Publ. RIMS, Kyoto Univ.
31 (1995), 645–665

Global Existence for Systems of Nonlinear Wave Equations in Two Space Dimensions, II

By

Soichiro KATAYAMA*

Abstract

We consider the Cauchy problem for the system of nonlinear wave equations

$$(*) \qquad (\partial_t^2 - \Delta)u_i = F_i(u,u',u'') \quad \text{in } (0,\infty)\times\mathbf{R}^2, i=1,\cdots,N$$

with initial data $u_i(0,x)=\varepsilon\phi_i(x)$, $(\partial_t u_i)(0,x)=\varepsilon\psi_i(x)$, where $F_i\,(i=1,\cdots,N)$ are smooth functions of degree 3 near the origin $(u,u',u'')=0$, $\phi_i, \psi_i \in C_0^\infty(\mathbf{R}^2)$ and ε is a small positive parameter. We assume that $F_i\,(i=1,\cdots,N)$ are independent of u_j'' for any $j\neq i$.

In the previous paper, the author showed the global existence of the small solution to the Cauchy problem (*) assuming that the cubic parts of the nonlinear terms satisfy Klainerman's null condition and that the nonlinear terms are independent of $u_j u_k u_l u_m$ for any $j,k,l,m=1,\cdots,N$. In this paper, we show the global existence without imposing further assumptions than the null condition on the cubic parts of the nonlinear terms.

§1. Introduction

We consider the Cauchy problem for the following system of nonlinear wave equations:

$$(1.1) \qquad \Box u_i = F_i(u,u',u'') \quad \text{in } (0,\infty)\times\mathbf{R}^n, \quad i=1,\cdots,N,$$

$$(1.2) \qquad u_i(0,x)=\varepsilon\phi_i(x), \quad \partial_t u_i(0,x)=\varepsilon\psi_i(x), \quad x\in\mathbf{R}^n,\ i=1,\cdots,N,$$

where $\Box=\partial_t^2-\sum_{i=1}^n\partial_i^2$ is the d'Alembertian, $\partial_0=\partial_t=\partial/\partial t$, $\partial_k=\partial/\partial x_k$ $(k=1,\cdots,n)$, $u=(u_j)$, $u'=(u_{j,a})=(\partial_a u_j)$, $u''=(u_{j,ab})=(\partial_a\partial_b u_j)$ with $j=1,\cdots,N$ and $a,b=0,1,\cdots,n$. ε is a small positive parameter. We assume that the nonlinear term $F=(F_j)_{j=1,\cdots,N}$ is a smooth function in its arguments and satisfies

$$F(u,u',u'')=O(|u|^\alpha+|u'|^\alpha+|u''|^\alpha)$$

near $(u,u',u'')=0$ with some integer $\alpha\geq 2$. Suppose that the initial data $\phi=(\phi_j)_{j=1,\cdots,N}$ and $\psi=(\psi_j)_{j=1,\cdots,N}$ belong to $C_0^\infty(\mathbf{R}^n;\mathbf{R}^N)$.

Throughout this paper we always assume

(H1) $F_i\,(i=1,\cdots,N)$ are independent of $u_{j,ab}$ for any $j\neq i$, i.e.,

---

Communicated by T. Kawai, September 28, 1994.
1991 Mathematics Subject Classification(s): 35L70.
* Department of Applied Mathematics and Physics, Kyoto University, Kyoto 606–01, Japan.

$$\frac{\partial F_i}{\partial u_{j,ab}} = 0 \text{ for any } j \neq i \text{ and } a, b = 0, \cdots, n,$$

and furthermore F_i are independent of $u_{i,00} (= \partial_t^2 u_i)$ for $i = 1, \cdots, N$.

We state some known results, restricting ourselves to the cases of n = 2 and of n = 3. (For the results of $n \geq 4$, see Hörmander [4], Katayama [7], Klainerman [10], Klainerman - Ponce [12], Li - Yu [14], Li - Zhou [16], Shatah [21], etc.) Define the lifespan T_ε by the maximal existence time of the C^∞-solution to (1.1) - (1.2), namely

$$\begin{aligned} T_\varepsilon &= T_\varepsilon(\phi, \psi, F) \\ &= \sup\{T \in (0, \infty);\ \text{there exists a (unique) solution} \\ &\qquad u \in C^\infty([0, T) \times \mathbf{R}^n) \text{ to (1.1) - (1.2)}\}. \end{aligned}$$

We say that there exists a global solution when $T_\varepsilon = +\infty$.

First assume that F does not depend on u explicitly, i.e., $F = F(u', u'')$. Using the energy estimate of wave equations and generators of the Lorentz group (which are concerned with the Lorentz invariance of wave equations) introduced by Klainerman [10], one can show the following provided that ε is sufficiently small:

When $n = 3$,

$$(1.3) \qquad \begin{cases} T_\varepsilon \geq \exp(c\varepsilon^{-1}), & \alpha = 2, \\ T_\varepsilon = +\infty, & \alpha \geq 3, \end{cases}$$

where c is a positive constant depending on ϕ, ψ and F.

When $n = 2$,

$$(1.4) \qquad \begin{cases} T_\varepsilon \geq \exp(c\varepsilon^{-2}), & \alpha = 3, \\ T_\varepsilon = +\infty, & \alpha \geq 4. \end{cases}$$

(See Klainerman [10] for $n = 3$ and Kovalyov [13] for $n = 2$.)

When F depends explicitly on u, i.e., $F = F(u, u', u'')$, circumstances become more complicated, because there is no natural estimate for the L^2-norm of u itself. In this case, the following results are known when ε is sufficiently small:

When $n = 3$,

$$(1.5) \qquad \begin{cases} T_\varepsilon \geq c\varepsilon^{-2}, & \alpha = 2 \text{ (in general)}, \\ T_\varepsilon \geq \exp(c\varepsilon^{-1}), & \alpha = 2 \text{ (if } F(u,0,0) = O(|u|^3) \text{ near } u = 0), \\ T_\varepsilon = +\infty, & \alpha \geq 3. \end{cases}$$

When $n = 2$,

$$(1.6)\qquad \begin{cases} T_\varepsilon \geq c\varepsilon^{-6}, & \alpha = 3 \text{ (in general)}, \\ T_\varepsilon \geq \exp(c\varepsilon^{-2}), & \alpha = 3 \text{ (if } F(u,0,0) = O(|u|^5) \text{ near } u = 0), \\ T_\varepsilon = +\infty, & \alpha \geq 4. \end{cases}$$

(See Lindblad [20] for n = 3, Li - Yu - Zhou [15], Li - Zhou [17] and [19] for n = 2. See also Li - Zhou [18] for the results when n = 2 and $\alpha = 2$.)

Remark 1. Strictly speaking, these results are proved for single equations (namely, N = 1). But all the above results hold also for systems by the same proofs, except the cases of n = 3, $\alpha = 2$ and $F(u,0,0) = O(|u|^3)$ in (1.5) and of n = 2, $\alpha = 3$ and $F(u,0,0) = O(|u|^5)$ in (1.6), because some rewriting of the nonlinear terms is used in the proofs for these two cases, and it only works for single equations. It is not known whether the same results hold or not for these two cases when we consider systems.

Concerning the global existence for the case of n = 3 and $\alpha = 2$, Klainerman introduced some sufficient condition, which is called the null condition. Here we recall the definition of the null condition.

Definition 1.1. *Let* $G = G((u_j);(u_{j,a});(u_{j,ab}))$ *be a smooth function of* $u_j (j = 1,\cdots,N)$, $u_{j,a} (j = 1,\cdots,N, a = 0,\cdots,n)$ *and* $u_{j,ab} (j = 1,\cdots,N, a,b = 0,\cdots,n)$.

We say that G satisfies the null condition when

$$G((\lambda_j);(\mu_j X_a);(\nu_j X_a X_b)) = 0$$

for all $\lambda,\mu,\nu \in \mathbb{R}^N$ *and all* $X = (X_0, X_1,\cdots,X_n) \in \mathbb{R}^{n+1}$ *satisfying* $X_0^2 - X_1^2 - \cdots - X_n^2 = 0$.

Klainerman showed in [11] that if the quadratic part of the Taylor expansion of F around the origin satisfies the null condition, then $T_\varepsilon = +\infty$, provided that ε is sufficiently small (see also Christodoulou [1] and John [6]).

For $n = 2$ and $\alpha = 3$, Godin [2] proved that if $F = F(u')$ and the cubic part of F satisfies the null condition, then $T_\varepsilon = +\infty$ for sufficiently small ε. (See also Hoshiga [5]. He showed the same result for $F = \sum_{a,b} f_{ab}(u')\partial_a\partial_b u$.) When $n = 2$, $\alpha = 3$ and F depends explicitly on u, the author showed in [8] that if we assume
(H2) The cubic part of F satisfies the null condition,
(H3) $F(u,0,0) = O(|u|^5)$ near $u = 0$,
then $T_\varepsilon = +\infty$ for small ε (observe that the condition (H3) appears also in (1.6)). If we compare this result for $n = 2$ with Klainerman's for $n = 3$, the condition (H3) seems removable. In fact, there are some examples which do not satisfy (H3), but admit global solutions. For instance, if we consider the Cauchy problem for the single equation

$$(1.7)\qquad \Box u = u^4 \text{ in } (0,\infty)\times\mathbb{R}^2,$$

we can see from Li - Zhou's result (1.6) that there exists a global solution for sufficiently small initial data. Observe that $F = u^4$ satisfies (H2) but does not satisfy (H3). Another example is

$$\square u = cu((\partial_t u)^2 - |\nabla_x u|^2) + H(u,u') \quad \text{in } (0,\infty)\times\mathbf{R}^2, \tag{1.8}$$

where $c \in \mathbf{R}$ and $H(u,u') = O(|u|^4 + |u'|^4)$ near $(u,u') = 0$. Simple calculations yield

$$\square\left(u - \frac{cu^3}{6}\right) = H(u,u') - \frac{cu^2}{2}\{cu((\partial_t u)^2 - |\nabla_x u|^2) + H(u,u')\}. \tag{1.9}$$

Define $w = u - cu^3/6$. Then, because the right-hand side of (1.9) becomes a function of degree 4 with respect to w and w', we can show that $T_\varepsilon = +\infty$ for small ε, again from Li - Zhou's result (1.6). This example also satisfies (H2), but not (H3) in general, because $H(u,u')$ may contain u^4. These examples suggest us that the assumption (H3) is not needed for the global existence. Our aim in this paper is to establish the global existence under (H1) and (H2) without (H3). The main theorem is the following:

Theorem 1.2. *Let* $n = 2$ *and* $F = O(|u|^3 + |u'|^3 + |u''|^3)$ *near the origin. Assume that F satisfies* (H1) *and*

(H2) *The cubic part of the Taylor expansion of F satisfies the null condition, that is,* F_i *can be written in the form*

$$F_i(u,u',u'') = G_i(u,u',u'') + H_i(u,u',u''), \quad i = 1,\cdots,N$$

in some neighborhood of $(u,u',u'') = 0$, *where* G_i $(i = 1,\cdots,N)$ *are homogeneous polynomials of degree* 3 *satisfying the null condition and* H_i $(i = 1,\cdots,N)$ *are smooth functions with* $H_i(u,u',u'') = O(|u|^4 + |u'|^4 + |u''|^4)$ *near* $(u,u',u'') = 0$.

Then, for any $\phi, \psi \in C_0^\infty(\mathbf{R}^2;\mathbf{R}^N)$ *there exists a positive constant* ε_0 *such that the Cauchy problem* (1.1) - (1.2) *has a unique solution* $u = (u_i(t,x))_{i=1,\cdots,N} \in C^\infty([0,\infty)\times\mathbf{R}^2;\mathbf{R}^N)$ *for any* $\varepsilon \le \varepsilon_0$.

For the proof of this theorem, it would suffice to get some *a priori* estimates, because we have the local existence theorem. It seems reasonable to state the difference between the proof of the former result in [8] and that of our present one here. To state it clearly, first we introduce some notations. In the rest of this paper, we assume that $n = 2$.

Notations. Following Klainerman [10], we introduce $\Gamma_0 = t\partial_t - \sum_{i=1}^2 x_i\partial_i$, $\Omega_{0j} = x_j\partial_t + t\partial_j$ $(j = 1,2)$ and $\Omega_{12} = x_1\partial_2 - x_2\partial_1$. Then simple calculations yield

$$[\Gamma_0, \square] = -2\square,\ [\Omega_{ab}, \square] = 0 \text{ for any } 0 \le a < b \le 2. \tag{1.10}$$

Let $\eta = (\eta_{ab})_{a,b=0,1,2} = \mathrm{diag}(-1,1,1)$, then we can show that

$$[\Omega_{ab}, \partial_c] = \eta_{ac}\partial_b - \eta_{bc}\partial_a, \tag{1.11}$$

$$[\Omega_{ab}, \Omega_{cd}] = \eta_{ac}\Omega_{bd} - \eta_{bc}\Omega_{ad} + \eta_{ad}\Omega_{bc} - \eta_{bd}\Omega_{ac}, \tag{1.12}$$

$$[\Omega_{ab}, \Gamma_0] = 0, \tag{1.13}$$

$$[\Gamma_0, \partial_a] = -\partial_a \tag{1.14}$$

for all $a,b,c,d = 0,1,2$. Define $\Gamma_1 = \Omega_{01}$, $\Gamma_2 = \Omega_{02}$, $\Gamma_3 = \Omega_{12}$, $\Gamma_4 = \partial_t$, $\Gamma_5 = \partial_1$ and $\Gamma_6 = \partial_2$. For any multi-index $I = (I^0, I^1, \cdots, I^6)$ we write $\Gamma^I = \Gamma_0^{I^0}\Gamma_1^{I^1}\cdots\Gamma_6^{I^6}$. We can easily verify from (1.10) that if v satisfies $\square v = f$, then $\Gamma^I v$ satisfies

$$\square(\Gamma^I v) = \sum_{|J|\le|I|} C_{IJ}\Gamma^J f$$

with appropriate constants C_{IJ}. Note that $C_{II} = 1$, and $C_{IJ} = 0$ for any multi-index J with $|J| = |I|$ and $J \neq I$. For any smooth function $v = v(t,x)$, (1.10) - (1.14) imply that

$$\Gamma^I\Gamma^J v = \Gamma^{I+J}v + \sum_{|K|\le|I|+|J|-1} C_K^{IJ}\Gamma^K v \tag{1.15}$$

and especially that

$$\Gamma^I\partial_a v = \partial_a\Gamma^I v + \sum_{b=0}^{2}\sum_{|J|\le|I|-1} C_{Jb}^{Ia}\partial_b\Gamma^J v \quad \text{for } a = 0,1,2 \tag{1.16}$$

with some appropriate constants C_K^{IJ} and C_{Jb}^{Ia}.

For any non-negative integer s and for any scalar or vector-valued function $v(t,x)$, we set

$$|v(t,x)|_s = \sum_{|I|\le s}|\Gamma^I v(t,x)|, \tag{1.17}$$

$$\|v(t)\|_{p,s} = \left(\int_{\mathbf{R}^2}|v(t,x)|_s^p dx\right)^{\frac{1}{p}} \quad \text{for } 1 \le p \le \infty, \tag{1.18}$$

$$\|v(t)\|_{\infty,s} = \operatorname*{esssup}_{x\in\mathbf{R}^2}|v(t,x)|_s. \tag{1.19}$$

When $s = 0$ we write $\|v(t)\|_p$ for $\|v(t)\|_{p,0}$ $(1 \le p \le \infty)$. Note that $\|v(t)\|_p$ coincides with the usual L^p-norm of $v(t,\cdot)$.

To show the global existence theorem under the assumptions (H1), (H2) and (H3) in the former paper [8], we got *a priori* bounds for $\|u(t)\|_{2,2k}$, $\|u'(t)\|_{2,2k}$ and $|u(t,x)|_{k+2}$ with some decay with respect to time and space variables, where k is a sufficiently large positive integer. To treat the cubic parts of the nonlinear terms in the estimates of $|u(t,x)|_{k+2}$ and of $\|u(t)\|_{2,2k}$, we use the pointwise estimates for the functions satisfying the null condition, which were first derived by Klainerman [11] (see Lemma 3.2 below). Then we can get an extra decay with respect to time and we can regard the cubic terms as if they were the terms of degree 5 from the point of the decay rates. In this case, the worst terms which cause the singularity of the solution are terms like $u_a u_b u_c u_d$. More precisely, because even the L^2-norm of the solution to the linear equation $\Box u = 0$ is not expected to be bounded in two space-dimensional case, terms of the form $u_a u_b u_c u_d$ have insufficient decay and cause the problem in the estimate of $\|u(t)\|_{2,2k}$. Therefore we must exclude such terms and this is the reason why we assumed (H3) in [8].

In this paper, we will get control of $\|u(t)\|_{\frac{2}{1-\gamma},2k}$ with some small $\gamma > 0$, instead of $\|u(t)\|_{2,2k}$. In the estimate for $\|u(t)\|_{\frac{2}{1-\gamma},2k}$, terms like $u_a u_b u_c u_d$ cause no trouble, but as we must make utility of Lemma 3.2 to treat the cubic terms, we need *a priori* bound for $\|u'(t)\|_{2,2k+1}$ (a bound for $\|u(t)\|_{p,2k+1}$ with some $p \geq 1$ is not needed, because of Corollary 3.5). Note that when we try to get a bound for $\|u(t)\|_{2,2k}$, we only need a bound for $\|u'(t)\|_{2,2k}$ and not $\|u'(t)\|_{2,2k+1}$ although we use Lemma 3.2, because we can estimate $\|u(t)\|_{2,2k}$ in terms of some norms of $|\Box u(t,x)|_{2k-1}$ (see Proposition 3.4 in [8]).[†] This causes the difficulty. Namely, when we use the energy inequality (Lemma 2.4) to get a bound for $\|u'(t)\|_{2,2k+1}$, we have to get control of $\|u(t)\|_{p,2k+1}$ with some $p \geq 1$ to estimate $\|u_a u_b u_c u_d(t)\|_{2,2k+1}$ when the nonlinear terms contain $u_a u_b u_c u_d$ (otherwise, we do not need it because of Corollary 3.5). Summing up, we need some information about $|u(t,\cdot)|_{2k+1}$ to get control of $\|u(t)\|_{\frac{2}{1-\gamma},2k}$. In other words, we meet a kind of the loss of derivative. Of course we can estimate $\|u(t)\|_{2,2k+1}$ in terms of $\|u'(t)\|_{2,2k+1}$ using Poincaré's inequality or the relation $u(t) = u(0) + \int_0^t \partial_t u(s,x)ds$, but these estimates give the loss of the factor $1+t$ in the decay rate of the nonlinear terms. This is the main difficulty in the proof of our theorem.

We will overcome this difficulty as follows. The solution of linear wave equation behaves like $(1+t+|x|)^{-1/2}(1+|t-|x||)^{-1/2}$ (see Lemma 2.2). If the behavior of the solution to (1.1) - (1.2) is similar to that of the solution to the linear problems, we have

$$\|u_a u_b u_c u_d(t)\|_{2,2k+1} \leq C(1+t)^{-\frac{3}{2}} \left\| \frac{|u(t,\cdot)|_{2k+1}}{(1+|t-|\cdot||)^{\frac{3}{2}}} \right\|_2 .$$

[†] Roughly speaking, this comes from the fact that $\|u'(t)\|_2$ can be estimated in terms of $\|\Box u\|_2$ by the energy inequality. Estimates like this do not hold for L^p-norms of u' when $p \neq 2$ and generally we need some norms of $(\Box u)'$ to estimate $\|u'(t)\|_p$.

Then, since $u(t,x)$ is compactly supported for any fixed t, we can estimate the right-hand side of the above inequality in terms of $\|u'(t)\|_{2,2k+1}$ (see Lemma 3.3 below, which is more precise than the Poincaré's inequality in some sense) and this shows that

$$\|u_a u_b u_c u_d(t)\|_{2,2k+1} \le C(1+t)^{-\frac{3}{2}}\|u'(t)\|_{2,2k+1}.$$

Therefore no information about $|u(t,x)|_{2k+1}$ is needed and no loss of the decay rate appears. This is the main idea of our proof.

Our plan in this paper is as follows. In Section 2, we state some known estimates for linear wave equations in two space-dimensions almost without proofs. We get estimates of nonlinear terms in Section 3, and derive some estimates to overcome the difficulty mentioned above. Finally in Section 4, we prove Theorem 1.2 by deriving *a priori* estimates.

§2. Preliminary Results for Linear Wave Equations

In this section we recall some estimates for the linear wave equations. First, using Γ's, we get $L^1 - L^\infty$ estimate for wave equations, which was first derived by Klainerman [11] in three space-dimensions and was extended by Hörmander [3] to arbitrary space-dimensions. We state here only the two space-dimensional case.

Lemma 2.1. *Let v be a solution of $\square v(t,x) = f(t,x)$ in $(0,\infty)\times\mathbf{R}^2$ with initial data* 0. *Suppose that $0 \le \kappa \le 1$. Then*

$$(1+t+|x|)^{\frac{1}{2}}(1+|t-|x||)^{\frac{1-\kappa}{2}}|u(t,x)| \le C\int_0^t \frac{\|f(\tau)\|_{1,1}}{(1+\tau)^{\kappa/2}}d\tau, \tag{2.1}$$

provided that f is sufficiently smooth and the right-hand side of (2.1) *is finite. Here C is a constant depending only on κ.*

Proof. See Hörmander [3]. See also Katayama [8] for the above expression of the assertion. □

For the Cauchy problem with non-zero initial data, we get the following.

Lemma 2.2. *Let v be a solution of*

$$\square v(t,x) = 0 \quad in\ (0,\infty)\times\mathbf{R}^2,$$

$$v(0,x) = \varepsilon\phi(x),\ (\partial_t v)(0,x) = \varepsilon\psi(x).$$

Suppose that $\phi,\psi \in C_0^\infty(\mathbf{R}^2)$. Then we have

$$\text{(i)} \qquad |v(t,x)| \le C\varepsilon(1+t+|x|)^{-\frac{1}{2}}(1+|t-|x||)^{-\frac{1}{2}} \ in \ (0,\infty)\times\mathbf{R}^2,$$

$$\text{(ii)} \quad \|v(t)\|_p \le C_p\varepsilon(1+t)^{-\frac{1}{2}+\frac{1}{p}} \ \textit{for any } p>2 \textit{ with some positive constant } C_p.$$

Proof. (i) is a consequence of Lemma 2.1. (For the direct proof, see Kovalyov [13] for instance.) Observing that

$$\left\|(1+t+|\cdot|)^{-\frac{1}{2}}(1+|t-|\cdot||)^{-\frac{1}{2}}\right\|_p \le C(1+t)^{-\frac{1}{2}+\frac{1}{p}}$$

for $p>2$, we obtain (ii) immediately from (i). ∎

The next lemma is due to Li - Zhou [17] (see also Li - Yu - Zhou [15]). We state it without the proof.

Lemma 2.3. *Suppose that v is a solution to*

$$\square v(t,x)=f(t,x) \quad in \ (0,\infty)\times\mathbf{R}^2$$

with initial data 0. *Let* $p>2$ *be given. Then there exists a positive constant C such that*

$$\|v(t)\|_p \le C\int_0^t \|f(\tau)\|_q d\tau \tag{2.2}$$

holds for $t\ge 0$, *where q satisfies*

$$\frac{1}{q}=\frac{1}{p}+\frac{1}{2}.$$

We conclude this section with the well-known energy estimates.

Lemma 2.4. *Let v be a smooth function satisfying*

$$\square v(t,x)-\sum_{a+b>0}\gamma_{ab}(t,x)\partial_a\partial_b v(t,x)=f(t,x) \quad in \ (0,\infty)\times\mathbf{R}^2.$$

Suppose that $\Sigma_{a,b}|\gamma_{ab}(t,x)|\le 1/2$ *for all* $(t,x)\in[0,\infty)\times\mathbf{R}^2$. *Define*

$$\|v(t)\|_E^2=\int_{\mathbf{R}^2}\left\{|\partial_t v|^2+\sum_{i,j=1}^{2}\beta_{ij}\partial_i v\partial_j v\right\}(t,x)dx,$$

where $\beta_{ij}=\delta_{ij}+\gamma_{ij}$ *for any* $i,j=1,2$ *with* $\delta_{ij}=1\,(i=j)$ *and* $\delta_{ij}=0\,(i\ne j)$. *Then there is a positive constant* C_1 *such that*

$$\frac{d}{dt}\|v(t)\|_E \le C_1\|\gamma'(t)\|_\infty \|v(t)\|_E + \|f(t)\|_2\,, \; t \ge 0\,, \tag{2.3}$$

where $\|\gamma'(t)\|_\infty = \max_{a,b,c} \|\partial_c \gamma_{ab}(t,\cdot)\|_\infty$. *Furthermore,*

$$\frac{1}{C_2}\|v(t)\|_E \le \|v'(t)\|_2 \le C_2\|v(t)\|_E \tag{2.4}$$

holds with some positive constant C_2.

Proof. See Klainerman [9] for instance. ∎

§3. Estimates of Nonlinear Terms

Now we turn our attention to the Cauchy problem (1.1) - (1.2). For any smooth functions f and g, define

$$Q_0(f,g) = \partial_t f\, \partial_t g - \sum_{j=1}^{2} \partial_j f\, \partial_j g\,, \tag{3.1}$$

$$Q_{ab}(f,g) = \partial_a f\, \partial_b g - \partial_b f\, \partial_a g \quad \text{for } a,b = 0,1,2\,. \tag{3.2}$$

These forms are closely connected to the null condition. In fact, if $G(u,u',u'')$ is a homogeneous polynomial of degree 3 satisfying the null condition, then G is a linear combination of $v_1 Q(v_2,v_3)$, where v_1 is any of u_j $(j=1,\cdots,N)$, $\partial_c u_j$ $(c=0,1,2,\ j=1,\cdots,N)$ or $\partial_c\partial_d u_j$ $(c,d=0,1,2,\ j=1,\cdots,N)$, v_2 and v_3 are any of u_j $(j=1,\cdots,N)$ or $\partial_c u_j$ $(c=0,1,2,\ j=1,\cdots,N)$, and Q is any of the forms (3.1) – (3.2) (see Katayama [8]).

Simple calculations yield the following (see Klainerman [11]):

Lemma 3.1. *For any smooth functions f and g, we have*

(i) $$Q_0(f,g)(t,x) = \frac{1}{t}\left(\partial_t f\, \Gamma_0 g - \sum_{j=1}^{2}\Omega_{0j} f\, \partial_j g\right)(t,x)\,,$$

(ii) $$Q_{0j}(f,g)(t,x) = \frac{1}{t}(\partial_t f\, \Omega_{0j} g - \Omega_{0j} f\, \partial_t g)(t,x)\,,\ j=1,2\,,$$

(iii) $$Q_{12}(f,g)(t,x) = \frac{1}{t}(\Omega_{01} f\, \partial_2 g - \Omega_{02} f\, \partial_1 g - \partial_t f\, \Omega_{12} g)(t,x)$$

for $t > 0$.

From (1.11) and (1.14), we can verify that

$$\Gamma^I Q_0(f,g) = Q_0(\Gamma^I f, g) + Q_0(f, \Gamma^I g) + \sum_{|J|+|K|\le|I|-1} C^I_{JK} Q_0(\Gamma^J f, \Gamma^K g)\,, \tag{3.3}$$

(3.4) $$\Gamma^I Q_{ab}(f,g) = Q_{ab}(\Gamma^I f, g) + Q_{ab}(f, \Gamma^I g) + \sum_{c,d=0}^{2} \sum_{|J|+|K|\le|I|-1} C_{JK,cd}^{I,ab} Q_{cd}(\Gamma^J f, \Gamma^K g)$$

for any $a,b = 0,1,2$ with appropriate constants C_{JK}^I and $C_{JK,cd}^{I,ab}$. Observing these facts, we can obtain basic estimates of nonlinear terms for our proof of Theorem 1.2.

Lemma 3.2. *Assume that $F(u,u',u'') = (F_i(u,u',u''))_{i=1,\dots,N}$ satisfies the assumption* (H2), *namely we assume that F_i can be written as*

$$F_i(u,u',u'') = G_i(u,u',u'') + H_i(u,u',u''),\ i = 1,\cdots,N$$

near the origin, where G_i $(i=1,\dots,N)$ are homogeneous polynomials of degree 3 *satisfying the null condition, H_i $(i=1,\dots,N)$ are smooth functions of degree* 4. *Let s be an integer ≥ 0. Suppose that $|u(t,x)|_{\left[\frac{s}{2}\right]+2}$ is sufficiently small ($|u(t,x)|_{\left[\frac{s}{2}\right]+2} \le 1$, say). Here $[m]$ represents the largest integer which does not exceed m. Then we have*

(3.5) $$|G_i(u,u',u'')(t,x)|_s \le C_s(1+t)^{-1} \sum_{S_1(s)} \left|\prod_{j=1}^{3} \partial^{m_j}\Gamma^{I_j} u(t,x)\right|,\ i = 1,\cdots,N,$$

where $m_j = (m_j^0, m_j^1, m_j^2)$, $I_j = (I_j^0,\cdots,I_j^6)$ are multi-indices, $\partial^{m_j} = \partial_0^{m_j^0}\partial_1^{m_j^1}\partial_2^{m_j^2}$, $\Gamma^{I_j} = \Gamma_0^{I_j^0}\cdots\Gamma_6^{I_j^6}$ and the summation above is taken over the set

$$S_1(s) = \left\{|I_j| \le \left[\frac{s+3}{2}\right] (j=1,2), |I_3| \le s+1, |m_j| \le 1\ (j=1,2,3), \sum_{j=1}^{3}|m_j| \ne 0\right\},$$

and

(3.6) $$|H_i(u,u',u'')(t,x)|_s \le C_s \sum_{S_2(s)} \left|\prod_{j=1}^{4} \partial^{n_j}\Gamma^{J_j} u(t,x)\right|,\ i = 1,\cdots,N,$$

where $n_j = (n_j^0, n_j^1, n_j^2)$, $J_j = (J_j^0,\cdots,J_j^6)$ are multi-indices, and the summation above is taken over the set

$$S_2(s) = \left\{|J_j| \le \left[\frac{s}{2}\right] (j=1,2,3), |J_4| \le s, |m_j| \le 2\ (j=1,2,3,4)\right\}.$$

Proof. Because $G_i(u,u',u'')$ satisfies the null condition, it follows from Definition 1.1 that $G_i(u,0,0)=0$ for $1 \le i \le N$. Therefore, if we apply Leibniz' formula, we can see that $\Gamma^I G_i(u,u',u'')$ is a linear combination of

$\prod_{j=1}^3 \Gamma^{I_j}(\partial^{m_j} u_{k(j)})$ with $|m_j| \le 2$, $\sum_{j=1}^3 |m_j| \ne 0$, $\sum_{j=1}^3 |I_j| = |I|$ and $k(j) \in \{1, \cdots, N\}$ for $j = 1,2,3$. By (1.16), this can be written as a linear combination of $\prod_{j=1}^3 \partial^{m_j} \Gamma^{I_j} u_{k(j)}$ with $|m_j| \le 2$, $\sum_{j=1}^3 |m_j| \ne 0$ and $\sum_{j=1}^3 |I_j| \le |I|$. Since we may assume $|I_1| \le |I_2| \le |I_3|$, (3.5) holds for $t \le 1$.

On the other hand, noting (3.3) and (3.4), we obtain from Lemma 3.1 that $\Gamma^I Q_0(f,g)$ and $\Gamma^I Q_{ab}(f,g)$ are linear combinations of $\frac{1}{t}\partial^{m_1}\Gamma^{I_1} f \Gamma^{I_2} g$ or $\frac{1}{t}\partial^{m_1}\Gamma^{I_1} g \Gamma^{I_2} f$ with $|m_1| = 1$, $|I_1| + |I_2| \le |I| + 1$, and $|I_1| \le |I|$. Since G_i is a linear combination of $v_1 Q(v_2, v_3)$ with $v_1 = u_l, \partial_c u_l, \partial_c \partial_d u_l$, $v_2, v_3 = u_l, \partial_c u_l$, this implies (3.5) for $t \ge 1$.

Similarly, since H_i is a function of degree 4 and $|u(t,x)|_{\left[\frac{s}{2}\right]+2} \le 1$, it follows from Leibniz' formula that $|H_i(u,u',u'')|_s \le C_s \sum |\prod_{j=1}^4 \Gamma^{J_j} \partial^{n_j} u_{k(j)}|$, $i = 1, \cdots, N$ with $\sum_{j=1}^4 |J_j| \le s$ and $|n_j| \le 2$. Since we may assume $|J_1| \le |J_2| \le |J_3| \le |J_4|$, we immediately obtain (3.6) using (1.16). ◻

The following two lemmas will be used to conquer the difficulty we mentioned in the introduction. These lemmas are essentially due to Lindblad [20], but since they play important roles in our proof of Theorem 1.2, we recall the proofs here.

Lemma 3.3. *Let* $v \in C^1([0,T) \times \mathbb{R}^2)$. *Suppose that*

$$\operatorname{supp} v(t,\cdot) \subset \{x \in \mathbb{R}^2; |x| \le t + \rho\}$$

with some $\rho > 0$. *Then*

$$\left\| \frac{v(t,\cdot)}{1 + |t - |\cdot||} \right\|_2 \le C_\rho \|\partial_r v(t,\cdot)\|_2 \quad \text{for } 0 < t < T, \tag{3.7}$$

where $\partial_r = \sum_{j=1}^2 (x_j/|x|)\partial_j$ *and* C_ρ *is a constant depending only on* ρ.

Proof. First let $w = w(x) \in C^1(\mathbb{R}^2)$ and $\operatorname{supp} w \subset \{x \in \mathbb{R}^2; |x| \le R\}$ with some $R > 0$. We claim that

$$\int_{\mathbb{R}^2} \frac{|w(x)|^2}{(1 + R - |x|)^2} dx \le 4 \int_{\mathbb{R}^2} |\partial_r w(x)|^2 dx. \tag{3.8}$$

In fact, switching to the polar coordinates, we get

$$\int_{\mathbb{R}^2} \frac{|w(x)|^2}{(1 + R - |x|)^2} dx = \int_0^{2\pi} \int_0^R \frac{|w(r,\theta)|^2}{(1 + R - r)^2} r dr d\theta.$$

By integration by parts, we have

$$\int_0^R \frac{|w|^2 r}{(1+R-r)^2}dr = -\int_0^R \frac{w^2+2w\partial_r wr}{1+R-r}dr$$

$$\leq -2\int_0^R \frac{w\partial_r w}{1+R-r}rdr\,.$$

By the Schwarz inequality, we get

$$\int_0^R \frac{|w|^2 r}{(1+R-r)^2}dr \leq 2\left(\int_0^R \frac{|w|^2 r}{(1+R-r)^2}dr\right)^{\frac{1}{2}}\left(\int_0^R |\partial_r w|^2 rdr\right)^{\frac{1}{2}}.$$

This implies (3.8) immediately.

Applying (3.8) to $v(t,x)$, we get

$$\int_{\mathbf{R}^2} \frac{|v(t,x)|^2}{(1+\rho+t-|x|)^2}dx \leq 4\int_{\mathbf{R}^2} |\partial_r v(t,x)|^2 dx\,.$$

Since $1+\rho+t-|x| \leq (1+\rho)(1+|t-|x||)$, this completes the proof. ∎

Lemma 3.4. *Let $v \in C^1([0,t]\times\mathbf{R}^2)$. Then we have*

$$(1+|t-|x||)\,|\partial_a v(t,x)| \leq C|v(t,x)|_1 \quad \textit{for } a=0,1,2. \tag{3.9}$$

Proof. By direct calculations we have

$$(t-|x|)\partial_t v(t,x) = \frac{1}{t+|x|}\left(t\Gamma_0 v(t,x) - x_1\Omega_{01}v(t,x) - x_2\Omega_{02}v(t,x)\right),$$

$$(t-|x|)\partial_1 v(t,x) = \frac{1}{t+|x|}\left(t\Omega_{01} v(t,x) - x_1\Gamma_0 v(t,x) + x_2\Omega_{12}v(t,x)\right),$$

$$(t-|x|)\partial_2 v(t,x) = \frac{1}{t+|x|}\left(t\Omega_{02} v(t,x) - x_1\Omega_{12} v(t,x) - x_2\Gamma_0 v(t,x)\right).$$

Therefore (3.9) holds when $|t-|x|| \geq 1$. On the other hand, it is clear that (3.9) holds when $|t-|x|| < 1$. ∎

Lemmas 3.3 and 3.4 give us the next estimate.

Corollary 3.5. *Let v_1, v_2 and v_3 be smooth functions. Suppose that* supp $v_3(t,\cdot) \subset \{x \in \mathbf{R}^2; |x| \leq t+\rho\}$ *with some positive constant ρ. Let $1 \leq p \leq 2$. Then we have*

$$\|v_1(\partial_a v_2)v_3(t)\|_p \leq C_\rho \|\,|v_1(t)|\,|v_2(t)|_1\|_q \|v_3'(t)\|_2 \quad \textit{for } a=0,1,2,$$

where $1/q = 1/p - 1/2$.

Proof. By Hölder's inequality, we have

$$\|v_1(\partial_a v_2)v_3(t)\|_p \le \|(1+|t-|\cdot\||)v_1\partial_a v_2(t)\|_q \left\|(1+|t-|\cdot\||)^{-1}v_3\right\|_2 .$$

Lemma 3.3 shows that $\left\|(1+|t-|\cdot\||)^{-1}v_3(t)\right\|_2 \le C_p\|v_3'(t)\|_2$. On the other hand, as it follows from Lemma 3.4 that

$$(3.10) \qquad |(1+|t-|\cdot\||)v_1\partial_a v_2(t,x)| \le C|v_1(t,x)|\,|v_2(t,x)|_1 ,$$

we get

$$\|(1+|t-|\cdot\||)v_1\partial_a v_2\|_q \le C\|\,|v_1(t)|\,|v_2(t)|_1\|_q .$$

This completes the proof. ■

For the later convenience, we prepare the following lemma before concluding this section.

Lemma 3.6. *Assume that a smooth function v(t,x) satisfies*

$$\sup_{x\in\mathbf{R}^2}(1+t+|x|)^{\frac{1}{2}}(1+|t-|x||)^{\frac{1-\kappa}{2}}|v(t,x)| \le M$$

with some $M>0$. *If* α *and p are positive constants satisfying* $\alpha p(1-\kappa)>2$, *then*

$$(3.11) \qquad \|\,|v(t,\cdot)|^\alpha\|_p \le C_{\alpha,p,\kappa}M^\alpha(1+t)^{-\frac{\alpha}{2}+\frac{1}{p}}$$

holds for $0\le t<T$ *with some positive constant* $C_{\alpha,p,\kappa}$.

Proof. By straightforward calculations. See Katayama [8; Lemma 3.6] for the details of the proof. ■

§4. Proof of Theorem 1.2

In order to prove Theorem 1.2, it suffices to get the following:

Proposition 4.1. *Let* $u\in C^\infty([0,T)\times\mathbf{R}^2;\mathbf{R}^N)$ *be a solution to the Cauchy problem* (1.1) - (1.2), *i.e.*,

$$\begin{cases}\Box u = F(u,u',u'') & \text{in } (0,\infty)\times\mathbf{R}^2,\\ u(0,x)=\varepsilon\phi(x),\ \partial_t u(0,x)=\varepsilon\psi(x).\end{cases}$$

Define

$$(4.1)\ D_\varepsilon(t)=\sup_{x\in\mathbf{R}^2}\left\{(1+t+|x|)^{\frac{1}{2}}(1+|t-|x||)^{\frac{1}{3}}|u(t,x)|_{k+2}\right\}+(1+t)^{-\mu}\left(\|u(t)\|_{\frac{2}{1-\gamma},2k}+\|u'(t)\|_{2,2k+1}\right),$$

where k *is some fixed integer with* $k \geq 5$, γ *and* μ *are constants satisfying* $0 < \gamma < 1$ *and* $0 < \mu < 1/10$ *respectively.*

If we choose sufficiently small γ *and* μ, *then there exists a positive constant* $M_0 \leq 1$ *such that for any* $M \leq M_0$,

$$\sup_{0 \leq t < T} D_\varepsilon(t) \leq M$$

implies that

$$\sup_{0 \leq t < T} D_\varepsilon(t) \leq C_0(\varepsilon + M^3),$$

where C_0 *is a positive constant which is independent of* T, ε *and* $M(\leq 1)$.

Once we establish Proposition 4.1, we can obtain *a priori* estimates. Choose $M_1(\leq M_0)$ and ε_1 to satisfy $C_0 M_1^2 \leq 1/4$ and $C_0 \varepsilon_1 \leq M_1/4$. We can find some ε_2 such that $D_\varepsilon(0) \leq M_1/2$ holds for any $\varepsilon \leq \varepsilon_2$. By the classical local existence theorem, there exists a solution $u \in C^\infty([0,T) \times \mathbb{R}^2; \mathbb{R}^N)$ to (1.1) - (1.2) with some $T > 0$. Let $\varepsilon \leq \varepsilon_0 = \max\{\varepsilon_1, \varepsilon_2\}$. Then, from the continuity of $D_\varepsilon(t)$ with respect to t,

$$\sup_{0 \leq t < T_0} D_\varepsilon(t) \leq M_1$$

holds with some $T_0 \in (0, T]$. Let T_0 be the maximal of such $T_0 \in (0, T]$. Now assume that $T_0 < T$. Then, since Proposition 4.1 implies that

$$\sup_{0 \leq t < T_0} D_\varepsilon(t) \leq C_0(\varepsilon + M_1^3) \leq \frac{M_1}{2},$$

it follows from the continuity of $D_\varepsilon(t)$ that

$$\sup_{0 \leq t < T_0 + \delta} D_\varepsilon(t) \leq M_1$$

holds with some $\delta > 0$. This contradicts the definition of T_0. Therefore $T_0 = T$. This means that $D_\varepsilon(t) \leq M_1$ holds as long as the solution to (1.1) - (1.2) exists, provided that $\varepsilon \leq \varepsilon_0$. Combining this *a priori* estimate with the local existence theorem, we get the global existence of the solution to (1.1) - (1.2). This shows Theorem 1.2.

Now we prove Proposition 4.1 to complete the proof of Theorem 1.2.

Proof of Proposition 4.1. Since $\phi, \psi \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^N)$, there exists a positive constant ρ such that

$$\text{supp } \phi \cup \text{ supp } \psi \subset \{x \in \mathbb{R}^2; |x| \leq \rho\}.$$

Then it is well-known that

$$\operatorname{supp} u(t,\cdot) \subset \{x \in \mathbf{R}^2; |x| \le t+\rho\}$$

holds for the solution $u(t,x)$ to (1.1) - (1.2). In what follows, we assume that M is so small that the assumption of Lemma 3.2 is fulfilled.

*Step*1: L^∞*-estimates*. Let $|I| \le k+2$. Since $\Gamma^I u$ satisfies

$$\Box(\Gamma^I u) = \sum_{|J| \le k+2} C_{IJ} \Gamma^J F$$

for $0 \le t < T$, from Lemma 2.1 with $\kappa = 1/3$ and Lemma 2.2, we have

$$(4.2) \qquad (1+t+|x|)^{\frac{1}{2}}(1+|t-|x||)^{\frac{1}{3}}|\Gamma^I u(t,x)|$$

$$\le C\left(\varepsilon + \int_0^t \frac{\|F(u,u',u'')(\tau)\|_{1,k+3}}{(1+\tau)^{\frac{1}{6}}}d\tau\right)$$

for $0 \le t < T$. From Lemma 3.2 we have

$$(4.3) \qquad \|F(u,u',u'')(\tau)\|_{1,k+3} \le C(1+\tau)^{-1} \sum_{S_1(k+3)} \left\| \prod_{l=1}^{3} |\partial^{m_l}\Gamma^{I_l} u(\tau)| \right\|_1$$

$$+ C \sum_{S_2(k+3)} \left\| \prod_{l=1}^{4} |\partial^{n_l}\Gamma^{J_l} u(\tau)| \right\|_1 ,$$

where $S_1(\cdot)$ and $S_2(\cdot)$ are defined as in Lemma 3.2.

When $|m_3| = 1$, from the Schwarz inequality and Lemma 3.6 we get

$$\left\| \prod_{l=1}^{3} |\partial^{m_l}\Gamma^{I_l} u(\tau)| \right\|_1 \le \left\| |u(\tau,\cdot)|^2_{[\frac{k+6}{2}]+1} \right\|_2 \|u'(\tau)\|_{2,k+3}$$

$$\le CM^3(1+\tau)^{\mu-\frac{1}{2}} \quad \text{for } 0 \le \tau < T,$$

because $\left[\dfrac{k+6}{2}\right]+1 \le k+2$ and $k+3 \le 2k+1$ for $k \ge 3$.

If $|m_3| = 0$, $|m_1|$ or $|m_2| \ne 0$ because $\sum_{l=1}^{3} |m_l| \ne 0$ in $S_1(k+3)$. Therefore we can apply Corollary 3.5 to get

$$\left\| \prod_{l=1}^{3} |\partial^{m_l}\Gamma^{I_l} u(\tau)| \right\|_1 \le \left\| |u(\tau,\cdot)|^2_{[\frac{k+6}{2}]+1} \right\|_2 \|u'(\tau)\|_{2,k+3}$$

as above. Summing up, we obtain

$$(4.4) \qquad \sum_{S_1(k+3)} \left\| \prod_{l=1}^{3} |\partial^{m_l}\Gamma^{I_l} u(\tau)| \right\|_1 \le C(1+\tau)^{\mu-\frac{1}{2}} M^3 \quad \text{for } 0 \le \tau < T.$$

On the other hand, observing that $|\partial^{n_4}\Gamma^{J_4}u(\tau,x)|\le C|u(\tau,x)|_{k+5}$ in $S_2(k+3)$ and that $k+5\le 2k$, from Hölder's inequality and Lemma 3.6 we get

$$\left\|\prod_{j=1}^{4}|\partial^{n_j}\Gamma^{J_j}u(\tau)|\right\|_1 \le \left\||u(\tau,\cdot)|^3_{[\frac{k+2}{2}]+2}\right\|_{\frac{2}{1+\gamma}}\|u(\tau)\|_{\frac{2}{1-\gamma},k+5} \tag{4.5}$$

$$\le CM^4(1+\tau)^{\mu-\frac{3}{2}+\frac{1+\gamma}{2}} \quad \text{for } 0\le\tau<T,$$

because $\left[\dfrac{k+2}{2}\right]+2\le k+2$ for $k\ge 3$. From (4.2) - (4.5) we get

$$(1+t+|x|)^{\frac{1}{2}}(1+|t-|x||)^{\frac{1}{3}}|u(t,x)|_{k+2} \tag{4.6}$$

$$\le C\left(\varepsilon+\int_0^t\left\{M^3(1+\tau)^{\mu-\frac{3}{2}}+M^4(1+\tau)^{\mu+\frac{\gamma}{2}-1}\right\}(1+\tau)^{-\frac{1}{6}}d\tau\right)$$

$$\le C\left(\varepsilon+M^3\int_0^\infty(1+\tau)^{-1+\left(\mu-\frac{2}{3}\right)}+(1+\tau)^{-1+\left(\mu+\frac{\gamma}{2}-\frac{1}{6}\right)}d\tau\right)$$

$$\le C_1(\varepsilon+M^3) \quad \text{for } 0\le t<T,$$

provided that we choose sufficiently small γ and μ to satisfy $\mu+\dfrac{\gamma}{2}-\dfrac{1}{6}<0$.

Step 2: $L^{\frac{2}{1-\gamma}}$ *- estimates. Let* $|I|\le 2k$. *Then* $\Gamma^I u$ *satisfies*

$$\Box(\Gamma^I u)=\sum_{|J|\le 2k}C_{IJ}\Gamma^J F(u,u',u'') \text{ for } 0\le t<T.$$

From Lemma 2.2 and Lemma 2.3 with $p=\dfrac{2}{1-\gamma}$, we get

$$\|\Gamma^I u(t)\|_{\frac{2}{1-\gamma}}\le C\left(\varepsilon+\int_0^t\|F(u,u',u'')(\tau)\|_{\frac{2}{2-\gamma},2k}d\tau\right) \tag{4.7}$$

for $0\le t<T$. From Lemma 3.2 it follows that

$$\|F(u,u',u'')(\tau)\|_{\frac{2}{2-\gamma},2k}\le C(1+\tau)^{-1}\sum_{S_1(2k)}\left\|\prod_{j=1}^{3}|\partial^{m_j}\Gamma^{I_j}u(\tau)|\right\|_{\frac{2}{2-\gamma}} \tag{4.8}$$

$$+C\sum_{S_2(2k)}\left\|\prod_{j=1}^{4}|\partial^{n_j}\Gamma^{J_j}u(\tau)|\right\|_{\frac{2}{2-\gamma}}$$

Using Hölder's inequality when $|m_3| \neq 0$ and Corollary 3.5 when $|m_3| = 0$, from Lemma 3.6 we get

$$\left\| \prod_{j=1}^{3} |\partial^{m_j} \Gamma^{l_j} u(\tau)| \right\|_{\frac{2}{2-\gamma}} \le C \| |u(\tau)|_{k+2}^2 \|_{\frac{2}{1-\gamma}} \|u'(\tau)\|_{2,2k+1} \tag{4.9}$$

$$\le CM^3(1+\tau)^{-1+\frac{1-\gamma}{2}+\mu} \quad \text{for } 0 \le \tau < T.$$

When $|n_4| \neq 0$ in $S_2(2k)$, Hölder's inequality and Lemma 3.6 imply that

$$\left\| \prod_{j=1}^{4} |\partial^{n_j} \Gamma^{J_j} u(\tau)| \right\|_{\frac{2}{2-\gamma}} \le C \| |u(\tau)|_{k+2}^3 \|_{\frac{2}{1-\gamma}} \left(\|u'(\tau)\|_{2,2k} + \|u''(\tau)\|_{2,2k} \right) \tag{4.10}$$

$$\le C \| |u(\tau)|_{k+2}^3 \|_{\frac{2}{1-\gamma}} \|u'(\tau)\|_{2,2k+1}$$

$$\le CM^4(1+\tau)^{\mu-\frac{3}{2}+\frac{1-\gamma}{2}} \le CM^4(1+\tau)^{\mu-1}$$

$$\left\| \prod_{j=1}^{4} |\partial^{n_j} \Gamma^{J_j} u(\tau)| \right\|_{\frac{2}{2-\gamma}} \le C \| |u(\tau)|_{k+2}^3 \|_2 \|u(\tau)\|_{\frac{2}{1-\gamma},2k} \tag{4.11}$$

$$\le CM^4(1+\tau)^{\mu-1} \text{ for } 0 \le \tau < T.$$

Because it follows from (4.8) - (4.11) that

$$\|F(u,u',u'')(\tau)\|_{\frac{2}{2-\gamma},2k} \le C\left\{ M^3(1+\tau)^{\mu-\frac{3}{2}-\frac{\gamma}{2}} + M^4(1+\tau)^{\mu-1} \right\}$$

$$\le CM^3(1+\tau)^{\mu-1} \quad \text{for } 0 \le \tau < T,$$

from (4.7) we obtain

$$(1+t)^{-\mu} \|u(t)\|_{\frac{2}{1-\gamma},2k} \le C_2(\varepsilon + M^3) \quad \text{for } 0 \le t < T. \tag{4.12}$$

Step 3: *The energy estimates*. Finally let $|I| \le 2k+1$. Then $\Gamma^I u_i$ satisfies

$$\Box(\Gamma^I u_i) - \sum_{a+b>0} \frac{\partial F_i}{\partial u_{i,ab}} \partial_a \partial_b \Gamma^I u_i = R_{I,i}, \; i = 1, \cdots, N, \tag{4.13}$$

where

$$R_{I,i} = \sum_{|J| \le |I|} C_{IJ} \Gamma^J F_i(u,u',u'') - \sum_{a+b>0} \frac{\partial F_i}{\partial u_{i,ab}} \partial_a \partial_b \Gamma^I u_i \tag{4.14}$$

$$= \left(\Gamma^I F_\iota(u,u',u'') - \sum_{a+b>0} \frac{\partial F_\iota}{\partial u_{\iota,ab}} \partial_a \partial_b \Gamma^I u_\iota \right)$$

$$+ \sum_{|J| \le 2k} C_{IJ} \Gamma^J F_\iota(u,u',u'') .$$

Since

$$\left\| \frac{\partial F_\iota}{\partial u_{\iota,ab}} \right\|_\infty \le C \| |u(t,\cdot)|^2_{k+2} \|_\infty \le CM^2(1+t)^{-1}$$

for $0 \le t < T$, we can choose M_0 such that $\sum \left\| \frac{\partial F_\iota}{\partial u_{\iota,ab}} \right\|_\infty \le CM_0^2 < 1/2$. Then for any $M \le M_0$, applying Lemma 2.4 with $v = \Gamma^I u_\iota$, we get

$$\frac{d}{dt} \|\Gamma^I u_\iota\|_E \le CM^2(1+t)^{-1} \|\Gamma^I u_\iota\|_E + \|R_{I,\iota}\|_2 \tag{4.15}$$

$$\le CM^3(1+t)^{\mu-1} + \|R_{I,\iota}\|_2 \quad \text{for } 0 \le t < T,$$

because $\|\Gamma^I u_\iota\|_E \le C\|u'(t)\|_{2,2k+1} \le CM(1+t)^\mu$ holds for $0 \le t < T$. From the assumption (H1) and (1.16), we can show as in the proof of Lemma 3.2 that

$$|R_{I,\iota}| \le C \left(\sum \prod_{j=1}^{3} |\partial^{m_j} \Gamma^{I_j} u| + \sum \prod_{j=1}^{4} |\partial^{n_j} \Gamma^{J_j} u| \right), \tag{4.16}$$

where the summations are taken over the sets

$$\left\{ \sum_{j=1}^{3} |I_j| \le 2k+1, |m_j| \le 2, \sum_{j=1}^{3} |m_j| \ne 0, |I_j| \le 2k \text{ when } |m_j| = 2 \right\}$$

and

$$\left\{ \sum_{j=1}^{4} |J_j| \le 2k+1, |n_j| \le 2, |J_j| \le 2k \text{ when } |n_j| = 2 \right\}$$

respectively. We may assume that $|I_1| \le |I_2| \le |I_3|$ and that $|J_1| \le \cdots \le |J_4|$. Then we get

$$|I_1|, |I_2| \le k, |I_3| \le \begin{cases} 2k+1, & |m_3| \le 1, \\ 2k, & |m_3| = 2, \end{cases}$$

$$|J_j| \le k \, (j=1,2,3), |J_4| \le \begin{cases} 2k+1, & |n_4| \le 1, \\ 2k, & |n_4| = 2. \end{cases}$$

Using Corollary 3.5 if necessary, we get

$$\left\| \prod_{j=1}^{3} |\partial^{m_j} \Gamma^{I_j} u(\tau)| \right\|_2 \le \|u(\tau)\|^2_{\infty,k+2} \left(\|u'(\tau)\|_{2,2k+1} + \|u''(\tau)\|_{2,2k} \right) \tag{4.17}$$

$$\leq CM^3(1+\tau)^{\mu-1} \quad \text{for } 0 \leq \tau < T.$$

When $|n_4| \neq 0$, we have

$$\left\| \prod_{l=1}^{4} |\partial^{n_l}\Gamma^{J_l}u(\tau)| \right\|_2 \leq \|u(\tau)\|^3_{\infty,k+2}\left(\|u'(\tau)\|_{2,2k+1} + \|u''(\tau)\|_{2,2k}\right) \tag{4.18}$$

$$\leq C\|u(\tau)\|^3_{\infty,k+2}\|u'(\tau)\|_{2,2k+1}$$

$$\leq CM^4(1+\tau)^{\mu-\frac{3}{2}} \quad \text{for } 0 \leq \tau < T.$$

If $|n_4| = 0$, observing that $\left|(1+|\tau-|\cdot||)^{\frac{1}{3}}u(\tau,x)\right|_{k+2} \leq M(1+\tau)^{-\frac{1}{2}}$ for $0 \leq \tau < T$, we obtain from Lemma 3.3 that

$$\left\| \prod_{l=1}^{4} |\partial^{n_l}\Gamma^{J_l}u(\tau)| \right\|_2 \leq \left\|(1+|\tau-|\cdot||)^{\frac{1}{3}}|u(\tau,\cdot)|_{k+2}\right\|^3_\infty \left\| \frac{\Gamma^{J_4}u(\tau,\cdot)}{(1+|\tau-|\cdot||)} \right\|_2 \tag{4.19}$$

$$\leq CM^3(1+\tau)^{-\frac{3}{2}}\|u'(\tau)\|_{2,2k+1}$$

$$\leq CM^4(1+\tau)^{\mu-\frac{3}{2}} \quad \text{for } 0 \leq \tau < T.$$

From (4.16) - (4.19) we get

$$\|R_{I,t}(\tau)\|_2 \leq CM^3(1+\tau)^{\mu-1} \quad \text{for } 0 \leq \tau < T. \tag{4.20}$$

Therefore, integrating (4.15) with respect to t, we have

$$\|\Gamma^I u_t(t)\|_E \leq C\left(\varepsilon + M^3\int_0^t (1+\tau)^{\mu-1}d\tau\right) \tag{4.21}$$

$$\leq C\left(\varepsilon + \frac{1}{\mu}M^3(1+t)^\mu\right)$$

$$\leq C_\mu(1+t)^\mu(\varepsilon+M^3) \quad \text{for } 0 \leq t < T.$$

Since $\|u'(t)\|_{2,2k+1} \leq C\sum_{|I|\leq 2k+1}\|\Gamma^I u\|_E$, this means that

$$(1+t)^{-\mu}\|u'(t)\|_{2,2k+1} \leq C_3(\varepsilon+M^3) \quad \text{for } 0 \leq t < T. \tag{4.22}$$

Finally (4.6), (4.12) and (4.22) imply the assertion of Proposition 4.1 immediately. This completes the proof. ∎

Remark 2. Consider the Cauchy problem for single and semilinear wave equations of the type

(4.23) $$\Box u = F(u,u') \quad \text{in } (0,\infty)\times \mathbf{R}^2,$$

(4.24) $$u(0,x) = \varepsilon\phi(x),\ \partial_t u(0,x) = \varepsilon\psi(x),$$

where $F = O(|u|^2 + |u'|^2)$ in some neighborhood of $(u,u') = 0$, and $\phi, \psi \in C_0^\infty(\mathbf{R}^2)$.

Assume that

(H4) The quadratic and cubic parts of F satisfy the null condition.

Making some change of variable stated in Katayama [8; Section5] (see also Godin [2] and Klainerman [9]), the right-hand side of (4.23) becomes a function of degree 3 with respect to the new variable, whose cubic part satisfies the null condition. Therefore we can show from Theorem 1.2 that if F satisfies (H4), then there exists a global smooth solution to (4.23) - (4.24) provided that ε is sufficiently small. We omit the details here.

Acknowledgments

The author would like to express his sincere gratitude to Professors Y.Ohya and S.Tarama for their helpful suggestions and constant encouragement.

References

[1] Christodoulou, D., Global solutions of nonlinear hyperbolic equations for small initial data, *Comm. Pure Appl. Math.*, **39** (1986), 267–282.

[2] Godin, P., Lifespan of semilinear wave equations in two space dimensions, *Comm. Partial Differential Equations*, **18** (1993), 895 - 916.

[3] Hörmander, L., L^1, L^∞ estimates for the wave operator, in *Analyse Mathématique et Applications, Contributions en l'Honneur de J. L. Lions*, Gauthier - Villars, Paris, (1988), 211–234.

[4] ————, On the fully non-linear Cauchy problem with small data, II., in *Microlocal Analysis and Nonlinear Waves* (M. Beals, R. Melrose and J. Rauch, eds.), *IMA Vol. Math. Appl.*, Springer - Verlag, New York, **30** (1991), 51–81.

[5] Hoshiga, A., The initial value problems for quasi-linear wave equations in two space dimensions with small data, *preprint*.

[6] John, F., Existence for large times of strict solutions of nonlinear wave equations in three space dimensions for small initial data, *Comm. Pure Appl. Math.*, **40** (1987), 79–109.

[7] Katayama, S., Global existence for some fully nonlinear wave equations with small data, *Math. Japon.*, **38** (1993), 1103–1114.

[8] ————, Global existence for systems of nonlinear wave equations in two space dimensions, *Publ. RIMS, Kyoto Univ.*, **29** (1993), 1021–1041.

[9] Klainerman, S., Global existence for nonlinear wave equations, *Comm. Pure Appl. Math.*, **33** (1980), 43–101.

[10] ————, Uniform decay estimates and the Lorentz invariance of the classical wave equation, *Comm. Pure Appl. Math.*, **38** (1985), 321–332.

[11] ————, The null condition and global existence to nonlinear wave equations, *Lectures in Appl. Math.*, **23** (1986), 293–326.

[12] Klainerman, S. and Ponce, G., Global, small amplitude solutions to nonlinear evolution equations, *Comm. Pure Appl. Math.*, **36** (1983), 133–141.

[13] Kovalyov, M., Long-time behavior of solutions of a system of nonlinear wave equations, *Comm. Partial Differential Equations*, **12** (1987), 471–501.

[14] Li Ta-tsien, and Yu Xin, Life-span of classical solutions to fully nonlinear wave equations, *Comm. Partial Differential Equations*, **16** (1991), 909–940.

[15] Li Ta-tsien, Yu Xin, and Zhou Yi, Problème de Cauchy pour les équations des ondes non linéaires avec petites données initiales, *C. R. Acad. Sci. Paris*, **312**, *Série I* (1991), 337–340.

[16] Li Ta-tsien, and Zhou Yi, Lifespan of classical solutions to fully nonlinear wave equations-II, *Nonlinear Anal.*, **19** (1992), 833–853.

[17] ———, Nonlinear stability for two space dimensional wave equations with higher order perturbations, *Nonlin. World*, **1** (1994), 35–58.

[18] ———, Life-span of classical solutions to nonlinear wave equations in two space dimensions, *J. Math. Pures Appl.*, **73** (1994), 223–249.

[19] ———, Life-span of classical solutions to nonlinear wave equations in two space dimensions II, *J. Partial Diff. Eqs.*, **6** (1993), 17–38.

[20] Lindblad, H., On the lifespan of solutions of nonlinear wave equations with small initial data, *Comm. Pure Appl. Math.*, **43** (1990), 445–472.

[21] Shatah, J., Global existence of small solutions to nonlinear evolution equations, *J. Differential Equations*, **46** (1982), 409–425.