

# The Flow of Weights in Subfactor Theory

By

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## Abstract

We define a Connes-Takesaki type flow of weights for any inclusion of factors. It is shown that Popa's classification of strongly amenable subfactors as well as some new structural results on type III-subfactors can be stated in a unified way using this invariant. The invariant also separates certain "exotic" examples of subfactors which are not covered by that classification.

## §1. Introduction

The flow of weights on a factor of type III was defined by Connes and Takesaki in 1973 and has been studied ever since as a tool to classify factors of type III and their automorphisms. The purpose of this paper is to discuss how to adapt this approach to the classification and structure theory of subfactors.

Let us briefly review the single factor case [2]. Let  $M$  denote a properly infinite von Neumann algebra with separable predual. Then we can choose a dominant weight  $\omega$  on  $M$ . Further, there exists an action  $\theta: \mathbf{R} \rightarrow \text{Aut}(M_\omega)$  such that

$$(1) \quad (M, \sigma^\omega) \cong (M_\omega \rtimes_\theta \mathbf{R}, \hat{\theta})$$

where  $\hat{\theta}: \mathbf{R} \rightarrow M_\omega \rtimes_\theta \mathbf{R}$  is the dual action of  $\theta$ , cf. [23]; also there exists a normal semifinite faithful trace  $\tau$  on  $M_\omega$  with

$$(2) \quad \tau \circ \theta_s = e^{-s} \tau, \quad s \in \mathbf{R}.$$

In view of (1), the pair  $(M_\omega, \theta)$  is called a *continuous decomposition* of  $M$ .

Connes and Takesaki next defined the (*smooth*) *flow of weights* on  $M$ . In the above setting, it is the pair  $(\mathcal{P}^M, \mathcal{F}^M)$ , where

$$(3) \quad \mathcal{P}^M = \mathcal{Z}(M_\omega) \quad (:= M_\omega \cap M'_\omega),$$

$$(4) \quad \mathcal{F}^M: \mathbf{R}_+ \rightarrow \text{Aut}(\mathcal{P}^M), \quad \mathcal{F}_\lambda^M = \theta_{-\log \lambda}|_{\mathcal{Z}(M_\omega)}, \quad \lambda > 0.$$

Note here that since  $\omega$  and  $\theta$  are both essentially unique, the isomorphism class of  $(\mathcal{P}^M, \mathcal{F}^M)$  does not depend upon a choice of them. The terminology is justified

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by a theorem in [2], which roughly speaking states that the lattice of projections in  $\mathcal{P}^M$  can be identified with the set of equivalence classes of so-called integrable weights on  $M$ , on which  $\mathcal{F}^M$  then acts by simple scalar multiplication.

The flow of weights turns out to be a complete invariant for hyperfinite factors of type III. In particular, the Connes invariant

$$S_0(M) = \bigcap_{\phi} Sp(\Delta_{\phi}) \setminus \{0\}$$

(where  $\phi$  runs over all faithful normal states on  $M$ ) can be calculated as

$$(5) \quad S_0(M) = \text{Ker}(\mathcal{F}^M),$$

thanks to [23,9.6]. Then the classification [1] in types  $\text{III}_{\lambda}$  ( $0 \leq \lambda \leq 1$ ) is given by which subgroup of  $\mathbb{R}_+$  this yields; and there is precisely one hyperfinite factor of each of these types except in the type  $\text{III}_0$ -case, where all non-transitive ergodic flows are realized as the flow  $(\mathcal{P}^M, \mathcal{F}^M)$  for exactly one hyperfinite factor  $M$ .

This paper is an attempt to establish a similar unified picture of the wealth of classification results in subfactor theory which has been obtained during the last few years, and it is a natural continuation of our work in [24], [25]. Besides deepening our understanding of existing results, such a picture is supposed to provide directions for obtaining a complete classification, cf. Conjecture 4.1. In fact, we define in Sec. 2 a Connes-Takesaki type flow of weights for subfactors which is a simultaneous generalization of the standard invariant for subfactors of type  $\text{II}_1$  (cf. [17]) and the classical flow of weights for type III-factors (as described above). After treating the classification issue, we demonstrate (in Sec. 3) how this new invariant contains information on the structure of subfactors even in cases where no classification has yet been found. The invariant also separates some exotic (non-classified) examples of type III-subfactors, which are due to H. Kosaki, P. Loi and T. Sano, and which will be discussed together with further applications in Sec. 4.

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## §2. The Relative Flow of Weights

Let  $M \supseteq N$  be an inclusion of  $\sigma$ -finite factors with finite index. Let  $E$  be a faithful normal conditional expectation from  $M$  onto  $N$  chosen as follows: If  $M$  (hence  $N$ ) is of type II,  $E$  is the canonical trace-preserving expectation; otherwise it is the minimal expectation (cf. [5], [6]). Also let

$$N \subseteq M = M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$$

be the tower for  $M \supseteq N$ , with the canonical conditional expectations  $E_k : M_k \rightarrow M_{k-1}$  defined by  $E_0 = E$ , as usual.

Take any normal semifinite faithful weight  $\psi$  on  $M$  satisfying

$$\psi \circ E = \psi.$$

Also put  $\phi = \psi|_N$ ,  $\psi_0 = \psi$  and

$$\psi_k = \psi \circ E_1 \circ \dots \circ E_k, \quad k \geq 1.$$

Finally let

$$\tilde{N} = N \rtimes_{\sigma_\phi} \mathbb{R}; \quad \tilde{M} = M \rtimes_{\sigma_\psi} \mathbb{R}; \quad \tilde{M}_k = M_k \rtimes_{\sigma_{\psi_k}} \mathbb{R}, \quad k \geq 0.$$

We then assume that

$$\mathcal{Z}(\tilde{M}) = \mathcal{Z}(\tilde{N})$$

from which, in particular, it follows that  $(\mathcal{P}^M, \mathcal{F}^M) = (\mathcal{P}^N, \mathcal{F}^N)$ , so that  $M$  and  $N$  are isomorphic if they are hyperfinite by the classification mentioned in the introduction.

**2.1. Definition.** Let

$$\mathcal{P}^{M \supseteq N} = \{\tilde{M}_k \cap \tilde{N}' \supseteq \tilde{M}_k \cap \tilde{M}'\}_{k=0}^\infty$$

and define an action  $\mathcal{F}^{M \supseteq N}$  of  $S_0(N) \times \mathbb{R}_+$  on  $\mathcal{P}^{M \supseteq N}$  by

$$\mathcal{F}_{(g,s)}^{M \supseteq N} = \{(\sigma^{\psi_k})^\wedge_{-\log(gs)}|_{\tilde{M}_k \cap \tilde{N}'}\}_{k=0}^\infty, \quad (g,s) \in S_0(N) \times \mathbb{R}_+,$$

where  $(\sigma^{\psi_k})^\wedge$  is the dual action of  $\sigma^{\psi_k}$ . Then the pair  $(\mathcal{P}^{M \supseteq N}, \mathcal{F}^{M \supseteq N})$  is called the *flow of weights* of  $M \supseteq N$ .

Note that the fundamental homomorphism on  $M \supseteq N$ , defined in [24, 4.2] (cf. [13], [8], [4] for notation and background) by

$$\Upsilon(\alpha) = (\alpha_k|_{\tilde{M}_k \cap \tilde{N}'} )_{k=0}^\infty, \quad \alpha \in \text{Aut}(M, N)$$

is in the above setting a homomorphism  $\Upsilon : \text{Aut}(M, N) \rightarrow \text{Aut}(\mathcal{P}^{M \supseteq N}, \mathcal{F}^{M \supseteq N})$ , where  $\text{Aut}(\mathcal{P}^{M \supseteq N}, \mathcal{F}^{M \supseteq N})$  denotes the set of (nested) automorphisms on  $\mathcal{P}^{M \supseteq N}$  which commutes with  $\text{Im}(\mathcal{F}^{M \supseteq N}) = \mathcal{F}^{M \supseteq N}(S_0(N) \times \mathbb{R}_+)$ .

Also recall that Loi's invariant  $\Phi$  can in general be defined as

$$\Phi(\alpha) = (\alpha_k|_{M_k \supseteq N'} )_{k=0}^\infty, \quad \alpha \in \text{Aut}(M, N),$$

and  $\Phi$  can be identified with  $\Upsilon$  when  $M \supseteq N$  is of type  $\text{II}_1$  (but not otherwise).

A few immediate observations should be noticed here.

2.2. *Remarks.*

1. We need to define  $\mathcal{P}^{M \supseteq N}$  as a sequence of *inclusions* of algebras in order to get an invariant for subfactors, due to the possible discrepancy between principal and “dual” principal graphs; however, in an abstract setting, one can often just look at say  $\{\tilde{M}_k \cap \tilde{M}'\}_{k=0}^\infty$  for simplicity.

2. The isomorphism class of  $(\mathcal{P}^{M \supseteq N}, \mathcal{F}^{M \supseteq N})$  does not depend on the choice of the weight  $\psi$ , since (using the Connes’ unitary cocycle theorem [1, 1.2.1]) one gets easily

$$(M \rtimes_{\sigma_{x_1}} \mathbb{R}, (\sigma^{x_1})^\wedge) \cong (M \rtimes_{\sigma_{x_2}} \mathbb{R}, (\sigma^{x_2})^\wedge)$$

for any two normal faithful semifinite weights  $\chi_1, \chi_2$  on  $M$ .

3. For  $M = N$  properly infinite, we have  $(\mathcal{P}^{M \supseteq N}, \mathcal{F}^{M \supseteq N}) \cong (\mathcal{P}^M, \mathcal{F}^M)$  because of equation (5) and [4, 13.1].

4. For  $M \supseteq N$  of type II,  $S_0(N) = \{1\}$  and  $(\mathcal{P}^{M \supseteq N}, \mathcal{F}^{M \supseteq N})$  is isomorphic to the standard invariant ([17]) together with the trivial action, tensor  $L^\infty(\mathbb{R})$  with translation. The last factor in the tensor product being independent of  $M \supseteq N$ , we can ignore it, hence identify the flow with the standard invariant.

5. For  $M \supseteq N$  of type III<sub>1</sub>,  $(\mathcal{P}^{M \supseteq N}, \mathcal{F}^{M \supseteq N})$  is isomorphic to the standard invariant of the type II<sub>1</sub>-inclusion associated to  $M \supseteq N$  via the common continuous decomposition (see [12]), together with the  $\mathbb{R}_+ \times \mathbb{R}_+$ -action

$$\{(s, t) \mapsto (\sigma^{\psi_k})^\wedge|_{\tilde{M}_k \cap \tilde{M}'}\}_{k=0}^\infty$$

which is essentially the Loi invariant of the  $\mathbb{R}$ -action from the continuous decomposition of  $M \supseteq N$ , cf. Eq. (13).

6. For  $M \supseteq N$  of type III<sub>0</sub>,  $S_0(N) = \{1\}$  and our invariant reduces to restriction  $(\sigma^{\psi_k})^\wedge$  to  $\tilde{M}_k \cap \tilde{N}'$ , which has been previously considered by Kosaki in [9], [10], cf. Sec. 4.

Without further ceremonies, we state the theorem which was our first motivation for studying the above concepts:

**2.3. Theorem.** *The classification of strongly amenable subfactors of type II ([17]) and type III<sub>λ</sub> (0 < λ < 1) ([18], [19]), as well as the classification in [20] of finite depth type III<sub>1</sub>-subfactors, can be formulated as follows: All these subfactors are completely classified (up to isomorphism) by their flow of weights.*

By 2.2.4 above, the type II-case of the above theorem is obvious.

The proof of the following statement is routine and we leave it to the reader.

**2.4. Proposition.** *Let  $A \supseteq B$  and  $M$  be factors with  $A \cong B$  of type II with a common trace. Then*

$$(\mathcal{P}^{A \otimes M \supseteq B \otimes M}, \mathcal{F}_{(g, \psi)}^{A \otimes M \supseteq B \otimes M}) \cong (\mathcal{P}^{A \supseteq B} \otimes \mathcal{P}^M, 1 \otimes \mathcal{F}_{g\psi}^M)$$

for all  $(g, s) \in S_0(M) \times \mathbb{R}_+$ .

Then the type III<sub>1</sub>-case of the theorem follows from [20] and the following easy consequence of [17], the classification of hyperfinite factors and the above proposition:

**2.5. Corollary.** *Subfactors of the form  $A \otimes M \supseteq B \otimes M$ , where  $A \supseteq B$  is a strongly amenable subfactor of type II, and  $M$  is a hyperfinite factor, are completely classified by their flow of weights.*

So we only have to deal with the discrete type III case of the theorem. This will be done in the next section.

We end this section with a remark on the computability of flow of weights. For the type III<sub>0</sub>-case, cf. also [28].

In the single factor case, the flow of weights of crossed products by discrete amenable groups can be computed in general in terms of the invariants of the  $W^*$ -dynamical system we start with (see [22, 1.5], [21, 4.1]). In the subfactor case, we lack the invariants for a general computation as in [22], because we do not have a general classification for discrete amenable actions; in fact only the strongly free case is known (see [27] for a survey). Here, an automorphism  $\sigma \in \text{Aut}(M, N)$  is called strongly free if, for all  $k \geq 0$  and  $a \in \tilde{M}_k$  the property

$$\tilde{\sigma}(x)a = ax, \quad x \in \tilde{M}$$

entails  $a = 0$ . (Cf. [25] and [10] for further discussion of strong freeness.) Then we have

**2.6. Proposition.** *Let  $M \supseteq N$  be a subfactor and  $\alpha$  a strongly free action of a discrete group  $G$  on  $M \supseteq N$ . Put*

$$\hat{M} = M \rtimes_{\alpha} G, \quad \hat{N} = N \rtimes_{\alpha} G.$$

Then

$$(\mathcal{F}^{\hat{M} \supseteq \hat{N}}, \mathcal{G}^{\hat{M} \supseteq \hat{N}}) \cong ((\mathcal{F}^{\hat{M} \supseteq \hat{N}})^{\Upsilon(\alpha)}, \mathcal{F}^{\hat{M} \supseteq \hat{N}}|_{\mathcal{F}^{\hat{M} \supseteq \hat{N}} \Upsilon(\alpha)}),$$

*i.e. the flow of weights of the crossed product equals the fixed point algebra of the original flow by the image of the action under the fundamental homomorphism.*

*Proof.* By the argument of [18, 1.5], strong freeness of  $\alpha$  means

$$\tilde{M}_k \cap \tilde{M}' = \tilde{M}_k \cap \tilde{M}', \quad k \geq 0$$

where  $\hat{M}_k$  is the crossed product of  $\tilde{M}_k$  by  $\tilde{\alpha}_k$ . The proof in [21, 4.2] that the crossed products by  $\mathbb{R}$  and  $G$  can be interchanged in the obvious sense also works for inclusions. Hence we get

$$\tilde{M}_k \cap \tilde{M}' \cong \hat{M}_k \cap \hat{M}' = \tilde{M}_k \cap \hat{M}' = (\tilde{M}_k \cap \tilde{M}')^{\tilde{\alpha}_k}$$

for all  $k \geq 0$ , and the flow is as claimed. Q.E.D.

Note that [26, 5.2] is a special case of the above general statement.

**§3. The Discrete Type III-case.**

As an illuminating and fairly well understood class of type III-subfactors, we study in this section the subfactors  $M \supseteq N$  of type  $\text{III}_\lambda$  ( $0 \leq \lambda \leq 1$ ) with the property that there exists on  $N$  a  $\lambda$ -trace  $\phi$  satisfying

$$M_{\phi \circ E} \vee N = M$$

where  $E$  is the minimal conditional expectation. This condition is equivalent to the requirement that  $\psi = \phi \circ E$  is a  $\lambda$ -trace on  $M$ , i.e.  $M$  and  $N$  have a common discrete decomposition in the sense of [12]. Namely, if  $U \in \mathcal{U}(N)$  is such that  $\sigma_t^\phi(U) = \lambda^t U$ ,  $t \in \mathbb{R}$ , then also  $\sigma_t^\psi(U) = \lambda^t U$ ,  $t \in \mathbb{R}$ . Hence with  $P = M_\psi$ ,  $Q = N_\phi$  and  $\theta = \text{Ad}(U)|_P \in \text{Aut}(P, Q)$  we have

$$(M \supseteq N) \cong (P \supseteq Q) \rtimes_\theta \mathbb{Z}.$$

Note also that by definition we have  $S_0(N) = \{\lambda^n : n \in \mathbb{Z}\}$ .

**3.1. Theorem.** *In the above situation, there is an isomorphism*

$$(6) \quad I : \mathcal{F}^{M \supseteq N} \rightarrow \mathcal{F}^{P \supseteq Q} \otimes \mathcal{F}^N$$

such that

$$(7) \quad I \mathcal{F}_{(\lambda^n, s)}^{M \supseteq N} I^{-1} = \Phi(\theta^n) \otimes \mathcal{F}_s^N, \quad n \in \mathbb{Z}, \quad s \in \mathbb{R}_+$$

$$(8) \quad I \Upsilon(\alpha) I^{-1} = \Phi(\alpha|_P) \otimes \text{mod}(\alpha|_N), \quad \alpha \in \text{Aut}(M, N)$$

where  $\text{mod}$  is the Connes-Takesaki module ([2]) and  $\alpha'$  is a perturbation of  $\alpha$  by a unitary from  $N$  such that, for a  $\mu \in \mathbb{R}_+$  with  $\mathcal{F}_\mu^M = \text{mod}(\alpha)^{-1}$ ,

$$\phi \circ \alpha' = \mu \phi, \quad \alpha'(U) = U.$$

*Proof.* We construct below the isomorphism  $I$  and check that (7) is satisfied. It is left to the reader to check (8), but the proof of [24, 4.6] contains the essential calculations.

In the construction of  $(\mathcal{P}^{M \supset N}, \mathcal{F}^{M \supset N})$  we can use the  $\lambda$ -trace  $\psi$  by Remark 2.2.2. Thus  $\sigma^\psi$  is periodic with period  $t_0 = \frac{-2\pi}{\log \lambda}$  and we obtain an action  $\underline{\sigma}^\psi$  of  $\mathbb{R}/t_0\mathbb{Z}$  on  $M$  by putting

$$\underline{\sigma}_{\eta(t)}^\psi = \sigma_t^\psi, \quad t \in \mathbb{R}$$

where  $\eta: \mathbb{R} \rightarrow \mathbb{R}/t_0\mathbb{Z}$  is the identification map. By [4, 5.4] we obtain an isomorphism

$$J: \tilde{M} \rightarrow M \rtimes_{\underline{\sigma}^\psi} \mathbb{R}/t_0\mathbb{Z} \otimes L^\infty(0, \log \lambda^{-1})$$

given by

$$J(\pi(x)) = \pi_0(x) \otimes 1, \quad x \in M$$

$$J(\lambda(t)) = \lambda_0(\eta(t)) \otimes m(e^{it}), \quad t \in \mathbb{R}$$

where  $\pi: M \rightarrow \tilde{M}$  and  $\pi_0: M \rightarrow M \rtimes_{\underline{\sigma}^\psi} \mathbb{R}/t_0\mathbb{Z}$  are the usual injections,  $\lambda: \mathbb{R} \rightarrow \tilde{M}$  and  $\lambda_0: \mathbb{R}/t_0\mathbb{Z} \rightarrow M \rtimes_{\underline{\sigma}^\psi} \mathbb{R}/t_0\mathbb{Z}$  are the left regular representations, and  $m(e^{it}) \cdot \xi(s) = e^{its} \xi(s)$  for  $t \in \mathbb{R}$ ,  $s \in ]0, \log \lambda^{-1}[$  and  $\xi \in L^2(0, \log \lambda^{-1})$ . Hence, for  $x \in M$ ,  $s, t \in \mathbb{R}$  and  $n \in \mathbb{Z}$ , we find:

$$J(\sigma^\psi)_{s+n \log \lambda}^\wedge J^{-1}(\pi_0(x) \otimes 1) = \pi_0(x) \otimes 1$$

$$\begin{aligned} J(\sigma^\psi)_{s+n \log \lambda}^\wedge J^{-1}(\lambda_0(\eta(t)) \otimes m(e^{it})) &= J(e^{-its} \lambda^{-mt} \lambda(t)) \\ &= \lambda^{-mt} \lambda_0(\eta(t)) \otimes e^{-its} m(e^{it}) \\ &= \lambda^{-mt} \lambda_0(\eta(t)) \otimes \mathcal{F}_s^M(m(e^{it})), \end{aligned}$$

where the last equality comes from the special form of  $(\mathcal{P}^M, \mathcal{F}^M)$ , cf. [2, IV.1]. It follows that

$$J(\sigma^\psi)_{s+n \log \lambda}^\wedge J^{-1} = (\underline{\sigma}^\psi)_n^\wedge \otimes \mathcal{F}_s^M = \hat{\theta}_n \otimes \mathcal{F}_s^M$$

where we identify, as usual, the dual group of  $\mathbb{R}/t_0\mathbb{Z}$  with  $\hat{\mathbb{Z}}$  and  $\underline{\sigma}^\psi$  with  $\hat{\theta}$ . We thus have

$$(\tilde{M}, (\sigma^\psi)_{s+n \log \lambda}^\wedge) \cong ((P \rtimes_{\hat{\theta}} \mathbb{Z} \rtimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Z}}) \otimes \mathcal{P}^M, \hat{\theta}_n \otimes \mathcal{F}_s^M).$$

By Takesaki duality [23],

$$(P \rtimes_{\hat{\theta}} \mathbb{Z} \rtimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Z}}, \hat{\theta}) \cong (P \otimes \mathcal{B}(\ell^2(\mathbb{Z})), \theta \otimes \text{Ad}(\varrho)),$$

where  $\varrho$  is the right regular representation of  $\mathbb{Z}$  on  $B(\ell^2(\mathbb{Z}))$ . Hence

$$(\tilde{M}, (\sigma^\psi)_{s+n \log \lambda}^\wedge) \cong (P \otimes \mathcal{B}(\ell^2(\mathbb{Z})) \otimes \mathcal{P}^M, \theta^n \otimes \text{Ad}(\varrho) \otimes \mathcal{F}_s^M).$$

This argument carries through to the whole tower  $(\tilde{M}_k)_k$  with isomorphisms preserving the inclusions, so we can define  $I$  by the composition of  $J$ 's and Takesaki duality. Then we get (7) by restriction to the relative commutants.

Q.E.D.

**3.2. Corollary (Popa).** *Strongly amenable subfactors of type  $\text{III}_\lambda$  ( $0 \leq \lambda \leq 1$ ) are completely classified by their flow of weights.*

*Proof.* Popa's classification ([19]) says that these subfactors are classified by the standard invariant of the associated type II-inclusion together with the standard part of the automorphism from the discrete decomposition; these are contained as the first tensor component in the right hand side of (6) and (7). (In fact, the invariant is seen to contain the single factor flow and hence the "type" of  $M \cong N$ . cf. Eq. (5).)

Q.E.D.

**3.3. Corollary.** *With notations as above, let  $\alpha$  be a strongly free action of a discrete amenable group on  $M \supseteq N$ . Let*

$$\hat{M} = M \rtimes_\alpha G, \quad \hat{N} = N \rtimes_\alpha G.$$

*Then if  $\hat{M} \supseteq \hat{N}$  is strongly amenable, so is  $M \supseteq N$ . Conversely, if  $G$  is finite and  $M \supseteq N$  is strongly amenable, so is  $\hat{M} \supseteq \hat{N}$ .*

*Proof.* From Proposition 2.6 and the theorem, we have (with the notation from there):

$$\mathcal{P}^{M \supseteq N} \cong \mathcal{P}^{P \supseteq Q} \otimes \mathcal{P}^N$$

and

$$\mathcal{P}^{\hat{M} \supseteq \hat{N}} \cong (\mathcal{P}^{P \supseteq Q})^{\Phi(\alpha|_P)} \otimes (\mathcal{P}^N)^{\text{mod}(\alpha|_N)}.$$

Thus, by definition, strong amenability of  $M \supseteq N$  and  $\hat{M} \supseteq \hat{N}$  means strong amenability of  $\mathcal{P}^{P \supseteq Q}$  and  $(\mathcal{P}^{P \supseteq Q})^{\Phi(\alpha|_P)}$  respectively. By [25, 3.5],  $\alpha|_P$  is strongly outer on  $P \supseteq Q$ . Now the conclusion follows from [26, 6.1].

Q.E.D.

**3.4. Corollary.** *Let  $M \supseteq N$  be as in the theorem, assuming further that  $M \supseteq N$  has finite depth. Let  $\gamma$  denote the canonical endomorphism for  $M \supseteq N$  as in [16]. Then*

$$\text{Ker}(\mathcal{F}^{M \supseteq N}) = \{\lambda^{n_0} : n \in \mathbb{Z}\} \times S_0(N)$$

where  $n_0 = \max\{n \in \mathbb{N} : \sigma_{\frac{2\pi}{-n \log \lambda}}$  is contained in  $\gamma^k$  for some  $k \geq 0\}$ .



*Proof.* From the theorem,  $\mathcal{F}_{(\lambda', t)}^{M \supseteq N} = 1$  if and only if  $\theta_k^n|_{P_k \cap Q'} = 1$  for all  $k$  and  $\mathcal{F}_t^N = 1$ . By Eq. (5), the last condition means  $t \in S_0(N)$ . With  $n_0$  as above, assume  $n_0 > 1$ ; then by [10, Lemma 9] we have  $\theta_k^{n_0}|_{P_k \cap Q'} = 1$  for all  $k$ , but for any  $n < n_0$  there is a  $k \in \mathbb{N}$  such that  $\theta_k^n|_{P_k \cap Q'} \neq 1$ . Also, by definition  $n_0 = 1$  means that no other modular automorphisms than 1 are contained in the powers of  $\gamma$  (in the sector sense), and by [7] this means  $M \supseteq N$  is isomorphic to a type  $\text{II}_1$ -subfactor tensor  $M$ ; however, by [12, Sec. 3, Sec. 6] this happens if and only if  $\Phi(\theta) = 1$ , i.e.  $\theta_k^n|_{P_k \cap Q'} = 1$  for all  $k, n \in \mathbb{N}$ . Q.E.D.

A similar result can be obtained for type  $\text{III}_1$ -subfactors with core inclusion of finite depth, using [10, Lemma 11]. In particular defining

$$(9) \quad S_0(M, N) = \text{Ker}(\mathcal{F}^{M \supseteq N})$$

(which is natural in view of Eq. (5)), we get the following for such type  $\text{III}_1$ -subfactors as well as for subfactors of the kind considered in Corollary 3.4:

$$(10) \quad S_0(M, N) = S_0(N) \times S_0(N) \Leftrightarrow (M \supseteq N) \cong (M \supseteq N)^{\text{st}} \otimes M$$

where  $(M \supseteq N)^{\text{st}}$  is the standard part of the core inclusion. It would be interesting to have more information on  $S_0(M, N)$ : the possible values, further structural interpretations etc.

We now turn to the study of the image

$$\text{Im}(\mathcal{F}^{M \supseteq N}) = \{ \mathcal{F}_{(g,t)}^{M \supseteq N} : (g,t) \in S_0(N) \times \mathbb{R}_+ \}$$

of  $\mathcal{F}^{M \supseteq N}$ .

For a single, injective factor  $M$  of type  $\text{III}_\lambda$  ( $0 < \lambda < 1$ ), we have

$$(11) \quad \text{Im}(\text{mod}) = \text{Im}(\mathcal{F}^M)$$

as subgroups of  $\text{Aut}(\mathcal{P}^M, \mathcal{F}^M)$ . Here, the inclusion  $\subseteq$  is [2, IV .1.3], and in fact injectivity is not needed for this part. The reverse inclusion is also well known; for the convenience of the reader and the proof of Proposition 3.5, we outline the details. Let  $\beta$  be the one-parameter action on the injective factor  $R_{0,1}$  of type  $\text{II}_\infty$  given by  $\tau \circ \beta_s = e^{-s} \tau$  ( $s \in \mathbb{R}$ ) where  $\tau$  is a trace on  $R_{0,1}$ , then the dual weight  $\hat{\tau}$  of  $\tau$  is a  $\lambda$ -trace on  $M$ , and  $M$  can be represented as  $M = R_{0,1} \rtimes_{\beta_{-\log \lambda}} \mathbb{Z}$ . Let  $\bar{\beta}$  be the obvious extension of  $\beta$  to  $M$ . Then as  $\hat{\tau} \circ \bar{\beta}_s = e^{-s} \hat{\tau}$  we have  $\text{mod}(\bar{\beta}_s) = \mathcal{F}_{e^s}^M$  for all  $s \in \mathbb{R}$ . (In fact  $\bar{\beta}_s$  can easily be modified to an action of  $\mathbb{R}/(\mathbb{Z} \log \lambda)$  on  $M$  with these modules.)

In the subfactor-case, we shall show that

$$\text{Im}(\Upsilon) \neq \text{Im}(\mathcal{F}^{M \supseteq N})$$

can occur, though equality does hold for small indices. More precisely, we have:

**3.5. Proposition.** *Let notation be as in the beginning of this section. Further, let  $\mathcal{G} = \Phi(\text{Aut}(A, B))$  for the  $\text{II}_1$ -inclusion associated to the core  $P \supseteq Q$  of  $M \supseteq N$ , and assume that  $M \supseteq N$  (i.e. that  $A \supseteq B$ ) is strongly amenable. Then*

$$\text{Im}(\mathcal{F}^{M \supseteq N}) \subseteq \text{Im}(\Upsilon)$$

holds. Moreover, the following conditions are equivalent:

- (i)  $\text{Im}(\Upsilon) = \text{Im}(\mathcal{F}^{M \supseteq N})$
- (ii)  $\mathcal{G} \cap \{\Phi(\theta)\}'$  is isomorphic to  $\mathbb{Z}$  or  $\mathbb{Z}_n$  for some  $n \in \mathbb{N}$  where  $\Phi(\theta)$  is viewed as an element of  $\mathcal{G}$  in the obvious way.

*Proof.* To prove the first statement, we represent the subfactor as

$$(M \supseteq N) = (A \otimes R_{0,1} \supseteq B \otimes R_{0,1}) \rtimes_{\theta^s \otimes \beta_{-\log \lambda}} \mathbb{Z}$$

where  $\beta$  is as above, cf. [18]. Then observe that  $\theta^s \otimes \beta_s$  can be extended to  $M \supseteq N$  just as in the argument before the proposition, and that this produces all values of  $\Upsilon$  by Theorem 3.1 (cf. below).

Now let  $\alpha \in \text{Aut}(M, N)$  and form  $\alpha'$  as in Theorem 3.1. By (7) and (8) there, we have

$$\Upsilon(\alpha) \equiv \Phi(\alpha'|_p) \otimes \text{mod}(\alpha|_N)$$

and

$$\mathcal{F}_{(\lambda', t)}^{M \supseteq N} \equiv \Phi(\theta^n) \otimes \mathcal{F}_t^N.$$

Hence (i) is equivalent to the condition that, for all  $\alpha \in \text{Aut}(M, N)$ ,

$$(12) \quad \Phi(\alpha'|_p) = \Phi(\theta)^n \text{ for some } n \in \mathbb{N}.$$

Obviously, if (ii) holds, then (12) must hold as well, since by construction  $[\alpha', \theta] = 0$  and hence  $\Phi(\alpha'|_p) \in \mathcal{G} \cap \{\Phi(\theta)\}'$ .

Conversely assume (12) is true. Take any  $\sigma \in \text{Aut}(A, B)$  with  $\Phi(\sigma) \in \mathcal{G} \cap \{\Phi(\theta)\}'$ . Let  $\bar{\sigma}$  be the extension of  $\sigma \otimes 1$  on  $(A \supseteq B) \otimes R_{0,1}$  to  $M \supseteq N$ , where  $P \supseteq Q$  is identified with  $(A \supseteq B) \otimes R_{0,1}$ . Then with the above notation,  $\bar{\sigma}'|_p = \sigma \otimes 1$  since  $[\sigma \otimes 1, \theta] = 0$  and  $\phi \circ \bar{\sigma} = \phi$ . Now, by (12) we get

$$\Phi(\sigma) = \Phi(\bar{\sigma}'|_p) = \Phi(\theta)^n$$

for some  $n \in \mathbb{N}$ .

Q.E.D.

**3.6. Corollary.** *With notations as above, assume  $M \supseteq N$  has minimal index less than 4. Then  $\text{Im}(\Upsilon) = \text{Im}(\mathcal{F}^{M \supseteq N})$ .*

*Proof.* By the classification of the type  $\text{II}_1$ -subfactors, the principal graph of the  $\text{II}_1$ -inclusion associated to  $M \supseteq N$  is one of  $A_n$  ( $n \in \mathbb{N}$ ),  $D_{2n}$  ( $n \geq 2$ ),  $E_6, E_8$

and the corresponding value of  $\mathcal{G}$  is trivial in all these cases except for  $D_{2n}$  where it is  $\mathbb{Z}_2$ . Hence (ii) in the proposition is automatic. Q.E.D.

As we move up in index, the conclusion of the corollary fails. This is illustrated by the following.

**3.7. Examples.**

1. Let  $\sigma$  be the outer action of  $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$  on the hyperfinite  $\text{II}_1$ -factor  $R$ , and put  $B = R$ ,  $A = B \rtimes_{\sigma} G$ . Then  $\mathcal{G}$  for this inclusion is  $\text{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \cong S_3$  by [13], so with  $(M \supseteq N) = (A \supseteq B) \otimes R_{\lambda}$ , where  $R_{\lambda}$  is the Powers factor of type  $\text{III}_{\lambda}$ , we get  $\text{Im}(\mathcal{F}^{M \supseteq N}) \subsetneq \text{Im}(\Upsilon)$  by the proposition. This example is rather trivial because  $\Phi(\theta) = 1$  in the notation above.

2. A less trivial example is obtained e.g. by taking  $G = \mathbb{Z}_2 \oplus \mathbb{Z}_8$  and construct  $A \supseteq B$  as in 1; then  $\mathcal{G} \cong D_4 \oplus \mathbb{Z}_2$  and we let  $\sigma$  be the element generating  $\mathbb{Z}_2$ . Then put  $(M \supseteq N) = ((A \supseteq B) \otimes R_{0,1}) \rtimes_{\sigma \otimes \theta_{\lambda}} \mathbb{Z}$  where  $\theta_{\lambda}$  is given by  $\text{mod}(\theta_{\lambda}) = \lambda$ . Then  $D_4 \leq \mathcal{G} \cap \{\Phi(\theta)\}' = \mathcal{G} \cap \{\sigma\}'$  and hence this group is not singly generated, so that again  $\text{Im}(\mathcal{F}^{M \supseteq N}) \subsetneq \text{Im}(\Upsilon)$ .

3. Let  $n \geq 2$  and let  $B$  be the index 4 subfactor of  $A = R$  with principal graph  $A_n^{(1)}$  as in [3, 4.6.7, 4.7.d]. By [14, 4.4],  $\mathcal{G} = \mathbb{T} \rtimes_{\gamma} \mathbb{Z}_2$  for some  $\mathbb{Z}_2$ -action  $\gamma$  on the unit circle  $\mathbb{T}$ , and taking any nontrivial  $\sigma \in \mathbb{T} \leq \mathcal{G}$ , we have  $\mathbb{T} \leq \mathcal{G} \cap \{\sigma\}'$ , hence the corresponding (index 4) subfactor of type  $\text{III}_{\lambda}$  constructed as in 2 has  $\text{Im}(\mathcal{F}^{M \supseteq N}) \subsetneq \text{Im}(\Upsilon)$ , but is not splitting like the example in 1.

Note that from the list in [14] and the proposition, if  $M \supseteq N$  has index 4, then  $\text{Im}(\mathcal{F}^{M \supseteq N}) = \text{Im}(\Upsilon)$  unless we have trivial splitting as in 1, or the type II-graph is  $A_n^{(1)}$  like in 3.

**§4. A Look at the Remaining Cases**

The only known type I classification result for subfactors of type  $\text{III}_1$  comes from Popa’s general theorem [20] stating that strong amenability, central freeness and approximate innerness for a subfactor imply that it splits into a type  $\text{II}_1$ -subfactor tensor a single factor, so that classification follows from the  $\text{II}_1$ -case and the classification of single hyperfinite factors, and by Proposition 2.4 the resulting classification can also be expressed using the relative flow of weights. From a type III-viewpoint, subfactors splitting in this way are of course not so interesting. In fact, by Eq. (10), the splitting corresponds to “trivial relative flow”.

However, P. Loi showed in [14], [15] (cf. [29]) that there exists a non-splitting inclusion of type  $\text{III}_1$  with core of finite depth and that an uncountable

family of type  $\text{III}_1$ -subfactors with the same type II and type III standard invariants as such an inclusion can be obtained under rather weak assumptions. Namely, let  $M \supseteq N$  be such a subfactor (cf. [14, 5.5] for examples), with continuous decomposition  $(M \supseteq N) = (P \supseteq Q) \rtimes_{\theta} \mathbb{R}$ . Then as  $P \supseteq Q$  carry a common trace (the one satisfying Eq. (2)), it splits as  $(P \supseteq Q) = (A \supseteq B) \otimes R_{0,1}$  where  $A \supseteq B$  is a finite depth subfactor of type  $\text{II}_1$  and  $R_{0,1}$  is the hyperfinite type  $\text{II}_{\infty}$ -factor. Now, assume  $\theta$  splits as an  $\mathbb{R}$ -action in the following way:

$$\theta = \theta^{st} \otimes \theta^{0,1}$$

where  $\theta^{st} \in \text{Aut}(A, B)$  is given by  $\Phi(\theta)$  using the generating property (cf. [18]), and  $\theta^{0,1}$  is the unique tracescaling (as in Eq. (2))  $\mathbb{R}$ -action on  $R_{0,1}$ . Note here that this assumption holds for all Loi's examples and that it is conjectured to hold in general (cf. below). Then, for each  $\varepsilon > 0$ , define an action  $\theta^{\varepsilon}$  of  $\mathbb{R}$  on  $P \supseteq Q$  by

$$\theta_r^{\varepsilon} = \theta_r^{st} \otimes \theta_r^{0,1}, \quad r \in \mathbb{R}$$

and put

$$(M^{\varepsilon} \supseteq N^{\varepsilon}) = (P \supseteq Q) \rtimes_{\theta^{\varepsilon}} \mathbb{R}.$$

Clearly  $\text{Ker}(\Phi(\theta^{\varepsilon})) = \varepsilon^{-1} \cdot \text{Ker}(\Phi(\theta))$ . Assume further that  $\text{Ker}(\Phi(\theta)) \neq 0$ ; this is an extra condition on  $A \supseteq B$  to which we return in the next paragraph. Then as Loi's invariant maps cocycle conjugate actions on  $A \supseteq B$  to conjugate actions on  $\{A_k \cap B'\}$ , we conclude that

$$(M^{\varepsilon} \supseteq N^{\varepsilon}) \not\cong (M^{\varepsilon'} \supseteq N^{\varepsilon'}), \quad \varepsilon \neq \varepsilon'.$$

Now, as  $(\sigma^{\psi_k})_s^{\wedge}$  is easily seen to be the canonical extension of  $(\sigma^{\psi_k})_s^{\wedge}$  on the tower  $\tilde{N} \subseteq \tilde{M} \subseteq \tilde{M}_1 \subseteq \dots \subseteq \tilde{M}_k$  for all  $s \in \mathbb{R}$ , we can identify  $(\sigma^{\psi_k})^{\wedge}$  with  $(\theta_s)_k \otimes \text{Ad}(\varrho_s)$  via the Takesaki duality  $\tilde{M}_k \cong P_k \otimes \mathcal{B}(L^2(\mathbb{R}))$ , where  $\varrho$  is the right regular representation. Hence in the above situation,

$$(13) \quad (\mathcal{P}^{M \supseteq N}, \mathcal{F}_{(s,t)}^{M \supseteq N}) \cong (\{P_k \cap Q' \supseteq P_k \cap P'\}_{k=0}^{\infty}, \Phi(\theta_{-\log st}))$$

so the uncountable family  $(M^{\varepsilon} \supseteq N^{\varepsilon})_{\varepsilon > 0}$  is separated by (the kernel of) the relative flow of weights. In this context, Eq. (10) says that all  $M^{\varepsilon} \supseteq N^{\varepsilon}$  are non-splitting.

After the completion of this paper, it was pointed out to us by H. Kosaki that  $\text{Ker}(\Phi(\theta)) = 0$  does occur (cf. [10]) but that, even in this case, a finer invariant separates the inclusions  $M^{\varepsilon} \supseteq N^{\varepsilon}$  (cf. [29]). Also a construction similar to Loi's above then works without assuming finite depth of the core (or splitting of  $\theta$ ). The point here is that this new invariant, which is a relative version of Connes' T-set, is also defined in terms of the relative flow of weights, so this flow also separates this larger family of examples.

Finally, the area of subfactors of type  $\text{III}_0$  is completely different in flavor because the core inclusions are not factorial, so that ergodic theory appears in a

relative form. This was used by Kosaki [9] and Kosaki-Sano [11] to construct uncountably many examples of non-splitting finite depth inclusions with the same principal graphs and (integer) index. The uncountable number of such inclusions follows from the wealth of so-called finite-to-one ergodic extensions of ergodic flows, and the resulting subfactors are shown to be non-isomorphic by proving that these extensions are isomorphic to the inclusion data

$$(14) \quad ((1 - e_1 \vee \dots \vee e_k) \tilde{M}_k \cap \tilde{N}', (\sigma^{\psi_k})^\wedge)_{(1 - e_1 \vee \dots \vee e_k) \tilde{M}_k \cap \tilde{N}'}$$

(for some finite  $k$ ) as an extension of the single factor flow  $(\mathcal{Z}(\tilde{M}), (\sigma^\psi)^\wedge)_{\mathcal{Z}(\tilde{M})}$ . Moreover, due to the actual construction of  $M \supseteq N$ , (14) is obtained from  $(\tilde{M}_k \cap \tilde{N}', (\sigma^{\psi_k})^\wedge)_{\tilde{M}_k \cap \tilde{N}'}$  by deleting the “trivial ergodic component”. Hence again, the examples are separated by the relative flow of weights.

To obtain a classification for subfactors of type  $\text{III}_0$ , we first have to define a concept of strong amenability; this is done by demanding strong amenability of the (factorial) component inclusions in the central disintegration of  $\tilde{M} \supseteq \tilde{N}$ , cf. [28]. With this definition, strong amenability is defined for all subfactors of type  $\text{III}$  in a consistent way. We then pose the following

**4.1. Conjecture.** *Strongly amenable subfactors of type  $\text{III}_0$  and type  $\text{III}_1$  are completely classified by their relative flow of weights.*

We have been able to prove this in the type  $\text{III}_0$ - case, in particular we re-obtain the results due to Kosaki and Sano that were discussed above. The techniques used in the proof of that theorem are quite different from what we considered here, and will be presented in [28] together with other applications.

In type  $\text{III}_1$ -case, the above conjecture is well known and widely believed. Namely, by Remark 2.2.5 we see that the problem essentially is to show that trace-scaling one parameter actions in a strongly amenable type  $\text{II}_\infty$ -subfactor are classified by the Loi invariant. This problem is highly non-trivial even in the single factor case, where it asks for uniqueness, and is solved indirectly by Haagerups proof of the uniqueness of the hyperfinite type  $\text{III}_1$ -factor. Haagerups techniques were also crucial in [20], but it seems likely that they cannot be used directly beyond the finite depth case.

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