

Estimation of the Number of the Critical Values at Infinity of a Polynomial Function $f: \mathbf{C}^2 \rightarrow \mathbf{C}$

By

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§1. Introduction

Let $f(x, y)$ be a polynomial of degree d and we consider the polynomial function $f: \mathbf{C}^2 \rightarrow \mathbf{C}$. Let $\Sigma(f)$ be the critical values. The restriction

$$f: \mathbf{C}^2 - f^{-1}(\Sigma) \rightarrow \mathbf{C} - \Sigma$$

is not necessarily a locally trivial fibration. In general, we have to exclude a finite values $\Sigma_\infty \subset \mathbf{C}$ from the base space so that $f: \mathbf{C}^2 - f^{-1}(\Sigma \cup \Sigma_\infty) \rightarrow \mathbf{C} - (\Sigma \cup \Sigma_\infty)$ is a locally trivial fibration. We say that $\tau \in \mathbf{C}$ is a *regular value at infinity* of the function $f: \mathbf{C}^2 \rightarrow \mathbf{C}$ if there exist positive numbers R and ε so that the restriction of f , $f: f^{-1}(D_\varepsilon(\tau)) - B_R^+ \rightarrow D_\varepsilon(\tau)$, is a trivial fibration over the disc $D_\varepsilon(\tau)$ where $D_\varepsilon(\tau) = \{\eta \in \mathbf{C}; |\eta - \tau| \leq \varepsilon\}$ and $B_R^+ = \{(x, y); |x|^2 = |y|^2 \leq R\}$. Otherwise τ is called a *critical value at infinity* or an *atypical value*. We denote the set of the critical values at infinity by Σ_∞ . It is known that Σ_∞ is finite ([V], [H1]). This fact also results from Theorem (1.4). The purpose of this note is to give an estimation on the number of critical values at infinity.

We consider the canonical projective compactification $\mathbf{C}^2 \subset \mathbf{P}^2$. We denote the homogeneous coordinates of \mathbf{P}^2 by X, Y, Z so that $x = X/Z$ and $y = Y/Z$. Let L_∞ be the line at infinity: $L_\infty = \{Z = 0\}$. Write

$$f(x, y) = f_0 + f_1(x, y) + \cdots + f_d(x, y)$$

where $f_i(x, y)$ is a homogeneous polynomial of degree i for $i = 0, \dots, d$. We can write

$$(1.1) \quad f_d(x, y) = cx^{v_0} y^{v_{k+1}} \prod_{j=1}^k (y - \lambda_j x)^{v_j}$$

Communicated by K. Saito, July 12, 1993.

1991 Mathematics Subject Classification(s): 32S05, 32S15.

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where $c \in \mathbb{C}^*$ and $\lambda_1, \dots, \lambda_k$ are non-zero distinct complex numbers and we assume that $v_i > 0$ for $1 \leq i \leq k$ and $v_0, v_{k+1} \geq 0$. Note that we have the equality

$$(1.2) \quad v_0 + \dots + v_{k+1} = d.$$

Let C_τ be the projective curve which is the closure of the fiber $f^{-1}(\tau)$. Then C_τ is defined by $C_\tau = \{(X; Y; Z) \in \mathbb{P}^2; F(X, Y, Z) - \tau Z^d = 0\}$ where $F(X, Y, Z)$ is the homogeneous polynomial defined by

$$F(X, Y, Z) = f(X/Z, Y/Z)Z^d = f_0 Z^d + f_1(X, Y)Z^{d-1} + \dots + f_d(X, Y).$$

The intersection of C_τ and the line at infinity, $C_\tau \cap L_\infty$, is independent of $\tau \in \mathbb{C}^2$ and it is the base point locus of the family $\{C_\tau; \tau \in \mathbb{C}\}$. Obviously we have $C_\tau \cap L_\infty = \{Z = f_d(X, Y) = 0\}$. For brevity, let $A_i = (\alpha_i; \beta_i; 0) \in \mathbb{P}^2$ for $i = 0, \dots, k+1$ where $A_0 = (0; 1; 0), A_{k+1} = (1; 0; 0)$ and $\beta_i / \alpha_i = \lambda_i$ for $1 \leq i \leq k$. Then under the assumption (1.1), $C_0 \cap L_\infty = \{A_i; v_i > 0\}$. Note that $A_i \in C_0 \cap L_\infty$ for $i = 1, \dots, k$. We consider the family of germs of a curve at $A_j : \{(C_\tau, A_j); \tau \in \mathbb{C}\}$. Then it is known that τ is a regular value at infinity if and only if $\{(C_t, A_j); t \in \mathbb{C}\}$ is a topologically stable family near $t = \tau$ for any A_j with $v_j > 0$ ([H1]). This is the case if $f(x, y) - \tau$ is reduced and the local Milnor number μ of the family $\{(C_t, A_j); t \in \mathbb{C}\}$ is constant in a neighborhood U of $\tau \in \mathbb{C}$. Note that the regularity at infinity of a value τ for f has nothing to do with the regularity (as a variety) of the curve C_τ at infinity.

Definition (1.3). Let $v'_i = \max\{v_i, -1, 0\}$ and put $v_\infty^{pr}(f) = \sum_{i=0}^{k+1} v'_i$. We call $v_\infty^{pr}(f)$ the projective degeneracy at infinity.

Then we have the following estimation.

Theorem (1.4). The number of critical points at infinity $|\Sigma_\infty|$ is less than or equal to $v_\infty^{pr}(f)$. In particular, $|\Sigma_\infty| \leq d - 1$.

A precise description of the cardinality $|\Sigma_\infty|$ is given in Theorem (2.6.1). Note that C_0 intersects transversely with the line at infinity if and only if $v_i \leq 1$ for any $i = 0, \dots, k+1$. Thus we get the following well-known corollary.

Corollary (1.4.1). Assume that C_0 intersects transversely with the line at infinity L_∞ , i.e. $v_\infty^{pr}(f) = 0$. Then f has no critical value at infinity.

There are many papers which are related with this topics. See for instance [B, H1, H2, H3, L1, O2, O3, O5, V]. The projective degeneracy $v_\infty^{pr}(f)$ does not depend on the choice of a linear coordinate system (x, y) . On the other hand, by this reason, when the support of $f_d(x, y)$ is small, this estimation is not so sharp. We will sharpen this estimation in §4. In fact, we introduce the toric degeneracy

$v_\infty^{tot}(f)$ in a similar way in §4. For the definition, we use not only $f_d(x, y)$ but also the information from other outside faces. This number $v_\infty^{tot}(f)$ is in general smaller than $v_\infty^{pr}(f)$ (Proposition (4.18)). The estimation in Theorem (1.4) can be replaced by the toric degeneracy $v_\infty^{tot}(f)$ (Main Theorem (4.17)). For the proof we use the affine polar invariant and the toric compactification method.

§2. Affine Polar Quotients

Let $\ell(x, y) = \alpha y - \beta x$ be a linear form. The polar curve $\Gamma_\ell(f)$ for f with respect to ℓ is defined by the Jacobian $\Gamma_\ell(f) = \{(x, y) \in \mathbb{C}^2; J(f, \ell)(x, y) = 0\}$ where

$$J(f, \ell)(x, y) = \alpha \frac{\partial f}{\partial x}(x, y) + \beta \frac{\partial f}{\partial y}(x, y) = 0.$$

$\Gamma_\ell(f)$ is an affine curve of degree $d - 1$ and equal to the critical locus of the mapping $(f, \ell): \mathbb{C}^2 \rightarrow \mathbb{C}^2$. Let L_η be the projective line $\{\alpha Y - \beta X - \eta Z = 0\}$ which is the closure of the affine line $\ell^{-1}(\eta)$. The base point of this pencil $\{L_\eta; \eta \in \mathbb{C}\}$ is $B = (\alpha; \beta; 0)$ in the homogeneous coordinates. We say that ℓ is *generic at infinity* for the polynomial f if $B \notin C_0 \cap L_\infty$. This is the case if and only if $f_d(\alpha, \beta) \neq 0$. We assume the genericity of ℓ hereafter.

Let $\overline{\Gamma_\ell(f)}$ be the projective closure of $\Gamma_\ell(f)$ and let $\overline{\Gamma_\ell(f)} \cap L_\infty = \{Q_1, \dots, Q_\delta\}$. Let γ be a local analytic irreducible component of $\overline{\Gamma_\ell(f)}$ at Q_i . Consider an analytic parametrization $\Phi_\gamma: (D_\epsilon(0), 0) \rightarrow (\gamma, Q_i)$ in a local coordinate system in a neighborhood of Q_i . In the original affine coordinates, this can be written as $\Phi_\gamma(t) = (x_\gamma(t), y_\gamma(t))$ where $x_\gamma(t)$ and $y_\gamma(t)$ are Laurent series in t . Consider the rational number $v_\gamma(f, \tau)$ defined by

$$v_\gamma(f, \tau) = \frac{\text{val}_t(f(x_\gamma(t), y_\gamma(t)) - \tau)}{\text{val}_t(\ell(x_\gamma(t), y_\gamma(t)))}.$$

Here val_t is the standard valuation defined by the variable t . It is easy to see that this number depends only on τ , γ and f and it does not depend on the choice of the parametrization. So we call this number *the affine polar quotient of the fiber $f^{-1}(\tau)$ along γ* ([L1],[N-L]). In the case of $f(x_\gamma(t), y_\gamma(t)) - \tau \equiv 0$, the valuation $\text{val}_t(f(x_\gamma(t), y_\gamma(t)) - \tau)$ is $+\infty$ by definition. Let p be a positive integer. We use the convention $+\infty/\pm p = \pm\infty$ and $-\infty$ (resp. $+\infty$) is negative (resp. positive). This is an analogy of the local polar quotient defined in [L-M-W].

Take $Q_i \in \overline{\Gamma_\ell(f)} \cap L_\infty$ and assume that $Q_i \neq B$. Let γ be a local analytic irreducible component at Q_i and let $(x_\gamma(t), y_\gamma(t))$ be a parametrization of γ at Q_i . We consider their Laurent series:

$$\begin{pmatrix} x_\gamma(t) \\ y_\gamma(t) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} t^{-p} + (\text{terms of higher degree}), \begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

As $(x_\gamma(t), y_\gamma(t)) \rightarrow Q_i$ in \mathbb{P}^2 and $\|(x_\gamma(t), y_\gamma(t))\| \rightarrow \infty$ when $t \rightarrow 0$, we must have $p > 0$.

Lemma (2.1). *Under the above situation, we have $\text{val}_i(\ell(x_\gamma(t), y_\gamma(t))) = -p < 0$ and $\text{val}_i(f(x_\gamma(t), y_\gamma(t)) - \tau) \geq -pd$. The equality holds if and only if $Q_i \notin C_0 \cap L_\infty$. If this is the case, $v_\gamma(f, \tau) = d > 0$.*

Proof. Write $(x_\gamma(t), y_\gamma(t))$ as above. This implies that $(a; b; 0) = Q_i$ for some i . It is easy to see that

$$\ell(x_\gamma(t), y_\gamma(t)) = (\alpha b - \beta a)t^{-p} + (\text{higher terms})$$

$$f(x_\gamma(t), y_\gamma(t)) = f_d(a, b)t^{-pd} + (\text{higher terms})$$

and the assumption $B \neq Q_i$ implies $\alpha b - \beta a \neq 0$. Thus $\text{val}_i(\ell(x_\gamma(t), y_\gamma(t))) = -p < 0$ and $f_d(a, b) \neq 0$ if we assume further that $Q_i \notin C_0 \cap L_\infty$. Thus we have $v_\gamma(f, \tau) = (-pd)/(-p) = d$. \square

Lemma (2.2). *Assume that ℓ is generic at infinity. Then the base point B of the pencil $\{L_\eta, \eta \in \mathbb{C}\}$ is not contained in $\overline{\Gamma_i(f)} \cap L_\infty$ and $\overline{\Gamma_i(f)} \cap C_0 \cap L_\infty = \{A_i; v_i \geq 2\}$.*

Proof. Recall that $\Gamma_i(f) = \{(x, y) \in \mathbb{C}^2; \alpha \frac{\partial f}{\partial x}(x, y) + \beta \frac{\partial f}{\partial y}(x, y) = 0\}$. By the genericity of ℓ and the Euler equality, $\alpha \frac{\partial f_d}{\partial x}(\alpha, \beta) + \beta \frac{\partial f_d}{\partial y}(\alpha, \beta) = df_d(\alpha, \beta) \neq 0$. This implies that $\alpha \frac{\partial f_d}{\partial x}(x, y) + \beta \frac{\partial f_d}{\partial y}(x, y)$ is a non-zero homogeneous polynomial of degree $d - 1$ and

$$\{Q_1, \dots, Q_m\} = \{(X; Y; 0) \in L_\infty; \alpha \frac{\partial f_d}{\partial x}(X, Y) + \beta \frac{\partial f_d}{\partial y}(X, Y) = 0\}$$

and therefore $B \neq Q_1, \dots, Q_m$. This proves the first assertion. Let $A_i = (\alpha_i; \beta_i; 0) \in C_0 \cap L_\infty$ as in §1 and assume that $A_i \in \overline{\Gamma_i(f)} \cap L_\infty$ for some i with $v_i > 0$. Then we have

$$(2.2.1) \quad \alpha \frac{\partial f_d}{\partial x}(\alpha_i, \beta_i) + \beta \frac{\partial f_d}{\partial y}(\alpha_i, \beta_i) = 0, \quad f_d(\alpha_i, \beta_i) = 0.$$

By the Euler equation for $f_d(\alpha_i, \beta_i) = 0$ implies

$$(2.2.2) \quad \alpha_i \frac{\partial f_d}{\partial x}(\alpha_i, \beta_i) + \beta_i \frac{\partial f_d}{\partial y}(\alpha_i, \beta_i) = 0.$$

Combining (2.2.1) and (2.2.2), we obtain

$$f_d(\alpha_i, \beta_i) = \frac{\partial f_d}{\partial x}(\alpha_i, \beta_i) = \frac{\partial f_d}{\partial y}(\alpha_i, \beta_i) = 0.$$

This is the case if and only if $v_i \geq 2$. \square

Thus by Lemma (2.1) and Lemma (2.2), if the affine polar quotients $v_\gamma(f, \tau)$ for an irreducible component γ at Q_j is not positive, we must have $Q_j \in \{A_i; v_i \geq 2\}$.

We generalize the notion of a regular value at infinity. Let $A_i \in C_0 \cap L_\infty$ (so $v_i > 0$) and let $\tau \in \mathbb{C}$. We say that τ is a regular value at A_i for $f: \mathbb{C}^2 \rightarrow \mathbb{C}$ if there exists an open neighborhood U of A_i in \mathbb{P}^2 and a positive number ε such that $f: U \cap f^{-1}(D_\varepsilon(\tau)) \rightarrow D_\varepsilon(\tau)$ is a trivial fibration. Here $f^{-1}(D_\varepsilon(\tau)) \subset \mathbb{C}^2$ and therefore $U \cap f^{-1}(D_\varepsilon(\tau)) \subset U - L_\infty$. Now the importance of the affine polar quotients is the following lemma:

Lemma (2.3). *Assume that ℓ is generic. Then $\tau \in \mathbb{C}$ is a regular value at A_i for $f: \mathbb{C}^2 \rightarrow \mathbb{C}$ if either (i) $v_i = 1$ or (ii) $v_i \geq 2$ and the affine polar quotient $v_\gamma(f, \tau) \geq 0$ for any local irreducible component γ of $\Gamma_i(f)$ at A_i .*

Proof. Assume first that $v_i = 1$. In this case, C_i meets transversely with the line at infinity L_∞ for any $t \in \mathbb{C}$ and there is no polar curve near A_i . Thus the family of curves (C_t, A_i) are topologically stable and the assertion is immediate. Assume now that $v_i \geq 2$ and let $(x_\gamma(t), y_\gamma(t))$ be a parametrization of component γ of $\overline{\Gamma_i(f)}$ at A_i . Then the assumption $v_\gamma(f, \tau) \geq 0$ and Lemma (2.1) implies that $\text{val}_i(f(x_\gamma(t), y_\gamma(t)) - \tau) \leq 0$. Thus either $|f(x_\gamma(t), y_\gamma(t))|$ goes to infinity or $\lim_{t \rightarrow 0} f(x_\gamma(t), y_\gamma(t))$ exists and $\lim_{t \rightarrow 0} f(x_\gamma(t), y_\gamma(t)) \neq \tau$. We can choose an open neighborhood U and a positive number $\varepsilon > 0$ so that $U \cap f^{-1}(D_\varepsilon(\tau)) \cap \Gamma_i(f) = \emptyset$. Then we can choose a large enough positive number R_i so that for any $\eta \in \mathbb{C}$ with $|\eta| \geq R_i$, $L_\eta \cap f^{-1}(D_\varepsilon(\tau))$ is compact and its boundary is $L_\eta \cap f^{-1}(\partial D_\varepsilon(\tau))$. This results from the elementary fact that the restriction $f|_{L_\eta - \{B\}}$ is an open mapping.

At this point we need the genericity of ℓ . We define a holomorphic vector field $\chi = X_1 \frac{\partial}{\partial x} + X_2 \frac{\partial}{\partial y}$ on $f^{-1}(D_\varepsilon(\tau)) \cap U$ such that

$$(1) \chi(f) = X_1 \frac{\partial f}{\partial x} + X_2 \frac{\partial f}{\partial y} = 1 \text{ and } (2) \chi(\ell) = 0.$$

The condition (2) says that any integral curve of χ preserves each line L_η in U . So any integral curve does not approach to A_i and it can be extended as long as its image by f is still in the interior of $D_\varepsilon(\tau)$. Thus the trivialization follows from the standard argument using the integrals of the vector field χ . \square

Corollary (2.3.1)([N-L]). *Assume that ℓ is generic. Then $\tau \in \mathbb{C}$ is a regular value at infinity for the function $f: \mathbb{C}^2 \rightarrow \mathbb{C}$ if (and only if) the affine polar quotient satisfies $v_\gamma(f, \tau) \geq 0$, for any local irreducible component γ of $\overline{\Gamma}_i(f)$ at A_i with $v_i \geq 2$.*

We do not use the only if part in this paper. For other characterizations of the regularity at infinity, see [H1,2,3]. Now we are ready to prove Theorem (1.4). Let $A_i \in C_0 \cap L_\infty$ be as in §1.

Proposition (2.4). *The number of the local irreducible components of the polar curve $\overline{\Gamma}_i(f)$ at A_i for a generic ℓ is less than or equal to $v_i - 1$.*

Proof. Let (u, v) be a local coordinate centered at A_i so that $v_i = 0$ is the defining equation of L_∞ . In the case that $1 \leq i \leq k + 1$, we can take $u = Y/X - \lambda_i$ and $v = Z/X$. In the case of $i = 0$ (so $v_0 \geq 2$), we take $u = X/Y, v = Z/Y$. In any case, the local defining function of $\overline{\Gamma}_i(f)$ say $h(u, v)$ satisfies that $h(u, 0) = c_i u^{v_i - 1} +$ (higher terms) with $c_i \neq 0$. Let $\gamma_1, \dots, \gamma_{\delta_i}$ be the local irreducible components of $\overline{\Gamma}_i(f)$ at A_i and let $h_j(u, v)$ be the local defining function of the component γ_j . Then we must have that $h_j(0, 0) = 0$ and $h(u, v) = e(u, v) \prod_{j=1}^{\delta_i} h_j(u, v)$ with e a suitable unit. This is enough to conclude that $\delta_i \leq v_i - 1$. \square

The estimation of the number of the local irreducible components by Proposition (2.4) is usually very rough. Note also that the local irreducible components of $\overline{\Gamma}_i(f)$ near A_i is not necessarily unstable in the sense of (2.6). Now the proof of Theorem (1.4) will be completed by Lemma (2.3) and the following lemma.

Lemma (2.5). *Assume that ℓ is generic. Choose A_i with $v_i \geq 2$ and let $(x_\gamma(t), y_\gamma(t))$ be a parametrization of local irreducible component γ of $\overline{\Gamma}_i(f)$ at A_i .*

- (i) *If $v_\gamma(f; 0) > 0$, then $v_\gamma(f; \tau) > 0$ for any $\tau \in \mathbb{C}$.*
- (ii) *If $v_\gamma(f; 0) \leq 0$, there exists a unique $\xi \in \mathbb{C}$ so that $v_\gamma(f; \xi) < 0$. For any other $\tau \neq \xi, v_\gamma(f; \tau) = 0$.*

Proof. We consider the Laurent expansion

$$(2.5.1) \quad \begin{pmatrix} x_\gamma(t) \\ y_\gamma(t) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} t^{-p} + (\text{terms of higher degree}), \quad \begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Assume first that $v_\gamma(f; 0) > 0$. Then $\text{val}_t(f(x_\gamma(t), y_\gamma(t))) < 0$ by Lemma (2.1) and therefore $\text{val}_t(f(x_\gamma(t), y_\gamma(t)) - \tau) < 0$ for any τ .

Assume that $v_\gamma(f;0) \leq 0$. This implies that $\text{val}_t(f(x_\gamma(t), y_\gamma(t))) > 0$. Then $\lim_{t \rightarrow 0} f(x_\gamma(t), y_\gamma(t))$ is well defined. So we denote this limit by $\xi(\gamma)$. Then it is obvious that $\text{val}_t(f(x_\gamma(t), y_\gamma(t)) - \tau) = 0$ for any $\tau \neq \xi(\gamma)$. This completes the proof. \square

Definition (2.6). We call that a local irreducible component γ of $\overline{\Gamma}_t(f)$ at A_t is *stable* (respectively *unstable*) if $v_\gamma(f;0) > 0$ (resp. $v_\gamma(f;0) \leq 0$). By Lemma (2.1), γ is stable if and only if $\lim_{t \rightarrow 0} |f(x_\gamma(t), y_\gamma(t))| = \infty$ under the above notation. We denote the set of unstable local irreducible components of $\overline{\Gamma}_t(f)$ at infinity by $\mathcal{U}(\Gamma_t)$. Assume that γ is a unstable local irreducible component and let $\xi(\gamma)$ be the complex number characterized in (ii). We consider $\xi(\gamma)$ as a mapping $\xi: \mathcal{U}(\Gamma_t) \rightarrow \mathbb{C}$, $\gamma \mapsto \xi(\gamma)$. $\xi(\gamma)$ is called *the limit critical value of f along γ* .

Actually in the above proof, we have proved the following.

Theorem (2.6.1). *The number of the critical values at infinity $|\Sigma_\infty|$ is equal to the cardinality of the image $\xi(\mathcal{U}(\Gamma_t))$. In particular, it is less than or equal to the cardinality of $\mathcal{U}(\Gamma_t)$.*

This gives a precise description of the set of critical values at infinity. Therefore if we have enough informations about the local irreducible components at infinity, we can get a better estimation using Theorem (2.6.1). In fact, we use this in §4 to get a better estimation (Theorem (4.17)).

§3. Toric Compactification of \mathbb{C}^2

Let $f(x, y) = \sum_{(m,n)} a_{m,n} x^m y^n$ be a given polynomial of degree d and let

$$(3.1) \quad f_d(x, y) = cx^s y^r \prod_{j=1}^k (y - \lambda_j x)^{v_j}$$

be as in §1 where $\lambda_1, \dots, \lambda_k$ are mutually distinct non-zero numbers. Let $A_0 = (0; 1; 0)$ and $A_{k+1} = (1; 0; 0)$ as in §1. The estimation by Theorem (1.4) is not so sharp when r or s is greater than 1. So we would like to sharpen this estimation using the toric embedding method. In §2, we used the projective compactification to discuss the stability of the family $\{C_t, \tau \in \mathbb{C}\}$ at the infinity. It turns out that a suitable toric compactification is more convenient for this purpose. As we are interested in the estimation of the number of critical values at infinity, we may assume that $f(0,0) \neq 0$ by adding a constant if necessary. We consider the Newton polygon $\Delta(f)$ of f which is the convex hull of the integral point (m,n) such that $a_{m,n} \neq 0$. By the assumption $f(0,0) \neq 0$, we have $O \in \Delta(f)$. Let N be the space of covectors. Any covector P defines a linear function on $\Delta(f)$. For any

integral covector $P = \prime(p, q)$, let $\Delta(P; f) \subset \Delta(f)$ be the locus where the linear function $P|\Delta(f)$ takes the minimal value. We denote this minimal value by $d(P; f)$ as usual. Let $f_p(x, y)$ be the partial sum

$$f_p(x, y) := \sum_{(m,n) \in \Delta(P,f)} a_{m,n} x^m y^n$$

and we call f_p the face function of the covector P . The dual Newton diagram $\Gamma^*(f)$ is defined by the following equivalence relation in $N : P \sim Q$ if and only if $\Delta(P; f) = \Delta(Q; f)$. Here $\Delta(P; f)$ is the locus where linear function $P|\Delta(f)$ takes its minimal value. Let Σ^* be a regular simplicial cone subdivision of $\Gamma(f)$ and let X be the toric variety associated with Σ^* . Let $E_1 = \prime(1, 0)$, $E_2 = \prime(0, 1)$. It is easy to see that $\sigma_1 := \text{Cone}(E_1, E_2)$ is admissible with Σ . Thus we may assume that σ_1 is a simplicial cone in Σ^* . This implies in particular that X is a smooth compactification of the original affine space $\mathbb{C}^2 = \mathbb{C}_{\sigma_1}^2$. Let R_1, \dots, R_μ be the vertices of Σ^* in the counter-clockwise orientation where $R_1 = E_1$, $R_2 = E_2$. Thus $\sigma_i := \text{Cone}(R_i, R_{i+1})$, $i = 1, \dots, \mu$ be the two-dimensional simplicial cones in Σ^* where $R_{\mu+1} = R_1$. Here we assume $R_1 = E_1$, $R_2 = E_2$, $R_{\mu+1} = R_1$. Let $\sigma_1 = \text{Cone}(E_1, E_2)$. Recall that X is a smooth compact toric variety of dimension 2 whose affine charts are $\mathbb{C}_{\sigma_i}^2$, $i = 1, \dots, \mu$ and it has the canonical decomposition

$$X = \mathbb{C}^2 \coprod_{i=1}^{\mu} \hat{E}(R_i) = \mathbb{C}^2 \coprod_{i=3}^{\mu} \hat{E}(R_i)$$

where $\hat{E}(R_i)$ is a rational curve corresponding to each vertex $R_i \in \text{Vertex}(\Sigma^*)$. The divisor $\hat{E}(R_i)$ intersects with (and only with) $\hat{E}(R_{i-1})$ and $\hat{E}(R_{i+1})$. So the dual graph of the divisors $\hat{E}(R_i)$, $i = 1, \dots, \mu$ makes a cycle. Taking a subdivision if necessary, we may assume that $H := \prime(-1, -1) \in \text{Vertex}(\Sigma^*)$. Thus we assume that $H = R_\theta$ for some $3 \leq \theta \leq \mu$. The projective compactification corresponds to the smallest simplicial cone Σ_0^* which has three vertices $\{E_1, E_2, H\}$. There is a canonical morphism $\Psi : X \rightarrow \mathbb{P}^2$ so that

$$\Psi(\hat{E}(R_i)) = \begin{cases} (1; 0; 0) & 3 \leq i < \theta \\ L_\infty & i = \theta \\ (0; 1; 0) & \theta < i \leq \mu. \end{cases}$$

Let (u_i, v_i) be the corresponding coordinates of the chart $\mathbb{C}_{\sigma_i}^2$. Let us consider the unimodular matrix σ'_i corresponding to the vertices of the cone σ_i :

$$\sigma'_i = \begin{pmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{pmatrix}, \quad a_i b_{i+1} - a_{i+1} b_i = 1.$$

Then the original affine space is identified with the coordinate space $\mathbb{C}_{\sigma_1}^2$ with $x = u_1$, $y = v_1$. Recall that $\mathbb{C}_{\sigma_i}^2$ is glued with the original affine space \mathbb{C}^2 by

$$(3.2) \quad \begin{cases} x = u_i^{a_i} v_i^{a_{i+1}} \\ y = u_i^{b_i} v_i^{b_{i+1}} \end{cases} \quad \begin{cases} u_i = x^{b_{i+1}} y^{-a_{i+1}} \\ v_i = x^{-b_i} y^{a_i} \end{cases}$$

We consider the curve $C = \{(x, y) \in \mathbf{C}^2; f(x, y) = 0\}$ in the original affine space \mathbf{C}^2 and let \tilde{C} be the closure of C in X . The curve \tilde{C} is defined in $\mathbf{C}_{\sigma_i}^2$ by the equation $f_{\sigma_i}(u_i, v_i) = 0$ where $f_{\sigma_i}(u_i, v_i)$ is defined by

$$(3.3) \quad f_{\sigma_i}(u_i, v_i) := f(u_i^{a_i} v_i^{a_{i+1}}, u_i^{b_i} v_i^{b_{i+1}}) / u_i^{d(R_i, f)} v_i^{d(R_{i+1}, f)}.$$

In $\mathbf{C}_{\sigma_i}^2$, $\hat{E}(R_i)$ is defined by $u_i = 0$. It is easy to see that $f_{\sigma_i}(0, 0) \neq 0$ and

$$\tilde{C} \cap \hat{E}(R_i) = \{(0, v_i) \in \mathbf{C}_{\sigma_i}^2; f_{\sigma_i}(0, v_i) = 0\}.$$

Therefore $\tilde{C} \cap \hat{E}(R_i)$ is non-empty if and only if $\dim \Delta(R_i; f) \geq 1$. Let D_1, \dots, D_m be the 1-dimensional faces of $\Delta(f)$ in the counter-clockwise orientation so that D_1, D_m contains the original O . Let $P_i = (p_i, q_i)$ be the corresponding primitive integral covector of D_i . Note that each P_i must be a vertex of Σ and therefore we can write $P_i = R_{v_i}$ for some $1 \leq v_i \leq \mu$. Then we can write

$$(3.4) \quad f_p(x, y) = \delta_i x^{p_i} y^{q_i} \prod_{j=1}^{\ell_i} (y^{p_{i,j}} - \xi_{i,j} x^{q_{i,j}})^{v_{i,j}}$$

where $\delta_i \in \mathbf{C}$ and $\xi_{i,j}, 1 \leq j \leq \ell_i$ are mutually distinct non-zero complex numbers. By the above consideration, $\hat{E}(R_i) \cap \tilde{C} \neq \emptyset$ if and only if $i = v_j$ for some $1 \leq j \leq m$. We consider the toric coordinate chart $\sigma_{v_i} = \text{Cone}(R_{v_i}, R_{v_i+1})$. Then by (3.3) and (3.4),

$$(3.5) \quad f_{\sigma_{v_i}}(0, v_{v_i}) = \delta_i \prod_{j=1}^{\ell_i} (v_{v_i} - \xi_{i,j})^{v_{i,j}}.$$

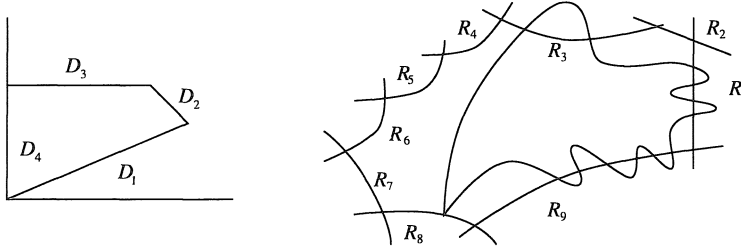
Thus $\hat{E}(R_{v_i}) \cap \tilde{C}$ consists of ℓ_i points $\{(0, \xi_{i,j}); j = 1, \dots, \ell_i\} \subset \mathbf{C}_{\sigma_{v_i}}^2$. Put $A_{i,j} := (0, \xi_{i,j}) \in \hat{E}(R_{v_i}) \cap \tilde{C}$ for $1 \leq i \leq m, 1 \leq j \leq \ell_i$. See [O5], [O1], [K-K-M-S] for further information about the toric compactification. The following example shows the situation.

Example (3.6). Let $f(x, y) = x^6 y^2 (x + y)^2 + y^4 + x^4 y + 1$. As Σ , we can take the regular simplicial cone subdivision with vertices $R_1 = E_1, R_2 = E_2, R_9 = -E_2$ and

$$R_3 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, R_4 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}, R_5 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, R_6 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, R_7 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, R_8 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

where R_1, R_3, R_8, R_9 correspond to the four faces. We have two transversal points in $\hat{E}(R_3) \cap \tilde{C}$ which are mapped to $(0; 1; 0)$ by Ψ and thus in the projective

compactification, we see that two local irreducible components are interesting at $(0; 1; 0)$.



$$(P_1 = R_3, P_2 = R_8, P_3 = R_9, P_4 = R_1)$$

Figure (3.6.A)

Now we consider the estimation of the value of the function f along an irreducible component γ of another curve D at infinity. (In §4, we take $\Gamma_r(f)$ as D .) Let $\Phi_\gamma(t)$ be a parametrization of γ in the coordinates (x, y) (namely in $\mathbb{C}_{\sigma_i}^2$) in the neighborhood of the infinity where $x_\gamma(t)$ and $y_\gamma(t)$ are Laurent series in the variable t . We assume that $x_\gamma(t), y_\gamma(t) \neq 0$ and write them as

$$(3.7) \quad \begin{cases} x_\gamma(t) = \alpha_\gamma t^{p_\gamma} + \text{(higher terms)} \\ y_\gamma(t) = \beta_\gamma t^{q_\gamma} + \text{(higher terms)}, \quad t \in D_\epsilon(0). \end{cases}$$

Let $Q_\gamma := (p_\gamma, q_\gamma) \in N$ and $A_\gamma := (\alpha_\gamma, \beta_\gamma)$. We assume that

$$(3.7.1) \quad \min(p_\gamma, q_\gamma) < 0, \quad \alpha_\gamma, \beta_\gamma \neq 0$$

so that $x_\gamma(t), y_\gamma(t) \neq 0$ and $\|(x_\gamma(t), y_\gamma(t))\| \rightarrow \infty$.

Proposition (3.8). (i) First we have $\text{val}_t f(x_\gamma(t), y_\gamma(t)) \geq d(Q_\gamma; f)$ and the inequality holds if and only if $Q_\gamma \sim P_i$ and $\beta_\gamma^{p_i} - \xi_{i,j} \alpha_\gamma^{q_i} = 0$ for some $i, 1 \leq i \leq m$ and $j, 1 \leq j \leq \ell_i$. (ii) The limit $\lim_{t \rightarrow 0} \Phi_\gamma(t)$ in X always exists and we have

$$\lim_{t \rightarrow 0} \Phi_\gamma(t) = \begin{cases} (0, 0) \in \mathbb{C}_{\sigma_i}^2 & \text{if } Q_\gamma \in \text{Int Cone}(R_j, R_{j+1}) \\ (0, \alpha_\gamma^{-b_i} \beta_\gamma^{a_i}) \in \mathbb{C}_{\sigma_i}^2 & \text{if } Q_\gamma = cR_j, \text{ for some } c > 0. \end{cases}$$

Here $\text{Int Cone}(R_j, R_{j+1})$ is the open cone generated by R_j and R_{j+1} . In particular, if $Q_\gamma \sim P_i$ and $\beta_\gamma^{p_i} - \xi_{i,j} \alpha_\gamma^{q_i} = 0$ for some $i, 1 \leq i \leq m$, $\lim_{t \rightarrow 0} \Phi_\gamma(t) = (0, \xi_{i,j}) \in \mathbb{C}_{\sigma_i}^2$.

Proof. We sketch the proof. First it is easy to see that

$$f(x_\gamma(t), y_\gamma(t)) = f_{Q_\gamma}(A_\gamma) t^{d(Q_\gamma, f)} + \text{(higher terms)}.$$

Assume that $Q_\gamma \neq P_1, \dots, P_m$. Then $f_{Q_\gamma}(x, y)$ is a monomial. Therefore $f_{Q_\gamma}(A_\gamma) = 0$ is possible if and only if $Q_\gamma = cP_i$ for some positive integer c and $\beta_\gamma^{p_i} - \xi_{i,j} \alpha_\gamma^{q_i} = 0$ for some j , $1 \leq j \leq \ell_i$. This proves the assertion (i).

We prove the assertion (ii). Assume first that $Q_\gamma \in \text{Int Cone}(R_i, R_{i+1})$. This is equivalent to

$$(3.8.1) \quad \det(R_i, Q_\gamma) = a_i q_\gamma - b_i p_\gamma > 0, \quad \det(Q_\gamma, R_{i+1}) = b_{i+1} p_\gamma - a_{i+1} q_\gamma > 0.$$

We use the coordinate $\mathbb{C}_{\sigma_i}^2$ to see the limit $\lim_{t \rightarrow 0} \Phi_\gamma(t)$. By the definition, we have

$$(3.8.2) \quad u_i(t) = x_\gamma(t)^{b_{i+1}} y_\gamma(t)^{-a_{i+1}} = C_1 t^{h_1} + (\text{higher terms})$$

$$(3.8.3) \quad v_i(t) = x_\gamma(t)^{-b_i} y_\gamma(t)^{a_i} = C_2 t^{h_2} + (\text{higher terms})$$

where

$$C_1 = \alpha_\gamma^{b_{i+1}} \beta_\gamma^{-a_{i+1}}, \quad C_2 = \alpha_\gamma^{-b_i} \beta_\gamma^{a_i}$$

$$h_1 = \det(Q_\gamma, R_{i+1}) > 0, \quad h_2 = \det(R_i, Q_\gamma) > 0.$$

Thus we see that $\lim_{t \rightarrow 0} \Phi_\gamma(t) = (0, 0) \in \mathbb{C}_{\sigma_i}^2$.

Assume that $Q_\gamma = cR_i$ for some positive integer c . Then the first inequality in (3.8.1) must be replaced by $a_i q_\gamma - b_i p_\gamma = 0$. Thus using (3.8.2) and (3.8.3), we get

$$\lim_{t \rightarrow 0} \Phi_\gamma(t) = (0, \alpha_\gamma^{-b_i} \beta_\gamma^{a_i}) \in \mathbb{C}_{\sigma_i}^2.$$

This completes the proof of the assertion (ii). \square

§4. Toric Generalization of Theorem (1.4)

Let $f(x, y)$ be as before. We will generalize Theorem (1.4) using Theorem (2.6.1) and the toric embedding theory. We assume that $\dim \Delta(f) = 2$ for brevity but every argument works even in the case $\dim \Delta(f) = 1$. See Remark (4.20). Let D_1, \dots, D_m be the faces of $\Delta(f)$ in the clockwise orientation so that D_1, D_m contain the origin. Let $P_i = (p_i, q_i)$ be the corresponding primitive integral covector of D_i . To get a better estimation, we first introduce the reduced polynomial $\tilde{f}(x, y) := f(x, y) - f(0, 0)$. Note that $\Delta(\tilde{f}) \subset \Delta(f)$ but $O \notin \Delta(\tilde{f})$. We factorize $\tilde{f}_p(x, y)$ as in (3.4):

$$(4.1) \quad \tilde{f}_p(x, y) = \delta_i x^{p_i} y^{q_i} \prod_{j=1}^{\ell_i} (y^{p_{ij}} - \xi_{i,j} x^{q_{ij}})^{v_{ij}}.$$

Note that $f_{P_i}(x, y) = f_{P_i}(x, y)$ for $i = 2, \dots, m - 1$.

Definition (4.2). We define two integers $\nu(D_i)$ and $\eta(D_i)$ by

$$v(D_i) = \sum_{i=1}^{l_i} (v_{i,i} - 1), \quad \eta(D_i) = \sum_{i=1}^{l_i} v_{i,i}$$

We say that $f(x, y)$ is *non-degenerate on the outside boundary* if $v(D_i) = 0$ for any $2 \leq i \leq m - 1$.

Thus the main aim of this section is to study unstable local irreducible components of $\overline{\Gamma}_i(f)$ and to get a better estimation about the cardinality of the set of critical values at infinity $\xi(\mathcal{W}(\Gamma_i))$ (see Definition (2.6) and Theorem (2.6.1) in §2) using the toric argument as in §3. Let γ be an irreducible component of $\overline{\Gamma}_i(f)$ at infinity and let $\Phi_\gamma(t)$ be a parametrization of γ in the coordinates (x, y) where $x_\gamma(t)$ and $y_\gamma(t)$ are Laurent series in the variable t . We assume first that

$$x_\gamma(t), y_\gamma(t) \neq 0 \text{ and } |x_\gamma(t)|^2 + |y_\gamma(t)|^2 \rightarrow \infty \text{ (} t \rightarrow 0 \text{)}$$

and we expand them in a Laurent series as

$$(4.3) \quad \begin{cases} x_\gamma(t) = a_\gamma t^{p_\gamma} + (\text{higher terms}), & a_\gamma \neq 0 \\ y_\gamma(t) = b_\gamma t^{q_\gamma} + (\text{higher terms}), & b_\gamma \neq 0 \\ f(x_\gamma(t), y_\gamma(t)) = f_{Q_\gamma}(A_\gamma) t^{d(Q_\gamma, f)} + (\text{higher terms}) \end{cases}$$

where $Q_\gamma := (p_\gamma, q_\gamma) \in N$ and $A_\gamma := (a_\gamma, b_\gamma)$. The case $x_\gamma(t)y_\gamma(t) \equiv 0$ or $\dim(\Delta(f)) = 1$ will be treated later. By the above assumption we have that

$$(4.4) \quad A_\gamma \in \mathbb{C}^{*2}, \quad \min(p_\gamma, q_\gamma) < 0.$$

We divide the situation in two cases.

Case I. $d(Q_\gamma; f) < 0$. **Case II.** $d(Q_\gamma; f) = 0$.

First Lemma (2.1) and Lemma (2.2) can be generalized by Proposition (3.8) as

Lemma (4.5). *We have $\text{val}_t f(\ell(x_\gamma(t), y_\gamma(t))) \geq d(Q_\gamma; f)$ and the inequality holds if and only if $Q_\gamma = cP_i, f_{P_i}(A_\gamma) = 0$ for some $c > 0$ and $1 \leq i \leq m$.*

Recall that $\Gamma_i(f)$ is defined by $\Gamma_i(f) = \{(x, y) \in \mathbb{C}^2; J(x, y) = 0\}$ where

$$J(x, y) = \alpha \frac{\partial f}{\partial x}(x, y) + \beta \frac{\partial f}{\partial y}(x, y) = \alpha \frac{\partial \tilde{f}}{\partial x}(x, y) + \beta \frac{\partial \tilde{f}}{\partial y}(x, y) = 0.$$

First we observe that the Newton boundary $\Delta(J)$ is slightly different from $\Delta(\tilde{f})$ (see Figure (4.19.A)) but the following is enough for our purpose.

$$(4.6) \quad J_{Q_\gamma}(x, y) = \begin{cases} \alpha \frac{\partial \tilde{f}_{Q_\gamma}}{\partial x}(x, y) + \beta \frac{\partial \tilde{f}_{Q_\gamma}}{\partial y}(x, y), & p_\gamma = q_\gamma < 0 \\ \alpha \frac{\partial \tilde{f}_{Q_\gamma}}{\partial x}(x, y), & p_\gamma > q_\gamma, \quad \tilde{f}_{Q_\gamma}(x, y) \neq \tilde{f}_{Q_\gamma}(0, y) \\ \beta \frac{\partial \tilde{f}_{Q_\gamma}}{\partial y}(x, y), & p_\gamma > q_\gamma, \quad \tilde{f}_{Q_\gamma}(x, y) \neq \tilde{f}_{Q_\gamma}(x, 0). \end{cases}$$

We assume that γ is an unstable irreducible component of $\overline{\Gamma}_i(f)$ at infinity.

Case I. We first assume that $d(Q_\gamma; f) < 0$. Then by Lemma (4.5), we must have $Q_\gamma = cP_i$ with $2 \leq i \leq m-1$. We call the face D_2, \dots, D_{m-1} the *outside faces* of $\Delta(f)$. We ask how many such components are possible for a fixed i . By an easy computation, we can see that the multiplicity of $y^{p_i} - \xi_{i,j} x^{q_i}$ in the factorization of $J_{P_i}(x, y)$ is exactly $v_{i,j} - 1$. Thus by the argument in §3, the local equation of $\overline{\Gamma}_i(f)$ in the toric coordinate chart $\mathbb{C}_{\sigma_i}^2$ is of the form

$$\delta_i \eta(v_{v_i}) \prod_{j=1}^{\ell_i} (v_{v_i} - \xi_{i,j})^{v_{i,j}-1} + u_{v_i} g(u_{v_i}, v_{v_i}) = 0$$

where $\delta_i \neq 0$, $\eta(v_{v_i})$ is a polynomial with $\eta(\xi_{i,j}) \neq 0$ for any $j = 1, \dots, \ell_i$. (Recall that $P_i = R_{v_i}$ and $\sigma_{v_i} = \text{Cone}(R_{v_i}, R_{v_i+1})$.) Let $A_{i,j} = (0, \xi_{i,j}) \in \mathbb{C}_{\sigma_{v_i}}^2$. By the same discussion as in Proposition (2.4) and by Proposition (3.8), we have that $C_\tau \cap \hat{E}(R_{v_i}) = \{A_{i,j}; j = 1, \dots, \ell_i\}$ and

Proposition (4.7). *The number of local irreducible components γ at infinity of $\overline{\Gamma}_i(f)$ such that $\lim_{t \rightarrow 0} (x_\gamma(t), y_\gamma(t)) = A_{i,j}$ is at most $v_{i,j} - 1$ for any $2 \leq i \leq m-1$. Thus the number of the unstable irreducible components γ such that the limit $\lim_{t \rightarrow 0} (x_\gamma(t), y_\gamma(t)) \in \hat{E}(P_i)$ is bounded by $v(D_i)$.*

Now we consider the second case: $d(Q_\gamma; f) = 0$. Then it is clear that $d(Q_\gamma; f) \geq 0$. We divide this case into two subcases.

Case II-1. $d(Q_\gamma; \tilde{f}) = 0$. **Case II-2.** $d(Q_\gamma; \tilde{f}) > 0$.

Recall that D_1 and D_m are the face which contains the origin O . Let $\tilde{D}_1 = \Delta(P_1; \tilde{f})$ and $\tilde{D}_m = \Delta(P_m; \tilde{f})$. We call D_1 and D_m (resp. \tilde{D}_1 and \tilde{D}_m) the *right conical face* and the *left conical face* of $f(x, y)$ (resp. of $\tilde{f}(x, y)$).

Note that $\tilde{D}_i \subset D_i$ and \tilde{D}_i might be a vertex for $i=1, m$. \tilde{D}_i for $p_i < 0$ (respectively \tilde{D}_m if $q_m < 0$) is called a bad face in [N-Z].

Assume for example $p_1 < 0$. It is more convenient to consider the factorizations :

$$(4.8.1) \quad \begin{cases} \tilde{f}_{P_1}(x, y) = \delta_1 (x^{q_1} y^{-p_1})^{e_1} \prod_{j=1}^{\ell'_1} (1 - \xi_{1,j} x^{q_1} y^{-p_1})^{v_{1,j}}, & e_1 > 0 \\ \frac{\partial \tilde{f}_{P_1}}{\partial y}(x, y) = d_1 x^{q_1 e_1} y^{-p_1 e_1 - 1} \prod_{j=1}^{\ell'_1} (x^{q_1} y^{-p_1} - \xi'_{1,j})^{v'_{1,j}}, & d_1 \in \mathbb{C}^*. \end{cases}$$

It is easy to see that the support of $J_{P_1}(x, y) = 0$ is the parallel translation of $\Delta(P_1; \tilde{f})$ by -1 to the direction of y -axis and $J_{P_1}(x, y) = \beta \frac{\partial \tilde{f}_{P_1}}{\partial y}(x, y)$. Thus we have $\sum_{j=1}^{\ell'_1} v'_{1,j} = \eta(D_1)$. In particular we have $\ell'_1 \leq \eta(D_1)$. Similarly if $q_m < 0$, we consider the factorizations :

$$(4.8.2) \quad \begin{cases} \tilde{f}_{P_m}(x, y) = \delta_m (x^{-q_m} y^{p_m})^{e_m} \prod_{j=1}^{\ell'_m} (x^{-q_m} y^{p_m} - \xi_{m,j})^{v_{m,j}}, & e_m > 0 \\ \frac{\partial \tilde{f}_{P_m}}{\partial y}(x, y) = d_m x^{-q_m e_m - 1} y^{p_m e_m} \prod_{j=1}^{\ell'_m} (x^{-q_m} y^{p_m} - \xi'_{m,j})^{v'_{m,j}}, & d_m \in \mathbb{C}^*. \end{cases}$$

We have also $\sum_{j=1}^{\ell'_m} v'_{m,j} = \eta(D_m)$ and $\ell'_m \leq \eta(D_m)$. For the convenience, we introduce the polynomials

$$\varphi_1(t) = \delta_1 t^{e_1} \prod_{j=1}^{\ell'_1} (1 - \xi_{1,j} t)^{v_{1,j}}, \quad \varphi_m(t) = \delta_m t^{e_m} \prod_{j=1}^{\ell'_m} (t - \xi_{m,j})^{v_{m,j}}$$

so that we have $\varphi_1(x^{q_1} y^{-p_1}) = \tilde{f}_{P_1}(x, y)$ and $\varphi_m(x^{-q_m} y^{p_m}) = \tilde{f}_{P_m}(x, y)$.

Definition (4.8.3). We define the contribution from the conical faces as follows.

$$\eta(D_1)' = \begin{cases} \ell'_1 & p_1 < 0 \\ 0 & p_1 = 0 \end{cases}, \quad \eta(D_m)' = \begin{cases} \ell'_m & q_m < 0 \\ 0 & q_m = 0 \end{cases}.$$

Note that $\eta(D_1)' \leq \eta(D_1)$. It is obvious that $p_1 = 0$ (respectively $q_m = 0$) implies $D_1 \subset \{y = 0\}$ (resp. $D_m \subset \{x = 0\}$).

Now we consider Case II-1 first. In this case, we must have either $Q_\gamma = cP_1$ or $Q_\gamma = cP_m$ for some $c > 0$. Let us consider the case $Q_\gamma = cP$ for instance. By the assumption $\min(p_i, q_i) < 0$ and $\dim(\Delta(f)) = 2$, we must have $p_1 < 0 < q_1$ if such a γ exists. The essential difference from Case I is that $\tilde{C}_\tau \cap \hat{E}(P_1)$ is not fixed as the constant term is on the initial term. Now we assert

Lemma (4.9). *Assume that $p_1 < 0$. The local irreducible components of $\overline{\Gamma}_l(f)$ of type Case II-1 with $Q_\gamma = cP_1$, $c > 0$ (respectively $Q_\gamma = cP_m$, $c > 0$) are all unstable. The corresponding critical values are $\{f(0,0) + \varphi_1(\xi'_{1,j}); 1 \leq j \leq \ell'_1\}$. Similarly if $q_m < 0$, the critical values from components γ such that $Q_\gamma = cP_m$, $c > 0$ are $\{f(0,0) + \varphi_m(\xi'_{m,j}); 1 \leq j \leq \ell'_m\}$.*

Proof. We consider the case $Q_\gamma = cP$, $c > 0$ and $p_1 < 0 < q_1$. As we have $J_{P_1}(x, y) = \beta \frac{\partial \tilde{f}_{P_1}}{\partial y}(x, y)$ and

$$0 \equiv J(x_\gamma(t), y_\gamma(t)) = J_{P_1}(a_\gamma, b_\gamma)^{-q_\gamma} + (\text{higher terms}),$$

we must have by (4.8.1) $a_\gamma^q b_\gamma^{-p_1} = \xi'_{1,j_0}$ for some $1 \leq j_0 \leq \ell'_1$. Thus by (4.3) we get

$$f(x_\gamma(t), y_\gamma(t)) = f(0,0) + \varphi_1(\xi'_{1,j_0}) + (\text{higher terms}).$$

Conversely for any $1 \leq j \leq \ell'_1$, there exists a local irreducible component γ of $\overline{\Gamma}_l$ such that $Q_\gamma = cP$, $c > 0$ and $a_\gamma^q b_\gamma^{-p_1} = \xi'_{1,j}$. This proves the assertion. \square

Let $\tilde{\Gamma}(\ell)$ be the compactification of $\Gamma(\ell)$ in X . Observe that $\tilde{C}_\tau \cap \tilde{\Gamma}(\ell) \cap \hat{E}(P_1) \neq \emptyset$ if and only if $\tau = f(0,0) + \varphi_1(\xi'_{1,j})$ for some j .

Case II-2. Now we consider the case $d(Q_\gamma; f) = 0$ and $d(Q_\gamma; \tilde{f}) > 0$. This is possible only if $\Delta(Q_\gamma; f) = \{O\}$ and $p_\gamma q_\gamma < 0$. So we assume for example

$$(4.10) \quad p_\gamma < 0 < q_\gamma.$$

By the assumption $d(Q_\gamma; \tilde{f}) > 0$, we have that $\tilde{f}_{Q_\gamma}(x, 0) \equiv 0$ (if not, we get a contradiction $d(Q_\gamma; \tilde{f}) < 0$) and $\tilde{f}_{Q_\gamma}(x, y) \neq \tilde{f}_{Q_\gamma}(x, 0)$. Thus by (4.6) $J_{Q_\gamma}(x, y) = \frac{\partial \tilde{f}_{Q_\gamma}}{\partial y}(x, y)$. The assumption $\Delta(Q_\gamma; f) = \{O\}$ implies that $p_1 < 0 < q_1$ and

$$(4.11) \quad \det(Q_\gamma, P_1) > 0.$$

From the equality $J(x_\gamma(t), y_\gamma(t)) \equiv 0$, we get

$$(4.12) \quad J_{Q_\gamma}(A_\gamma) = \frac{\partial \tilde{f}_{Q_\gamma}}{\partial y}(A_\gamma) = 0.$$

By (4.12), we must have $\dim \Delta(Q_\gamma; \tilde{f}) = 1$. Such a face $\Delta(Q_\gamma; \tilde{f})$ is called an *inside face with mixed weight vector* of $\tilde{f}(x, y)$. Geometrically the supporting line of such a face separates the Newton polygon $\Delta(\tilde{f})$ and the origin O . See the left side figure of Figure (4.14.A). We consider the right conical face \tilde{D}_1 . By the expression (4.1) or (4.8), the left edge of \tilde{D}_1 is $R := (r_1, s_1 + p_1 \sum_{j=1}^{\ell'_1} v_{1,j}) =$

$(q_1e_1, -p_1e_1)$. This gives a vertex $(q_1e_1, -p_1e_1 - 1)$ of the Newton polygon $\Delta(J)$ by the differential in y . If $-p_1e_1 = 1$, it is easy to see that there exists no inside face of mixed weight Q_γ with $p_\gamma < 0 < q_\gamma$. Therefore we assume

$$(4.13) \quad -p_1e_1 \geq 2.$$

In this case, it is not necessary to count the number of such local irreducible components. In fact, we have

Proposition (4.14). *Each local irreducible components γ of $\overline{\Gamma_r(f)}$ of Case II-2 gives the limit critical value $f(0,0)$.*

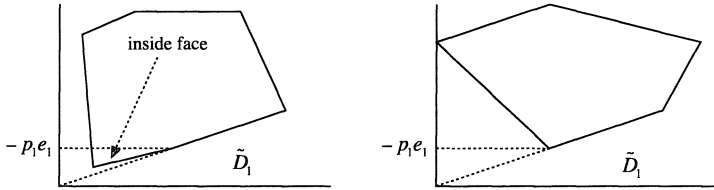


Figure (4.14.A)

The left side of Figure (4.14.A) show the situation where we have a inside face with a mixed covector. In the right side figure, we do not have any inside face.

Proof. By the assumption, we have

$$f(x_\gamma(t), y_\gamma(t)) = f(0,0) + (\text{higher terms}).$$

Thus the assertion is trivial. \square

Until now, we have assumed that $x_\gamma(t), y_\gamma(t) \neq 0$. Now we consider the exceptional case that $x_\gamma(t) \equiv 0$ or $y_\gamma(t) \equiv 0$. Assume for example

$$\gamma : x_\gamma(t) = 1/t, \quad y_\gamma(t) \equiv 0.$$

This implies that y divides $J(x, y)$. By the above argument, it is necessary that $-p_1e_1 \geq 2$. In this case, we can see that $f(x,0) \equiv f(0,0)$. Thus if this is the case, $\text{val}_t f(x_\gamma(t), y_\gamma(t)) = 0$ and γ is unstable and the corresponding limit critical value is again $f(0,0)$. Now summerizing the above argument, we have

Proposition (4.15). *Assume that $-p_1e_1 \geq 2$ in (4.8.1) (respectively $-q_m e_m \geq 2$ in (4.8.2)). Then either there exists an unstable local irreducible component of $\overline{\Gamma_r(f)}$ of type Case II-2 with $p_\gamma < 0 < q_\gamma$ (resp. $q_\gamma < 0 < p_\gamma$), or $y = 0$ (resp. $x = 0$) is a (global) component of $\overline{\Gamma_r(f)}$. In any case, the possible limit critical value is $f(0,0)$.*

Definition (4.16). Let us define

$$\varepsilon(f) = \begin{cases} 1, & \max\{-p_1 e_1, -q_m e_m\} \geq 2 \\ 0, & \max\{-p_1 e_1, -q_m e_m\} \leq 1 \end{cases}$$

and we define the toric degeneracy $v_\infty^{toi}(f)$ by

$$v_\infty^{toi}(f) = \sum_{i=2}^{m-1} v(D_i) + \eta(D_1)' + \eta(D_m)' + \varepsilon(f).$$

Recall that $\tilde{f}(x, y)$ is called convenient if $\Delta(\tilde{f})$ intersects with both axes. Note that the last three terms are zero if $\tilde{f}(x, y)$ is convenient. Now we are ready to state the main theorem.

Main Theorem (4.17). *The number of critical values at infinity of the function f is less than or equal to $v_\infty^{toi}(f)$.*

Corollary (4.17.1) ([O5]). *Assume that $\tilde{f}(x, y)$ is a convenient polynomial with non-degenerate on the outside Newton boundary. Then f has no critical value at infinity.*

The assertion is obvious as $v(D_i) = 0$ by non-degeneracy and $\eta(D_1)' = \eta(D_2)' = \varepsilon(f) = 0$ by the convenience. The following proposition says Theorem (4.18) is stronger than Theorem (1.4).

Proposition (4.18). *We have the inequality: $v_\infty^{toi}(f) \leq v_\infty^{pi}(f)$.*

Proof. We assume for brevity that D_h corresponds to the support of $f_d(x, y)$. So $P_h = (-1, -1)$. In the case of $k = 0$ in (3.1), D_h is a vertex but then $v(D_h) = 0$ and this does not make any problem. We divide the two degeneracy into three parts.

$$v_\infty^{pi}(f) = \max(r - 1, 0) + v(D_h) + \max(s - 1, 0) \\ v_\infty^{toi}(f) = \left(\sum_{i=2}^{h-1} v(D_i) + \eta(D_i) + \varepsilon(f) \right) + v(D_h) - \varepsilon(f) + \left(\sum_{i=h+1}^{m-1} v(D_i) + \eta(D_m) + \varepsilon(f) \right).$$

We will show that

$$(4.18.1) \quad \sum_{i=2}^{h-1} v(D_i) + \eta(D_i) + \varepsilon(f) \leq \max(s - 1, 0)$$

$$(4.18.2) \quad \sum_{i=h+1}^{m-1} v(D_i) + \eta(D_m) + \varepsilon(f) \leq \max(r - 1, 0).$$

The assertion (4.18) follows immediately from these inequalities. Let us show (4.18.2) for example. As the assertion is obvious in the case of $r \leq 1$, we assume that $r \geq 2$. Let us consider the factorization of the face function of $\tilde{f}(x, y)$:

$$\begin{aligned} \tilde{f}_{P_h}(x, y) &= \delta_h x^{h'} y^{h''} \prod_{j=1}^{l_h} (y^{-1} - \xi_{h,j} x^{-1})^{v_{h,j}}, \\ &= \delta_h x^{h'} y^{h''} (xy)^\theta \prod_{j=1}^{l_h} (x - \xi_{h,j} y)^{v_{h,j}}, \end{aligned}$$

where $\theta = -\sum_{j=1}^{l_h} v_{h,j}$. Comparing with (3.1), we have $r_h - \sum_{j=1}^{l_h} v_{h,j} = r$ and $s_h - \sum_{j=1}^{l_h} v_{h,j} = s$. Note that $q_j < 0$ and $q_j < p_j$ for $h+1 \leq j \leq m-1$. Thus the projection of the face D_j in x -axis for $h+1 \leq j \leq m-1$ is the interval $[r_j + q_j \sum_{i=1}^{l_j} v_{j,i}, r_j]$ and $r_j + q_j \sum_{i=1}^{l_j} v_{j,i} = r_{j+1}$. Therefore

$$(4.18.3) \quad r_j - r_{j+1} = -q_j \sum_{i=1}^{l_j} v_{j,i} \geq v(D_{j+1}) + 1, \quad h \leq j \leq m-1.$$

Taking $j = h$, we see that $r = r_{h+1}$. In the case of $j = m$, we have

$$(4.18.4) \quad r_m - r_{m+1} \geq -q_m \sum_{i=1}^{l_m} v_{m,i} \geq \eta(D_m)', \quad q_m < 0.$$

Here r_{m+1} is by definition $r_m + q_m \sum_{i=1}^{l_m} v_{m,i}$ is the x -coordinate of the left side edge of \tilde{D}_m . Taking the summation for $j = h+1, \dots, m$, we get

$$(4.18.5) \quad r - r_{m+1} = r_{h+1} - r_{m+1} \geq \sum_{j=h+1}^{m-1} v(D_j) + \eta(D_m)' + m - h - 1.$$

We can see easily that if $r_{m+1} = 0$, then $q_m = 0$ and as we have assumed $r \geq 2$, this implies that $m > h+1$. Therefore we have $r_{m+1} + m - h - 1 > 0$. By this inequality and (4.18.5), we get the inequality (4.18.2). \square

Now we give several examples.

Example (4.19). (A) Let $\tilde{f}(x, y) = y^{2n} + x^{3n}y^n(x+y)^n + x^4y$. Then $\Delta(\tilde{f})$ has four faces. See Figure (3.19.A). In this example, $d = 5n$ and $\tilde{f}_{5n} = x^{3n}y^n(x+y)^n$, and the projective degeneracy at infinity $v_\infty^p(f) = 5n - 3$. On the other hand, $\eta(D_1)' = n - 1$, $v(\Delta_2) = n - 1$ and $v(D_3) = 0$, $\eta(D_4)' = 0$ and $\varepsilon(f) = 0$. Thus we have $v_\infty^p(f) = 2n - 2$. In the left figure, the dotted region is the Newton polygon $\Delta(\tilde{f})$. The right figure is $\Delta(J)$.

(B) Let $\tilde{f}(x, y) = x^4y^4 + xy^3 + x^3y^2 + xy$. In this example, we have $v(\Delta_2) = v(\Delta_3) = 0$, $\eta(D_1)' = \eta(D_4)' = 0$ and $\varepsilon(f) = 1$ and $v_\infty^p(f) = 1$. In fact, $f(0, 0)$ is the only critical point at infinity.

(C) Let $\tilde{f}(x, y) = x + c_2x^2 + \dots + c_nx^n + x^m y$. Then $\Delta(\tilde{f})$ has three faces and $v_\infty^{tot}(\tilde{f}) = 1$. In fact, \tilde{f} has one critical value 0 from the infinity. This polynomial has no critical point ([O4]).

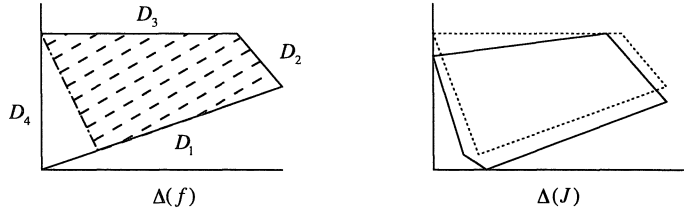


Figure (4.19.A)

Remark (4.20). (1) We have remarked that the toric degeneracy at infinity depends on the choice of the linear coordinate system. Let us see this by Example (4.19.A). Let $f(x, y) = y^{2n} + x^{3n}y^n(x + y)^n + x^4y$. We have seen that $v_\infty^{tot}(f) = 2n - 2$. We have seen the contribution from D_1 is effective by Lemma (4.19). In fact $\frac{\partial f}{\partial y}(x, y) = 0$ has $n - 1$ -distinct solutions and the corresponding critical values are also mutually distinct. However the contribution $v(D_2) = n - 1$ from D_2 is negligible. Namely these components give no critical values at infinity. In fact, let us consider the change of coordinates $u = x, v = x + y$. Then in $(u, v), f(x, y)$, is equal to $f'(u, v) = (v - u)^{2n} + u^{3n}v^n(v - u)^n + u^n(v - u)$. The corresponding Newton diagram is as in Figure (4.20.A).

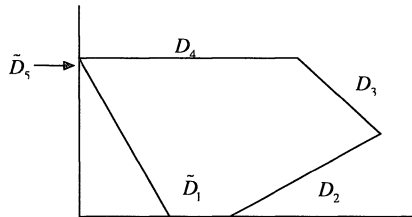


Figure (4.20.A)

Now we see that $f'(u, v)$ is convenient and $v(D_2) = v(D_4) = 0$ and $v(D_3) = n - 1$. Thus $v_\infty^{tot}(f') = n - 1$. Now note that the contribution from D_3 of $\Delta(f')$ is nothing but the contribution from D_1 of $\Delta(f)$. So we get $n - 1$ critical values at infinity from this face. Thus the function $f(x, y)$ has exactly $n - 1$ limit critical values.

(2) We consider the case that $\dim \Delta(\tilde{f}) = 1$.

(I) Assume that $\dim \Delta(f) = 2$. Then $\tilde{f}(x, y)$ is a weighted homogeneous polynomial of degree $d \neq 0$ with weight vector $P = '(p, q)$. $\Delta(f)$ has three faces. See Figure (4.20.B). It is convenient to consider the factorization:

$$\tilde{f}(x, y) = \begin{cases} cx'y \cdot \prod_{i=1}^k (y^{|p|} - \gamma_i x^{|q|})^{v_i}, & \text{if } pq \geq 0 \\ cx^r y^s \cdot \prod_{i=1}^k (y^{|p|} x^{|q|} - \gamma_i)^{v_i}, & \text{if } pq < 0. \end{cases}$$

It is well known that $\tilde{f}: \mathbb{C}^2 - \tilde{f}^{-1}(0) \rightarrow \mathbb{C}$ is a locally trivial fibration ([M]). In fact, the trivialization is explicitly given using the associated \mathbb{C}^1 -action defined by $t \circ (x, y) = (xt^p, yt^q)$ for $t \in \mathbb{C}^1$ and $(x, y) \in \mathbb{C}^2$. Therefore 0 is the only possible critical value of \tilde{f} from the infinity. The following is a corollary of Theorem (4.17) and Proposition (4.15).

Proposition (4.21). *Let $\tilde{f}(x, y)$ be as above. Then 0 is a regular value of \tilde{f} from the infinity if and only if $v_1 = \dots = v_k = 1$ and $\varepsilon(f) = 0$. $\varepsilon(f) = 0$ if and only if $r, s \leq 1$ and $pq \geq 0$.*

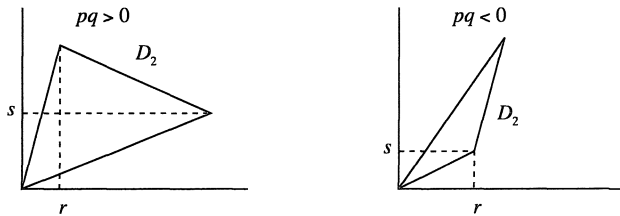


Figure (4.20.B)

(II) Assume that $\dim \Delta(f) = 1$. This implies that $d(P; f) = 0$. The factorization take the form:

$$\tilde{f}(x, y) = c(y^{|p|} x^{|q|})^e \prod_{i=1}^k (y^{|p|} x^{|q|} - \gamma_i)^{v_i}, \quad e > 0.$$

Let $\varphi(s) = cs^e \prod_{i=1}^k (s - \gamma_i)^{v_i}$. Then $\tilde{f}(x, y) = \varphi(y^{|p|} x^{|q|})$. Let $\{\rho_1, \dots, \rho_\delta\}$ be the critical values of φ . Note that $\delta \leq \sum_{i=1}^k v_i = \eta(\Delta(\tilde{f}))$. The corresponding critical locus of \tilde{f} is one-dimensional. Therefore they give the same critical values from the infinity as well. If $e \geq 2$, 0 is also a critical value of φ . If $e = 1$ but $|p| \geq 2$ or $|q| \geq 2$, $y = 0$ or $x = 0$ is a component of $\Gamma_c(f)$ and 0 is a critical value from the infinity of \tilde{f} . This can be interpreted by Main Theorem (4.17) as follows. $\Delta(f)$ has two identical conical faces $D_1 = D_m$. But the corresponding weight vectors are $P_1 = '(-|p|, |q|)$ and $P_2 = '(|p|, -|q|)$. In the case $pq < 0$, $\eta(D_1)' = \eta(D_2)' = \sum_{i=1}^k v_i$, but they give the same critical values. If $pq = 0$, say assume that $p = 0, q > 0$, then $\eta(D_1)' = 0$ and $\eta(D_2)' = \sum_{i=1}^k v_i$.

Example (4.22). (A) Let $\tilde{f}(x, y) = x \prod_{i=1}^k (y - \gamma_i)$. Then $\tilde{f}^{-1}(0)$ has k isolated critical points $(0, \gamma_1), \dots, (0, \gamma_k)$. As $v_\infty^{tot}(\tilde{f}) = 0$, \tilde{f} has no critical values

at infinity. In fact, the fibers $\{\tilde{f}^{-1}(t); t \in \mathbf{C}\}$ can be defined by the homogeneous polynomial $X \prod_{i=1}^k (Y - \gamma_i Z) - tZ^{k+1} = 0$. Thus in the neighborhood of $(0; 1; 0)$, this gives a family of smooth transversal curves:

$$u \prod_{i=1}^k (1 - \gamma_i v) - tv^{k+1} = 0, \quad u = X/Y, \quad v = Z/Y.$$

In the neighborhood of $(1; 0; 0)$, this gives a family of curves:

$$\prod_{i=1}^k (\zeta - \gamma_i \xi) - t\xi^{k+1} = 0, \quad \zeta = Y/Z, \quad v = Z/X.$$

This is also topological stable family as the local Milnor number is constantly $(k - 1)^2$.

(B) Let

$$\tilde{f}(x, y) = x \prod_{i=1}^k (x^p y^q - \gamma_i), \quad p, q > 0.$$

Then $\tilde{f}: \mathbf{C}^2 \rightarrow \mathbf{C}$ has no critical point but 0 is a critical value from the infinity as $\varepsilon(f) = 1$.

This work was done when the first author was visiting Tokyo Institute of Technology in October, 1992 and completed when the second author was visiting Institute of Mathematics in Hanoi for the International Workshop on Topology and Geometry in March, 1993. We would like to thank the both institutions for their support.

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