha_{ll} - \alpha_{kk})}$ of the variables α_{kk} has a convergent power series expansion in a neighbourhood of zero.

Lemma 2. *Let $f(a_1, \dots a_n)$ for $a_1, \dots a_n \in \mathbb{N}$ be the coefficient of $\alpha_{11}^{a_1} \dots \alpha_{nn}^{a_n}$ in the expansion of $\prod_{k<l} \frac{\alpha_{ll} - \alpha_{kk}}{sh(\alpha_{ll} - \alpha_{kk})}$. One has $|f(a_1, \dots a_n)| \leq (6n)^{a_1 + \dots + a_n}$.*

Proof. One has $\frac{x}{shx} = \sum_{k=0}^{\infty} (-1)^k x^{2k} \frac{E_{2k+1}}{(2k+1)!}$ where E_{2k+1} are Euler's numbers counting the number of alternating permutations, and $E_{2k+1} \le (2k+1)!$ hence the absolute value of the coefficient of $x^u y^v$ in $\frac{x-y}{sh(x-y)}$ is $\frac{E_{u+v+1}}{(u+v+1)!}\frac{(u+v)!}{u!v!}$ when $u+v$ is even, and 0 if $u+v$ is odd. This coefficient is thus smaller than $\frac{(u+v)!}{u!v!}$ and hence smaller than 2^{u+v}. We conclude from this that the coefficient of $\alpha_{11}^{a_1} \dots \alpha_{nn}^{a_n}$ in $\prod_{k<l} \frac{\alpha_{ll} - \alpha_{kk}}{sh(\alpha_{ll} - \alpha_{kk})}$ is smaller in absolute value than that of $\alpha_{11}^{a_1} \dots \alpha_{nn}^{a_n}$ in $\prod_{k<l} \sum_{uv} 2^u \alpha_{kk}^u 2^v \alpha_{ll}^v = \prod_k (1 - 2\alpha_{kk})^{-(n-1)}$. But this last coefficient is equal to $\prod_k 2^{a_k} \frac{(a_k + n)!}{a_k! n!}$ and we can use the elementary inequalities $\frac{(u+v)!}{u!v!} \le (ev)^u$ for $v \ge 1$, and $2e < 6$, to obtain the result.

Lemma 3. *For any permutation σ of $1, \dots, r$ one has*

$$|tr(\xi^\mu_{k_1 l_1} \cdots \xi^\mu_{k_r l_r}) - tr(\xi^\mu_{k_{\sigma(1)} l_{\sigma(1)}} \cdots \xi^\mu_{k_{\sigma(r)} l_{\sigma(r)}})| \le r(r-1)|\mu|^{r-1}.$$

Proof. Let τ be a transposition of the form $(j, j+1)$ for $1 \le j \le r-1$. One has

$$\begin{aligned} &|tr(\xi^\mu_{k_1 l_1} \cdots \xi^\mu_{k_r l_r}) - tr(\xi^\mu_{k_{\tau(1)} l_{\tau(1)}} \cdots \xi^\mu_{k_{\tau(r)} l_{\tau(r)}})| \\ &= |tr(\xi^\mu_{k_1 l_1} \cdots [\xi^\mu_{k_j l_j}, \xi^\mu_{k_{j+1} l_{j+1}}] \cdots \xi^\mu_{k_r l_r})| \\ &= |tr(\xi^\mu_{k_1 l_1} \cdots (\xi^\mu_{k_j l_{j+1}} \delta_{l_j k_{j+1}} - \xi^\mu_{k_{j+1} l_j} \delta_{k_j l_{j+1}}) \cdots \xi^\mu_{k_r l_r})| \end{aligned}$$

since dR_μ is a Lie algebra representation

$$\le 2|\mu|^{r-1} \text{ by Lemma 1}.$$

Since every permutation is a product of at most $\frac{r(r-1)}{2}$ such transpositions, the lemma follows.

2.3. We can now prove Proposition 1.

Thanks to the estimates of Lemmas 1 and 2 we can expand the two sides of Kirillov's formula (1) as power series of the variables α_{kl} in a neighbourhood of zero and equate the coefficients of both sides. Let $a = (a_{kl})_{1 \le k, l \le n}$ be a sequence of nonnegative integers then the coefficient of $\prod_{kl} \alpha_{kl}^{a_{kl}}$ in the left side of (1) is equal to

$$\sum_{(u_1, v_1), \dots (u_r, v_r) \in \mathscr{P}_a} i^r \beta((u_1, v_1), \dots (u_r, v_r)) tr(\xi^\mu_{u_1 v_1} \cdots \xi^\mu_{u_r v_r})$$

where the sum is taken over the set $\mathscr{P}_a$ of all sequences $((u_1, v_1), \ldots (u_r, v_r))$ such that the number of occurences of each pair (k, l) is exactly a_{kl} (so that $\sum_{kl} a_{kl} = r$), and the numbers $\beta((u_1, v_1), \ldots (u_r, v_r))$ are some positive universal coefficients. Moreover, one has

$$\sum_{u=((u_1,v_1),\ldots(u_r,v_r))\in\mathscr{P}_a} \beta(u) = \prod_{kl} (a_{kl}!)^{-1} .$$

In the right hand side of the formula, the coefficient of $\prod_{kl} \alpha_{kl}^{a_{kl}}$ is equal to

$$\sum_{\substack{0\le b_{kk}\le a_{kk}\\ for\ 1\le k\le n}} i^{r-\sum_k b_{kk}} f(b_{11}, \ldots, b_{nn}) E\left[\prod_{k\neq l} \frac{(X_{kl}^{\mu+\rho})^{a_{kl}}}{a_{kl}!} \prod_k \frac{(X_{kk}^{\mu+\rho})^{a_{kk}-b_{kk}}}{(a_{kk}-b_{kk})!}\right].$$

It follows that

$$\left|\sum_{(u_1,v_1),\ldots(u_r,v_r)\in\mathscr{P}_a} \beta((u_1, v_1), \ldots (u_r, v_r)) tr(\xi_{u_1v_1}^{\mu} \cdots \xi_{u_rv_r}^{\mu}) - E\left[\prod_{kl} \frac{(X_{kl}^{\mu+\rho})^{a_{kl}}}{a_{kl}!}\right]\right| \tag{2}$$

is equal to

$$\left|\sum_{\substack{0\le b_{kk}\le a_{kk}\\ \text{not all } b_{kk}=0}} i^{r-\sum_k b_{kk}} f(b_{11}, \ldots, b_{nn}) E\left[\prod_{k\neq l} \frac{(X_{kl}^{\mu+\rho})^{a_{kl}}}{a_{kl}!} \prod_k \frac{(X_{kk}^{\mu+\rho})^{a_{kk}-b_{kk}}}{(a_{kk}-b_{kk})!}\right]\right|$$

which is, according to Lemmas 1 and 2, less in absolute value than

$$\sum_{\substack{0\le b_{kk}\le a_{kk}\\ \text{not all } b_{kk}=0}} (|\mu| + n)^{r-\sum_k b_{kk}} \prod_k \frac{1}{(a_{kk}-b_{kk})!} (6n)^{\sum_k b_{kk}} .$$

In this expression one has $(|\mu| + n)^{r-\sum_k b_{kk}} (6n)^{\sum_k b_{kk}} \le 6n(|\mu| + 6n)^{r-1}$ since the b_{kk} are not all zero, so that it is less than

$$\begin{aligned}
6n(|\mu| + 6n)^{r-1} &\sum_{0\le b_{kk}\le a_{kk}} \prod_k \frac{1}{(a_{kk}-b_{kk})!} \\
&= 6n(|\mu| + 6n)^{r-1} \prod_k \left(\sum_{w=0}^{a_{kk}} \frac{1}{w!}\right) \\
&\le 6n(|\mu| + 6n)^{r-1} \prod_{\substack{k\\ a_{kk}\neq 0}} e \\
&\le 6n(|\mu| + 6n)^{r-1} e^r
\end{aligned}$$

since there are at most r numbers $a_{kk} \neq 0$.

For any sequence $(k_1, l_1), \ldots, (k_r, l_r)$ which belongs to $\mathscr{P}_a$ one has $E[X_{k_1l_1}^{\mu+\rho} \ldots X_{k_rl_r}^{\mu+\rho}] = E[\prod_{kl} (X_{kl}^{\mu+\rho})^{a_{kl}}]$ and, by Lemma 3,

$$\left|\prod_{kl}(a_{kl}!)^{-1}tr(\xi^{\mu}_{k_1l_1}\dots\xi^{\mu}_{k_rl_r})-\sum_{u=((u_1,v_1),\dots(u_r,v_r))\in\mathscr{P}_a}\beta(u)tr(\xi^{\mu}_{u_1v_1}\dots\xi^{\mu}_{u_rv_r})\right|$$
$$\leq\sum_{u=((u_1,v_1),\dots(u_r,v_r))\in\mathscr{P}_a}\beta(u)|tr(\xi^{\mu}_{k_1l_1}\dots\xi^{\mu}_{k_rl_r})-tr(\xi^{\mu}_{u_1v_1}\dots\xi^{\mu}_{u_rv_r})|$$
$$\leq\prod_{kl}(a_{kl}!)^{-1}r(r-1)|\mu|^{r-1}\,. \tag{3}$$

From the majorizations of the quantities (2) and (3) it follows that

$$|tr(\xi^{\mu}_{k_1l_1}\dots\xi^{\mu}_{k_rl_r})-E[X^{\mu+\rho}_{k_1l_1}\dots X^{\mu+\rho}_{k_rl_r}]|$$
$$\leq r(r-1)|\mu|^{r-1}+\prod_{kl}(a_{kl}!)6ne^r(|\mu|+6n)^{r-1}$$
$$\leq r(r-1)|\mu|^{r-1}+6nr!e^r(|\mu|+6n)^{r-1}$$

and Proposition 1 follows easily from this estimate.

2.4. We have the following corollary of Proposition 1.

Corollary. *Let μ^m be a sequence in P_+ and ε_m a sequence in $\mathbb{R}_+$ such that $\varepsilon_m\to 0$ and $\varepsilon_m\mu^m\to\nu\in\mathfrak{d}^*_n$ as $m\to\infty$. Then the family of random variables $(\varepsilon_m\xi^{\mu^m}_{kl})_{1\leq k,l\leq n}$ converge in distribution to $(X^{\nu}_{kl})_{1\leq k,l\leq n}$ as $m\to\infty$.*

Proof. It is enough to prove that for any sequence $(k_1,l_1),\dots(k_r,l_r)$ in $[1,n]^2$ one has

$$tr(\varepsilon_m\xi^{\mu^m}_{k_1l_1}\dots\varepsilon_m\xi^{\mu^m}_{k_rl_r})\to E[X^{\nu}_{k_1l_1}\dots X^{\nu}_{k_rl_r}]$$

as $m\to\infty$. From Proposition 1 we have

$$|tr(\varepsilon_m\xi^{\mu^m}_{k_1l_1}\dots\varepsilon_m\xi^{\mu^m}_{k_rl_r})-E[\varepsilon_mX^{\mu^m+\rho}_{k_1l_1}\dots\varepsilon_mX^{\mu^m+\rho}_{k_rl_r}]|\leq(\varepsilon_m)^r6nC_r(|\mu^m|+6n)^{r-1}$$

and the expression on the right goes to 0 as $m\to\infty$. Moreover

$$E[\varepsilon_mX^{\mu^m+\rho}_{k_1l_1}\dots\varepsilon_mX^{\mu^m+\rho}_{k_rl_r}]=\int_{\mathcal{O}^*_{\mu^m+\rho}}\varepsilon_mX^{\mu^m+\rho}_{k_1l_1}(\omega)\dots\varepsilon_mX^{\mu^m+\rho}_{k_rl_r}(\omega)d\omega$$
$$=\int_{\mathcal{O}^*_{\varepsilon_m\mu^m+\varepsilon_m\rho}}\omega(E_{k_1l_1})\dots\omega(E_{k_rl_r})d\omega$$

and since $\varepsilon_m\mu^m\to\nu$ this last expression converges as $m\to\infty$ to

$$\int_{\mathcal{O}^*_\nu}\omega(E_{k_1l_1})\dots\omega(E_{k_rl_r})d\omega=E[X^{\nu}_{k_1l_1}\dots X^{\nu}_{k_rl_r}]\,.$$

This proves the corollary.

§3

In this section we will let the dimension n go to infinity.

3.1. Let R be a finite dimensional unitary representation of $U(n)$. We will associate to R two objects, namely a probability measure on $\mathbb{Z}$ and a non commutative random variable.

We define the measure $\mathscr{M}(R)$ in the following way, if R is irreducible with highest weight $\mu = \sum \mu_k \eta_k$ then $\mathscr{M}(R) = \frac{1}{n}\sum_k \delta_{\mu_k}$. If R can be decomposed into a sum of two subrepresentations $R = R_1 \oplus R_2$ then $\mathscr{M}(R)$ is the convex combination $\mathscr{M}(R) = \frac{\dim(R_1)}{\dim(R)}\mathscr{M}(R_1) + \frac{\dim(R_2)}{\dim(R)}\mathscr{M}(R_2)$. Thanks to the complete reducibility of representations of $U(n)$, there is a uniquely defined map $R \mapsto \mathscr{M}(R)$ from finite dimensional representations of $U(n)$ to probability measures on $\mathbb{Z}$ satisfying these requirements.

Let V be the space of the representation of R, which is a Hilbert space. The C-random variable associated to R is the element of the non-commutative probability space $(M_n(\mathbb{C}) \otimes \mathscr{L}(V), tr)$ (where tr is the normalized trace) defined as $C(R) = \sum_{kl} E_{kl} \otimes dR(E_{kl})$ where dR is the representation of the Lie algebra $gl_n(\mathbb{C})$ corresponding to R. We use the letter C because of the obvious relation with Casimir operator.

It is easy to see that $C(R)$ is a self-adjoint element of $M_n(\mathbb{C}) \otimes \mathscr{L}(V)$, hence it has a distribution which is given by a probability measure on $\mathbb{R}$, which is a finite convex combination of Dirac measures. It turns out that this measure, although not equal to $\mathscr{M}(R)$ is closely related to it. In fact it follows easily from the corollary of Proposition 1, proved in Section 2 that if μ^m is a sequence in P_+ and ε_m a sequence in $\mathbb{R}_+$ such that $\varepsilon_m \to 0$ and $\varepsilon_m \mu^m \to \nu \in \mathfrak{h}_n^*$ as $m \to \infty$ if R_m is an irreducible representation with highest weight μ^m then the distribution of $\varepsilon_m C(R_m)$ and the measure $\mathscr{M}^{\varepsilon_m}(R_m)$ converge to the same limit which is $\frac{1}{n}\sum_k \delta_{\nu_k}$.

3.2. The explicit distribution of $C(R)$ can be obtained. Indeed, the computations of Zelobenko (see [8] Ch IX, 60) show that the operator $C(R)$ has the eigenvalues $\lambda_j = \mu_j + n - j$ for $j = 1, \ldots, n$ the multiplicity of λ_j being $\prod_{k \neq j} \frac{\mu_j - j - \mu_k + k - 1}{j - k}$.

Another method of obtaining this result was communicated to us by Patrick Polo [3] and is as follows. On the space $\mathbb{C}^n \otimes V$ let $U(n)$ act by the representation $\chi\colon g \mapsto \bar{g} \otimes R(g)$. A simple computation reveals that the matrix $C(R)$ commutes with all the operators of this representation. The representation χ is the tensor product of the representation R and of the conjugate of the basic representation of $U(n)$. It is well known (since the highest weight of the conjugate representation is a minuscule weight) that this representation decomposes into a sum of the irreducible representations with highest weights $\mu - \eta_k$ for the numbers $k \in \{1, \ldots, n\}$ such that $\mu - \eta_k$ is a dominant weight. From this we conclude that each of the subspaces of these representations is an eigenspace for $C(R)$. The corresponding eigenvalue can be computed by evaluating the image by $C(R)$ of a highest weight vector.

We could have defined the measure $\mathscr{M}(R)$ as being the distribution of $C(R)$, and the theorem that we prove in Section 3.4, as well as Proposition 3 would be true and have simpler proofs. However, the formula giving the law of $C(R)$ is complicated so that we have preferred the simpler definition of $\mathscr{M}(R)$ given in the text.

We will not use the result of this section in the sequel.

3.3. We shall now prove the following proposition which deals with the case when the dimension goes to infinity and the representations are allowed to be reducible.

Proposition 2. *Let n_m be a sequence of positive integers such that $n_m \to \infty$. For each $m \geq 1$, let R_m be a representation of $U(n_m)$. Let ε_m be a sequence in $\mathbb{R}_+$ such that*

i) $\varepsilon_m (n_m)^r \to 0$ for all $r \in \mathbb{N}$

ii) there is a constant c such that for every m and every highest weight μ occuring in the representation R_m one has $\varepsilon_m |\mu| \leq c$

iii) the measures $\mathscr{M}^{\varepsilon_m}(R_m)$ have a limit α as $m \to \infty$, then the distribution of $\varepsilon_m C(R_m)$ converges to α as $m \to \infty$.

Proof. First remark that by condition *ii)* the distribution α must come from a probability measure with compact support on $[-c, c]$.

We must show that for all $r \in \mathbb{N}$, $tr((\varepsilon_m C(R_m))^r) \to \int_{\mathbb{R}} x^r \alpha(dx)$.

For each m let $R_m = R_m^{(1)} \oplus R_m^{(2)} \oplus \cdots \oplus R_m^{(u_m)}$ be a decomposition of R_m into irreducible components with highest weights μ_m^w for $w = 1, \ldots, u_m$, then $C(R_m) = \pi_1 C_1 \pi_1 + \cdots + \pi_{u_m} C_{u_m} \pi_{u_m}$ where the π_w are orthogonal self-adjoint projections of trace $tr(\pi_w) = \dfrac{\dim (R_m^{(w)})}{\dim (R_m)}$, and each C_w is an operator with the same distribution as $C(R_m^{(w)})$, and so $tr(C(R_m)^r) = \sum_{w=1}^{u_m} \dfrac{\dim (R_m^{(w)})}{\dim (R_m)} tr(C(R_m^{(w)})^r)$. Since $\mathscr{M}(R_m) = \sum_{w=1}^{u_m} \dfrac{\dim R_m^{(w)}}{\dim (R_m)} \mathscr{M}(R_m^{(w)})$ it is enough to show that for each $w \in [1, u_m]$ one has $\varepsilon_m^r |tr(C(R_m^{(w)})^r) - \int_{\mathbb{R}} x^r \mathscr{M}(R_m^{(w)})(dx)| \leq v_m$ where v_m does not depend on w and converges to zero as $m \to \infty$.

One has

$$tr(C(R_m^{(w)})^r) = \frac{1}{n_m} \sum_{1 \le k_1, \dots, k_r \le n_m} tr_w(\xi_{k_1 k_2}^{\mu_m^w} \xi_{k_2 k_3}^{\mu_m^w} \cdots \xi_{k_r k_1}^{\mu_m^w})$$

where tr_w is the normalized trace on the representation space of $R_m^{(w)}$.

Let $M_{\mu_m^w+\rho} = \sum_{kl} E_{kl} X_{kl}^{\mu_m^w+\rho}$ where $X_{kl}^{\mu_m^w+\rho}$ are as in 1.2.

When computing $E[tr((M_{\mu_m^w+\rho})^r]$ one can expand the trace with respect to the variables $X_{kl}^{\mu_m^w+\rho}$ to get the expression

$$\frac{1}{n_m} \sum_{1 \le k_1, \dots, k_r \le n_m} E(X_{k_1 k_2}^{\mu_m^w+\rho} X_{k_2 k_3}^{\mu_m^w+\rho} \cdots X_{k_r k_1}^{\mu_m^w+\rho}).$$

It follows then from Proposition 1 that

$$|tr(C(R_m^{(w)})^r) - E[tr((M_{\mu_m^w+\rho})^r])| \le (n_m)^{r-1} 6 n_m C_r (|\mu_m^w| + 6n_m)^{r-1}.$$

Since $M_{\mu_m^w+\rho}$ is a uniform random matrix with spectrum $(\mu_m^w(iE_{jj}) + n_m - j + 1, j = 1, \dots, n_m)$, and since $\mathscr{M}(R_m^{(w)}) = \sum_j \delta_{\mu_m^w(iE_{jj})}$, using the elementary inequality $|(a+b)^r - a^r| \le r|b|(|a| + |b|)^{r-1}$ for $a, b \in \mathbb{R}$ one has

$$tr((M_{\mu_m^w+\rho})^r) = \frac{1}{n_m} \sum_j (\mu_m^w(iE_{jj}) + n_m - j + 1)^r = \int_{\mathbb{R}} x^r d\mathscr{M}(R_m^w)(x) + g(m, w)$$

with $|g(m, w)| \le r n_m (|\mu_m^w| + n_m)^{r-1}$. We see that for every w

$$\left| (\varepsilon_m)^r (tr(C(R_m^{(w)})^r) - \int x^r d\mathscr{M}(R_m^{(w)})(x)) \right|$$
$$\le (\varepsilon_m)^r (n_m)^{r-1} (6 n_m C_r (|\mu_m^w| + 6n_m)^{r-1} + r n_m (|\mu_m^w| + n_m)^{r-1})$$
$$\le \varepsilon_m (n_m)^{r-1} (6 n_m C_r (c + 6\varepsilon_m n_m)^{r-1} + r n_m (c + \varepsilon_m n_m)^{r-1}).$$

This quantity is independent of w and goes to 0 by hypothesis *i*). This finishes the proof.

3.4. We say that a family of representations $R(s)$ (indexed by some finite set S) of $U(n)$ on a finite dimensional space V is independent if there exists a tensor product decomposition $V = V_0 \otimes (\otimes_{s \in S} V_s)$ such that for every $t \in S$ and $g \in U(n)$ one has $R(t)(g) = Id_{V_0} \otimes (\otimes_{s \ne t} Id_{V_s}) \otimes \tilde{R}(t)(g)$ where $\tilde{R}(t)$ is a representation on V_t.

For independent representations one has $[dR(s)(E_{kl}), dR(s')(E_{k'l'})] = 0$ for all k, l, k', l' if $s \ne s'$, and the trace factorizes:

$$tr_V\left(\prod_{s \in S} \zeta_s\right) = \prod_{s \in S} tr_V(\zeta_s)$$

where ζ_s belongs to the algebra generated by the operators $dR(s)(E_{kl})$.

We are now in position to state and prove the main result of this paper.

Theorem. *Let S be a finite set. Let n_m be a sequence of positive integers such that $n_m \to \infty$. Let, for each $m \geq 0$, $R_m(s)$, $s \in S$ be a family of independent isotypic representations of $U(n_m)$ (i.e. the representations $\tilde{R}_m(s)$ are irreducible), and let ε_m, $m \geq 0$ be a sequence in $\mathbb{R}_+$ such that*

i) for every $r \in \mathbb{N}$ $\varepsilon_m(n_m)^r \to 0$

ii) there is a constant c such that $\varepsilon_m|\mu^m(s)| \leq c$ for all $m \in \mathbb{N}$, $s \in S$ where $\mu^m(s)$ is the highest weight of $R_m(s)$

iii) there exists probability distributions $\alpha(s)$ such that $\mathcal{M}^{\varepsilon_m}(R_m(s)) \to \alpha(s)$.

Let furthermore $D(t, m)_{t \in T}$ be families of $n_m \times n_m$ diagonal matrices which have a limit distribution as $m \to \infty$ and such that $\sup_m \|D(t, m)\| < \infty$ for all $t \in T$. We consider them as elements of $M_{n_m}(\mathbb{C}) \otimes \mathcal{L}(V)$ by tensoring with Id_V.

Then $(C(R_m(s))_{s \in S}, \{D(t, m)_{t \in T}\})$ is an asymptotically free family of sets of random variables.

Proof. As in [7] we can assume that the matrices $D(t, m)_{t \in T}$ form a multiplicative semi-group and that the identity is among them. We will study the asymptotic evaluation of quantities like

$$tr(D(t_1, m)(\varepsilon_m C(R_m(s_1))^{r_1} \ldots D(t_l, m)(\varepsilon_m C(R(s_l))^{r_l} D(t_{l+1}, m))$$

for $t_1, \ldots, t_{l+1} \in T$, $s_1, \ldots, s_l \in S$ and $r_1 \ldots r_l \in \mathbb{N}$.

Expanding this trace as in the proof of Proposition 2, and using the fact that the representations $R_m(s)$ are independent we see that it is equal to

$$\sum_{u \in \mathcal{S}} d(u)H(u)$$

where $\mathcal{S}$ is a certain set of maps of $S \cup T$ into sequences of the form

$$((u_1(s), v_1(s)) \ldots, (u_{z(s)}(s), v_{z(s)}(s)))$$

so that $z(s) = \sum_{s_k = s} r_k$ for $s \in \mathcal{S}$ and $z(t) = \sum_{t_k = t} 1$ for $t \in T$, one has card $(\mathcal{S}) \leq n_m^{r_1 + \cdots + r_l + l + 1}$ and for each $u \in \mathcal{S}$,

$$d(u) = \prod_{t \in T} \prod_{k=1}^{z(t)} D(t, m)_{u_k(t) v_k(t)}$$

and

$$H(u) = \prod_{s \in S} tr_V(\varepsilon_m \xi^{\mu_m(s)}_{u_1(s)v_1(s)} \ldots \varepsilon_m \xi^{\mu_m(s)}_{u_{z(s)}(s)v_{z(s)}(s)}) \,.$$

Let $M_{\mu^m(s)+\rho}$ be independent random matrices as in the proof of Proposition 2. Expanding the trace in the expression

$$tr(D(t_1, m)(\varepsilon_m M_{\mu^m(s_1)+\rho})^{r_1} \ldots D(t_l, m)(\varepsilon_m M_{\mu^m(s_l)+\rho})^{r_l} D(t_{l+1}, m))$$

we see that it is equal to

$$\sum_{u \in \mathcal{S}} d(u)G(u)$$

where $G(u)$ is the expression obtained from $H(u)$ by replacing $\xi_{kl}^{\mu_m(s)}$ by $M_{kl}^{\mu^m(s)+\rho}$. Using the estimates of Proposition 1 and Lemma 1 in the same way as in the proof of Proposition 2, and the fact that $\varepsilon_m|\mu_m(s)| \leq c$ and $\sup_m \|D(t, m)\| \leq c(t)$ for some constants $c(t)$, we can see that $|H(u) - G(u)|$ is bounded by $\varepsilon_m K_m$ where K_m is a polynomial in n_m whose degree and coefficients depend only on c, l, $r_1 \dots r_l$ and the constants $c(t)$. Details are straightforward and left to the reader. We deduce from this that

$$\sum_{u \in \mathscr{S}} d(u)|G(u) - H(u)| \leq \varepsilon_m n_m^{r_1 + \cdots + r_l + l + 1} K_m$$

and by hypothesis *i*) this quantity goes to zero as $m \to \infty$.

Thanks to the result of Section 1.4, the expression

$$tr(D(t_1, m)(\varepsilon_m M_{\mu^m(s_1)+\rho})^{r_1} \dots D(t_l, m)(\varepsilon_m M_{\mu^m(s_l)+\rho})^{r_l} D(t_{l+1}, m))$$

has a limit which is

$$\phi(D(t_1)X(s_1)^{r_1}D(t_2) \dots X(s_l)^{r_l}D(t_{l+1}))$$

where $(X(s)\ s \in S, \{D(t), t \in T\})$ is a free family in some non commutative probability space $(\mathscr{A}, \phi)$ and the law of $X(s)$ is $\alpha(s)$, the law of $\{D(t), t \in T\}$ is the limit of the law of $\{D(t, m), t \in T\}$. So we have proved that

$$tr(D(t_1, m)(\varepsilon_m C(R_m(s_1))^{r_1} \dots D(t_l, m)(\varepsilon_m C(R(s_l))^{r_l} D(t_{l+1}, m))$$

also converges towards

$$\phi(D(t_1)X(s_1)^{r_1}D(t_2) \dots X(s_l)^{r_l}D(t_{l+1}))$$

and this finishes the proof of the theorem.

We will now apply the preceding theorem to the problem of decompositions of tensor products and of restrictions.

Proposition 3. *Let n_m be a sequence of positive integers such that $n_m \to \infty$. For each $m \geq 1$ let $R_m(1)$ and $R_m(2)$ be two irreducible representations with highest weights $\mu^m(1)$ and $\mu^m(2)$, such that $\mathscr{M}^{\varepsilon_m}(R_m(s)) \to \alpha(s)$ as $m \to \infty$ for $s = 1, 2$, where $\alpha(1)$ and $\alpha(2)$ are two probability measures with compact support. Suppose that the sequences $\varepsilon_m|\mu^m(s)|$ are bounded.*

Then $\mathscr{M}^{\varepsilon_m}(R_m(1) \otimes R_m(2)) \to \alpha(1) \oplus \alpha(2)$.

Put $R_m = R_m(1)$ and $\alpha = \alpha(1)$. Let q_m be a sequence of integers such that $q_m \leq n_m$ and $\frac{q_m}{n_m} \to p \in]0, 1]$, and call T_m the restriction of R_m to the subgroup $U(q_m)$ of $U(n_m)$ (imbedded as acting on the q_m first vectors of the canonical basis of $\mathbb{C}^{n_m}$).

Then $\mathscr{M}^{\varepsilon_m}(T_m)$ converges towards the measure $\frac{1}{p}(\alpha \otimes (p\delta_1 + (1-p)\delta_0) - (1-p)\delta_0)$, where $\otimes$ denotes the free multiplicative convolution.

Proof. The representations $R_m(1)\otimes Id$ and $Id\otimes R_m(2)$ acting on the space of the representation $R_m(1)\otimes R_m(2)$ are independent representations, so that the random variable $C(R_m(1)\otimes R_m(2))$ can be written as $C_m(1)+C_m(2)$ where $\varepsilon_m C_m(1)$ and $\varepsilon_m C_m(2)$ have the same distribution respectively as $\varepsilon_m C(R_m(1))$ and $\varepsilon_m C(R_m(2))$ and are asymptotically free by the preceding theorem. We deduce from this that the law of $\varepsilon_m C(R_m(1)\otimes R_m(2))$ converges towards the free convolution of $\alpha(1)$ and $\alpha(2)$. Applying Proposition 2 to the sequence of random variables $\varepsilon_m C(R_m(1)\otimes R_m(2))$ we see that the sequence $\mathscr{M}^{\varepsilon_m}(R_m(1)\otimes R_m(2))$ converges towards $\alpha(1)\oplus\alpha(2)$ (we can apply Proposition 2 because all the highest weights occuring in $R_m(1)\otimes R_m(2)$ have components smaller than $|\mu^m(1)|+|\mu^m(2)|$).

Let $D(m)$ be the diagonal matrix $\sum_{k=1}^{q_m} E_{kk}$ then $D(m)$ has a limit distribution which is $(1-p)\delta_0+p\delta_1$ and by the theorem it is asymptotically free with the random variable $\varepsilon_m C(R_m)$. Now we see that $D(m)C(R_m)D(m)$ is the upper left corner embedding of the matrix $C(T_m)$ (which lies in $M_{q_m}(\mathbb{C})\otimes\mathscr{L}(V)$) in the space $M_{n_m}(\mathbb{C})\otimes\mathscr{L}(V)$. We deduce from this that the distribution of $C(T_m)$ can be obtained from that of $D(m)C(R_m)D(m)$ by substracting $\left(1-\frac{q_m}{n_m}\right)\delta_0$ and then multiplying by $\frac{n_m}{q_m}$. But the limit distribution of $D(m)C(R_m)D(m)$ is the free multiplicative convolution of $(1-p)\delta_0+p\delta_1$ and α, so the result follows again by an application of Proposition 2.

3.5. We will now give an explicit example. Let us consider the group $U(n)$ and the dominant weight $(K,\dots,K,0,\dots,0)$ where the first l coordinates are K's and the remaining are 0's. Proceeding recursively, and using Weyl's branching rule, we see that for m smaller than l and $n-l$, the restriction of the corresponding representation R_m to $U(n-m)$ is the sum of the representations with highest weights $(K,\dots,K,\varphi_1,\dots\varphi_m,0\dots,0)$ where $K\geq\varphi_1\geq\cdots\geq\varphi_m\geq 0$ and there are K's on the first $l-m$ coordinates and 0's on the last $n-l-m$, these representations occurring with multiplicity $\prod_{k<l}\frac{\varphi_k-\varphi_l+l-k}{l-k}$. This fact can also be established by decomposing the restriction of the representation to the subgroup $U(m)\times U(n-m)\subset U(n)$.

If $n\to\infty$ and $\frac{l}{n}\to p\in[0,1]$, $\frac{m}{n}\to q\in\,]0,1]$, and $Kn^{-r}\to 0$ then the measure $\mathscr{M}^{1/K}(R_n)$ converges to $(1-p)\delta_0+p\delta_1$ and by Proposition 3, the measure $\mathscr{M}^{1/K}(T_n)$, where T_n is the restriction of R_n to $U(n-m)$ converges towards

$$\frac{1}{q}[((1-p)\delta_0+p\delta_1)\otimes(q\delta_0+(1-q)\delta_1)-(1-q)\delta_0].$$

In particular let us take $n = 2l = 2m$ then the limit measure $\left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) \otimes \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) - \frac{1}{2}\delta_0$ is the arcsine law given by the distribution function $F(t) = \frac{2}{\pi}\text{Arcsine}\sqrt{t}$ on $[0, 1]$ (see [1] Example 3.6.7). Since we have convergence of all moments and the limit distribution function is continuous, we have convergence of the distribution function of the measure $\mathscr{M}^{1/K}(T_n)$. The distribution function of the measure $\mathscr{M}^{1/K}(T_n)$ can be computed. Indeed, we know that T_n decomposes into the sum of the irreducible representations of highest weights $(\varphi_1, \ldots, \varphi_m)$ with multiplicity $\prod_{1 \le k < l \le m} \frac{\varphi_k - \varphi_l + l - k}{l - k}$. By Weyl's formula, the dimension of this representation is also $\prod_{1 \le k < l \le m} \frac{\varphi_k - \varphi_l + l - k}{l - k}$ and the dimension of R_n is $\prod_{1 \le k, l \le m} \frac{K + m + l - k}{m + l - k}$, hence

$$\mathscr{M}^{1/K}(T_n) = \left(\prod_{1 \le k, l \le m} \frac{m + l - k}{K + m + l - k}\right) \sum_{K \ge \varphi_1 \ge \cdots \ge \varphi_m \ge 0} \left(\prod_{k<l} \frac{\varphi_k - \varphi_l + l - k}{l - k}\right)^2 \frac{1}{m} \sum_{j=1}^{m} \delta_{\varphi_k/K} .$$

Since $Kn^{-3} \to 0$ one has $\prod_{1 \le k, l \le m} \frac{m + l - k}{K + m + l - k} \sim K^{-m^2}(\prod_{1 \le k, l \le n} m + l - k)$ as $n = 2m \to \infty$ and the convergence of the distribution function shows that for every $t \in [0, 1]$

$$\sum_{K \ge \varphi_1 \ge \cdots \ge \varphi_m \ge 0} \left(\prod_{k<l} \frac{\varphi_k - \varphi_l + l - k}{l - k}\right)^2 \frac{1}{m} \operatorname{card}\{j | \varphi_j \le tK\} = \left(\frac{K^{m^2}}{\prod_{1 \le k, l \le m} m + l - k}\right)\left(\frac{2}{\pi}\text{Arcsine}\sqrt{t} + o(1)\right)$$

as $m \to \infty$.

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