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The Chern Character of the Symmetric Space *EI*

Dedicated to Professor Seiya Sasao on his 60th birthday

By

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Abstract

The purpose of this paper is to describe the Chern character homomorphism of the compact symmetric space *EI*.

§1. Introduction

Let E_6 be the compact, 1-connected, exceptional Lie group of rank 6 and $PSp(4) = Sp(4)/\{\pm I_4\}$ the coset space of the symplectic group of degree 4 by its center (where I_n denotes the unit matrix of degree n). There is an involutive automorphism $\rho : E_6 \to E_6$ whose fixed point set E_6^ρ is $PSp(4)$ (see [9]). So the coset space $E_6/PSp(4)$ is a compact, 1-connected, irreducible symmetric space which is denoted by *EI* in É. Cartan's notation. The cohomology and *K*-theory of $EI = E_6/PSp(4)$ are known (see [3], [4] and [6]). In this paper we describe the Chern character homomorphism of *EI*.

According to Ishitoya [3], the cohomology of *EI* is as follows. Its rational cohomology ring is given by

$$H^*(EI;\mathbf{Q}) = \mathbf{Q}[e_8]/(e_8^3) \otimes \Lambda_{\mathbf{Q}}(e_9, e_{17}),$$

where $e_i \in H^i(EI;\mathbf{Q})$. Note that $\dim EI = 42$. The integral cohomology of *EI* has only 2-torsion, and

$$H^*(EI;\mathbf{Z})/\mathrm{Tors}.H^*(EI;\mathbf{Z})$$
$$= \mathbf{Z}\{1, e_8, e_9, e'_{16}, e_{17}, e'_{17}, e_9e'_{16}, e'_{25}, e_9e_{17}, e'_{16}e_{17}, e'_{34}, e_9e'_{16}e_{17}\}.$$

where the relations $4e_8^2 = e'_{16}$, $2e_8e_9 = e'_{17}$, $2e_8e_{17} = e'_{25}$ and $4e_8e_9e_{17} = e'_{34}$ hold.

According to Minami [6], the *K*-theory of *EI* is as follows. The complex representation ring of E_6 is given by

$$R(E_6) = \mathbf{Z}[\varphi_1, \varphi_2, \lambda^2\varphi_1, \lambda^3\varphi_1, \lambda^2\varphi_6, \varphi_6], \tag{1.1}$$

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where λ^k denotes the k-th exterior power operation, $\dim\varphi_1 = \dim\varphi_6 = 27$, $\dim\varphi_2 = 78$ and the relation $\lambda^3\varphi_1 = \lambda^3\varphi_6$ holds. Let $\chi_1 = [(\mathbf{H}^4)_{\mathbf{C}}] \in R(Sp(4))$. Then

$$R(Sp(4)) = \mathbb{Z}[\chi_1, \lambda^2\chi_1, \lambda^3\chi_1, \lambda^4\chi_1], \tag{1.2}$$

where $\dim\lambda^k\chi_1 = \binom{8}{k}$. As a subring of $R(Sp(4))$,

$$R(PSp(4)) = \mathbb{Z}[\lambda^2\chi_1, \lambda^4\chi_1, \chi_1^2, (\lambda^3\chi_1)^2, \chi_1\lambda^2\chi_1]. \tag{1.3}$$

The element $\chi_1^2 - 64$ belongs to the augmentation ideal $I(PSp(4))$. Let $\alpha(\tilde{\chi}_1^2) \in \tilde{K}^0(EI)$ denote the image of $\chi_1^2 - 64$ under the composite

$$R(PSp(4)) \xrightarrow{\alpha} K^0(BPSp(4)) \xrightarrow{j_6} K^0(EI),$$

where α is the λ-ring homomorphism of [1], and $j_6 : EI \to BPSp(4)$ is the map induced from the inclusion $i_6 : PSp(4) \to E_6$. Let $(I(E_6))$ be the ideal in $R(PSp(4))$ generated by the image of $i_6 : I(E_6) \to I(PSp(4))$. Then the above composite factors to give

$$R(PSp(4))/(I(E_6)) \to K^0(EI),$$

where by [6, II, Theorem 5.3],

$$R(PSp(4))/(I(E_6)) = \mathbb{Z}[\chi_1^2]/((\chi_1^2 - 64)^3). \tag{1.4}$$

The homomorphism $\rho^* : R(E_6) \to R(E_6)$ satisfies

$$\rho^*(\varphi_1) = \varphi_6,\ \rho^*(\varphi_2) = \varphi_2,\ \rho^*(\lambda^2\varphi_1) = \lambda^2\varphi_6 \text{ and } \rho^*(\lambda^3\varphi_1) = \lambda^3\varphi_1. \tag{1.5}$$

Let U be the infinite unitary group and $\iota_n : U(n) \to U$ the canonical injection. Since $E_6^\rho = PSp(4)$, if $\rho^*(\lambda) = \mu$ and $\dim\lambda = n$, there is a map $f_\lambda : EI \to U(n)$ defined by

$$f_\lambda(xPSp(4)) = \lambda(x)\mu(x)^{-1} \text{ for } xPSp(4) \in EI.$$

Denote by $\beta(\lambda - \mu)$ the homotopy class of the composite $\iota_n f_\lambda$. Thus we have $\beta(\varphi_1 - \varphi_6)$, $\beta(\lambda^2\varphi_1 - \lambda^2\varphi_6) \in [EI, U] = \tilde{K}^{-1}(EI)$. Elements $\beta(\varphi_1)$, $\beta(\lambda^2\varphi_1) \in \tilde{K}^{-1}(E_6)$ are defined in a similar manner. By [6, I, Proposition 7.3], the K-theory of EI is torsion-free and

$$K^*(EI) = K^*(pt) \otimes \mathbb{Z}[\alpha(\tilde{\chi}_1^2)]/(\alpha(\tilde{\chi}_1^2)^3) \otimes \Lambda_{\mathbf{Z}}(\beta(\varphi_1 - \varphi_6), \beta(\lambda^2\varphi_1 - \lambda^2\varphi_6)).$$

With the above notation, our main result is

Theorem 1. *The Chern character* $ch : \tilde{K}^*(EI) \to \tilde{H}^*(EI; \mathbf{Q})$ *is given by*

$$ch(\alpha(\tilde{\chi}_1^2)) = e_8 + \frac{1}{60}e'_{16},$$

$$ch(\beta(\varphi_1 - \varphi_6)) = \frac{1}{2}e_9 + \frac{1}{480}e_{17},$$

$$ch(\beta(\lambda^2\varphi_1 - \lambda^2\varphi_6)) = \frac{11}{2}e_9 - \frac{229}{480}e_{17}.$$

§2. Root Systems

According to [9], there is an involutive outer automorphism $\tau : E_6 \to E_6$ whose fixed point set E_6^τ is the compact exceptional group F_4 of rank 4, and there is an inner automorphism γ of E_6 such that $\rho = \gamma\tau = \tau\gamma$ (where our notation is different from [9]). Since $\rho\tau = \tau\rho$, it follows that

$$(E_6^\rho)^\tau = (E_6^\tau)^\rho = E_6^\rho \cap E_6^\tau.$$

It is known to be $\tilde{S}^3 \cdot Sp(3) = (S^3 \times Sp(3))/\mathbf{Z}_2$, where $\mathbf{Z}_2 = \{(1, I_3), (-1, -I_3)\}$. We denote it by D. So $D = F_4 \cap PSp(4)$. Let $C = T^1 \cdot Sp(3)$ be as in [4]. Then $C \subset D$. If $\tilde{T}'$ is a maximal torus of $Sp(4)$, then $T' = \tilde{T}'/\{\pm I_4\}$ is a maximal torus of C, D, F_4 and $PSp(4)$. Choose a maximal torus T of E_6 so that $T' \subset T$. Thus we have an inclusion $i_1 : C \to D$ and a diagram of inclusions

$$\begin{array}{ccc} D & \xrightarrow{i_2} & F_4 \\ i_4 \downarrow & & \downarrow i_8 \\ PSp(4) & \xrightarrow[i_6]{} & E_6. \end{array}$$

We also have inclusions $i_3 = i_2 i_1 : C \to F_4$ and $i_{10} = i_6 i_4 = i_8 i_2 : D \to E_6$ etc.

For details of the following argument, see [4, §2]. F_4 has a system of simple roots $\{\alpha_i \mid i = 1, 2, 3, 4\}$. The corresponding fundamental weights ω_i are given by

$$\begin{aligned} \omega_1 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \\ \omega_2 &= 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4, \\ \omega_3 &= 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4, \\ \omega_4 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4. \end{aligned}$$

D has a system of simple roots $\{\beta_i \mid i = 0, 1, 2, 3\}$. The corresponding fundamental weights ϕ_i are given by

$$\begin{aligned} \phi_0 &= \frac{1}{2}\beta_0. \\ \phi_1 &= \beta_1 + \beta_2 + \frac{1}{2}\beta_3, \\ \phi_2 &= \beta_1 + 2\beta_2 + \beta_3, \\ \phi_3 &= \beta_1 + 2\beta_2 + \frac{3}{2}\beta_3. \end{aligned}$$

$PSp(4)$ has a system of simple roots $\{\gamma_i \mid i=1,2,3,4\}$. The corresponding fundamental weights χ_i are given by

$$\begin{aligned}
\chi_1 &= \gamma_1 + \gamma_2 + \gamma_3 + \frac{1}{2}\gamma_4,\\
\chi_2 &= \gamma_1 + 2\gamma_2 + 2\gamma_3 + \gamma_4,\\
\chi_3 &= \gamma_1 + 2\gamma_2 + 3\gamma_3 + \frac{3}{2}\gamma_4,\\
\chi_4 &= \gamma_1 + 2\gamma_2 + 3\gamma_3 + 2\gamma_4.
\end{aligned}$$

E_6 has a system of simple roots $\{\delta_j \mid j=1,2,3,4,5,6\}$. The corresponding fundamental weights φ_j are given by

$$\begin{aligned}
\varphi_1 &= \frac{4}{3}\delta_1 + \delta_2 + \frac{5}{3}\delta_3 + 2\delta_4 + \frac{4}{3}\delta_5 + \frac{2}{3}\delta_6,\\
\varphi_2 &= \delta_1 + 2\delta_2 + 2\delta_3 + 3\delta_4 + 2\delta_5 + \delta_6,\\
\varphi_3 &= \frac{5}{3}\delta_1 + 2\delta_2 + \frac{10}{3}\delta_3 + 4\delta_4 + \frac{8}{3}\delta_5 + \frac{4}{3}\delta_6,\\
\varphi_4 &= 2\delta_1 + 3\delta_2 + 4\delta_3 + 6\delta_4 + 4\delta_5 + 2\delta_6,\\
\varphi_5 &= \frac{4}{3}\delta_1 + 2\delta_2 + \frac{8}{3}\delta_3 + 4\delta_4 + \frac{10}{3}\delta_5 + \frac{5}{3}\delta_6,\\
\varphi_6 &= \frac{2}{3}\delta_1 + \delta_2 + \frac{4}{3}\delta_3 + 2\delta_4 + \frac{5}{3}\delta_5 + \frac{4}{3}\delta_6.
\end{aligned}$$

These $\{\alpha_i\}, \{\beta_i\}, \{\gamma_i\}$ can be regarded as bases for $H^2(BT';\mathbf{Q})$ and $\{\delta_j\}$ a basis for $H^2(BT;\mathbf{Q})$. We may suppose that $i_2(T') \subset T', i_4(T') \subset T', i_6(T') \subset T$ and $i_8(T') \subset T$. The theory of Lie algebras for symmetric spaces [5] tells us the following facts. The dominant root with respect to the root system of F_4 is $\tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. As to $Bi_2 : BT' \to BT'$, we have

$$Bi_2(\alpha_2) = \beta_3,\ Bi_2(\alpha_3) = \beta_2,$$
$$Bi_2(\alpha_4) = \beta_1,\ Bi_2(-\tilde{\alpha}) = \beta_0$$

and so

$$Bi_2(\alpha_1) = -\frac{1}{2}\beta_0 - \beta_1 - 2\beta_2 - \frac{3}{2}\beta_3.$$

The dominant root with respect to the root system of $PSp(4)$ is $\tilde{\gamma} = 2\gamma_1 + 2\gamma_2 + 2\gamma_3 + \gamma_4$. As to $Bi_4 : BT' \to BT'$, we have

$$Bi_4(\gamma_2) = \beta_1,\ Bi_4'(\gamma_3) = \beta_2,$$
$$Bi_4(\gamma_4) = \beta_3,\ Bi_4(-\tilde{\gamma}) = \beta_0$$

and so

$$Bi_4'(\gamma_1) = -\frac{1}{2}\beta_0 - \beta_1 - \beta_2 - \frac{1}{2}\beta_3.$$

As to $Bi_8 : BT' \to BT$, we have

$$Bi_8(\delta_1) = Bi_8(\delta_6) = \alpha_4, \qquad Bi_8(\delta_2) = \alpha_1,$$
$$Bi_8(\delta_3) = Bi_8(\delta_5) = \alpha_3, \qquad Bi_8(\delta_4) = \alpha_2.$$

Using these equalities and $i_6 i_4 = i_8 i_2$, as to $Bi_6 : BT' \to BT$, we conclude that

$$Bi_6(\delta_1) = Bi_6(\delta_6) = \gamma_2,$$
$$Bi_6(\delta_3) = Bi_6(\delta_5) = \gamma_3,$$
$$Bi_6(\delta_4) = \gamma_4$$

and so

$$Bi_6(\delta_2) = \gamma_1 - \gamma_3 - \gamma_4.$$

From the above, it follows that

$$(2.1) \qquad \begin{aligned} Bi_2(\omega_1) &= -2\phi_0, & Bi_4(\chi_1) &= -\phi_0, \\ Bi_2(\omega_2) &= \phi_3 - 3\phi_0, & Bi_4(\chi_2) &= \phi_1 - \phi_0, \\ Bi_2(\omega_3) &= \phi_2 - 2\phi_0, & Bi_4(\chi_3) &= \phi_2 - \phi_0, \\ Bi_2(\omega_4) &= \phi_1 - \phi_0, & Bi_4(\chi_4) &= \phi_3 - \phi_0. \end{aligned}$$

and

$$Bi_8(\varphi_1) = Bi_8(\varphi_6) = \omega_4, \quad Bi_8(\varphi_2) = \omega_1,$$
$$Bi_8(\varphi_3) = Bi_8(\varphi_5) = \omega_3, \quad Bi_8(\varphi_4) = \omega_2.$$

Furthermore,

$$(2.2) \qquad \begin{aligned} Bi_6(\varphi_1) &= Bi_6(\varphi_6) = \chi_2, & Bi_6(\varphi_2) &= 2\chi_1, \\ Bi_6(\varphi_3) &= Bi_6(\varphi_5) = \chi_1 + \chi_3, & Bi_6(\varphi_4) &= 2\chi_1 + \chi_4. \end{aligned}$$

This result is restated in terms of representations. In fact, φ_j of (1.1) is defined as the irreducible representation with highest weight φ_j, and χ_1 of (1.2) is just the irreducible representation with highest weight χ_1. Using (2.2), we see that $i_6 : R(E_6) \to R(PSp(4))$ satisfies

$$i_6(\varphi_1) = \lambda^2\chi_1 - 1 \text{ and } i_6(\varphi_2) = \lambda^4\chi_1 + \chi_1^2 - 2\lambda^2\chi_1.$$

Then (1.4) follows from this, (1.1) and (1.3). (For details, see [6, II, §5].)

By [4, p. 234],

$$H\ (BT';\mathbb{Z}) = \mathbb{Z}[\omega_1, \omega_2, \omega_3, \omega_4] = \mathbb{Z}[t, y_1, y_2, y_3]$$

where $\omega_i \in H^2(BT';\mathbb{Z})$ and

$$t = \omega_1,$$
$$y_1 = \omega_2 - \omega_3,$$
$$y_2 = \omega_3 - \omega_4,$$
$$y_3 = \omega_4.$$

On the other hand,

$$H^{*}(B\tilde{T}';\mathbb{Z}) = \mathbb{Z}[\chi_1, \chi_2, \chi_3, \chi_4] = \mathbb{Z}[t_1', t_2', t_3', t_4']$$

where $\chi_i \in H^2(B\tilde{T}';\mathbb{Z})$ and

$$t_1' = \chi_1,$$
$$t_i' = -\chi_{i-1} + \chi_i \text{ for } i = 2,3,4.$$

(Note that $\{\pm t_i' \mid i = 1,2,3,4\}$ is the set of weights of χ_1.) For i = 1, 2, 3, 4 let

$$p_i = \sigma_i(t_1'^2, t_2'^2, t_3'^2, t_4'^2),$$

the i-th elementary symmetric function in the indicated variables. As is well known, the map $B\tilde{T}' \to BSp(4)$ coming from the inclusion $\tilde{T}' \to Sp(4)$ induces the following isomorphism

$$H^{*}(BSp(4);\mathbb{Z}) \cong H^{*}(B\tilde{T}';\mathbb{Z})^{W(Sp(4))} = \mathbb{Z}[p_1, p_2, p_3, p_4]$$

where the middle notation stands for the subalgebra of $H^{*}(B\tilde{T}';\mathbb{Z})$ consisting of invariants under the action of the Weyl group $W(Sp(4))$. We may identify $H^2(BT';\mathbb{Q})$ with $H^2(B\tilde{T}';\mathbb{Q})$. Since $W(PSp(4)) \cong W(Sp(4))$, we have

(2.3) $$H^{*}(BT';\mathbb{Q})^{W(PSp(4))} = \mathbb{Q}[p_1, p_2, p_3, p_4].$$

Next, by [7, p. 266],

$$H^{*}(BT;\mathbb{Z}) = \mathbb{Z}[\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6]$$
$$= \mathbb{Z}[t_1, \cdots, t_6, x]/(t_1 + \cdots + t_6 - 3x)$$

where $\varphi_j \in H^2(BT;\mathbb{Z})$ and

$$t_6 = \varphi_6,$$
$$t_5 = \varphi_5 - \varphi_6,$$
$$t_4 = \varphi_4 - \varphi_5,$$
$$t_3 = \varphi_2 + \varphi_3 - \varphi_4,$$
$$t_2 = \varphi_1 + \varphi_2 - \varphi_3,$$
$$t_1 = -\varphi_1 + \varphi_2,$$
$$x = \varphi_2.$$

If we put

$$x_j = 2t_j - x \text{ for } j = 1, 2, \cdots, 6,$$

the set

$$S = \{x_i + x_j, x - x_i, -x - x_i \mid 1 \le i < j \le 6\}$$

is invariant under the action of $W(E_6)$ (see [7, §4(B) and §5(B)]). For $n \ge 1$ let

$$I_n = \sum_{y \in S} y^n \in H^{2n}(BT;\mathbf{Q})^{W(E_6)}$$

By [7, Lemma 5.2],

$$H^*(BT;\mathbf{Q})^{W(E_6)} = \mathbf{Q}[I_2, I_5, I_6, I_8, I_9, I_{12}].$$

Consider the set

$$S' = \{\pm t_i' \pm t_j' \mid 1 \le i < j \le 4\}.$$

Then, by (2.2),

$$\{Bi_6^*(y) \mid y \in S\} = \{2y' \mid y' \in S'\}.$$

For $n \ge 1$ let

$$I_n' = \sum_{y' \in S'} y'^n \in H^{2n}(BT';\mathbf{Q}).$$

By [7, §5(A)], each I_n' is written as a polynomial of the p_i, and the ideal generated by I_n''s is given by

$$(2.4) \qquad \begin{aligned}(I_n' \mid n \ge 1) &= (I_2', I_6', I_8', I_{12}') \\ &= (p_1, p_3, 12p_4 + p_2^2, p_2^3).\end{aligned}$$

Let $(H^+(BT;\mathbf{Q})^{W(E_6)})$ be the ideal in $H^*(BT';\mathbf{Q})^{W(PSp(4))}$ generated by the image of

$$Bi_6 : H^+(BT;\mathbf{Q})^{W(E_6)} \to H^+(BT';\mathbf{Q})^{W(PSp(4))},$$

where $H^+(X;\mathbf{Q}) = \sum_{q>0} H^q(X;\mathbf{Q})$. By (2.3) and (2.4),

$$(2.5) \qquad \begin{aligned}&H^*(BT';\mathbf{Q})^{W(PSp(4))}/(H^+(BT;\mathbf{Q})^{W(E_6)}) \\ &\quad = \mathbf{Q}[p_1, p_2, p_3, p_4]/(p_1, p_3, 12p_4 + p_2^2, p_2^3) \\ &\quad = \mathbf{Q}[p_2]/(p_2^3).\end{aligned}$$

§3. Some Observation in Cohomology

Let $i : T \to E_6$ and $i' : T' \to PSp(4)$ be the inclusions respectively. The following commutative diagram

$$\begin{array}{ccccccc}
R(E_6) & \xrightarrow{\alpha} & K^0(BE_6) & \xrightarrow{ch} & H^*(BE_6;\mathbf{Q}) & \xrightarrow[\cong]{Bi} & H^*(BT;\mathbf{Q})^{W(E_6)} \\
{\scriptstyle i_6}\downarrow & & \downarrow{\scriptstyle Bi_6} & & \downarrow{\scriptstyle Bi_6^*} & & \downarrow{\scriptstyle Bi_6} \\
R(PSp(4)) & \xrightarrow{\alpha} & K^0(BPSp(4)) & \xrightarrow{ch} & H^*(BPSp(4);\mathbf{Q}) & \xrightarrow[\cong]{Bi'} & H^*(BT';\mathbf{Q})^{W(PSp(4))} \\
 & & \downarrow{\scriptstyle j_6^*} & & \downarrow{\scriptstyle j_6^*} & & \\
 & & K^0(EI) & \xrightarrow{ch} & H^*(EI;\mathbf{Q}) & &
\end{array}$$

yields a commutative square

$$(3.1)\quad \begin{array}{ccc} R(PSp(4))/(I(E_6)) & \xrightarrow{ch\alpha} & H^{\cdot}(BT';\mathbf{Q})^{W(PSp(4))}/(H^{+}(BT;\mathbf{Q})^{W(E_6)}) \\ j_6^{\cdot}\alpha\downarrow & & \downarrow j_6^{\cdot} \\ K^0(EI) & \xrightarrow{ch} & H^{*}(EI;\mathbf{Q}). \end{array}$$

Note that the vertical homomorphisms are injective. We need to describe the image of the right vertical homomorphism $j_6^{\cdot}$ in terms of the generator $e_8 \in H^8(EI;\mathbf{Z})$.

By (2.1) and definitions in §2, the following relations hold in $H^2(BT';\mathbf{Q})$:

$$t_1' = \frac{1}{2}t \text{ and } t_i' = \frac{1}{2}t + y_{5-i} \text{ for } i = 2, 3, 4.$$

As in [4, p. 235], put

$$z_i = (t - y_i)y_i \text{ for } i = 1, 2, 3$$

and let $q_i = \sigma_i(z_1, z_2, z_3)$. Then, by [4, p. 236],

$$H^{\cdot}(BC;\mathbf{Z}) \cong H^{\cdot}(BT';\mathbf{Z})^{W(C)} = \mathbf{Z}[t, q_1, q_2, q_3].$$

By definition,

$$\begin{aligned}\sum_{i=0}^{4} p_i &= \prod_{i=1}^{4}(1 + t_i'^2) \\ &= (1 + \frac{1}{4}t^2)\prod_{i=1}^{3}(1 + \frac{1}{4}t^2 - ty_i + y_i^2) \\ &= z\prod_{i=1}^{3}(z - z_i) \qquad \text{where } z = \frac{1}{4}t^2 + 1 \\ &= z(z^3 - q_1z^2 + q_2z - q_3).\end{aligned}$$

Therefore

$$(3.2)\qquad p_1 = -q_1 + t^2 \textit{ and } p_2 = q_2 - \frac{3}{4}q_1t^2 + \frac{3}{8}t^4 \textit{ in } H^{\cdot}(BT';\mathbf{Q}).$$

It is known that

$$H\ (E_6/F_4;\mathbf{Z}) = \Lambda_{\mathbf{Z}}(e_9, e_{17})$$

where $e_i \in H^i(E_6/F_4;\mathbf{Z})$. As in [3, §3],

$$\begin{aligned}H\ (F_4/D;\mathbf{Z}) = \mathbf{Z}[\chi, f_4, f_8, f_{12}]/(2\chi, f_4\chi, \chi^3, \\ f_4^3 - 12f_4f_8 + 8f_{12}, f_4f_{12} - 3f_8^2, f_8^3 - f_{12}^2)\end{aligned}$$

where $\chi \in H^3(F_4/D;\mathbf{Z})$ and $f_i \in H^i(F_4/D;\mathbf{Z})$. By [4, Theorem 4.4],

$$H\ (F_4/C;\mathbf{Z}) = \mathbf{Z}[t, u, v, w]/(t^3 - 2u, u^2 - 3t^2v + 2w, 3v^2 - t^2w, v^3 - w^2)$$

where $t \in H^2(F_4/C;\mathbf{Z})$, $u \in H^6(F_4/C;\mathbf{Z})$, $v \in H^8(F_4/C;\mathbf{Z})$, $w \in H^{12}(F_4/C;\mathbf{Z})$ and then $j_3 : H^{\cdot}(BC;\mathbf{Z}) \to H\ (F_4/C;\mathbf{Z})$ satisfies

$$(3.3)\qquad j_3^{\cdot}(t) = t,\ j_3(q_1) = t^2,\ j_3(q_2) = 3v \text{ and } j_3(q_3) = w.$$

Moreover, by [3, Proposition 1],

$$H^{*}(E_6/D;\mathbf{Z}) \cong H^{*}(F_4/D;\mathbf{Z}) \otimes H^{*}(E_6/F_4;\mathbf{Z}).$$

Consider the commutative diagram

$$\begin{array}{ccccccc} F_4/C & \xrightarrow{\pi_1'} & F_4/D & \xrightarrow{i_8'} & E_6/D & \xrightarrow{\pi_4'} & EI \\ j_3 \downarrow & & & & & & \downarrow j_6 \\ BC & & & \xrightarrow{Bi_5} & & & BPSp(4). \end{array}$$

Then

$$\pi_1'^{*} i_8'^{*} \pi_4'^{*}(e_8) = \pi_1'^{*}(-8f_8 + f_4^2) = -8v + t^4 \in H^8(F_4/C;\mathbf{Z})$$

(see [3, §3]). Therefore

$$\begin{aligned} \pi_1'^{*} i_8'^{*} \pi_4'^{*}(p_2) &= j_3^{*} Bi_5^{*}(p_2) \\ &= j_3^{*}(q_2 - \frac{3}{4} q_1 t^2 + \frac{3}{8} t^4) && \text{by (3.2)} \\ &= 3v - \frac{3}{8} t^4 && \text{by (3.3)} \\ &= -\frac{3}{8}(-8v + t^4) \end{aligned}$$

in $H^8(F_4/C;\mathbf{Q})$. Thus we have

Lemma 2. *$j_6^{*} : H^{*}(BT';\mathbf{Q})^{W(PSp(4))}/(H^{+}(BT;\mathbf{Q})^{W(E_6)}) \to H^{*}(EI;\mathbf{Q})$ of (3.1) is given by*

$$j_6^{*}(p_2) = -\frac{3}{8} e_8.$$

§4. Proof of Theorem 1

There exist elements $x_j \in H^j(E_6;\mathbf{Z})$ for j = 3, 9, 11, 15, 17, 23 such that $\langle x_3 x_9 x_{11} x_{15} x_{17} x_{23}, [E_6] \rangle = 1$ up to sign, where $[E_6]$ is the fundamental homology class, and

$$H^{*}(E_6;\mathbf{Q}) = \Lambda_{\mathbf{Q}}(x_3, x_9, x_{11}, x_{15}, x_{17}, x_{23}),$$

where each $x_j \in H^j(E_6;\mathbf{Q})$ is primitive.

Lemma 3. *Let $\pi_6 : E_6 \to EI$ be the projection. Then $\pi_6^{*} : H^{*}(EI;\mathbf{Z}) \to H^{*}(E_6;\mathbf{Z})$ satisfies*

$$\pi_6^{*}(e_9) = 2x_9 \text{ and } \pi_6^{*}(e_{17}) = 2x_{17}.$$

Proof. Let p be a prime and consider the Serre spectral sequence for the mod p cohomology of the fibration

$$E_6 \xrightarrow{\pi_6} EI \xrightarrow{j_6} BSp(4).$$

If $p \geq 5$,

$$H^*(E_6;\mathbf{Z}/(p)) = \Lambda_{\mathbf{Z}/(p)}(x_3, x_9, x_{11}, x_{15}, x_{17}, x_{23}),$$

where $x_i \in H^i(E_6;\mathbf{Z}/(p))$, and if $p = 3$,

$$H^*(E_6;\mathbf{Z}/(3)) = \mathbf{Z}/(3)[x_8]/(x_8^3) \otimes \Lambda_{\mathbf{Z}/(3)}(x_3, x_7, x_9, x_{11}, x_{15}, x_{17}),$$

where $x_i \in H^i(E_6;\mathbf{Z}/(3))$. If $p \geq 3$,

$$H^*(BPSp(4);\mathbf{Z}/(p)) = \mathbf{Z}/(p)[y_4, y_8, y_{12}, y_{16}],$$

where $y_i \in H^i(BPSp(4);\mathbf{Z}/(p))$. If $p \geq 3$,

$$H^*(EI;\mathbf{Z}/(p)) = \mathbf{Z}/(p)[e_8]/(e_8^3) \otimes \Lambda_{\mathbf{Z}/(p)}(e_9, e_{17}),$$

where $e_i \in H^i(EI;\mathbf{Z}/(p))$. By a routine spectral sequence argument, we see that if $p \geq 3$, $\pi_6 : H^*(EI;\mathbf{Z}/(p)) \to H^*(E_6;\mathbf{Z}/(p))$ satisfies

$$\pi_6(e_9) = x_9 \text{ and } \pi_6(e_{17}) = x_{17}.$$

Let $\pi_4' : E_6/D \to EI$ be the projection. In view of [3, §5], $\pi_4'^* : H^*(EI;\mathbf{Z}) \to H^*(E_6/D;\mathbf{Z})$ satisfies

$$\pi_4'^*(e_9) = 2e_9 \text{ and } \pi_4'^*(e_{17}) = 2e_{17}.$$

Consider the Serre spectral sequence for the mod 2 cohomology of the fibration

$$E_6 \xrightarrow{\pi_{10}} E_6/D \xrightarrow{j_{10}} BD.$$

If we denote by $\Delta_{\mathbf{Z}/(2)}$ a graded ring over $\mathbf{Z}/(2)$ with a simple system of generators,

$$H^*(E_6;\mathbf{Z}/(2)) = \Delta_{\mathbf{Z}/(2)}(x_3, x_5, x_6, x_9, x_{15}, x_{17}, x_{23}),$$

where $x_i \in H^i(E_6;\mathbf{Z}/(2))$. By [4, Corollary 4.8],

$$H^*(BD;\mathbf{Z}/(2)) = \mathbf{Z}/(2)[u_2, u_3, u_4, u_8, u_{12}],$$

where $u_i \in H^i(BD;\mathbf{Z}/(2))$. By [3, §3],

$$\begin{aligned} &H^*(E_6/D;\mathbf{Z}/(2)) \\ &\quad = \mathbf{Z}/(2)[e_2, e_3, e_8, e_{12}]/(e_2^3 + e_3^2, e_2^2e_3, e_8^2 + e_2^2e_{12}, e_{12}^2 + e_2^2e_8e_{12}) \\ &\qquad \otimes \Lambda_{\mathbf{Z}/(2)}(e_9, e_{17}), \end{aligned}$$

where $e_i \in H^i(E_6/D;\mathbf{Z}/(2))$. Similarly we see that $\pi_{10} : H^*(E_6/D;\mathbf{Z}/(2)) \to H^*(E_6;\mathbf{Z}/(2))$ satisfies

$$\pi_{10}^*(e_9) = x_9 \text{ and } \pi_{10}^*(e_{17}) = x_{17}.$$

Since $\pi_6 = \pi_4'\pi_{10}$, the lemma follows.

Let us compute the image of χ_1^2 under the composite

$$R(PSp(4)) \xrightarrow{i'^!} R(T') \xrightarrow{\alpha} K^0(BT') \xrightarrow{ch} H^{\cdot}(BT';\mathbf{Q}).$$

Since χ_1 has weights $\pm t_i', i = 1,2,3,4$ (see §2), χ_1^2 has weights

$$\pm 2t_i', \pm t_i' \pm t_j', \pm t_i' \pm t_j', \overbrace{0,\dots,0}^{8}$$

where $1 \le i < j \le 4$. For $n \ge 1$ let

$$s_n = \sum_{i=1}^{4} t_i'^n \in H^{2n}(BT';\mathbf{Z}).$$

Then

$$ch\alpha i'^!(\chi_1^2) = \sum_{n\ge 0}\sum_{i=1}^{4} \frac{(2t_i')^n + (-2t_i')^n}{n!} + \sum_{n\ge 0} \frac{2I_n'}{n!} + 8.$$

Denoting by ch^q the $2q$-dimensional component, we have

$$ch^{2n}\alpha i'^!(\chi_1^2) = \frac{2}{(2n)!}(I_{2n}' + 2^{2n}s_{2n}).$$

Now we use the expressions of I_{2n}' and s_{2n} in terms of the p_i given in [7, p. 271]. For $n = 2$ we have

$$ch^4\alpha i'^!(\chi_1^2) = \frac{1}{12}(12p_1^2 + 16(-2p_2 + p_1^2)) \equiv -\frac{8}{3}p_2 \mod(p_1).$$

By this, (1.4) and (2.5), the upper horizontal homomorphism $ch\alpha$ of (3.1) is given by

$$ch\alpha(\chi_1^2) = 64 - \frac{8}{3}p_2.$$

Combining this, Lemma 2 and (3.1), we find that the coefficient of e_8 in $ch(\alpha(\chi_1^2))$ is 1. Similarly for $n = 4$ we have

$$\begin{aligned} ch^8\alpha i'^!(\chi_1^2) &\equiv \frac{1}{20160}(80(12p_4 + p_2^2) + 256(-4p_4 + 2p_2^2)) \mod(p_1, p_3) \\ &\equiv \frac{1}{1260}(-4p_4 + 37p_2^2) \mod(p_1, p_3) \\ &\equiv \frac{4}{135}p_2^2 \mod(p_1, p_3, 12p_4 + p_2^2). \end{aligned}$$

Combining this, Lemma 2 and (3.1), we find that the coefficient of e_8^2 in $ch(\alpha(\tilde{\chi}_1^2))$ is 1/240. Since $4e_{16}' = e_8^2$, that of e_{16}' in $ch(\alpha(\tilde{\chi}_1^2))$ is 1/60. Thus we obtain the first equality of Theorem 1.

Let us define a map $\xi_p : EI \to E_6$ by

$$\xi_\rho(xPSp(4)) = x\rho(x)^{-1} \text{ for } xPSp(4) \in EI.$$

By the definition of $\beta(\lambda - \mu)$ and (1.5), it is easy to see that

$$\xi_\rho^{'}(\beta(\varphi_1)) = \beta(\varphi_1 - \varphi_6) \text{ and } \xi_\rho^{'}(\beta(\lambda^2\varphi_1)) = \beta(\lambda^2\varphi_1 - \lambda^2\varphi_6).$$

By [2, §2], if $x \in H^J(E_6;\mathbf{Q})$ is primitive, then

$$\pi_6\xi_\rho^{'}(x) = x - \rho\ (x)$$

and

$$\rho\ (x_j) = \begin{cases} x_j & \text{for } j = 3, 11, 15, 23 \\ -x_j & \text{for } j = 9, 17 \end{cases}$$

in $H^J(E_6;\mathbf{Z})$ (see also [8, (1.5)]. Therefore, by Lemma 3,

$$\xi_\rho(x_j) = \begin{cases} 0 & \text{for } j = 3, 11, 15, 23 \\ e_j & \text{for } j = 9, 17 \end{cases}$$

in $H^J(EI;\mathbf{Z})$. We quote from [8, Theorem 1] that

$$ch(\beta(\varphi_1)) = 6x_3 + \frac{1}{2}x_9 + \frac{1}{20}x_{11} + \frac{1}{168}x_{15} + \frac{1}{480}x_{17} + \frac{1}{443520}x_{23},$$

$$ch(\beta(\lambda^2\varphi_1)) = 150x_3 + \frac{11}{2}x_9 - \frac{1}{4}x_{11} - \frac{101}{168}x_{15} - \frac{229}{480}x_{17} - \frac{2021}{443520}x_{23}.$$

Then, by applying ξ_ρ to these equalities, the second and third equalities of Theorem 1 follow. This completes the proof.

References

[1] Atiyah, M. F. and Hirzebruch, F., Vector bundles and homogeneous spaces, In: *Proc. Sympos. Pure Math.*, **III**, 7–38, (Amer. Math. Soc., Providence, 1961).

[2] Harris, B., Suspensions and characteristic maps for symmetric spaces, *Ann. of Math.*, **76** (1962), 295–305.

[3] Ishitoya, K., Cohomology of the symmetric space *EI*, *Proc. Japan Acad.*, **53**, *Ser.* **A** (1977), 56–60.

[4] Ishitoya, K. and Toda, H., On the cohomology of irreducible symmetric spaces of exceptional type, *J. Math. Kyoto Univ.*, **17** (1977), 225–243.

[5] Loos, O., *Symmetric Spaces, I and II,* (Math. Lecture Note Ser., W. A. Benjamin, New York-London, 1969).

[6] Minami, H., *K*-groups of symmetric spaces I, II, *Osaka J. Math.*, **12** (1975), 623–634, **13** (1976), 271–287.

[7] Toda, H. and Watanabe, T., The integral cohomology rings of F_4/T and E_6/T, *J. Math. Kyoto Univ.*, **14** (1974), 257–286.

[8] Watanabe, T., The Chern character homomorphism of the compact simply connected exceptional group E_6, *Osaka J. Math.*, **28** (1991), 663–681.

[9] Yokota, I., Realizations of involutive automorphisms σ and G^σ of exceptional linear Lie groups G, Part I, $G = G_2, F_4$ and E_6, *Tsukuba J. Math.*, **14** (1990), 185–223.