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# Perturbation Formulas for Traces on $C^*$-algebras

By

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**Abstract**

We introduce the Fréchet differential of operator functions on $C^*$-algebras obtained via spectral theory from ordinary differentiable functions. In the finite-dimensional case this differential is expressed in terms of Hadamard products of matrices. A perturbation formula with applications to traces is given.

## § 1. The Fréchet Differential

**Definition 1.1.** *If $\mathscr{X}$ and $\mathscr{Y}$ are Banach spaces, and $\mathscr{D}$ is an open subset of $\mathscr{X}$, we say that a function $F: \mathscr{X} \to \mathscr{Y}$ is Fréchet differentiable, if for each $x$ in $\mathscr{D}$ there is a bounded linear operator $F_x^{[1]}$ in $B(\mathscr{X}, \mathscr{Y})$ such that*

$$\lim_{h \to 0} \|h\|^{-1}(F(x+h) - F(x) - F_x^{[1]}(h)) = 0 .$$

*If the differential map $x \to F_x^{[1]}$ is continuous from $\mathscr{D}$ to $B(\mathscr{X}, \mathscr{Y})$, we say that $F$ is continuously Fréchet differentiable.*

Straightforward computations give the following result, which we list for easy reference.

**Proposition 1.2.** *If $F: \mathscr{X} \to \mathscr{Y}$ and $G: \mathscr{Y} \to \mathscr{Z}$ are continuously Fréchet differentiable maps between Banach spaces $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$, then $G \circ F$ is also continuously Fréchet differentiable, and*

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$$(G \circ F)_x^{[1]}(h) = G_{F(x)}^{[1]}(F_x^{[1]}(h))$$

*for every x, h in* $\mathscr{X}$.

The next result is well known to mathematical physicists, who would derive it from the socalled Dyson Expansion, cf. [10, 10.69] and [11, 1.15]. A complete and stringent formulation is found in Araki's paper [3], that "contains a powerful computational tool, which does not seem to be widely known among mathematicians." We shall only need a fraction of this tool, and include a simple proof for the convenience of the reader.

**Proposition 1.3.** *If* $\mathscr{A}$ *is a Banach algebra, then the exponential function* $A \to \exp(A)$ *is continuously Fréchet differentiable with*

$$\exp_A^{[1]}(B) = \int_0^1 \exp(sA)B\exp((1-s)A)ds$$

*for all* $A$, $B$ *in* $\mathscr{A}$.

*Proof.* By elementary calculus we have

$$\int_0^1 s^k(1-s)^m ds = \frac{k!m!}{(k+m+1)!}$$

and we can prove either by direct calculation or by induction that

$$(A+B)^n - A^n = \sum_{k=0}^{n-1}(A+B)^k BA^{n-(k+1)}.$$

Combining these two expressions we establish the Dyson formula

$$(*) \qquad \exp(A+B) - \exp(A) = \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{1}{n!}(A+B)^k BA^{n-k-1}$$
$$= \sum_{k,m=0}^{\infty}\frac{1}{k!m!}(A+B)^k BA^m \int_0^1 s^k(1-s)^m ds$$
$$= \int_0^1 \exp(s(A+B))B\exp((1-s)A)ds,$$

where we rearranged the sums by setting $m = n-k-1$. It is clear that the proposed expression for $\exp_A^{[1]}$ is a bounded linear operator that depends continuously on $A$, and by subtraction we get from $(*)$ that

$$\|\exp(A+B) - \exp(A) - \exp_A^{[1]}(B)\|$$
$$= \left\| \int_0^1 (\exp(s(A+B)) - \exp(sA))B\exp((1-s)A)ds \right\|$$
$$\le \|B\| e^{\|A\|} \int_0^1 \|\exp(s(A+B) - \exp(sA)\| ds.$$

From Lebesgues theorem of dominated convergence we see that the last integral converges to zero as $B \to 0$. We can thus conclude that exp is continuously Fréchet differentiable with the desired differential. QED

**Definition 1.4.** *We denote by $C^1_F(\mathbf{R})$ the set of real $C^1$-functions $f$ of the form*

$$f(t) = \int_{-\infty}^{\infty} e^{ixt} d\mu(x),$$

*where $\mu$ is a finite, symmetric, signed measure on $\mathbf{R}$, such that the moment*

$$m_1(\mu) = \int_{-\infty}^{\infty} |x| d|\mu|(x) < \infty .$$

The derivative $f'$ of a function $f$ in $C^1_F(\mathbf{R})$ is given by

$$f'(t) = \int_{-\infty}^{\infty} e^{ixt} ix d\mu(x) .$$

Symbolically, at least, we can write $\mu = \hat{f}$, so that the moment requirement can be restated as $\|\hat{f}'\|_1 < \infty$.

Note that $C^1_F(\mathbf{R})$ is an algebra of functions containing the Schwartz class; so its restriction to any finite interval $I$ is dense in $C^1(I)$ with respect to the $C^1$-norm. We are indebted to U. Haagerup for suggesting this class of functions as the most convenient carrier of a theory of Fréchet differentiability. Its use in the theory of unbounded derivations is evident from [11, 3.3.6].

If $\mathscr{A}$ is a $C^*$-algebra, and $\mathscr{A}_{\text{sa}}$ denotes the self-adjoint part of $\mathscr{A}$, then each bounded, continuous real function $f$ on $\mathbf{R}$ defines a continuous operator function $T \to f(T)$ on $\mathscr{A}_{\text{sa}}$ via the spectral theorem.

**Theorem 1.5.** *Let $\mathscr{A}$ be a $C^*$-algebra, and take $f$ in $C^1_F(\mathbf{R})$. Then the function $T \to f(T)$ is continuously Fréchet differentiable on $\mathscr{A}_{\text{sa}}$ with*

$$f_T^{[1]}(S) = \int_{-\infty}^{\infty} ix \int_0^1 e^{ixyT} S e^{ix(1-y)T} dy \; d\mu(x)$$

*for all $T$, $S$ in $\mathscr{A}_{\text{sa}}$. Moreover, the norm of the differential is $\|f_T^{[1]}\| \leq \|\hat{f}'\|_1$.*

*Proof.* Note first that the proposed expression of the Fréchet differential certainly is bounded—independent of $T$—by $\|\hat{f}'\|_1$, because $e^{ixyT}$ and $e^{ix(1-y)T}$ are unitary operators. We then apply the spectral theorem to obtain

$$f(T+S) - f(T) - \int_{-\infty}^{\infty} ix \int_0^1 e^{ixyT} S e^{ix(1-y)T} dy \, d\mu(x)$$

$$= \int_{-\infty}^{\infty} \left( e^{ix(T+S)} - e^{ixT} - ix \int_0^1 e^{ixyT} S e^{ix(1-y)T} dy \right) d\mu(x)$$

$$= \int_{-\infty}^{\infty} ix \int_0^1 (e^{ixy(T+S)} - e^{ixyT}) S e^{ix(1-y)T} dy \, d\mu(x),$$

where we used (*) from the proof of Proposition 1.3. The norm of this expression is bounded by

$$\|S\| \int_{-\infty}^{\infty} |x| \int_0^1 \|e^{ixy(T+S)} - e^{ixyT}\| dy \, d|\mu|(x),$$

and even after division by $\|S\|$ this does tend to zero as $S \to 0$ by Lebesgues theorem of dominated convergence. QED

As a first application of Fréchet differentiability we give the next result. More will follow in section 2.

**Proposition 1.6.** *Assume that a state $\varphi$ of a $C^*$-algebra $\mathscr{A}$ is definite on some element $T$ in $\mathscr{A}_{sa}$, i.e. $\varphi(T^2) = \varphi(T)^2$. Then for each $f \in C_F^1(\mathbf{R})$ there is a function $o_f \colon \mathscr{A}_{sa} \to \mathbf{R}$, with $\|S\|^{-1} o_f(S) \to 0$ as $S \to 0$, such that*

$$\varphi(f(T+S)) = f(\varphi(T)) + f'(\varphi(T))\varphi(S) + o_f(S)$$

*for every $S$ in $\mathscr{A}_{sa}$.*

*Proof.* Let $(\pi_\varphi, H_\varphi, x_\varphi)$ denote the GNS-representation of $\mathscr{A}$ associated with $\varphi$. We see from the Cauchy-Schwarz inequality that $\varphi$ is definite on $T$, if and only if $x_\varphi$ is an eigenvector for $\pi_\varphi(T)$, i.e. $\pi_\varphi(T)x_\varphi = \varphi(T)x_\varphi$. It follows that $\varphi$ is also definite on $g(T)$ for each $g$ in $C(Sp(T))$, and that $\varphi(g(T)) = g(\varphi(T))$. Moreover,

$$\varphi(TA) = \varphi(AT) = \varphi(T)\varphi(A)$$

for every $A$ in $\mathscr{A}$. If $f \in C_F^1(\mathbf{R})$, then we know from Theorem 1.5 that

$$f(T+S) = f(T) + f_T^{[1]}(S) + \|S\| R(T,S),$$

where $R(T,S) \to 0$ as $S \to 0$. The formula given for $f_T^{[1]}(S)$ shows that

$$\varphi(f(T+S)) = f(\varphi(T)) + \int_{-\infty}^{\infty} e^{ix\varphi(T)} ix \, d\mu(x)\varphi(S) + \|S\| \varphi(R(T,S))$$

$$= f(\varphi(T)) + f'(\varphi(T))\varphi(S) + o_f(S),$$

where we define

$$o_f(S) = \|S\| \varphi(R(T, S))$$

for $S$ in $\mathscr{A}_{\mathrm{sa}}$. QED

## § 2. Perturbation Formulas

**Theorem 2.1.** *Let $\mathscr{A}$ be a $C^*$-algebra, and let $t \to A(t)$ be a continuously differentiable function from the interval* $[0, 1]$ *into $\mathscr{A}_{\mathrm{sa}}$. Then we have*

$$f(A(1)) - f(A(0)) = \int_0^1 f_{A(t)}^{[1]}(A'(t))dt$$

*for each $f$ in $C_F^1(\mathbf{R})$.*

*Proof.* The composed map $t \to f(A(t))$ is continuously Fréchet differentiable, cf. Proposition 1.2. We divide the unit interval $0 = t_0 < t_1 < \cdots < t_n = 1$ and write

$$f(A(1)) - f(A(0)) = \sum_{k=1}^{n} f(A(t_k)) - f(A(t_{k-1}))$$

$$= \sum_{k=1}^{n} (t_k - t_{k-1})(f_{A(t_k)}^{[1]}(A'(t)) + R_n(t_k)),$$

where $R_n(t_k) \to 0$ as $t_k - t_{k-1} \to 0$. We can to each $\varepsilon > 0$ find a $\delta > 0$ such that $\|R_n(t_k)\| < \varepsilon$ for all $k$ with $t_k - t_{k-1} < \delta$, cf. Theorem 1.5. The difference $f(A(1)) - f(A(0))$ is thus obtained as the limit of a Riemann sum and the assertion follows. QED

**Theorem 2.2.** *If $\tau$ is a finite trace on a $C^*$-algebra $\mathscr{A}$, and $f \in C_F^1(\mathbf{R})$, then*

$$\tau(f_T^{[1]}(S)) = \tau(f'(T)S)$$

*for all $S$, $T$ in $\mathscr{A}_{\mathrm{sa}}$.*

*Proof.* By Theorem 1.5 we have

$$\tau(f_T^{[1]}(S)) = \int_{-\infty}^{\infty} \int_0^1 \tau(e^{ixT}S)ix \, d\mu(x)dy$$

$$= \int_{-\infty}^{\infty} \tau(ixe^{ixT}S)d\mu(x)$$

$$= \tau(f'(T)S),$$

since

$$f'(t) = \int_{-\infty}^{\infty} e^{ixt}ix \, d\mu(x). \qquad \text{QED}$$

**Theorem 2.3.** *If $\tau$ is a finite trace on a $C^*$-algebra $\mathscr{A}$, then*

$$S \leq T \Rightarrow \tau(f(S)) \leq \tau(f(T))$$

*for all $S$, $T$ in $\mathscr{A}_{sa}$ and every monotone increasing, continuous function $f$ on an interval containing the spectra of $S$ and $T$.*

*Proof.* Fix an (finite) interval $I$ containing the spectra of $S$ and $T$. Define the $C^1$-curve

$$A(t) = tT + (1-t)S \qquad t \in [0, 1],$$

and note that $A'(t) = T - S \geq 0$. Combining Theorems 2.1 and 2.2 we get

$$\tau(f(T) - f(S)) = \int_0^1 \tau(f^{[1]}_{A(t)}(T - S))dt$$
$$= \int_0^1 \tau((T-S)^{1/2} f'(tT + (1-t)S)(T-S)^{1/2})dt$$

for each $f$ in $C^1_F(\mathbf{R})$. This difference is positive, as claimed, whenever $f$ is increasing on $I$.

It thus follows that the theorem is true for any increasing function $f$ in $C(I)$ that has an extension to an element in $C^1_F(\mathbf{R})$. Since $C^1_F(\mathbf{R})$ contains the class of Schwartz functions, we see that the theorem holds for any increasing function in $C^\infty(I)$.

In the general case consider an increasing function $f$ in $C(I)$ and extend it to an increasing function $\tilde{f}$ in $C_b(\mathbf{R})$. Then with

$$e_n(t) = \sqrt{\frac{n}{2\pi}} \exp(-nt^2/2)$$

define

$$f_n(t) = \int_{-\infty}^{\infty} \tilde{f}(s) e_n(t-s) ds = \int_{-\infty}^{\infty} \tilde{f}(s+t) e_n(-s) ds.$$

Clearly $f_n \in C^\infty(\mathbf{R})$ and $f_n$ is increasing. Moreover, $f_n \to f$ uniformly on $I$. Consequently

$$\tau(f(T) - f(S)) = \lim_{n\to\infty} \tau(f_n(T) - f_n(S)) \geq 0$$

and the assertion is proved. QED

**Theorem 2.4.** *If $\tau$ is a normal, semi-finite trace on a von Neumann algebra $\mathscr{A}$, then*

$$S \leq T \Rightarrow \tau(f(S)) \leq \tau((f(T))$$

*for all $S$, $T$ in $\mathscr{A}_{sa}$ and every positive, monotone increasing function $f$ on an interval containing the spectra of $S$ and $T$.*

*Proof.* Take elements $S$, $T$ in $\mathscr{A}_{sa}$ with $S \leq T$ and fix an (finite) interval $I$ containing the spectra of $S$ and $T$. Possibly after translation and scaling we may assume that $I = [0, 1]$, and that $f(0) = 0$ and $f(1) = 1$. We set

$$\mathscr{F} = \{f: I \to I \mid f \text{ is increasing, } f(0) = 0,\ f(1) = 1\}$$

and note that $\mathscr{F}$ is convex and compact. In particular, it is closed under monotone (increasing or decreasing) limits. We consider the subset

$$\mathscr{F}_\tau = \{f \in \mathscr{F} \mid \tau(f(S)) \leq \tau((f(T))\},$$

and must show that $\mathscr{F}_\tau = \mathscr{F}$.

Let $E_\lambda$ denote the spectral projection of $T$ corresponding to the open interval $]\lambda, \infty[$. If $E_0 \in \mathscr{A}^\tau$, then $\tau$ is bounded on the set $E_0 \mathscr{A} E_0$, which contains $S$ and $T$. By Theorem 2.3 this implies that

$$\mathscr{F} \cap C(I) \subset \mathscr{F}_\tau.$$

But the boundedness and normality of $\tau$ also implies that $\mathscr{F}_\tau$ is closed under monotone limits, and therefore $\mathscr{F}_\tau = \mathscr{F}$.

If $E_0 \notin \mathscr{A}^\tau$, but $E_\varepsilon \in \mathscr{A}^\tau$ for every $\varepsilon > 0$, we may replace the interval $[0, 1]$ by $[\varepsilon, 1]$ to prove that

$$\{f \in \mathscr{F} \mid f(t) = 0 \quad \text{for} \quad t \leq \varepsilon\} \subset \mathscr{F}_\tau.$$

For every $f \in \mathscr{F}$ we choose an increasing sequence $(f_n)$ of functions in $\mathscr{F}$ vanishing in a neighborhood of zero, such that $f_n \nearrow f$. Since $f_n \in \mathscr{F}_\tau$ we derive that

$$\tau(f_n(S)) \leq \tau((f_n(T)),$$

and the normality of $\tau$ then shows that $f \in \mathscr{F}_\tau$.

In the general case let

$$\lambda = \inf\{\mu \mid E_\mu \in \mathscr{A}^\tau\}.$$

Then $\tau(f(T)) = \infty$, which makes the theorem trivially true, unless $f$ vanishes on $[0, \lambda]$. But if it does we are back in case one, if $E_\lambda \in \mathscr{A}^\tau$, or in case two, if $E_\mu \in \mathscr{A}^\tau$ for every $\mu > \lambda$, of the previous argument, replacing $[0, 1]$ by $[\lambda, 1]$. We conclude that $f \in \mathscr{F}_\tau$ and that $\mathscr{F}_\tau = \mathscr{F}$ in general. QED

*Remark* 2.5. In [5, 4] the previous result is obtained for continuous functions using the theory of spectral domination. It is amusing to note that—using our $C^*$-algebraic proof above—one may conversely deduce spectral domination.

Indeed, if $S \leq T$ in a von Neumann algebra $\mathscr{A}$, and $f$ is an increasing function on $\mathbf{R}$, let $1_\lambda$ denote the characteristic function for the half-line $]\lambda, \infty[$, and set

$$E_\lambda = 1_\lambda(f(S)), \qquad F_\lambda = 1_\lambda(f(T)).$$

That $f(S)$ spectrally dominates $f(T)$ means exactly that $E_\lambda \preceq F_\lambda$, for all $\lambda$ in $\mathbf{R}$, i.e. $E_\lambda$ is Murray-von Neumann equivalent to a subprojection of $F_\lambda$, cf. [1]. But this is immediate from Theorem 2.4, because $1_\lambda \circ f$ is an increasing function, whence

$$\tau(E_\lambda) \leq \tau(F_\lambda)$$

for every semi-finite, normal trace $\tau$ on $\mathscr{A}$, and thus $E_\lambda \preceq F_\lambda$. For $\mathscr{A} = B(H)$ and $S \leq T$ compact operators, these relations were established by Powers already in [9, 5.4].

## § 3. The Perturbation Formula for Matrices

Let $T$ be a self-adjoint $n \times n$ matrix with (not necessarily distinct) eigenvalues $\lambda_1, \dots, \lambda_n$ and let

$$(\#) \qquad (e_1, \dots, e_n)$$

be an orthonormal basis of (corresponding) eigenvectors, whence

$$T = \sum_{i=1}^{n} \lambda_i e_{ii}$$

where $\{e_{ij}\}_{i,j=1}^{n}$ is the associated system of matrix units.

**Definition 3.1.** *Let $f$ be a differentiable function defined on the spectrum of $T$. The Löwner matrix*

$$f^{[1]}(T) = \sum_{i,j=1}^{n} c_{ij} e_{ij}$$

*is defined by setting*

$$c_{ij} = \begin{cases} f'(\lambda_i) & \text{for} \quad \lambda_i = \lambda_j \\ \dfrac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} & \text{for} \quad \lambda_i \neq \lambda_j \end{cases}$$

**Theorem 3.2.** *Let $T$, $S$ be self-adjoint $n \times n$ matrices and let $f \in C_F^1(\mathbf{R})$. Then*

$$f_T^{[1]}(S) = f^{[1]}(T) * S$$

*where we identify the Fréchet differential with its matrix representation in the basis $(\#)$ and $*$ denotes the Hadamard product.*

*Proof.* By Theorem 1.5 we have

$$\begin{aligned} e_i f_T^{[1]}(S) e_j &= \int_{-\infty}^{\infty} \int_0^1 e_i e^{ixyT} S e^{ix(1-y)T} e_j \, dy \, ix \, d\mu(x) \\ &= \int_{-\infty}^{\infty} \int_0^1 e^{ixy\lambda_i} e^{ix(1-y)\lambda_j} ix \, dy \, d\mu(x) e_i S e_j \\ &= c_{ij} e_i S e_j \end{aligned}$$

*for* $i, j = 1, \ldots, n$. QED

*Remark* 3.3. The Löwner matrix—essentially due to K. Löwner—was used in [7] to give a streamlined version of the theory of operator monotone and operator convex functions. The key observation, [7, 3.4] is that a function $f$ is operator monotone on an interval $I$, if and only if $f^{[1]}(T)$ is a positive definite matrix for every self-adjoint matrix $T$ (of arbitrary order) with spectrum in $I$. For the proof of this result we needed a lemma, [7, 3.3] which is the finite-dimensional version of Theorem 2.1—but phrased in the terminology of Theorem 3.2. While essentially correct, this lemma nevertheless entails a measurable selection of orthonormal bases for a curve of matrices (in order to define the Hadamard products), and the authors feel that the present version is less ambiguous.

*Remark* 3.4. Each non-constant operator monotone function $f$ on the interval $]-1, 1[$, normalised such that $f(0) = 0$ and $f'(0) = 1$, has the representation

$$f(t) = \int_{-1}^{1} t(1 - \alpha t)^{-1} d\mu(t),$$

where $\mu$ is a unique probability measure on $[-1, 1]$, cf. [7, 4.4]. Elementary calculations show that

$$f_T^{[1]}(S) = \int_{-1}^{1} (1 - \alpha T)^{-1} S (1 - \alpha T)^{-1} d\mu(\alpha).$$

Likewise, each operator monotone function $f \colon \mathbf{R}_+ \to \mathbf{R}_+$, normalised such that $f(1) = 1$, has the representation

$$f(t) = \int_0^{\infty} \frac{t(1 + \lambda)}{t + \lambda} d\mu(\lambda),$$

where $\mu$ is a unique probability measure on the extended half-line $[0, \infty]$. Again we obtain the Fréchet differential

$$f_T^{[1]}(S) = \int_0^{\infty} \lambda(1 + \lambda)(T + \lambda)^{-1} S (T + \lambda)^{-1} d\mu(\lambda)$$

by elementary calculations.

*Remark* 3.5. It is not, in general, possible to describe the differential $f_T^{[1]}(S)$ as an Hadamard product in the infinite dimensional case. However, if $H = L_\nu^2(I)$ for some probability measure $\nu$ on $I$ and $T\varphi(x) = x\varphi(x)$ for every $\varphi$ in $H$—so that $T$ is "diagonalised"—then for each $f$ in $C_F^1(I)$ and every Hilbert-Schmidt operator $S$ on $H$ given by a self-adjoint kernel $k$, i.e.

$$S\varphi(x) = \int k(x, y)\varphi(y)d\nu(y),$$

we find $f_T^{[1]}(S)$ to be the Hilbert-Schmidt operator with product kernel $f^{[1]}(T)k$, where $f^{[1]}(T) \in C(I \times I)$, given by

$$f^{[1]}(T)(x, y) = \begin{cases} f'(x) & \text{for} \quad x = y \\ \dfrac{f(x) - f(y)}{x - y} & \text{for} \quad x \neq y\,. \end{cases}$$

The computations are straightforward and left to the reader.

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