Publ. RIMS, Kyoto Univ.
31 (1995), 443–464

WKB Analysis to Global Solvability and Hypoellipticity

Dedicated to Professor Kenjiro Okubo for his 60th Birthday

By

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§1. Introduction

This paper studies the global regularity and solvability of operators which could change their type at every point of the domain. Our object is to understand such operators from the viewpoints of a WKB analysis.

To be more precise, let X be a compact manifold or an open domain in R^n. We denote by $C^\infty(X)$ and $C_0^\infty(X)$ the set of smooth functions on X and the set of smooth functions with compact supports respectively. We also denote the set of distributions on X by $\mathscr{D}'(X)$. We say that a differential operator P is globally solvable (resp. globally hypoelliptic) in X if for every $f \in C_0^\infty(X)$ there exists $u \in \mathscr{D}'(X)$ satisfying $Pu = f$. (resp. $u \in C^\infty(X)$ when $Pu \in C^\infty(X)$ and $u \in \mathscr{D}'(X)$). The operator P is said to be locally solvable (resp. locally hypoelliptic) at a point $p \in X$ if there exists a neighborhood U of p such that for every $f \in C_0^\infty(U)$, there exists $u \in \mathscr{D}'(U)$ satisfying $Pu = f$ in U (resp. $p \notin singsupp(Pu)$ implies $p \notin singsupp(u)$). By definition local hypoellipticity at each point $p \in X$ implies the global hypoellipticity in X, and the global solvability implies the local solvability at each point $p \in X$, while the corresponding inverse implications are not true. ([7]).

Because the operators which we want to study are in general of mixed type the structures of local solutions may change drastically in every part of the domain. Therefore most of the methods such as those for degenerate elliptic operators, and for weakly hyperbolic operators are not applicable to such operators. (cf. [9], [17]). Moreover, because the structure of the characteristics is so complicated that the usual characteristic geometry does not seem adequate to

Communicated by T. Kawai, July 5, 1993. Revised April 11, 1994.
1991 Mathematics Subject Classifications: 35H05, 34E20
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apply to our situation. On the other hand, there are examples of constant coefficients operators on the torus showing that Siegel type conditions are essential to describe global hypoellipticity and the global solvability (cf. [7]). The above methods are, as far as the authors know, not adequate to say when such diophantine analysis enters in the theory for operators with variable coefficients of arbitrary order. In fact, it is not even clear how to define a Siegel condition for variable coefficients operators invariantly under change of variables. We note that the diophantine phenomena in [7] are quite discontinuous, hence, the situations could change completely under very small perturbations.

This paper gives an answer to these problems by using WKB formal solutions. We note that WKB solutions are formal power series with respect to a large parameter whose coefficients may have poles with arbitrarily large order. In spite of this we realize such formal solutions according to the situations which we consider. The important points are that the results do not depend on the realizations and that though WKB solutions are constructed algebraically, they can explain transcendental phenomena. This implies that such formal quantities play important roles in describing global phenomena precisely.

This paper is organized as follows. In §2 we give a fundamental necessary and sufficient condition for the global hypoellipticity for first order systems. Siegel conditions are invariantly defined under realizations and changes of variables in terms of formal solutions which are substitutes of WKB solutions for single equations. In §3 we shall give the proof of the theorem of §2. In §4 we study single equations of variable multiplicity with complex coefficients. The key is the existence of a smooth formal solution to a Riccati equation. (cf. §5). The proof of the theorem in §4 is given in §5. In §6 we consider second order operators and we study how diophantine analysis enters in the theory. In fact, it will be shown that WKB formal solutions are useful to decide when a diophantine condition is necessary. (cf. the cases (a), (b) and (c) in §6). In this paper we mainly state the results on global hypoellipticity. Global solvability and Fredholm property will be treated similarly.

Finally the authors would like to express their sincere thanks to the referee for useful suggestions in preparing this paper.

§2. Systems

Let us consider an $m\times m$ system of equations with smooth periodic coefficients on $\mathbb{T}^2$

$$Pu := (\frac{\partial}{\partial x})u - (\sum_{j=0}^{\infty} A_j(x)D_y^{1-j})u = f(x,y) \tag{2.1}$$

where $u = {}^t(u_1,\dots,u_m)$ and $f(x,y)$ is an m vector smooth function. Here $A_j(x)$ are smooth $m\times m$ matrix-valued functions and the negative powers of D_y denote pseudodifferential operators on $\mathbb{T}$ with the symbol η^{1-i} with modifications near $\eta=0$. We assume

(C.1) The eigenvalues of $A_0(x)$ are distinct and 2π periodic functions.

By a partial Fourier transform with respect to y we get from (2.1)

$$\hat{P}\hat{u} := (\frac{\partial}{\partial x})\hat{u} - (\sum_{i=0}^{\infty} A_i(x)\eta^{1-i})\hat{u} = \hat{f}(x,\eta). \tag{2.2}$$

Here $\hat{u}$ (resp. $\hat{f}$) denotes the partial Fourier transformation of u (resp. f) with respect to y. By a standard formal reduction procedure and (C.1) we can construct a formal fundamental solution $X(x,\eta)$ to the equation (2.2) in the form

$$X(x,\eta) \sim (Y_0(x) + Y_1(x)\eta^{-1} + \cdots)e^{\eta\Lambda(x,\eta)}. \tag{2.3}$$

Here $Y_j(x), j=0,1,\ldots$ are 2π periodic, smooth functions and $Y_0(x)$ is invertible and the dots denote negative powers of η with coefficients smooth 2π periodic in x. The matrix $\Lambda(x,\eta)$ is a diagonal matrix given by

$$\Lambda(x,\eta) = \mathrm{diag}(\int_0^x \lambda_1(t,\eta)dt, \ldots, \int_0^x \lambda_m(t,\eta)dt) \tag{2.4}$$

in this order. Here $\lambda_j(t,\eta)$ $(1 \le j \le m)$ is a formal power series of η^{-1} with coefficients 2π periodic in $t, \lambda_j(t,\eta) \sim \lambda_j^0(t) + \eta^{-1}\lambda_j^1(t) + \cdots$. (cf. [19]) where $\lambda_j^0 (1 \le j \le m)$ denote the eigenvalues of $A_0(x)$. We take any realizations $\tilde{\lambda}_j'(t,\eta)$ of formal power series $\lambda_j(t,\eta) - \lambda_j^0(t)$ and we define $\tilde{\Lambda}(x,\eta)$ by replacing λ_j by $\tilde{\lambda}_j = \tilde{\lambda}_j' + \lambda_j^0$ in (2.4). (For the precise definition, see Lemma 1 in the appendix). Then we have

Theorem 2.1. *Suppose that* (C.1) *is satisfied. Then P is globally hypoelliptic if and only if the following conditions are satisfied.*

(I) $\mathrm{Re}\,\lambda_j^0(x)$ $(1 \le j \le m)$ *do not change their sign on the interval* $0 \le x \le 2\pi$.

(II) *There exists $N > 0$ such that for all* j, $1 \le j \le m$

$$\liminf_{\eta\to\infty, \eta\in\mathbf{Z}} \left|1 - \exp(\int_0^{2\pi} \tilde{\lambda}_j(t,\eta)\eta dt)\right| |\eta|^N > 0 . \tag{2.5}$$

Remarks 2.2. (i) We point out that in view of the construction in Lemma 1 in the appendix if $\tilde{\lambda}_{j1}(x,\eta)$ and $\tilde{\lambda}_{j2}(x,\eta)$ are two realizations of $\lambda_j(x,\eta)$, then for every $N \in \mathbf{Z}_+$ we have

$$\sup_{x\in[0,2\pi]} |\tilde{\lambda}_{j1}(x,\eta) - \tilde{\lambda}_{j2}(x,\eta)| = O(|\eta|^{-N}) \text{ as } |\eta| \to \infty .$$

Hence the condition (2.5) is invariant under the choice of the realizations.

(ii) We note that the quantity $d\Lambda/dx$ in (2.3) is formally invariant, that is, invariant under formal change of unknown functions. (cf. Lemma 2 in the appendix). Because such formal transformations correspond to the ones by elliptic pseudodifferential operators on the torus this implies, in view of $\Lambda(0,\eta)=0$ and (i), the invariance of (2.5) under such transformations. Especially, though the formal solution (2.3) itself has an arbitrariness the condition (2.5) does not depend on such arbitrariness. We also remark that formal solutions (2.3) play the role of WKB solutions for single equations. (cf. Lemma 3 in the appendix).

Example 2.3. We will consider the third order operator

$$\hat{P} := \partial_x^3 - \eta^3 a(x)^3, \ a(x) \neq 0, \ \partial_x = \partial/\partial x. \tag{2.6}$$

Clearly the equation (2.6) is written in the form (2.1). Smooth WKB solutions are constructed because $a(x)\neq 0$. Let $S=\sum_{j=-1}^{\infty}\eta^{-j}S_j(x)$ be any WKB solution. Then, by substitution we have $S_{-1}(x)=\omega a(x), \omega^3=1$ for some ω. Theorem 2.1 implies that the global hypoellipticity is equivalent to the following conditions:

1) $\mathrm{Re}\,\omega a(x)$ do not change their sign on $\mathbb{T}$ for all ω such that $\omega^3=1$.
2) For every ω such that $\mathrm{Re}\,\omega a(x)\equiv 0$ Siegel conditions are satisfied.

The first condition is equivalent to say that $A=\{a(x); 0\le x\le 2\pi\}$ is contained in one of the six sectors bounded by three lines $L_1=\{\mathrm{Re}\,z=0\}$, $L_2=\{\mathrm{Re}\,\omega z=0\}$ and $L_3=\{\mathrm{Re}\,\omega^2 z=0\}$ where $\omega^3=1$, $\omega\neq 1$. Suppose that $A\subset\Lambda$ where Λ is one of these six sectors. If A is not contained in the boundary $\partial\Lambda$ of Λ then we are in a simple situation, no Siegel condition enters. In fact the global hypoellipticity follows. Note that the operators are of mixed type in general.

On the other hand if $A\subset\partial\Lambda$ $\hat{P}$ contains a hyperbolic factor. We remark that A is contained in one of the three lines L_1, L_2 and L_3 because $a(x)\neq 0$. For simplicity let us assume that $A\subset L_1$. Note $b(x)=\mathrm{Im}\,a(x)\neq 0$ on $\mathbb{T}$ because $a(x)\neq 0$. Hence $b(x)$ does not change its sign on $\mathbb{T}$. Because $\mathrm{Re}\,a(x)\equiv 0$ we have, for $\omega\neq 1$,

$$\mathrm{Re}\int_0^{2\pi}\omega i b(x)dx = -\mathrm{Im}\,\omega\int_0^{2\pi} b(x)dx = -(\mathrm{Im}\,\omega)\int_0^{2\pi} b(x)dx \neq 0. \tag{2.7}$$

Therefore the diophantine condition is necessary only for the WKB solution corresponding to L_1, i.e., $\omega=1$.

We again find a natural extension of Greenfield's example [7] when $a(x)$ is constant. We also note that the diophantine condition is replaced by algebraic conditions if a certain finiteness condition is fulfilled. (cf §6).

The similar assertions are true for general n-th order equations of the following type

$$\hat{P} = \partial_x^n - \eta^n a(x)^n, \ a(x)\neq 0. \tag{2.8}$$

By the same considerations as above we see that WKB solutions $S=\sum_{j=-1}^{\infty}\eta^{-j}S_j(x)$ are of the form, $S_{-1}=a(x)\omega$ with $\omega^n=1$. The global hypoellipticity is equivalent to the following conditions.

(a) $A=\{a(x);0\leq x\leq 2\pi\}$ is contained in one of the sectors bounded by n lines $\mathrm{Re}\,\omega z=0$ where $\omega^n=1$. We denote the sector containing A by Λ.

(b) If $A\subset\partial\Lambda$ then the Siegel conditions of the form (2.5) corresponding to the line containing $\partial\Lambda$ are satisfied.

For more detailed treatment of single equations we refer to §§4 and 6.

We note that among n different WKB solutions the Siegel conditions are necessary for those which correspond to the line containing A. By the formal invariance under the transformations stated above the conditions (a) and (b) are invariant. (cf. [19]).

§3. Proof of Theorem 2.1

Proof of the sufficiency of Theorem 2.1. By the general theory of ordinary differential equations we can construct a formal solution to (2.2) in the form (2.3). (cf [20, Theorem 2.8-1]). We note that because the eigenvalues $\lambda_j^0(x)(j=1,\ldots,m)$ of $A_0(x)$ are 2π periodic smooth functions of x by the assumption (C.1) we can construct the matrix $Y_0(x)$ as a 2π periodic smooth function of x. Hence we can inductively determine the matrices $Y_j(x)(j=1,2,\ldots)$ and $\lambda_j^k(x)(j=1,\ldots,m;k=1,\ldots)$ as smooth functions on $\mathbb{T}$. Hence by Lemma 1 in the appendix we can construct an asymptotic fundamental solution $X(x)(=X(x,\eta))$ of the equation (2.2), that is $X(x)$ satisfies the equation

$$\hat{P}X(x)=R(x,\eta) \tag{3.1}$$

where for every $\alpha\in\mathrm{Z}_+$, $\partial_x^\alpha R(x,\eta)$ is rapidly decreasing in η, when $\eta\to\infty$ i.e., a smoothing operator.

By the method of variation of constant an asymptotic solution to (2.2) is given by

$$u(x)=X(x)c+X(x)\int_0^x X^{-1}(t)\hat{f}(t,\eta)dt \tag{3.2}$$

where the constant vector c is determined so that $u(x)$ is 2π periodic, $u(0)=u(2\pi)$. This yields the equation

$$X(0)c=X(2\pi)c+X(2\pi)\int_0^{2\pi} X^{-1}(t)\hat{f}(t,\eta)dt. \tag{3.3}$$

Because we can write $X(x)=\Phi(x,\eta)e^{\eta\tilde{\Lambda}(x,\eta)}$ with $\Phi(x,\eta)$ being a realization of the formal sum, $Y_0(x)(I+Y_1(x)\eta^{-1}+\ldots)$ it follows that $\Phi(0,\eta)=\Phi(2\pi,\eta)$ and $\tilde{\Lambda}(0,\eta)=0$. Therefore we have

$$\Phi(0,\eta)c=\Phi(0,\eta)e^{\eta\tilde{\Lambda}(2\pi,\eta)}c+\Phi(0,\eta)e^{\eta\tilde{\Lambda}(2\pi,\eta)}\int_0^{2\pi}e^{-\eta\tilde{\Lambda}(t,\eta)}\Phi(t,\eta)^{-1}\hat{f}(t,\eta)dt.$$

Because $\Phi(0,\eta)$ is invertible for large η by definition we have

$$(I-e^{\eta\tilde{\Lambda}(2\pi,\eta)})c=e^{\eta\tilde{\Lambda}(2\pi,\eta)}\int_0^{2\pi}X^{-1}(t)\hat{f}(t,\eta)dt. \tag{3.4}$$

Let us assume that $(I-e^{\eta\tilde{\Lambda}(2\pi,\eta)})$ is invertible for large η. Then we get, from (3.2) and (3.4)

$$u(x)=X(x)((I-e^{\eta\tilde{\Lambda}(2\pi,\eta)})^{-1}e^{\eta\tilde{\Lambda}(2\pi,\eta)}\int_0^{2\pi}X^{-1}(t)\hat{f}(t,\eta)dt+\int_0^{x}X^{-1}(t)\hat{f}(t,\eta)dt). \tag{3.5}$$

We shall prove the sufficiency of (I) and (II). Without loss of generality, we may assume that $\operatorname{Re}\lambda_j(x)\geq 0$ for $1\leq j\leq j_0$, $\operatorname{Re}\lambda_j(x)\leq 0$ for $j_0<j\leq m$ for some $j_0\geq 1$. It follows from (3.5) that

$$\begin{aligned}u(x)&=\Phi(x,\eta)(e^{(\tilde{\Lambda}(x,\eta)+\tilde{\Lambda}(2\pi,\eta))\eta}(1-e^{\tilde{\Lambda}(2\pi,\eta)\eta})^{-1}\\&\quad\times\int_0^{2\pi}e^{-\eta\tilde{\Lambda}(t,\eta)}\Phi(t,\eta)^{-1}\hat{f}(t,\eta)dt+e^{\tilde{\Lambda}(x,\eta)\eta}\int_0^{x}e^{-\tilde{\Lambda}(t,\eta)\eta}\Phi(t,\eta)^{-1}\hat{f}(t,\eta)dt)\\&=\Phi(x,\eta)(e^{\tilde{\Lambda}(2\pi,\eta)\eta}(1-e^{\tilde{\Lambda}(2\pi,\eta)\eta})^{-1}\int_0^{2\pi}e^{\eta(\tilde{\Lambda}(x,\eta)-\tilde{\Lambda}(t,\eta))}\Phi(t,\eta)^{-1}\hat{f}(t,\eta)dt\\&\quad+\int_0^{x}e^{\eta(\tilde{\Lambda}(x,\eta)-\tilde{\Lambda}(t,\eta))}\Phi(t,\eta)^{-1}\hat{f}(t,\eta)dt)\\&=\Phi(x,\eta)((1-e^{\eta\tilde{\Lambda}(2\pi,\eta)})^{-1}\int_0^{2\pi}e^{\eta(\tilde{\Lambda}(x,\eta)-\tilde{\Lambda}(t,\eta))}\Phi(t,\eta)^{-1}\hat{f}(t,\eta)dt\\&\quad-\int_x^{2\pi}e^{\eta(\tilde{\Lambda}(x,\eta)-\tilde{\Lambda}(t,\eta))}\Phi(t,\eta)^{-1}\hat{f}(t,\eta)dt).\end{aligned} \tag{3.6}$$

Let π_0 be the projection to the linear subspace spanned by the vectors whose last $m-j_0$ components are zero and let us set $\pi_1=I-\pi_0$. Then we have

$$\begin{aligned}u(x)&=\Phi(x,\eta)(e^{\tilde{\Lambda}(2\pi,\eta)\eta}(1-e^{\tilde{\Lambda}(2\pi,\eta)\eta})^{-1}\int_0^{2\pi}e^{\eta(\tilde{\Lambda}(x,\eta)-\tilde{\Lambda}(t,\eta))}\pi_1\Phi(t,\eta)^{-1}\hat{f}(t,\eta)dt\\&\quad+\int_0^{x}e^{\eta(\tilde{\Lambda}(x,\eta)-\tilde{\Lambda}(t,\eta))}\pi_1\Phi(t,\eta)^{-1}\hat{f}(t,\eta)dt\\&\quad+(1-e^{\tilde{\Lambda}(2\pi,\eta)\eta})^{-1}\int_0^{2\pi}e^{\eta(\tilde{\Lambda}(x,\eta)-\tilde{\Lambda}(t,\eta))}\pi_0\Phi(t,\eta)^{-1}\hat{f}(t,\eta)dt\\&\quad-\int_x^{2\pi}e^{\eta(\tilde{\Lambda}(x,\eta)-\tilde{\Lambda}(t,\eta))}\pi_0\Phi(t,\eta)^{-1}\hat{f}(t,\eta)dt).\end{aligned} \tag{3.7}$$

Suppose that η tends to $+\infty$. Then (3.7) and the definitions of π_0 and π_1 imply that $u(x)$ is rapidly decreasing when η tends to $+\infty$. On the other hand if η tends to $-\infty$ we have the same conclusion by changing the parts of π_0 and π_1. We note that if $\mathrm{Re}\,\lambda_j^0 \equiv 0$ for some j, then we use the assumption (II) in order to estimate the growth of denominators in (3.7). This proves the sufficiency of (I) and (II).

Proof of the necessity of Theorem 2.1. We shall prove the necessity of (I). For each $j, 1 \le j \le m$ we denote by $\tilde{\Lambda}_j(t) \equiv \tilde{\Lambda}_j(t,\eta)$ the j-th component of the matrix $\tilde{\Lambda}(t,\eta)$. We first prove that for each $j = 1,\dots m, 1 - e^{\tilde{\Lambda}_j(2\pi,\eta)\eta} \ne 0$ except for a finite number of η's.

Indeed, if this is not true for some j then it follows from the definition of $\tilde{\Lambda}$ that $\int_0^{2\pi} \mathrm{Re}\,\lambda_j^0(x)dx = 0$. On the other hand, we get from (3.4) that the homogeneous equation (2.2) with $\hat{f} = 0$ has infinitely many nontrivial asymptotic solutions $X(x)c_j \equiv X(x,\eta_k)c_j$, for some $\eta = \eta_k, (k = 1,2,\dots)$, where $c_j = {}^t(0,\dots,1,\dots,0)$ is the j-th unit vector. Hence we have $R_j(x,\eta_k) = \hat{P}X(x,\eta_k)c_j$, where for every $\alpha \in \mathbb{Z}_+$ $\partial_x^\alpha R_j(x,\eta)$ is 2π periodic in x and is rapidly decreasing function of η when η tends to infinity.

If $\mathrm{Re}\,\lambda_j^0(x)$ vanishes identically then $X(x)c_j$ gives a distribution which is not smooth such that $\hat{P}X(x)c_j$ is rapidly decreasing. This contradicts to the global hypoellipticity.

Next we consider the case where $\mathrm{Re}\,\lambda_j^0$ changes its sign. By taking a subsequence we may assume that $\eta_k > 0$ or $\eta_k < 0$ for all k, $k = 1,2,\dots$. Since the argument is the same let us consider the case $\eta_k > 0$. By the periodicity we may assume that $\mathrm{Re}\,\lambda_j^0(0) > 0$. Hence we have $\kappa = \max \int_0^x \mathrm{Re}\,\lambda_j^0(t)dt > 0$. Let the maximum be taken at $x = x_1$. Then the function $\exp(-\eta_k\,\kappa)X(x,\eta_k)c_j$ defines a distribution solution which is not rapidly decreasing at $x = x_1$ when $\eta = \eta_k$ tends to infinity. This contradicts to the global hypoellipticity. This proves the assertion.

In order to prove the necessity of (I) suppose that $\mathrm{Re}\,\lambda_j^0(x)$ changes its sign for some $x = x_0$. First we consider the case $\mathrm{Re}\,\tilde{\Lambda}_j(2\pi) > 0$. We take a point x_0 such that $\mathrm{Re}\,\lambda_j^0\eta$ changes its sign from positive to negative at $x = x_0$. Let π_j be a projection to the linear subspace spanned by j-th unit vector. We take a $\hat{f}(t,\eta)$ such that the support of the function $\pi_j\Phi(t,\eta)^{-1}\hat{f}(t,\eta)$ is contained in a small neighborhood of x_0 such that $t > x_0$ and is rapidly decreasing when $\eta \to \infty$. Then the function

$$(3.8)\qquad \Phi(x_0,\eta)e^{\tilde{\Lambda}(2\pi,\eta)\eta}(I - e^{\tilde{\Lambda}(2\pi,\eta)\eta})^{-1}\int_0^{2\pi} e^{(\tilde{\Lambda}(x_0,\eta)-\tilde{\Lambda}(t,\eta))\eta}\pi_j\Phi(t,\eta)^{-1}\hat{f}(t,\eta)dt$$

is not rapidly decreasing when η tends to $+\infty$ if we choose a rapidly decreasing function $\pi_j \Phi(t,\eta)^{-1} \hat{f}(t,\eta)$ appropriately. Then the second expression of (3.6) shows that $u(x)$ is not rapidly decreasing, which is a contradiction.

If $\mathrm{Re}\,\tilde{\Lambda}_j(2\pi) < 0$ we take a point x_0 such that $\mathrm{Re}\,\lambda_j^0(x)$ changes its sign from negative to positive at x_0. Then we take $\hat{f}$ such that the support of the function $\pi_j \Phi(t,\eta)^{-1} \hat{f}(t,\eta)$ is contained in a small neighborhood of x_0 such that $t > x_0$. By the same argument as above we see that $u(x)$ is not rapidly decreasing when η tends to $-\infty$. If $\mathrm{Re}\,\tilde{\Lambda}_j(2\pi) = 0$ then by the same way as above the integral (3.8) is not rapidly decreasing when $\eta \to \infty$ if $\mathrm{Re}\,\lambda_j^0(x)$ changes its sign at $x = x_0$. Hence we have proved the necessity of (I).

We shall prove the necessity of (II). Let us assume that (II) is not true for some j. By the argument as above we may assume (I) and that for every $k, k = 1, \ldots, m, 1 - e^{\tilde{\Lambda}_k(2\pi,\eta)\eta} \neq 0$ except for a finite number of η's. Because the integral $\int_0^{2\pi} \mathrm{Re}\,\lambda_j^0(t)dt$ vanishes for such j it follows from (I) that $\mathrm{Re}\,\lambda_j^0(x)$ vanishes identically. Let $\eta_n \in \mathbb{Z} (n = 1, 2, \ldots)$ be such that $|1 - e^{\tilde{\Lambda}_j(2\pi,\eta_n)\eta_n}||\eta_n|^N \to 0$ when $n \to \infty$ for every $N = 1, 2\ldots.$ We define $\hat{f}(t,\eta)$ such that $\hat{f}(t,\eta) = 0$ for $\eta \neq \eta_n$ and for $\eta = \eta_n, n = 1, 2, \ldots$ the components of the function $\Phi(t,\eta)^{-1} \hat{f}(t,\eta)$ except for the j-th one are zero and the j-th one is given by $(1 - e^{\tilde{\Lambda}_j(2\pi,\eta)\eta}) e^{\tilde{\Lambda}_j(t,\eta)\eta}$. Then the function $\hat{f}(t,\eta)$ is a rapidly decreasing function of η because $\mathrm{Re}\,\tilde{\Lambda}_j(t,\eta)\eta$ is bounded in η. It follows from (3.6) that $u(x)$ is not a rapidly decreasing function of η for this $\hat{f}$, which contradicts to the global hypoellipticity. This proves the necessity of (II) . Hence we have proved Theorem 2.1.

§4. Single Equations with Variable Multiplicities

We want to weaken the condition (C.1) in §2 for single equations. In this section we consider operators of order $m \geq 3$. Second order operators will be studied in the next section. Let us consider

$$(4.1) \qquad Pu = f(x,y), \quad P := \prod_{j=1}^{m} (\partial_x - b_j(x) D_y) + \sum_{\alpha + j \leq m-1} a_{\alpha j}(x) \partial_x^j D_y^\alpha,$$

on $(x,y) \in \mathbb{T}^2$, where $m \geq 3, b_j(x)(1 \leq j \leq m), a_{\alpha j}(x)$ are 2π periodic complex-valued functions and $\partial_x = \partial / \partial x, D_y = i^{-1} \partial / \partial y$. Let us consider a WKB approximate solution ψ

$$(4.2) \qquad \psi(x,\eta) := \exp\left(\int_0^x S(t,\eta)dt\right), \; S(t,\eta) := \sum_{j=-1}^{\infty} \eta^{-j} S_j(t)$$

to the partial Fourier transform $\hat{P}$ of P with respect to y

$$\hat{P}\hat{u} = \hat{p}_m \hat{u} + \sum_{\alpha+j\le m-1} \eta^\alpha a_{\alpha j}(x)\partial_x^j \hat{u},\ \hat{p}_m = \prod_{j=1}^{m}(\partial_x - b_j(x)\eta), \tag{4.3}$$

where η is covariable of y. We easily see that S satisfies a Riccati equation

$$\begin{aligned} R(S) &:= \prod_{j=2}^{m}(\partial_x + S - b_j(x)\eta)(S - b_1\eta) \\ &\quad + \sum_{\alpha+j\le m-1, j\ge 1} a_{\alpha j}(x)\eta^\alpha(\partial_x + S)^{j-1}S + \sum_{\alpha\le m-1} a_{\alpha 0}(x)\eta^\alpha = 0. \end{aligned} \tag{4.4}$$

If we substitute the expansion of S, (4.2) into (4.4) we have

$$R(S) = \prod_{j=1}^{m}(S - b_j(x)\eta) + (\text{ terms with power } \le m-1) = 0. \tag{4.5}$$

By substituting the expansion of S in (4.5) and by equating the terms with power η^m in (4.5), we have that $\prod_{j=1}^{m}(S_{-1}(x) - b_j(x)) = 0$. Suppose that $S_{-1}(x) = b_k(x)$ for some k. Then by comparing the coefficients of η^{m-1} in (4.5) we have

$$S_0(x)\prod_{j\ne k}(b_k(x) - b_j(x)) = \text{ terms containing } S_{-1}(x). \tag{4.6}$$

We easily see that $S_j(x)\,(j = 1,2,\ldots)$ satisfy the same type of equations as (4.6). Now we assume

(C.2) For every $S_{-1}(x) = b_k(x)\ (1 \le k \le m)$ $S_j(x)$ are smoothly defined on $\mathbf{T}$ for all $j = 0,1,2,\ldots$.

We write a formal solution $S(x,\eta)$ corresponding to $b_k(x)$ by $S^k(x,\eta)$. We assume

(C.3) There exists a point $x_0 \in \mathbb{T}$ such that $b_k(x_0)\ (1 \le k \le m)$ are distinct.

In order to state our theorem we take realizations $\tilde{S}^k(x,\eta)$ of formal WKB solutions $S^k(x,\eta)\ (1 \le k \le m)$ where $S^k(x,\eta) = \sum_{j=-1}^{\infty} S_j^k(x)\eta^{-j}$. Indeed, we take smooth functions $\chi_j(\eta)$ with supports being contained in $|\eta| \ge c_j$ where the constants c_j are taken so large that the following sum converges:

$$\tilde{S}^k(x,\eta) := \sum_{j=-1}^{\infty} \eta^{-j}\chi_j(\eta)S_j^k(x). \tag{4.7}$$

For more precise definition of $\chi_j(\eta)$ we refer to Lemma 1 in the appendix. Then we have

Theorem 4.1. *Assume that* (C.2) *and* (C.3) *are satisfied. Then the operator P is globally hypoelliptic on $\mathbb{T}^2$ if and only if the following conditions are satisfied.*

(I) *The functions* $\operatorname{Re} b_k(x)\ (k = 1,\ldots,m)$ *do not change their sign.*

(II) *There exists* $N \ge 0$ *such that the Siegel conditions,*

$$\liminf_{\eta\to\infty, \eta\in\mathbf{Z}} |\eta|^N \left| \exp\left(\int_0^{2\pi} G(x,\eta)dx\right) - 1 \right| > 0 \tag{4.8}$$

are satisfied for $G(x,\eta)=\tilde{S}^k(x,\eta)\ (1\le k\le m)$.

Remarks 4.2. (i) The condition (C.3) is unnecessary for the proof of the necessity of (4.8).

(ii) We point out that in view of the constructions above if $\tilde{S}_1(x,\eta)$ and $\tilde{S}_2(x,\eta)$ are two realizations of any formal solution $S(x,\eta)$, then for every $N\in\mathbb{Z}_+$ we have

$$\sup_{x\in[0,2\pi]}|\tilde{S}_1(x,\eta)-\tilde{S}_2(x,\eta)|=O(|\eta|^{-N}) \text{ as } |\eta|\to\infty .$$

Hence we see that the condition (4.8) is invariant under the choice of the realizations of the formal power series.

We note that we do not assume any condition on the behavior of characteristic point. Hence it may be degenerate elliptic and weakly hyperbolic with infinite degeneracy at some points in the domain.

§5. Proof of Theorem 4.1

Because $b_k(x)$ are periodic, we may assume that $x_0=0$ in (C.3). Hence the condition (C.3) implies

$$b_k(0)\ne b_j(0) \text{ for every } k \text{ and } j\ (1\le k,j\le m). \tag{5.1}$$

We want to write the equation $\hat{P}u=\hat{f}$ in the vector-matrix notation, where $\hat{f}$ denotes the partial Fourier transformation of f as in (4.3). For the sake of brevity we write

$$\hat{P}=(\frac{d}{dx})^m+\sum_{j=1}^m p_j(x,\eta)(\frac{d}{dx})^{j-1}$$

and set

$$U:=\begin{pmatrix}\hat{u}\\ \hat{u}'\\ \\ \vdots\\ \\ \hat{u}^{(m-1)}\end{pmatrix},\ A(x):=\begin{pmatrix}0 & 1 & & & \\ & 0 & 1 & & \\ & \cdots & & \cdots & \\ & & \cdots & & \cdots\\ & & & 0 & 1\\ -p_1 & -p_2 & & \cdots & -p_m\end{pmatrix},\ F(x):=\begin{pmatrix}0\\ \vdots\\ \\ 0\\ \hat{f}\end{pmatrix}. \tag{5.2}$$

Then the equation $\hat{P}\hat{u}=\hat{f}$ can be written in the form

$$LU:=U'-A(x)U=F(x). \tag{5.3}$$

Let $\tilde{\psi}_k=\exp(\int\tilde{S}^k dt)$ be a realization of a WKB formal solution. We set

(5.4)
$$Z(x) := \begin{pmatrix} \tilde{\psi}_1 & \dots & \tilde{\psi}_m \\ \tilde{\psi}_1' & \dots & \tilde{\psi}_m' \\ \dots & & \dots \\ \tilde{\psi}_1^{(m-1)} & \dots & \tilde{\psi}_m^{(m-1)} \end{pmatrix}.$$

Because $\tilde{\psi}_j$ is an approximate solution to the homogeneous equation $\hat{P}\hat{u}=0$, it follows that $\hat{P}\tilde{\psi}_j = R(\tilde{S}_j)\tilde{\psi}_j$ with R given by (4.4). Hence $Z(x)$ is an approximate fundamental solution to (5.3)

(5.5)
$$LZ(x) = Z'(x) - AZ(x) = R(x)D(x),$$

where $D(x)$ is a diagonal matrix given by

(5.6)
$$D(x) := \operatorname{diag}(\tilde{\psi}_1(x), \dots, \tilde{\psi}_m(x))$$

and $R(x) := R(x,\eta)$ is a matrix whose components are rapidly decreasing functions of η when η tends to infinity.

In order to write the equation (5.3) in the form (2.1) we define the matrix $B(\eta)$ by

(5.7)
$$B(\eta) := \operatorname{diag}(1, \eta, \eta^2, \dots, \eta^{m-1}).$$

Suppose that $\eta \neq 0$. If we set $B(\eta)X(x,\eta) = Z(x,\eta)$ and if we multiply (5.5) with $B(\eta)^{-1}$ from the left the equation (5.5) can be written in the form (2.1), where the matrix $A(x)$ is replaced by $B(\eta)^{-1}A(x)B(\eta)$.

We want to check that the formal solution $X(x,\eta)$ can be written in the form (2.3). To this end we set $X(x,\eta) = \Xi(x,\eta)D(x)$. We want to determine the matrix $\Xi(x,\eta)$. By the relation $\tilde{\psi}_j' = \tilde{S}_j\tilde{\psi}_j$ we have

(5.8)
$$Z(x,\eta) = (\tilde{\psi}_j^{(k)}(x,\eta))_{\substack{j\to 1,\dots,m \\ k\downarrow 0,\dots,m-1}} = ((\tilde{S}_j(x)^k + O(\eta^{k-1}))\tilde{\psi}_j(x,\eta)) \equiv B(\eta)\Xi(x,\eta)D(x).$$

Hence we have

(5.9)
$$\Xi(x,\eta) = B(\eta)^{-1}(\tilde{S}_j(x)^k + O(\eta^{k-1}))_{\substack{j\to 1,\dots,m \\ k\downarrow 0,\dots,m-1}}.$$

On the other hand we have $\tilde{S}_j(x) = (b_j(x) + O(\eta^{-1}))\eta$. Hence if we expand $\Xi(x,\eta)$ in the descending power of $\Xi(x,\eta) = Y_0(x) + Y_1(x)\eta^{-1} + \dots$ we see that the matrix $Y_0(x)$ is given by

(5.10)
$$Y_0(x) = (b_j(x)^k)_{\substack{j\to 1,\dots,m \\ k\downarrow 0,\dots,m-1}}.$$

We shall show that the matrix $Y_0(x)$ is invertible. To this end, we set $\Psi(x) := \det Z(x,\eta)$. Then, by (5.4) the function Ψ satisfies the differential equation

$$\Psi'(x) = -p_m(x,\eta)\Psi(x) + \Lambda(x,\eta)\tilde{\psi}_1(x)\cdots\tilde{\psi}_m(x), \tag{5.11}$$

where the function $\Lambda(x,\eta)$ is a rapidly decreasing function of η. Hence we obtain

$$\begin{aligned}\Psi(x) = {}&\Psi(0)\exp(-\int_0^x p_m(t,\eta)dt)\\ &+\exp(-\int_0^x p_m(t,\eta)dt)\int_0^x \exp(\int_0^t p_m(s,\eta)ds)\Lambda(t,\eta)\tilde{\psi}_1(t)\cdots\tilde{\psi}_m(t)dt.\end{aligned} \tag{5.12}$$

On the other hand, we can easily see that

$$p_m(x,\eta) = a_{0,m-1}(x) - \eta\sum_{j=1}^m b_j(x).$$

Hence we have

$$\exp(\int_0^t p_m(s,\eta)ds)\tilde{\psi}_1(t)\cdots\tilde{\psi}_m(t) = e^{\int_0^t a_{0,m-1}(s)ds}\tilde{\psi}_1(t)e^{-\eta\int_0^t b_1(s)ds}\cdots\tilde{\psi}_m(t)e^{-\eta\int_0^t b_m(s)ds}.$$

We note that $\tilde{\psi}_k(t)\exp(-\eta\int_0^t b_k(s)ds) = O(1)\,(k=1,\dots,m)$ when $\eta\to\infty$. Because the function $\Lambda(t,\eta)$ is a rapidly decreasing function of η it follows that the integrand in the second term of (5.12) is a rapidly decreasing function of η. We have

$$\begin{aligned}\det\Xi(x,\eta) &= \det(Y_0(x)+Y_1(x)\eta^{-1}+\dots) = (\det D(x)\det B(\eta))^{-1}\det Z(x,\eta)\\ &= (\det D(x)\det B(\eta))^{-1}\Psi(0)\exp(-\int_0^x p_m(t,\eta)dt)\\ &\quad +(\det D(x)\det B(\eta))^{-1}\exp(-\int_0^x p_m(t,\eta)dt)O(\eta^{-\infty}),\end{aligned} \tag{5.13}$$

where $O(\eta^{-\infty})$ denotes a term which is rapidly decreasing in η. We can easily see that $(\det D(x))^{-1}\exp(-\int_0^x p_m(t,\eta)dt) = O(1)$ when $|\eta|\to\infty$. Hence the second term of the right-hand side of (5.13) is a rapidly decreasing function of η. In order to get the expansion of the first term in the descending power of η we note by (5.9)

$$\begin{aligned}(\det B(\eta)^{-1})\Psi(0) &= \det(B(\eta)^{-1}Z(0,\eta)) = \det X(0,\eta)\\ &= \det(\Xi(0,\eta)D(0)) = \det(Y_0(0)+Y_1(0)\eta^{-1}+\dots)\det D(0).\end{aligned}$$

By (5.10) and the assumption (C.3) the first term in the expansion of $(\det B(\eta)^{-1})\Psi(0)$ in the ascending power of η^{-1} does not vanish. It follows that the right-hand side of (5.13) can be written in $C(x)+O(\eta^{-1})$, where $C(x)\neq 0$. In view of (5.10) this implies that $\det Y_0(x)$ does not vanish.

If we can construct a fundamental solution $X(x,\eta)$ then all arguments in the proof of Theorem 2.1 is valid in the present case. Hence we have Theorem 4.1.

In the following we shall give an essentially different proof of the necessary part of Theorem 4.1 based on factorization of differential operators. More precisely we shall prove the necessity of (4.8) under (I). The advantage of this new proof is that we do not assume (C.3) or (5.1).

Let $\tilde{S}$ be any realization of a phase S of a WKB solution to $\hat{P}$, which exists by Lemma 1 in the appendix. We first show that if we divide the operators $\hat{P}$ by $d/dx-\tilde{S}$ from the right then the remainder is given by $R(\tilde{S})$ where R is a Riccati operator given by (4.4). Indeed, by writing $d/dx=\partial$ we denote by $a\equiv b$ if $a-b$ is divisable from the right by $\partial-\tilde{S}$ in the class of differential operators in x with coefficients smooth in x and η. Then we have, for $u\in C^\infty$

$$
\begin{aligned}
&(\partial-b_m\eta)\dots(\partial-b_1\eta)u\\
&=(\partial-b_m\eta)\dots(\partial-\tilde{S}+\tilde{S}-b_1\eta)u\equiv(\partial-b_m\eta)\dots(\partial-b_2\eta)(\tilde{S}-b_1\eta)u\\
&=(\partial-b_m\eta)\dots(\partial-\tilde{S}+\tilde{S}-b_2\eta)(\tilde{S}-b_1\eta)u=(\partial-b_m\eta)\dots(\tilde{S}-b_2\eta)(\tilde{S}-b_1\eta)u\\
&\quad+(\partial-b_m\eta)\dots(\tilde{S}-b_1\eta)(\partial-\tilde{S})u+(\partial-b_m\eta)\dots[\partial-\tilde{S},\tilde{S}-b_1\eta]u\\
&\quad\equiv(\partial-b_m\eta)\dots(\tilde{S}-b_2\eta)(\tilde{S}-b_1\eta)u+(\partial-b_m\eta)\dots(\tilde{S}-b_2\eta)[\partial,\tilde{S}-b_1\eta]u\\
&=(\partial-b_m\eta)\dots(\partial-b_3\eta)T_2u,
\end{aligned}
$$

where $[a,b]:=ab-ba$ is a commutator of a and b, and where the function T_2 is given by $T_2:=(\partial+\tilde{S}-b_2\eta)(\tilde{S}-b_1\eta)$. By the same calculations the right-hand side of the above expression is equal to the term

$$
\begin{aligned}
&(\partial-b_m\eta)\dots(\partial-\tilde{S}+\tilde{S}-b_3\eta)T_2u\\
&=(\partial-b_m\eta)\dots(\tilde{S}-b_3\eta)T_2u+(\partial-b_m\eta)\dots T_2(\partial-\tilde{S})u+(\partial-b_m\eta)\dots[\partial-\tilde{S},T_2]u\\
&\quad\equiv(\partial-b_m\eta)\dots(\tilde{S}-b_3\eta)T_2u+(\partial-b_m\eta)\dots\partial T_2u=(\partial-b_m\eta)\dots(\partial-b_4\eta)T_3u,
\end{aligned}
$$

where $T_3:=(\partial+\tilde{S}-b_3\eta)T_2$. By repeating this calculation we have

$$
\begin{aligned}
(5.14)\qquad &(\partial-b_m\eta)\dots(\partial-b_1\eta)u=(\partial-b_m\eta)\dots(\partial-b_2\eta)(\tilde{S}-b_1\eta)u\\
&\equiv\cdots\equiv T_mu,\quad T_m:=(\partial+\tilde{S}-b_m\eta)\dots(\partial+\tilde{S}-b_2\eta)(\tilde{S}-b_1\eta).
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
(5.15)\qquad &\partial^lu=\partial^{l-1}(\partial-\tilde{S})u+\partial^{l-1}\tilde{S}u\equiv\partial^{l-1}\tilde{S}u=\partial^{l-2}(\partial-\tilde{S}+\tilde{S})\tilde{S}u\\
&=\partial^{l-2}(\partial-\tilde{S})\tilde{S}u+\partial^{l-2}\tilde{S}^2u=\partial^{l-2}[\partial-\tilde{S},\tilde{S}]u+\partial^{l-2}\tilde{S}(\partial-\tilde{S})u+\partial^{l-2}\tilde{S}^2u\\
&\equiv\partial^{l-2}[\partial-\tilde{S},\tilde{S}]u+\partial^{l-2}\tilde{S}^2u=\partial^{l-2}\partial\tilde{S}u+\partial^{l-2}\tilde{S}^2u=\partial^{l-2}A_2u,
\end{aligned}
$$

where $A_2:=(\partial+\tilde{S})\tilde{S}$. By repeating this argument we finally obtain

$$\partial^l\equiv(\partial+\tilde{S})^{l-1}\tilde{S}.$$

This proves the assertion in view of (4.4) and Lemma 1.

We shall prove the necessity of (4.8). By the above calculations we have

$$(5.16)\qquad \hat{P}=Q(d/dx-\tilde{S})+R(\tilde{S})$$

for some differential operator Q and R given by (4.4). For the sake of simplicity we denote the pseudodifferential operators determined by Q, $R(\tilde{S})$ and $\tilde{S}$ via inverse partial Fourier transform with the same letters. Note that R is a Riccati smoothing operator.

Suppose that (4.8) is not fulfilled for some $\tilde{S}=\tilde{S}_j$. Then by an evident adjustment of the arguments of Hounie (cf. [9]) we can find a distribution $u \in C^\infty(\mathbb{T}^1_x : \mathscr{D}'(\mathbb{T}_y)) \setminus C^\infty(\mathbb{T}^2)$ such that $(\partial_x - \tilde{S})u \in C^\infty(\mathbb{T}^2)$. By (5.16) this contradicts to the global hypoellipticity of P, because $R(\tilde{S})u \in C^\infty(\mathbb{T}^2)$ by definition.

§6. Second Order Equations and Diophantine Phenomena

We shall study equations of second order. We want to make clear how the Siegel conditions enter in the theory. This will be done in terms of WKB solutions. (cf. the case (a), (b) and (c) which follow.) Equivalent expressions to the Siegel condition (4.8) are also presented.

Let us consider the equation

$$Pu=(D_x+ia(x)D_y)(D_x+ib(x)D_y)u+\gamma(x)D_y u \tag{6.1}$$

on $(x,y)\in\mathbb{T}^2$, where $a(x)$, $b(x)$ and $\gamma(x)$ are 2π periodic complex-valued functions. By the partial Fourier transform with respect to y we have

$$\hat{P}\hat{u}=-(\frac{\partial}{\partial x})^2\hat{u}+(a(x)+b(x))\eta(\frac{\partial}{\partial x})\hat{u}+(b'(x)\eta+\gamma(x)\eta-a(x)b(x)\eta^2)\hat{u}=0, \tag{6.2}$$

where η is a covariable of y and $\hat{u}$ denotes the partial Fourier transformation of u with respect to y. Let ψ be a WKB solution to (6.2) given by (4.2). We denote by $\tilde{S}$ a realization of S in (4.2). Then we have

Theorem 6.1. *Suppose that there exists a smooth* WKB *solution* (4.2) *to the equation* (6.2). *Then the operator P in* (6.1) *is globally hypoelliptic on* $\mathbb{T}^2$ *if and only if* $\mathrm{Re}\,a(x)$ *and* $\mathrm{Re}\,b(x)$ *do not change their sign and that there exists* $N\geq 0$ *such that the condition* (4.8) *is satisfied for* $G(x,\eta)=\tilde{S}(x,\eta)$ *and* $G(x,\eta)=(a(x)+b(x))\eta\ -\tilde{S}(x,\eta)$.

We note that S satisfies a Riccati equation

$$R(S):=-S^2-S'+(a+b)\eta S+b'\eta+\gamma\eta-ab\eta^2=0. \tag{6.3}$$

A WKB solution exists if we assume

(C.2)′ Either $(a(x)-b(x))^{-l}D_x^k\gamma(x)$ or $(a(x)-b(x))^{-l}D_x^k(\gamma(x)-a'(x)+b'(x))$ is defined as a smooth function on $\mathbb{T}$ for $k=0,1,2,\ldots$ and $l=1,2,\ldots$.

One gets from (6.3) that $S_{-1}(x)$ verifies $S_{-1}^2-(a+b)S_{-1}+ab=0$ i.e. S_{-1} is either $a(x)$ or $b(x)$. The term $S_0(x)$ is given by

$$S_0(x) = -\frac{\gamma(x) + b'(x) - S'_{-1}(x)}{a(x) + b(x) - 2S_{-1}(x)}$$

and clearly in order $S_0(x)$ to be smooth function we must choose $S_{-1}(x) = b(x)$ if the former half of (C.2)′ holds while in case the latter half of (C.2)′ is true we take $S_{-1}(x) = a(x)$. Then we proceed by induction (cf. [6]).

The condition (C.2)′ is fulfilled if the function $a - b \not\equiv 0$ is analytic and if, either γ or $\gamma - a' + b'$ is flat on the set $\{x : a(x) = b(x)\}$. In case $a - b$ is not necessarily analytic then (C.2)′ is satisfied if either γ or $\gamma - a' + b'$ vanished in some neighborhood of the set $\{x : a(x) = b(x)\}$. We note that if $\gamma \equiv 0$ or $\gamma = a' - b'$ then (C.2)′ holds without any restriction on a and b. The condition is, so to speak, a Levi condition.

Let us assume that a WKB solution $S(x,\eta) = \sum_{j=-1}^{\infty} \eta^{-j} S_j(x)$ ($S_{-1} = a$, or b) to (6.2) is smoothly defined on the torus. Three cases are distinguished.

(a) $\mathrm{Re}\, a(x) \not\equiv 0$ and $\mathrm{Re}\, b(x) \not\equiv 0$.

(b) Either $\mathrm{Re}\, a(x) \equiv 0$ or $\mathrm{Re}\, b(x) \equiv 0$ holds. There exists $j \geq 0$ such that

$$\int_0^{2\pi} \mathrm{Re}\, S_j(x) dx \neq 0.$$

(c) Either $\mathrm{Re}\, a(x) \equiv 0$ or $\mathrm{Re}\, b(x) \equiv 0$ holds. In addition the following condition

$$\int_0^{2\pi} \mathrm{Re}\, S_j(x) dx = 0$$

holds for all $j = 1, 2, \ldots$.

We note that one of the three cases (a), (b) and (c) occurs. As we shall see later, the diophantine analysis may enter only in the case (c). Note that WKB solutions give a criterion for this.

Suppose that there exists a smooth WKB solution. Moreover, assume that either the condition (a) or (b) is satisfied. Then the operator P is globally hypoelliptic on $\mathbb{T}^2$ and globally Fredholm solvable if and only if $\mathrm{Re}\, a(x)$ and $\mathrm{Re}\, b(x)$ do not change their sign. In the case (a), the sufficient part is also true for the perturbed operator $P + \delta(x)$ with sufficiently small supremum norm of the zeroth order term $\delta(x)$. Here we say that P is globally Fredholm solvable if there exists a finite dimensional subspace $\mathscr{B}$ of $C^\infty(\mathbb{T}^2)$ such that the equation $Pu = f$ has always a solution $u \in C^\infty$ provided $\iint f\phi\, dxdy = 0$ for all $\phi \in \mathscr{B}$.

Indeed, if (a) or (b) is fulfilled then (4.8) automatically follows. Hence the former half of the above assertion is a special case of Theorem 6.1. The latter half is almost clear from the proof of Theorem 6.1. (cf. [6]).

Remark 6.1. The above result can be viewed as a result for degenerate elliptic operators. We recall that D. Fujiwara and H. Omori [3] established global hypoellipticity for $D_x^2+\varphi(x)D_y^2$, where φ is $C^\infty(\mathbb{R})\,2\pi$ periodic real-valued nonnegative function, identically equal to 0 and 1 on some subintervals of $[0,2\pi]$. Concerning this we note that among the operators satisfying the condition (a) there are operators which may be identically degenerate elliptic in some sub-domain and weakly hyperbolic and of mixed type in other regions.

The remaining case (c) is the one where diophantine analysis enters. In order to see this we begin with a rather special case, where the diophantine condition can be replaced by an algebraic condition. We set

$$\tau_a := \int_0^{2\pi} \operatorname{Im} a(x)dx, \quad \tau_b := \int_0^{2\pi} \operatorname{Im} b(x)dx. \tag{6.4}$$

Then we have the following assertion.

Suppose that the condition (c) is satisfied. Moreover, suppose that $\operatorname{Re} a(x)$ and $\operatorname{Re} b(x)$ do not change their sign. If $\operatorname{Re} a(x)\equiv 0$ we assume the following condition

$$\tau_a/(2\pi) \text{ is a rational number} \tag{6.5}$$

(respectively $\operatorname{Re} b(x)\equiv 0$ implies that $\tau_b/(2\pi)$ is a rational number).
Then in view of (6.5) if

$$\int_0^{2\pi} \operatorname{Im} S_0(x)dx \neq -\eta\tau_a \bmod 2\pi\mathbb{Z} \text{ (respectively, } \neq -\eta\tau_b \bmod 2\pi\mathbb{Z}) \tag{6.6}$$

for all $\eta\in\mathbb{Z}$ the operator P is globally hypoelliptic.

In order to see this we shall show that $|\exp(\int Sdt)-1|\ge c>0$ for some $c>0$ independent of η, $|\eta|>>1$. Indeed, we have that $\exp(\int Sdt)=\exp(i\tau_a\eta+i\int \operatorname{Im} S_0dt + \int \operatorname{Re} S_0 dt+O(\eta^{-1}))$. Hence the estimate is clear if $\int \operatorname{Re} S_0dt\neq 0$. Let us assume that $\int \operatorname{Re} S_0dt=0$. By (6.5) we can write $\tau_a=2\pi q/p$ for mutually prime integers p and q. It follows that the quantity $\tau_a\eta+\int \operatorname{Im} S_0dt$ takes only finite number of values mod $2\pi\mathbb{Z}$ when η moves on $\mathbb{Z}$. Because of (6.6) we have $|\exp(i\tau_a\eta+i\int \operatorname{Im} S_0dt)-1|\ \ge c'>0$ for some $c'>0$ independent of η. This proves the assertion. Hence we have (4.8). Now the result follows from Theorem 6.1.

Suppose further that $\int \operatorname{Re} Sdt=0$ and (6.6) is not true. Then P is globally hypoelliptic if and only if

$$\text{there exists } j\ge 1 \text{ such that } \int_0^{2\pi} \operatorname{Im} S_j(x)dx\neq 0. \tag{6.7}$$

Indeed, for the set of η satisfying the relation (6.6) we have the estimate $|\exp(\int Sdt)-1|\ge c>0$ for some $c>0$ independent of η. If η does not satisfy (6.6) we have the inequality $|\exp(\int Sdt)-1|\ge c|\eta|^{-j}>0$ for some $c>0$ if η is large. Hence we have (4.8). Conversely, we can easily show that (4.8) does not hold if (6.7) is not true because of the defintion of S. We note that if (6.5) is fulfilled then no diophantine phenomena occur because $e^{\tau_a \eta}$ takes only finite number of values when $\eta \in \mathbf{Z}$.

In the case where the finiteness condition (6.5) is not true one encounters a diophantine condition of the form (4.8). More precisely, by Theorem 6.1 we have the following assertion. Assume that there exists a smooth WKB solution and that the condition (c) is satisfied. Then P is globally hypoelliptic on $\mathbf{T}^2$ if and only if the functions $\operatorname{Re} a(x)$ and $\operatorname{Re} b(x)$ do not change their sign and there exists $N \geq 0$ such that the Siegel condition

$$\liminf_{n \to \infty, \eta \in \mathbf{Z}} |\eta|^N |\exp(\int_0^{2\pi} G(x,\eta) dx) - 1| > 0, \tag{6.8}$$

is fulfilled for $G(x,\eta) = \tilde{S}(x,\eta)$ and $G(x,\eta) = (a(x)+b(x))\eta - \tilde{S}(x,\eta)$ where $\tilde{S}(x,\eta)$ is a realization of WKB solution to (6.2).

In order to see the transcendency of the phenomena more clearly, let us consider a rather special operator (6.1) such that $\gamma \equiv 0$ or $\gamma = a' - b'$. We denote the operator by P_0. (cf. [5]). First we note that an irrational number τ is called a non Liouville number if and only if there exist $C > 0$ and $N > 0$ such that $|\tau - p/q| \geq C/q^N$, $p \in \mathbf{Z}$, $q \in \mathbf{N}$.

We assume that $\operatorname{Re} a(x)$ and $\operatorname{Re} b(x)$ do not change their sign. Then if $\operatorname{Re} a(x) \equiv 0$ (resp. $\operatorname{Re} b(x) \equiv 0$) the equation $P_0 u = f$ has a solution $u \in \mathscr{D}'(\mathbf{T}^2)$ for every $f \in C^\infty(\mathbf{T}^2)$ such that $\int_0^{2\pi}\int_0^{2\pi} f(x,y) dx dy$ if and only if

(6.9) $\tau_a/(2\pi)$ (resp. $\tau_b/(2\pi)$) is an irrational non Liouville number.

Moreover the condition (6.9) is necessary and sufficient for the global hypoellipticity of P_0.

First we note that, $S(x,\eta) = \eta S_{-1}(x)$ for P_0. Then the condition (6.9) is equivalent to the corresponding Siegel condition in Theorem 6.1. The global solvability follows from the result of [9] and the observation; The operator P_0 can be factored as a product of first order operators (cf. the proof of Theorem 6.1). In order to prove the necessity, let us suppose that $P_0 = L_1 L_2$ where $L_1 = D_x + ib(x) D_y$ and $L_2 = D_x + ia(x) D_y$. If $\tau_b/(2\pi)$ is an irrational Liouville number or a rational number, it follows that the equation $L_1 v = f$ is not globally solvable for some $f \in C^\infty$. Hence we see that the equation $P_0 = L_1 L_2 u = f$ is not globally solvable. If $\tau_a/(2\pi)$ is an irrational Liouville number or a rational number and $\tau_b/(2\pi)$ is an irrational non Liouville number it follows that $L_2 v = f$ is not globally solvable for some f orthogonal to the constant function 1 and L_1 is a globally solvable operator. Hence the equation $P_0 u = L_1 f$ is not globally solvable. Indeed, if otherwise we have $L_1(L_2 u - f) = 0$ and therefore $L_2 u - f$ is constant. This constant is equal to zero in view of the orthogonality condition on f, which is a contradiction.

In case P_0 is the following constant coefficients operator L we find a classical result of [7].

$$(6.10)\qquad L := (\partial_{x_1} - \tau\partial_{x_2})(\partial_{x_1} - \mu\partial_{x_2}),\ \text{on } \mathbb{T}^2, \partial_{x_j} = \frac{\partial}{\partial x_j},\ \tau > 0, \mu > 0.$$

Then we can easily see that $\tau_a = 2\pi\tau, \tau_b = 2\pi\mu$. Hence the above results read:

$Lu = f$ is solvable for any f such that $\int_0^{2\pi}\int_0^{2\pi} f(x,y)dxdy = 0$ if and only if τ and μ are irrational non Liouville numbers. (Siegel condition).

Remark 6.2. In 1974, S. Greenfield and N. Wallach [7] showed that a scalar constant coefficients differential operator $P(D)$ on the n-dimensional torus is globally hypoelliptic if and only if its full symbol satisfies a Siegel type condition. An interesting example is a globally hypoelliptic hyperbolic operator on $\mathbb{T}^2$, $D_x + cD_y$, where $c \in R \setminus 0$ is an irrational non Liouville number. This result is extended for linear systems with constant coefficients on $\mathbb{T}^n$ by P. Popivanov (cf. [14]). Later on he used Siegel type conditions in order to establish regularity in Schwartz class $\mathscr{S}(\mathbb{R}^n)$ of rapidly decreasing functions on $\mathbb{R}^n$ for certain second order operators with polynomial coefficients (cf. [15]). J. Hounie [9] gave a necessary and sufficient condition for global solvability for first order systems $\partial_t u + b(t)A$, where A is an essentially self-adjoint operator on a compact manifold. The second author has also obtained results for global hypoellipticity on tori for operators not satisfying Siegel type conditions [20], and for the Mathieu operator [21].

Proof of Theorem 6.1. Let S be a phase of a WKB solution and let $\tilde{S}$ be a realization of S. We note that the existence of a WKB solution implies the complete factorization of P to a product of first order operators mod a Riccati smoothing operator

$$(6.11)\qquad \begin{aligned} P &= (D_x + i(a(x) + b(x))D_x - i\tilde{S}(x, D_x))(D_x + i\tilde{S}(x, D_x)) + R(\tilde{S})(x, D_x) \\ &\equiv L_1 L_2 + R(\tilde{S})(x, D_x), \end{aligned}$$

where $R(\tilde{S})(x, D_x)$ is a pseudodifferential operator with smoothing kernel given by Riccati equation and the realization of S. Because the global hypoellipticity of P is equivalent to that of the operator L_1L_2 we may assume that $P = L_1L_2$. Suppose that the conditions of Theorem 6.1 are satisfied. Assume that $L_1L_2u \in C^\infty$ and $u \in \mathscr{D}'$. Then by the result of Hounie (or by simple modifications of the arguments in §3) the operators L_1 and L_2 are globally hypoelliptic. Hence it follows that $u \in C^\infty$.

Conversely, suppose that one of the conditions of Theorem 6.1 is not true. For the sake of simplicity we assume that the top term in $S(x,\eta)$ is $a(x)\eta$. Then either L_1 or L_2 is not globally hypoelliptic. If L_2 is not globally hypoelliptic we can find $u \in \mathscr{D}' \setminus C^\infty$ such that $L_2u \in C^\infty$. Since $Pu = L_1L_2u \in C^\infty$ this implies that $P = L_1L_2$ is not globally hypoelliptic.

Suppose that L_1 is not globally hypoelliptic and L_2 is globally hypoelliptic. Then the Siegel condition (4.8) to the operator L_2 is valid and $\mathrm{Re}\, a$ does not change its sign. We can find a $v \in \mathscr{D}' \setminus C^\infty$ such that $L_1 v \in C^\infty$. Let $\hat{v}(x,\eta)$ be a partial Fourier transformation of v with respect to y. By assumption $\hat{v}(x,\eta)$ is a smooth function of x such that it is of polynomial growth in η. Let $\hat{L}_2$ be a partial Fourier transformation of L_2 with respect to y. In terms of the conditions on L_2 it follows from the argument of Section 3 (or by a formula for first order ordinary differential equations) that the equation $\hat{L}_2\hat{u} = \hat{v}$ has a periodic solution $\hat{u}(x,\eta)$ if η is sufficiently large. Moreover, $\hat{u}(x,\eta)$ is of polynomial growth in η. We write $\hat{v} = \hat{v}_1 + \hat{v}_2$, where the function $\hat{v}_2(x,\eta)$ vanish identically if η is sufficiently large and that the equation $\hat{L}_2\hat{u} = \hat{v}_1$ has a periodic solution $\hat{u}(x,\eta)$. It follows that we can find a distribution $u \in \mathscr{D}' \setminus C^\infty$ such that $L_2 u = v_1$, where v_1 is the inverse Fourier transformation of $\hat{v}_1$. If we denote the inverse Fourier transformation of $\hat{v}_2$ by v_2 the function v_2 is a smooth function. Hence the function $L_1 v_1 = L_1 v - L_1 v_2$ is a smooth function because $L_1 v \in C^\infty$ by assumption. It follows that $Pu = L_1 L_2 u = L_1 v_1$ is smooth. Because $u \in \mathscr{D}' \setminus C^\infty$ this implies that the operator P is not globally hypoelliptic. This proves Theorem 6.1.

Appendix

We shall show the next lemma.

Lemma 1. *Let $U(x,\eta) = \sum_{j=-1}^{\infty} \eta^{-j} U_j(x)$ be a formal power series of η such that $U_j(x)$ $(j = -1, 0, 1, 2, \ldots)$ are smooth functions on $\mathbb{T}$. Then there exist $\chi_j(\eta)$ $(j = -1, 0, 1, 2, \ldots, \chi_{-1} \equiv 1)$ such that the sum*

$$\tilde{U}(x,\eta) := \sum_{j=-1}^{\infty} \eta^{-j} \chi_j(\eta) U_j(x)$$

defines a smooth function $\tilde{U}(x,\eta)$ of x on $\mathbb{T}$ satisfying for $k = 1, 2, \ldots,$ and $\beta \in \mathbb{N}$.

$$D_x^\beta\Big(\tilde{U}(\eta,x) - \sum_{j=-1}^{k} \eta^{-j} U_j(x)\Big) = O(\eta^{-k/2+1/2}) \text{ when } \eta \to \infty. \tag{A.1}$$

If the function $\tilde{U}$ is given by (4.7), $\tilde{U} = \tilde{S}$ then $\tilde{S}$ satisfies (4.5) asymptotically, $R(\tilde{S})$ is rapidly decreasing in η when $\eta \to \infty$.

Proof. We note that all U_j's are 2π periodic function of x. We put

$$m_j := \sup_{x\in\mathbf{T},\beta\le j} |D_x^\beta U_j(x)|, \; j \ge -1. \tag{A.2}$$

For $j \ge 0$, we take $\chi_j(\eta) \in C^\infty(\mathbb{R}), 0 \le \chi_j \le 1$ whose support is contained in $|\eta| \ge (m_j)^{2/j} + j^\varepsilon - 1$ and that

$$\chi_j \equiv 1 \text{ on } |\eta| \ge (m_j)^{2/j} + j^\varepsilon$$

where $\varepsilon > 0$ is a constant taken sufficiently small. Then the function

$$\tilde{U}(\eta, x) := \eta U_{-1}(x) + \chi_0(\eta) U_0(x) + \sum_{j=1}^{\infty} \chi_j(\eta)\eta^{-j} U_j(x) \tag{A.3}$$

is defined as a smooth function of η and x which is 2π periodic with respect to x. Indeed, the right hand side of (A.3) is a finite sum for each η and x. Also it is bounded when $\eta \to \infty$ except for the term $\eta U_{-1}(x)$. Furthermore, for each $k \geq 1$ we have

$$\begin{aligned} &\tilde{U}(\eta, x) - \sum_{j=-1}^{k} \eta^{-j} U_j(x) \\ &= \sum_{j=0}^{k} \eta^{-j} U_j(x)(\chi_j(\eta) - 1) + \sum_{j=k+1}^{\infty} \chi_j(\eta)\eta^{-j} U_j(x). \end{aligned} \tag{A.4}$$

The first term in the right-hand side is bounded when $\eta \to \infty$ because $\chi_j(\eta) - 1$ has a compact support. On the other hand,

$$\sum_{j=k+1}^{\infty} \chi_j(\eta)\eta^{-j} U_j(x) = \eta^{(-k+1)/2} \sum_{j=k+1}^{\infty} \chi_j(\eta)\eta^{(-j+k-1)/2}\eta^{-j/2} U_j(x).$$

$|\eta^{-j/2} U_j(x)|$ is bounded in x and η, as $\eta \to \infty$ by the condition on $\chi_j(\eta)$. Because $k - 1 - j \leq -2$ and $|\eta| > 1$, the sum is bounded for all η and x. This proves the case $\beta = 0$. The general case $\beta \neq 0$ will be proved similarly.

The latter half of the lemma follows from the definition of the formal sum $S(x, \eta)$. This ends the proof of Lemma 1.

The following results are taken from Chapter 4 of [19] with modifications necessary in this paper. Let us consider the equation of the form (2.2).

$$v' = \left(\sum_{j=0}^{\infty} A_j(x)\eta^{-j+1}\right)v := Av. \tag{A.5}$$

The equation (2.2) can be written in (A.5) by an appropriate choice of the coefficients matrices in (A.5). In the following we assume that the eigenvalues of $A_0(x)$ are distinct. We take a formal solution $\hat{Y}(x, \eta)e^{\eta\Lambda(x,\eta)}$ to (A.5)

$$\hat{Y}(x, \eta)e^{\eta\Lambda(x,\eta)} := \left(\sum_{j=0}^{\infty} Y_j(x)\eta^{-j}\right)e^{\eta\Lambda(x,\eta)}, \tag{A.6}$$

where the diagonal matrix $\Lambda(x, \eta)$ is given by (2.4) and where $\det Y_0(x) \neq 0$. We note that the coefficients $Y_j(x)$ are smooth functions on $\mathbb{T}$. We say that the formal matrix solution (A.6) is a *basic formal matrix solution* of (2.2) if the formal power series $\hat{Y}(x, \eta)$ satisfies the formal differential equation

$$\hat{Y}' = A\hat{Y} - \eta\hat{Y}\Lambda'. \tag{A.7}$$

We note that (A.7) is equivalent to that $\hat{Y}e^{\eta\Lambda}$ satisfies the equation (A.5) formally. Let us consider a differential equation of the same type as (2.2)

$$w' = (\sum_{j=0}^{\infty} B_j(x)\eta^{-j+1})w. \tag{A.8}$$

Then the equations (A.5) and (A.8) are said to be formally equivalent if there exists a matrix $P(x,\eta) \sim \sum_{j=0}^{\infty} P_j(x)\eta^{-j}$ with $P_0(x)$ being invertible and all P_j's being smooth functions on $\mathbf{T}$ such that (A.5) is transformed into (A.8) by the change of variables $v = P(x,\eta)w$. Under these preparations we have the following

Lemma 2. *Let Λ be given in formula* (A.6) *for a basic formal matrix solution to the equation* (A.5). *Then the diagonal entries of the matrix $d\Lambda/dx$ are formal invariants of the differential equation.*

Lemma 2 is proved in §4.2 of [19]. By this lemma if we normalize $\Lambda(x,\eta)$ so that $\Lambda(0,\eta)=0$ then $\Lambda(x,\eta)$ is invariantly defined under formal change of variables (A.8).

Lemma 3. *Every formal equivalence class contains differential equations for whose basic formal matrix solutions $\hat{Y}e^{Q}$, the series $\hat{Y}$ is independent of x.*

This lemma is proved in [19]. (cf. [18, Theorem 4.3-1]). Note that the above formal solution plays the role of a WKB solution for a single equation.

Acknowledgement

A part of this work was done when the second author was visiting Bulgarian Academy of Sciences, Sofia, Bulgaria with the support of JSPS. The authors would like to thank Prof. P. Popivanov for interesting discussions. The second author also would like to thank JSPS (Japan) and GNAFA, CNR (Italy) for financial supports and Prof. T. Kawai in RIMS Kyoto University (Japan) for useful and interesting discussions.

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