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Hermitian and Positive C-Semigroups on Banach Spaces†

By

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Abstract

Two classes of operator families, namely n-times integrated C-semigroups of hermitian and positive operators on Banach spaces, are studied. By using Gelfand transform and a theorem of Sinclair, we prove some interesting special properties of such C-semigroups. For instances, every hermitian nondegenerate n-times integrated C-semigroup on a reflexive space is the n-times integral of some hermitian C-semigroup with a densely defined generator; an exponentially bounded C-semigroup on $L^p(\mu)(1 < p < \infty)$ dominates C (a positive injective operator) if and only if its generator A is bounded, positive , and commutes with C; when C has dense range, the latter assertion is also true on $L^1(\mu)$ and $C_0(\Omega)$.

§1. Introduction

Let X be a (complex) Banach space. We denote by X the dual space of X and by $B(X)$ the space of all bounded linear operators on X. Let $C \in B(X)$, and let $T(\cdot) \equiv \{T(t); t \geq 0\}$ be a strongly continuous family in $B(X)$. For $n \geq 1$, $T(\cdot)$ is called an *n-times integrated C-semigroup* on X ([10], [11]) if it satisfies: $T(t)C = CT(t)$, $T(0) = \mathbf{0}$, and

$$(1.1)\quad T(s)T(t)x = \frac{1}{(n-1)!}\left(\int_t^{s+t} - \int_0^s\right)(s+t-r)^{n-1}T(r)Cx\,dr \text{ for } x \in X,\ s, t \geq 0.$$

(see also [1], [15], [20] for the case $C = J$). $T(\cdot)$ is called a (0-times integrated) *C-semigroup* (see [5], [6], [13], [21]) on X if it satisfies: $T(0) = C$, and

$$(1.2)\qquad T(s)T(t) = T(s+t)C \text{ for s, } t \geq 0.$$

The classical C_0-semigroups are C-semigroups with C equal to the identity operator I.

$T(\cdot)$ is said to be *nondegenerate* if $T(t)x = \mathbf{0}$ for all $t > 0$ implies $x = \mathbf{0}$. In order that $T(\cdot)$ be nondegenerate it is necessary (and sufficient in case $n = 0$) that C is injective. The *generator A* of a nondegenerate n-times integrated C-semigroup $T(\cdot)$ is the closed operator A defined as:

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$$x \in D(A) \text{ and } Ax = y \Leftrightarrow T(t)x - t^n Cx / n! = \int_0^t T(s)y\, ds \text{ for } t \geq 0.$$

We know that $R(\int_0^t T(s)ds) \subset D(A)$ and

$$\int_0^t T(s)dsA \subset A\int_0^t T(s)ds = T(t) - \frac{t^n}{n!}C \text{ for } t \geq 0.$$

When $n = 0$, the generator A is identical to the *infinitesimal generator*, which is defined as

$$\begin{cases} D(A) := \{x \in X; \lim_{t \to 0^+} t^{-1}(T(t)x - Cx) \in R(C)\}, \\ Ax := C^{-1} \lim_{t \to 0^+} t^{-1}(T(t)x - Cx) \text{ for } x \in D(A). \end{cases}$$

Furthermore, if a nondegenerate n-times integrated C-semigroup $T(\cdot)$ is *exponentially bounded* in the sense that there are M > 0 and $\omega \geq 0$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, we have the following equivalent definition of generator:

$$\begin{cases} D(A) := \{x \in X; Cx \in R(R_n(\lambda))\}, \\ Ax := (\lambda - R_n(\lambda)^{-1}C)x \text{ for } x \in D(A), \end{cases}$$

where $R_n(\lambda) := \int_0^\infty \lambda^n e^{-\lambda t}T(t)dt$ for $\lambda > \omega$. It is known [10] that an exponentially bounded, strongly continuous family $\{T(t); t \geq 0\}$ of operators is an n-times integrated C-semigroup with generator A if and only if $C^{-1}AC = A$ and for all large λ, $\lambda - A$ is injective and $(\lambda - A)\int_0^\infty e^{-\lambda t}T(t)xdt = Cx$ for all $x \in X$.

The (algebra) *numerical range* of an operator $T \in B(X)$ is defined as the nonempty compact convex set

$$V(T) := \{F(T); F \in B(X)', \|F\| = F(I) = 1\}.$$

An equivalent expression due to J.P. Williams is: $V(T) = \{\lambda \in \mathbb{C}; \|T - zI\| \geq |\lambda - z|$ for all $z \in C\}$ (see [4, Lemma 22.1]), from which it is clear that both the dual operator T and the left multiplication operator L_T *by* T have the same numerical range $V(T)$.

T is called *hermitian* if $V(T)$ is contained in the real line $\mathbb{R}$, or equivalently, if $\|\exp(itT)\| = 1$ for all $t \in \mathbb{R}$. T is said to be *positive* (in the sense of numerical range), in notation $T \geq \mathbb{O}$, if $V(T) \subset [0, \infty)$. Since $V(T)$ is equal to the closed convex hull of the spatial numerical range

$$W(T) := \{\langle Tx, x' \rangle; x \in X, x' \in X', \|x\| = \|x'\| = \langle x, x' \rangle = 1\}$$

(see [3, p. 83]), the set of all hermitian (resp. positive) operators is clearly closed with respect to the weak operator topology. It is well-known that $V(T)$ always contains the spectrum $\sigma(T)$ of T, and when T is a hermitian operator, a theorem of Sinclair shows that $V(T)=\overline{co}\,\sigma(T)$ which is equivalent to that $r(\alpha T+\beta I) =\|\alpha T+\beta I\|$ for all complex α and β, where $r(T)$ denotes the spectral radius of T (see [4], §26).

An n-times integrated C-semigroup $T(\cdot)$ is said to be *hermitian* (resp. *positive*) if $T(t)$ is hermitian (resp. positive) for all $t\geq 0$. The purpose of this paper is to investigate some interesting properties of hermitian and positive n-times integrated C-semigroups. Section 2 is concerned with hermitian ones, Section 3 concentrates on positive C-semigroups which dominate the operator C, and Section 4 consists of some illustrating examples.

As is well known, a classical C_0-semigroup $T(\cdot)$ is always exponentially bounded, and its generator A is bounded if and only if $T(\cdot)$ is uniformly continuous on $[0,\infty)$. For a C-semigroup with $C\neq I$, the situation is quite different, even when it is positive. For instance, in Section 4 we give an example (Example 3) of a C-semigroup $T(\cdot)$ on ℓ_1, which satisfies $T(t)\geq C\geq 0$ for all $t\geq 0$, is not exponentially bounded, is uniformly continuous on $[0,\infty)$, but has an unbounded generator.

Our main theorem (Theorem 3.3) about positive C-semigroups states that a closed operator A generates a C-semigroup $T(\cdot)$ satisfying $T(t)\geq C\geq 0$ if and only if $C^{-1}AC=A$, $R(C)\subset D(A^n)$, $A^nC\in B(X)$ and $A^nC\geq 0$ for all $n\geq 1$, so that $T(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}A^nC$. In case the space is a Lebesgue space $L^p(\mu)$ with $1<p<\infty$ (*or* L^1, C_0 under the additional assumption that $\overline{R(C)}=X$), the above condition becomes that A and C are commuting bounded positive operators (Corollary 3.4). The proof of it for spaces $C_0(\Omega)$ and $L^p(\mu), 1\leq p\leq\infty, p\neq 2$, uses the fact that a bounded linear operator on any one of these spaces is hermitian (resp. positive) if and only if it is a multiplication operator by a bounded, real (resp. positive) valued function (see [22], [12], and [19]). It is unknown whether a similar statement as Corollary 3.4 holds for C-semigroups of positivity preserving operators on Banach lattices, although it is true for the special case: $C=I$ [14].

It is known that an $(n+1)$-times integrated semigroup need not be the integral of some n-times integrated semigroup (see e.g. [1]), that is, it is not necessarily differentiable. A hermitian n-times integrated C-semigroup $T(\cdot)$ turns out to possess better regularity. In fact, $T(\cdot)$ is n-th continuously differentiable in norm on $(0,\infty)$ and $T^{(n)}(\cdot)$ is locally bounded on $[0,\infty)$; in case $n\geq 1$, $T(\cdot)$ is $(n-1)$-th continuously differentiable in norm on $[0,\infty)$ (Theorem 2.3, (b)-(e)). If $T(\cdot)$ is a hermitian nondegenerate n-times integrated C-semigroup with generator A, where $n\geq 1$, then $T^{(n)}(t)x$ converges to Cx as $t\to 0^+$ for x in $\overline{D(A)}$ (Corollary 2.4). Thus, if A has dense domain (this is the case in particular when the space X is reflexive), then $T(\cdot)$ is n-th strongly differentiable on $[0,\infty)$ and $T^{(n)}(\cdot)$ is a hermitian C-semigroup (Theorem 2.5). Nevertheless, this conclusion is not true in general (see Example 2.) We also deduce that if $T(\cdot)$ is a hermitian n-times integrated C-semigroup on a reflexive space X with generator A, then $T\,(\cdot)$ is a hermitian n-times integrated C'-semigroup on X' with genetator A' (Corollary 2.6). It is unknown to us whether the same property is shared by nonhermitian C-semigroups, although the affirmative answer for the case $C = I$ is well known.

Another interesting phenomenon is that every hermitian n-times integrated semigroup is exponentially bounded (Theorem 2.3 (g)). This is similar to Arendt's result [2, Proposition 6.7] for positivity preserving integrated semigroups on Banach lattices. In general, integrated semigroups are not necessarily exponentially bounded (see [9] and [7, p. 110]).

§2. Hermitian C-semigroups

In this section we study some properties of n-times integrated C-semigroups of hermitian operators on a Banach space X.

Lemma 2.1. (i) *If $f: [0,\infty) \to \mathbb{C}$ is a continuous function satisfying*

$$f(t)f(s) = f(t+s) \text{ for } t,s \geq 0, \tag{2.1}$$

then either $f \equiv 0$ or there is a complex number α such that $f(t) = e^{\alpha t}$ for all $t \geq 0$; $f(t) \in \mathbb{R}$ (resp. $f(t) \geq 1$) for all $t \geq 0$ if and only if $\alpha \in \mathbb{R}$ (resp. $\alpha \geq 0$).

(ii) *Let n be a positive integer. If g: $[0,\infty) \to \mathbb{C}$ is a locally integrable function satisfying*

$$g(t)g(s) = \frac{1}{(n-1)!}\left\{\int_t^{t+s} - \int_0^s\right\}(s+t-r)^{n-1} g(r)dr \text{ for } t,s \geq 0, \tag{2.2}$$

then either $g \equiv 0$ or there is an $\alpha \in \mathbb{C}$ such that $g(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} e^{\alpha s} ds$ for all $t \geq 0$.

Proof. (i) is well-known. We deduce (ii) from (i). It follows from (2.2) that $g(s_0) \neq 0$ for some $s_0 \geq 0$ implies $g, g', \cdots, g^{(n)}$ are continuous, $g(0) = \cdots = g^{(n-1)}(0) = 0$, and $g^{(n)}$ satisfies (2.1). The result then follows from (i).

Lemma 2.2. *Let Ω be a nonempty set, and let p, q be two real-valued functions on Ω such that $\beta_t := \sup\{|\exp(q(\omega)s)p(\omega)|; 0 \leq s \leq t, \omega \in \Omega\} < \infty$ for all $t \geq 0$. Then*

$$\begin{aligned} &\lim_{h\to 0} \frac{1}{(n-1)!h}\{\int_0^{t+h} (t+h-r)^{n-1} \exp(q(\omega)r)p(\omega)dr \\ &\qquad - \int_0^t (t-r)^{n-1} \exp(q(\omega)r)p(\omega)dr\} \\ &= \frac{1}{(n-1)!}\frac{\partial}{\partial t}\int_0^t (t-r)^{n-1} \exp(q(\omega)r)p(\omega)dr \end{aligned} \tag{2.3}$$

uniformly for (t,ω) *in* $J\times\Omega$, *where* J *is a compact subset of* $(0,\infty)$ *in case* $n=1$, *or of* $[0,\infty)$ *in case* $n\geq 2$.

Proof. Clearly, β_t is increasing in t and $\beta_0=\sup\{|p(\omega)|;\omega\in\Omega\}$. First, we consider the case $n\geq 2$. Fix any $b>0$. We have for $\omega\in\Omega$ and $t\geq 0$

$$\frac{1}{(n-1)!}\frac{\partial}{\partial t}\int_0^t (t-r)^{n-1}e^{q(\omega)t}p(\omega)dr=\frac{1}{(n-2)!}\int_0^t (t-r)^{n-2}e^{q(\omega)t}p(\omega)dr.$$

Moreover, for $0\leq t\leq b$ and $|h|<1$ with $t+h\geq 0$ we have

$$\begin{aligned}
&|\frac{1}{(n-1)!h}\left\{\int_0^{t+h}(t+h-r)^{n-1}e^{q(\omega)t}p(\omega)dr-\int_0^t (t-r)^{n-1}e^{q(\omega)r}p(\omega)dr\right\}\\
&\qquad-\frac{1}{(n-2)!}\int_0^t (t-r)^{n-2}\exp(q(\omega)r)p(\omega)dr|\\
&=\frac{1}{(n-2)!}|h^{-1}\int_t^{t+h}\left\{\int_0^s (s-r)^{n-2}\exp(q(\omega)r)p(\omega)dr\right.\\
&\qquad\left.-\int_0^t (t-r)^{n-2}\exp(q(\omega)r)p(\omega)dr\right\}ds|\\
&=\frac{1}{(n-2)!}|h^{-1}\int_t^{t+h}\left\{\int_t^s (s-r)^{n-2}\exp(q(\omega)r)p(\omega)dr\right.\\
&\qquad\left.+\int_0^t[(s-r)^{n-2}-(t-r)^{n-2}]\exp(q(\omega)r)p(\omega)dr\right\}ds|\\
&\leq\frac{1}{(n-2)!}[|h|^{n-1}\beta_{b+1}+b\beta_b\sup\{|(t+|h|)^{n-2}-t^{n-2}|;0\leq t\leq b\}],
\end{aligned}$$

which converges to 0 (as $h\to 0$) uniformly for (t,ω) in $[0,b]\times\Omega$.

Next, we consider the case $n=1$. Without loss of generality we assume $J=[t_1,t_2]$ for some $0<t_1<t_2<\infty$. Let $\varepsilon>0$ be arbitrary. There are numbers $R_1<0$ and $R_2>0$ such that

$$\exp(\alpha t_3)<\varepsilon/(1+2\beta_0)\text{ for all }\alpha<R_1$$

and

$$\exp(-\alpha t_2)<\varepsilon/(1+2\beta_{2t_1}^{\frac{1}{2}}\beta_{4t_2}^{\frac{1}{2}})\text{ for all }\alpha>R_2,$$

where $t_3=t_1/2$. Let $S_1:=\{\omega\in\Omega;q(\omega)<R_1\}$, $S_2:=\{\omega\in\Omega;R_1\leq q(\omega)\leq R_2\}$ and $S_3:=\{\omega\in\Omega;q(\omega)>R_2\}$. Thus for $\omega\in S_1$ and $t\in J$

$$\begin{aligned}
&\left|\frac{1}{h}\int_t^{t+h}\exp(q(\omega)r)p(\omega)dr-\exp(q(\omega)t)p(\omega)\right|\\
&\leq\left|\frac{1}{h}\int_{t_3}^{t_3+h}\exp(q(\omega)r)dr-\exp(q(\omega)t_3)\right|\cdot|p(\omega)|\cdot\exp(q(\omega)(t-t_3))\\
&\leq 2\beta_0\exp(q(\omega)(t-t_3))\leq 2\beta_0\exp(q(\omega)t_3)<\varepsilon
\end{aligned}$$

for all $0<|h|<t_3$. For $\omega \in S_2$ and $t \in J$ we have

$$\left|\frac{1}{h}\int_t^{t+h}\exp(q(\omega)r)p(\omega)dr-\exp(q(\omega)t)p(\omega)\right|$$

$$=\left|\frac{1}{h}\int_0^{h}\exp(q(\omega)r)dr-1\right|\exp(q(\omega)t)|\,p(\omega)|$$

$$\leq \beta_{t_2}\cdot\left|\frac{1}{h}\int_0^{h}\exp(q(\omega)r)dr-1\right|\to 0$$

uniformly for (t,ω) in $J\times S_2$ as $h\to 0$. Finally, we have for $\omega\in S_3$, $t\in J$, and $0<|h|<t_3$,

$$\left|\frac{1}{h}\int_t^{t+h}\exp(q(\omega)r)p(\omega)dr-\exp(q(\omega)t)p(\omega)\right|$$

$$\leq\left|\frac{1}{h}\int_{t_3}^{t_3+h}\exp(q(\omega)r)|p(\omega)|^{1/2}dr-\exp(q(\omega)t_3)|p(\omega)|^{1/2}\right|\exp(q(\omega)(t-t_3)|p(\omega)|^{1/2}$$

$$\leq 2\beta_{2t_1}^{1/2}\cdot\exp(q(\omega)t_2)|p(\omega)|^{1/2}\leq 2\beta_{2t_1}^{1/2}\cdot\beta_{4t_2}^{1/2}/\exp(q(\omega)t_2)<\varepsilon.$$

This proves the lemma for $n = 1$, and completes the proof.

Theorem 2.3. *Let $T(\cdot)$ be a hermitian n-times integrated C-semigroup on a Banach space X.*

(a) *If $C\geq \mathbb{0}$, then $T(t)\geq \mathbb{0}$ for all $t\geq 0$.*
(b) *If $n\geq 1$, then $T(\cdot)$ is norm continuous on $[0,\infty)$.*
(c) *If $n\geq 2$, then $T(\cdot)$ is norm differentiable on $[0,\infty)$ and $T'(\cdot)$ is a norm continuous hermitian $(n-1)$- times integrated C-semigroup.*
(d) *If $n = 1$, then $T(\cdot)$ is norm differentiable on $(0,\infty)$, $T'(\cdot)$ is hermitian, locally bounded on $[0,\infty)$, and norm continuous on $(0,\infty)$, and $T'(t+s)C=T'(t)T'(s)$ for all $t,s\geq 0$, where we define $T'(0)=C$.*
(e) *If $n = 0$, then $T(\cdot)$ is norm continuous on $(0,\infty)$.*
(f) *If $n = 0$ and $T(t)\geq C\geq \mathbb{0}$ for all $t\geq 0$, then $T(\cdot)$ is norm continuous on $[0,\infty)$.*
(g) *If $C = I$, then there exist $M > 0$ and $\omega\in\mathbb{R}$ such that $\|T(t)\|\leq Me^{\omega t}$ for all $t\geq 0$; in case $n = 0$, one can take $M = 1$.*

Proof. Let A_T be the Banach subalgebra of $B(X)$ generated by $T(\cdot)$, C and I, the identity operator on X. Let $\mathcal{M}_T$ be the carrier space of A_T, i.e. the space of all nonzero multiplicative linear functionals on A_T.

Let $\phi\in\mathcal{M}_T$ be arbitrary. By (1.1) and (1.2) we have for all t, $s\geq 0$

$$\phi(T(t))\phi(T(s))=\phi(T(t+s))\phi(C)$$

if $n = 0$, and

$$\phi(T(t))\phi(T(s)) = \frac{1}{(n-1)!}\phi\left\{\left(\int_t^{t+s} - \int_0^s\right)(t+s-r)^{n-1}T(r)dr\right\}\phi(C)$$

if $n \geq 1$. It follows that $\phi(T(\cdot)) \equiv 0$ if (and also only if for the case $n = 0$) $\phi(C) = 0$.

Let $\phi \in m'_T := \{\phi \in m_T; \phi(T(\cdot)) \not\equiv 0\}$. If $n = 0$, Lemma 2.1(i) implies that

(2.4) $$\phi(T(t)) = \exp(\alpha_\phi t)\phi(C), t \geq 0$$

for some $\alpha_\phi \in \mathbb{R}$. For the case $n \geq 1$ we temporarily assume that $T(\cdot)$ is norm continuous on $[0,\infty)$. Then one can move ϕ inside the integral so that

$$\phi(T(t))\phi(T(s)) = \frac{1}{(n-1)!}\left(\int_t^{t+s} - \int_0^s\right)(t+s-r)^{n-1}\phi(T(r))dr \cdot \phi(C)$$

for all $t, s \geq 0$. It follows from Lemma 2.1(ii) that there is an $\alpha_\phi \in \mathbb{R}$ such that

(2.5) $$\phi(T(t)) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}\exp(\alpha_\phi s)ds \cdot \phi(C) \text{ for all } t \geq 0.$$

Next, we show that the nondecreasing function

(2.6) $$\beta_t := \sup\{|\exp(\alpha_\phi s)\phi(C)|; \phi \in m'_T, 0 \leq s \leq t\} < \infty \text{ for all } t > 0.$$

If a $\phi \in m'_T$ has $\alpha_\phi \leq 0$, then $|\exp(\alpha_\phi t) \cdot \phi(C)| \leq \|C\|$ for all $t \geq 0$. Suppose $\beta_\tau = \infty$ for some $\tau \geq 0$. Then we have

$$\beta_\tau := \sup\{|\exp(\alpha_\phi \tau)\phi(C)|; \phi \in m'_T, \alpha_\phi > 0\} < \infty,$$

and so for every $r > 0$ there is a $\phi_\tau \in m'_T$ such that $\alpha_{\phi_t} > 0$ and $|\exp(\alpha_{\phi_t} r)\phi_t(C)| > r$. Then, since a hermitian element has norm equal to its spectral radius, we have for $t > \tau$

$$\begin{aligned}
\|T(t)\| &= \sup_{\phi \in m_T}|\phi(T(t))| = \sup_{\phi \in m'_t}|\phi(T(t))| \\
&= \sup_{\phi \in m'_t}\left|\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}\exp(\alpha_\phi s)ds \cdot \phi(C)\right| \\
&\geq \frac{1}{(n-1)!}\int_\tau^t (t-s)^{n-1}\exp(\alpha_{\phi_t} s)ds|\phi_t(C)| \\
&\geq \frac{1}{(n-1)!}\int_\tau^t (t-s)^{n-1}ds \cdot r
\end{aligned}$$

in case $n \geq 1$, and $\|T(t)\| \geq |\exp(\alpha_{\phi_t}\tau)\phi_t(C)| > r$ in case $n = 0$. Since r can be arbitrarily large, this is a contradiction.

To prove (c) and (d) we define $A_h(t) := h^{-1}[T(t+h) - T(t)]$ for $t \geq 0, h \neq 0$ with $t + h \geq 0$. Since $T(\cdot)$ is hermitian, $A_h(t)$ is hermitian for all $t \geq 0, h \neq 0$ with $t + h \geq 0$. Since

$$\phi(A_h(t)) = \frac{1}{(n-1)!h}\Big[\int_0^{t+h} (t+h-r)^{n-1}\exp(\alpha_\phi r)\phi(C)dr$$
$$-\int_0^t (t-r)^{n-1}\exp(\alpha_\phi r)\phi(C)dr\Big]$$

for $\phi \in \mathcal{M}'_T$ and $t \geq 0, h \neq 0$ with $t+h \geq 0$, we can apply Lemma 2.2 to $\Omega = \mathcal{M}'_T$, $p(\phi) = \phi(C)$, and $q(\phi) = \alpha_\phi (\phi \in \mathcal{M}'_T)$, and it follows that the limit

$$\lim_{h\to 0^+} \phi(A_h(t)) = \frac{1}{(n-1)!}\frac{\partial}{\partial t}\int_0^t (t-r)^{n-1}\exp(\alpha_\phi r)\phi(C)dr$$

converges uniformly for (t,ϕ) in $J \times \mathcal{M}'_T$, where J can be any compact set in $(0,\infty)$ (resp. $[0,\infty)$) in case n = 1 (resp. $n \geq 2$). Hence we have for such set J

$$\sup\{\|A_{h_1}(t) - A_{h_2}(t)\|; t \in J\}$$
$$\leq \sup\{|\phi(A_{h_1}(t)) - \frac{1}{(n-1)!}\frac{\partial}{\partial t}\int_0^t (t-r)^{n-1}\exp(\alpha_\phi r)\phi(C)dr|; t \in J, \phi \in \mathcal{M}'_T\}$$
$$+\sup\{|\phi(A_{h_2}(t)) - \frac{1}{(n-1)!}\frac{\partial}{\partial t}\int_0^t (t-r)^{n-1}\exp(\alpha_\phi r)\phi(C)dr|; t \in J, \phi \in \mathcal{M}'_T\}$$
$$\to 0 \text{ as } h_1, h_2 \to 0^+.$$

This implies that $T(\cdot)$ is norm differentiable on J and $T'(t) = \lim_{h\to 0^+} A_h(t)$ uniformly for t in J. Hence $T'(\cdot)$ is norm continuous on J. When $n \geq 2$, J can be any compact subset of $[0,\infty)$ so that $T'(\cdot)$ is a norm continuous hermitian (n-1)-times integrated C-semigroup, i.e. (c) is true. When n = 1, $J \subset (0,\infty)$, so $T'(\cdot)$ is norm continuous on $(0,\infty)$. Also, we have

$$\phi(T'(t)) = \exp(\alpha_\phi t)\phi(C) \text{ for } t > 0 \text{ and } \phi \in \mathcal{M}'_T.$$

This implies that $\|T'(t)\| \leq \beta_t$ for $t > 0$ and hence $T'(\cdot)$ is bounded on [0,t] for any $t > 0$. If we define $T'(0) = C$, then $T'(\cdot)$ satisfies $T'(t+s)C = T'(t)T'(s)$ for all $t, s \geq 0$, and is locally bounded on $[0,\infty)$. This shows (d).

We have shown (2.5), and assertions (c), (d) under the assumption that $T(\cdot)$ is norm continuous on $[0,\infty)$. It turns out that this assumption is superfluous. Indeed, applying (c) to the norm continuous (n + 1)-times integrated C-semigroup $S(t) := \int_0^t T(s)ds$ implies that $T(\cdot) = S'(\cdot)$ is norm continuous on $[0,\infty)$. Hence (2.5), (2.6), (c) and (d) hold for any hermitian n-times integrated C-semigroup.

Clearly, (a) can be seen from (2.4) and (2.5), and (b) and (e) follow by applying (c) and (d), respectively, to $S(\cdot)$. When n = 1 and C = I, we have $\beta_1 = \sup\{\exp(\alpha_\phi s); \phi \in \mathcal{M}'_T, 0 \leq s \leq 1\} < \infty$. Hence there is a $\omega \in \mathbb{R}$ such that $\alpha_\phi \leq \omega$ for all $\phi \in \mathcal{M}'_T$. This means that $\|T'(t)\| \leq \exp(\omega t)$ for all t > 0, which shows assertion (g) for the cases n = 0 and n = 1. To show (g) for $n \geq 2$ one can apply (g) to the integrated C-semigroup $T^{(n-1)}(\cdot)$ and then take integration n times.

Finally, we show assertion (f). If $\phi \in \mathcal{M}_T'$, then $\phi(T(t)) \geq \phi(C) > 0$ for all $t \geq 0$ implies $\alpha_\phi \geq 0$. Let $\varepsilon > 0$ be arbitrary. Let $r > 0$ be such that $\frac{2\beta_1}{\exp(\alpha)} < \varepsilon$ for all $\alpha > \frac{1}{2}r$. Then we have that as $t \to 0^+$, $\phi(T(t) - C) = [\exp(\alpha_\phi t) - 1]\phi(C) \to 0$ uniformly for those $\phi \in \mathcal{M}_T'$ with $0 \leq \alpha_\phi \leq r$. For those $\phi \in \mathcal{M}_T'$ with $\alpha_\phi > r$ and for $0 < t < \frac{1}{2}$ one has

$$
\begin{aligned}
|\phi(T(t) - C) &= |[\exp(\alpha_\phi t) - 1]\phi(C)| \\
&= \frac{|\exp(\alpha_\phi (t + \frac{1}{2})) - \exp(\frac{1}{2}\alpha_\phi)|}{\exp(\frac{1}{2}\alpha_\phi)} \cdot |\phi(C)| \\
&\leq \frac{2\beta_1}{\exp(\frac{1}{2}\alpha_\phi)} < \varepsilon.
\end{aligned}
$$

Therefore we have proved that $\|(T(t) - C)\| = \sup_{\phi \in \mathcal{M}_T'} |\phi(T(t)) - \phi(C)| \to 0$ as $t \to 0^+$. The proof is complete.

Corollary 2.4. *Let $T(\cdot)$ be a hermitian nondegenerate integrated C-semigroup on a Banach space X, and let A be its generator. Then $T'(\cdot)|X_1$ is a hermitian $C|X_1$-semigroup on $X_1 := \overline{D(A)}$ and it is norm continuous on $(0, \infty)$.*

Proof. (d) of Theorem 2.3 asserts that $T'(\cdot)$ is hermitian, locally bounded, and norm continuous on $(0, \infty)$, and $CT'(t+s) = T'(t)T'(s)$ for all $t, s \geq 0$. Since $T(t)x - tCx = \int_0^t T(s)\, Axds$ for all $x \in D(A)$ and $t \geq 0$, one has that as $t \to 0^+$, $\|T'(t)x - Cx\| = \|T(t)Ax\| \to 0$ for all $x \in D(A)$, and hence for all x in X_1, due to the local boundedness of $T'(\cdot)$ on $[0, \infty)$. Restricting $T(\cdot)$ to the invariant subspace X_1 we come to the conclusion.

Theorem 2.5. *Let $n \geq 1$. If the generator A of a hermitian nondegenerate n-times integrated C-semigroup $T(\cdot)$ on X is densely defined, then $T(\cdot)$ is n-th strongly differentiable on $[0, \infty)$ and $T^{(n)}(\cdot)$ is a hermitian C-semigroup with generator A. In particular, every hermitian nondegenerate n-times integrated C-semigroup on a reflexive space is the n-times integral of some C-semigroup with a densely defined generator.*

Proof. The first part of the theorem follows from Theorem 2.3 (c) and Corollary 2.4. For the second part we need only to show that the generator A of a hermitian nondegenerate integrated C-semigroup $T(\cdot)$ on a reflexive space must have dense domain.

Since $T'(t)$ exists for all $t > 0$ and A is closed, from the identity: $T(t)x - tCx = A\int_0^t T(s)xds$, $x \in X$, $t \geq 0$ one sees that $R(T(t)) \subset D(A)$ and $AT(t)x = T'(t)x - Cx$ for all $x \in X$ and $t > 0$. Since X is reflexive, the local boundedness of $T'(\cdot)$ implies that for any $x \in X$ and for any sequence $t_n \to 0$, the sequence $\{t_n^{-1}T(t_n)x = t_n^{-1}\int_0^{t_n} T'(s)xds\}$ has a weakly convergent subsequence, say $t_{n_k}^{-1}T(t_{n_k})x \to y$ weakly, so that $A(t_{n_k}^{-1}\int_0^{t_{n_k}} T(s)xds) = t_{n_k}^{-1}T(t_{n_k})x - Cx$ converges weakly to $y \to Cx$. This, together with the facts that $t_{n_k}^{-1}\int_0^{t_{n_k}} T(s)xds \to 0$ and A is closed implies that $y \to Cx = A\mathbb{0} = \mathbb{0}$, and so $Cx = \text{w}-\lim t_{n_k}^{-1}T(t_{n_k})x$. Since $\{t_n\}$ is arbitrary, we must have that $Cx = \text{w}-\lim_{t\to 0^+} t^{-1}T(t)x \in \overline{D(A)}$. Hence C is a hermitian operator with $R(C) \subset \overline{D(A)}$. Since $\{\exp(itC); t \in \mathbb{R}\}$ is a unitary group, it follows from the mean ergodic theorem for semigroups on reflexive spaces (see e.g. [18]) that $X = N(C) \oplus \overline{R(C)}$. The nondegeneracy of $T(\cdot)$ implies that C is injective, and so we have $X = \overline{R(C)} = \overline{D(A)}$.

Corollary 2.6. *Let $T(\cdot)$ be a hermitian n-times integrated C-semigroup on a Banach space X. Under each of the conditions*: (1) $n \geq 1$; (2) $n = 0$ *and* $T(t) \geq C \geq \mathbb{0}$ *for all* $t \geq 0$; (3) $n = 0$, $T(\cdot)$ *is nondegenerate and X is reflexive, the dual family* $\{T\ (t); t \geq 0\}$ *is a hermitian n-times integrated C -semigroup on X .*

Proof. Since the dual operator of a hermitian operator is also hermitian, the assertion for the cases (1) and (2) follows from (b) and (f) of Theorem 2.3. For the case (3) one can apply case (1) and then Theorem 2.5 to the integral of $T(\cdot)$.

Theorem 2.7. *Let $T(\cdot)$ be a nondegenerate hermitian C-semigroup on a Banach space X with an infinitesimal generator A. Then*

(a) $R(T(t)) \subset D(A^n)$ *for* $n = 0, 1, \cdots$ *and* $t > 0$;

(b) $A^nT(t) \in B(X)$ *is hermitian for* $n = 0, 1, \cdots$ *and* $t > 0$;

(c) $A^nT(\cdot)$ *is norm continuous on* $(0, \infty)$ *for* $n = 0, 1, \cdots$

(d) $\| A^nT(t) \| \leq \max\{t^{-n}M_n\| C \|, (\beta_{2t}\beta_1 M_{2n})^{1/2}\}$ *for $t > 0$ and* $n = 0, 1, \cdots$, *where* $\beta_t := \sup\{|\exp(\alpha_\phi s)\phi(C)|; \phi \in \mathcal{M}'_T, 0 \leq s \leq t\}, t \geq 0$, *and* $M_n := \sup\{a^n e^{-a}; a \geq 0\} = n^n e^{-n}$, $n = 0, 1, \cdots$.

Proof. From the proof of Theorem 2.3 we see that β_t is finite and increasing in t. Let $S_+ := \{\phi \in \mathcal{M}'_T; \alpha_\phi \geq 0\}$ and $S_- := \mathcal{M}'_T - S_+$. Since β_1 is finite, if $\phi \in S_+$, we have for all $n = 0, 1, \cdots$, and $t \geq 0$

$$0 \leq \alpha_\phi^n |\phi(C)| \leq \alpha_\phi^n \exp(-\alpha_\phi)\beta_1 \leq M_n\beta_1 \ ,$$

and

$$|\alpha_\phi^n \exp(\alpha_\phi t)\phi(C)| \le [\exp(2\alpha_\phi t)|\phi(C)|]^{1/2}[\alpha_\phi^{2n}|\phi(C)|]^{1/2} \le (\beta_{2t}\beta_1 M_{2n})^{1/2}.$$

If $\phi \in S_-$, then $|\alpha_\phi^n \exp(\alpha_\phi t)\phi(C)| \le t^{-n}M_n\|C\|$. Hence we have

$$|\alpha_\phi^n \exp(\alpha_\phi t)\phi(C)| \le \max\{t^{-n}M_n\|C\|, (\beta_{2t}\beta_1 M_{2n})^{1/2}\} \tag{2.7}$$

for all $t > 0$ and $n = 0, 1, \cdots$. Thus (d) follows from (2.7), condition (b), and the following assertion:

(d') If $\phi \in \mathcal{M}_T \setminus \mathcal{M}_T'$ then $\phi(A^n T(\cdot)) \equiv 0$, $n \ge 0$; if $\phi \in \mathcal{M}_T'$, then $\phi(A^n T(t)) = \alpha_\phi^n \exp(\alpha_\phi t)\phi(C)$ for $t > 0$, $n \ge 0$.

We shall prove (a)-(c) and (d') by induction on n. (c) for $n = 0$ is (e) of Theorem 2.3, and (a), (b), and (d') are obviouus for $n = 0$. Suppose they are true for $n = 0, 1, \cdots, k$. We show that they are also true for $n = k + 1$. Since A is the generator of $T(\cdot)$, we have $A\int_0^t T(s)ds = T(t) - C$ for $t \ge 0$. By the induction hypothesis for $n = k$ we have for $t, h > 0$

$$h^{-1}A^{k+1}\int_t^{t+h} T(s)ds = h^{-1}[A^k T(t+h) - A^k T(t)]$$

are hermitian operators, and $\phi(h^{-1}A^{k+1}\int_t^{t+h} T(s)ds) \equiv 0$ for all $\phi \in \mathcal{M}_T \setminus \mathcal{M}_T'$, $t \ge 0$, $h \ne 0$ with $t + h \ge 0$. Let $J = [t_1, t_2]$ be an arbitrary close subinterval of $(0, \infty)$. We claim that

$$R(T(t)) \subset D(A^{k+1}) \text{ and } \lim_{h\to 0^+} h^{-1}A^{k+1}\int_t^{t+h} T(s)ds = A^{k+1}T(t)$$

in operator norm uniformly for t in J.

To show this we let $\Omega = \mathcal{M}_T', q(\phi) = \alpha_\phi$, and $p(\phi) = \alpha_\phi^{k+1}\exp(\alpha_\phi t_3)\phi(C)$ for $\phi \in \mathcal{M}_T'$, where $t_3 = t_1/2$. Then by (2.7) we have for $t \in J$ and $\phi \in \mathcal{M}_T'$

$$|\alpha_\phi^{k+1}\exp(\alpha_\phi t)\phi(C)| \le \max\{t_3^{-k-1}M_{k+1}\|C\|, (\beta_{t_1}\beta_1 M_{2(k+1)})^{1/2}\}.$$

Thus we can apply Lemma 2.2 to obtain that

$$\begin{aligned}
&\lim_{h\to 0^+} \phi(h^{-1}A^{k+1}\int_t^{t+h} T(s)ds) \\
&= \lim_{h\to 0^+} h^{-1}[\alpha_\phi^k \exp(\alpha_\phi(t+h)\phi(C) - \alpha_\phi^k \exp(\alpha_\phi t)\phi(C)] \\
&= \lim_{h\to 0^+} h^{-1}\int_{t-t_3}^{t-t_3+h} \exp(\alpha_\phi r)[\alpha_\phi^{k+1}\exp(\alpha_\phi t_3)\phi(C)]dr \\
&= \exp(\alpha_\phi(t-t_3))\alpha_\phi^{k+1}\exp(\alpha_\phi t_3)\phi(C)
\end{aligned}$$

uniformly in (t,ϕ) on $J\times \mathcal{M}_T'$, and which is equal to $\alpha_\phi^{k+1}\exp(\alpha_\phi t)\phi(C)$. Since each $h^{-1}A^{k+1}\int_t^{t+h} T(s)ds$ is hermitian, this shows that for $h_1, h_2>0$

$$\begin{aligned}
\sup &= \{\|h_1^{-1}A^{k+1}\int_t^{t+h_1}T(s)ds - h_2^{-1}A^{k+1}\int_t^{t+h_2}T(s)ds\|; t\in J\}\\
&= \sup\{|\phi(h_1^{-1}A^{k+1}\int_t^{t+h_1}T(s)ds)-\phi(h_2^{-1}A^{k+1}\int_t^{t+h_2}T(s)ds)|; t\in J, \phi\in\mathcal{M}_T'\}\\
&\le \sup\{|\phi(h_1^{-1}A^{k+1}\int_t^{t+h_1}T(s)ds)-\alpha_\phi^{k+1}\exp(\alpha_\phi t)\phi(C)|; t\in J,\phi\in\mathcal{M}_T'\}\\
&+ \sup\{|\phi(h_2^{-1}A^{k+1}\int_t^{t+h_2}T(s)ds)-\alpha_\phi^{k+1}\exp(\alpha_\phi t)\phi(C)|; t\in J,\phi\in\mathcal{M}_T'\}\\
&\to 0+0 \text{ as } h_1, h_2\to 0^+.
\end{aligned}$$

Since J is an arbitrary compact interval in $(0,\infty)$, the closedness of A and the induction assumption for $n = k$ show that (a)-(c) are true for $n = k + 1$, and we have for $\phi\in\mathcal{M}_T'$

$$\begin{aligned}
\phi(A^{k+1}T(t)) &= \lim_{h\to 0^+} h^{-1}[\phi(A^kT(t+h))-\phi(A^kT(t))]\\
&= \lim_{h\to 0^+} h^{-1}[\alpha_\phi^k\exp(\alpha_\phi(t+h)\phi(C)-\alpha_\phi^k\exp(\alpha_\phi t)\phi(C)]\\
&= \alpha_\phi^{k+1}\exp(\alpha_\phi t)\phi(C)
\end{aligned}$$

for $t>0$. For $\phi\in\mathcal{M}_T\setminus\mathcal{M}_T'$ we have

$$\phi(A^{k+1}T(t)) = \lim_{h\to 0^+} h^{-1}[\phi(A^kT(t+h))-\phi(A^kT(t))] = 0,\ t\ge 0.$$

Therefore (d') is true for $n = k + 1$, and the proof is complete.

Corollary 2.8. *Let $T(\cdot)$ be a hermitian C_0-semigroup on a Banach space X with infinitesimal generator A.*
(a) *If $A\in B(X)$, then A^n is hermitian for $n=0,1,\cdots$.*
(b) *If $A\in B(X)$ and $A\ge \mathbb{0}$, then $A^n\ge\mathbb{0}$ for $n=0,1,\cdots$.*

Proof. (a) By Theorem 2.3 (g) there is a $\omega\in R$ such that $\|T(t)\|\le e^{\omega t}$ for all $t\ge 0$. Since $T(\cdot)$ is hermitian, so is $(\lambda-A)^{-1}:=\int_0^\infty e^{-\lambda t}T(t)dt$ for all $\lambda>\omega$. Since $A\in B(X)$, we have for all $n=0,1,\cdots$,

$$\|\lambda A^n(\lambda-A)^{-1}-A^n\| = \|A^{n+1}(\lambda-A)^{-1}\|\le \frac{\|A\|^{n+1}}{\lambda-\|A\|}\to 0 \text{ as } \lambda\to\infty.$$

Using this fact and the equality $A^n(\lambda-A)^{-1}=\lambda A^{n-1}(\lambda-A)^{-1}-A^{n-1}$, one can easily show by induction that $A^n(\lambda-A)^{-1}$ and A^n are hermitian for all $\lambda>\omega$ and $n\ge 0$.

(b) By (a) we have that A^n is hermitian for $n=1,2,\cdots$. The result follows from the spectral mapping theorem.

Proposition 2.9. *Let $T(\cdot)$ be a hermitian C_0-semigroup on a Banach space X with infinitesimal generator A. Then $A\in B(X)$ if and only if there is a real number ω such that $e^{\omega t}T(t)\geq I$ for all $t\geq 0$.*

Proof. The sufficiency follows from Theorem 2.3 (f) and the fact that a uniformly continuous C_0-semigroup has a bounded generator. To see the necessity we apply Corollary 2.8 (b) to the hermitian C_0-semigroup $e^{\|A\|t}T(t)$. It follows that

$$e^{\|A\|t}T(t)=\sum_{n=0}^{\infty}(A+\|A\|)^n t^n/n!\geq I \text{ for all } t\geq 0.$$

§3. Positive C-Semigroups which Dominate C

The following theorem presents some properties of positive C-semigroups.

Theorem 3.1. *Let $T(\cdot)$ be a nondegenerate C-semigroup on a Banach space X such that $T(t)\geq C\geq \mathbf{0}$ for all $t\geq 0$, and let A be its generator. Then*
(a) *$R(T(t))\subset D(A^n)$ for $n=0,1,\cdots$, and $t\geq 0$;*
(b) *$A^nT(t)\in B(X)$ and $A^nT(t)\geq \mathbf{0}$ for $n=0,1,\cdots$, and $t\geq 0$;*
(c) *$A^nT(\cdot)$ is norm continuous on $[0,\infty)$ for $n=0,1,\cdots$;*
(d) *$\|A^nT(t)\|\leq(\beta_{2t}\beta_1 M_{2n})^{1/2}$ for $t\geq 0$ and $n=0,1,\cdots$, where $\beta_t:=\sup\{|\exp(\alpha_\phi s)\phi(C)|;\ \phi\in\mathcal{M}_T', 0\leq s\leq t\}$, $t\geq 0$, and $M_n:=\sup\{a^n e^{-a}; a\geq 0\}=n^n e^{-n}$, $n=0,1,\cdots$.*

Proof. From the proof of Theorem 2.3 we see that β_t is finite and increasing in t. The hypothesis: $T(t)\geq C\geq \mathbf{0}$, $t\geq 0$, implies that $\phi(C)=0$ for $\phi\in\mathcal{M}_T\setminus\mathcal{M}_T'$ and $\alpha_\phi\geq 0$ for $\phi\in\mathcal{M}_T'$ (Lemma 2.1 (i)) so that

$$0\leq\alpha_\phi^n\phi(C)\leq\alpha_\phi^n\exp(-\alpha_\phi)\beta_1\leq M_n\beta_1.$$

The estimation in the proof of Theorem 2.7 yields

$$|\alpha_\phi^n\exp(\alpha_\phi t)\phi(C)|\leq(\beta_{2t}\beta_1 M_{2n})^{1/2}. \tag{3.1}$$

Thus (d) follows from (3.1), condition (b), and the following assertion:
(d') If $\phi\in\mathcal{M}_T\setminus\mathcal{M}_T'$, then $\phi(A^nT(\cdot))\equiv 0$, $n\geq 0$; if $\phi\in\mathcal{M}_T'$, then $\phi(A^nT(t))=\alpha_\phi^n\exp(\alpha_\phi t)\,\phi(C)$ for $t\geq 0$, $n\geq 0$.

We shall prove (a)-(c) and (d') by induction on n. (c) for $n=0$ is (f) of Theorem 2.3, and (a), (b), and (d') are obviouus for $n=0$. Suppose they are true for $n=0,1,\cdots,k$. We show that they are also true for $n=k+1$. Since A is the generator of $T(\cdot)$, we have $A\int_0^t T(s)\,ds=T(t)-C$ for $t\geq 0$. By the induction assumption for $n = k$ we have for all $t \geq 0$, $h > 0$, $\phi(h^{-1}A^{k+1}\int_t^{t+h} T(s)ds) = 0$ if $\phi \in \mathscr{M}_T \setminus \mathscr{M}'_T$ and

$$\begin{aligned}\phi(h^{-1}A^{k+1}\int_t^{t+h} T(s)ds) &= h^{-1}[\phi(A^k T(t+h)) - \phi(A^k T(t))] \\ &= h^{-1}(\alpha_\phi^k \exp(\alpha_\phi(t+h))\phi(C) - \alpha_\phi^k \exp(\alpha_\phi t)\phi(C)) \\ &\geq 0\end{aligned}$$

if $\phi \in \mathscr{M}'_T$. Hence $h^{-1}A^{k+1}\int_t^{t+h} T(s)ds$ is positive for all $t \geq 0$ and $h > 0$.

Let $b > 0$ be arbitrary. We claim that

$$R(T(t)) \subset D(A^{k+1}) \text{ and } \lim_{h\to 0^+} h^{-1}A^{k+1}\int_t^{t+h} T(s)ds = A^{k+1}T(t)$$

in operator norm uniformly for t on $[0, b]$. First, using integration by parts we write

$$\begin{aligned}&\phi(h^{-1}A^{k+1}\int_t^{t+h} T(s)ds) \\ &= h^{-1}[\exp(\alpha_\phi(t+h)) - \exp(\alpha_\phi t)]\alpha_\phi^k \phi(C) \\ &= h^{-1}[\int_0^{t+h} \exp(\alpha_\phi r)dr - \int_0^t \exp(\alpha_\phi r)dr]\alpha_\phi^{k+1}\phi(C) \\ &= \alpha_\phi^{k+1}\phi(C) + h^{-1}[\int_0^{t+h} (t+h-r)\exp(\alpha_\phi r)\alpha_\phi^{k+2}\phi(C)dr \\ &\qquad -\int_0^t (t-r)\exp(\alpha_\phi r)\alpha_\phi^{k+2}\phi(C)dr].\end{aligned}$$

Applying Lemma 2.2 with $\Omega = \mathscr{M}'_T, p(\phi) = \alpha_\phi^{k+2}\phi(C)$, and $q(\phi) = \alpha_\phi$, we obtain that

$$\begin{aligned}&\lim_{h\to 0^+} \phi(h^{-1}A^{k+1}\int_t^{t+h} T(s)ds) \\ &= \alpha_\phi^{k+1}\phi(C) + \frac{\partial}{\partial t}\int_0^t (t-r)\exp(\alpha_\phi r)\alpha_\phi^{k+2}\phi(C)dr \\ &= \alpha_\phi^{k+1}\exp(\alpha_\phi t)\phi(C)\end{aligned}$$

uniformly for (t, ϕ) in $[0, b] \times \mathscr{M}'_T$. This shows for $h_1, h_2 > 0$

$$\begin{aligned}&\sup\{\|h_1^{-1}A^{k+1}\int_t^{t+h_1} T(s)ds - h_2^{-1}A^{k+1}\int_t^{t+h_2} T(s)ds\|; 0 \leq t \leq b\} \\ &= \sup\{|\phi(\frac{1}{h_1}A^{k+1}\int_t^{t+h_1} T(s)ds - \frac{1}{h_2}A^{k+1}\int_t^{t+h_2} T(s)ds)|; 0 \leq t \leq b, \phi \in \mathscr{M}'_T\} \\ &\to 0 + 0 \text{ as } h_1, h_2 \to 0_+.\end{aligned}$$

Since $b > 0$ is an arbitrary, the closedness of A and the induction assumption for $n=k$ show that (a)-(c) are true for $n = k + 1$, and we have for $\phi \in \mathcal{M}_T$

$$\begin{aligned}\phi(A^{k+1}T(t)) &= \lim_{h\to 0^+} h^{-1}[\phi(A^k T(t+h)) - \phi(A^k T(t))] \\ &= \lim_{h\to 0^+} h^{-1}[\alpha_\phi^k \exp(\alpha_\phi(t+h))\phi(C) - \alpha_\phi^k \exp(\alpha_\phi t)\phi(C)] \\ &= \alpha_\phi^{k+1}\exp(\alpha_\phi t)\phi(C)\end{aligned}$$

for $t \geq 0$. Therefore (d') is true for $n = k + 1$, and the proof is complete.

For exponentially bounded positive C-semigroups we have the next theorem.

Theorem 3.2. *Let $T(\cdot)$ be a nondegenerate exponentially bounded C-semigroup, say $\|T(t)\| \leq Me^{\omega t}$ for some constants $M > 0$, $\omega \in R$ and all $t \geq 0$. If $T(t) \geq C \geq \mathbb{O}$, then the infinitesimal generator A of $T(\cdot)$ has the following properties*:

(a) *$R(T(t)) \subset D(A^n)$ for $n = 0, 1, \cdots$, and $t \geq 0$;*

(b) *$A^n T(t) \in B(X)$ and $A^n T(t) \geq \mathbb{O}$ for $n = 0, 1, \cdots$, and $t \geq 0$;*

(c) *$R(R(\lambda)) \subset D(A^n)$ for $n = 0, 1, \cdots$, $\lambda > \omega$, where $R(\lambda) := \int_0^\infty e^{-\lambda t}T(t)dt$;*

(d) *$\lim_{\lambda\to\infty} \lambda A^n R(\lambda) = A^n C$ in operator norm for $n = 0, 1, \cdots$;*

(e) *$A^n R(\lambda) \in B(X)$ and $A^n R(\lambda) \geq \mathbb{O}$ for all $\lambda > \omega$ and $n = 0, 1, \cdots$;*

(f) *$\|A^n C\| \leq \omega^n \|C\|$ and $\|A^n R(\lambda)\| \leq \dfrac{\omega^n}{\lambda - \omega}\|C\|$ for all $\lambda > \omega$ and $n = 0, 1, \cdots$.*

Proof. (a) and (b) have been proved in Theorem 3.1, and the first part of (f) follows from (d), (e), and the second part of (f). Note also that (f) follows from the following assertion:

(f')

$$\phi(A^n C) = \alpha_\phi^n \phi(C) \text{ and } \phi(A^n R(\lambda)) = \frac{\alpha_\phi^n}{\lambda - \alpha_\phi}\phi(C) \text{ for } \phi \in \mathcal{M}_T,\ \lambda > \omega, 0, 1, 2, \cdots,$$

where α_ϕ is treated as zero whenever $\phi \in \mathcal{M}_T \setminus \mathcal{M}_T'$.

We shall prove (c)-(e) and (f') by induction on n. First, let us consider $n = 0$. Since $\|T(t)\| \leq Me^{\omega t}$ and $T(t) \geq C \geq \mathbb{O}$ for all $t \geq 0$, it follows from Lemma 2.1 (i) that for every $\phi \in m_T$ there is a positive number $\alpha_\phi \leq \omega$ such that $\phi(T(t)) = \phi(C)\exp(\alpha_\phi t)$ for all $t \geq 0$. Thus, by Theorem 2.3 (e), we have for $\lambda > \omega$

$$\begin{aligned}\phi(R(\lambda)) &= \phi\left(\int_0^\infty e^{-\lambda t}T(t)dt\right) = \int_0^\infty e^{-\lambda t}\phi(T(t))dt \\ &= \int_0^\infty e^{-\lambda t}\exp(\alpha_\phi t)dt\,\phi(C) = \frac{1}{\lambda - \alpha_\phi}\phi(C),\end{aligned}$$

and

$$|\phi(\lambda R(\lambda)-C)|=\frac{\alpha_\phi}{\lambda-\alpha_\phi}\phi(C)\le\frac{\omega}{\lambda-\omega}\|C\|\to 0$$

uniformly for ϕ on m_T, as $\lambda\to\infty$. Since $\lambda R(\lambda)-C$ is hermitian, we have $\|\lambda R(\lambda)-C\|=\sup\{|\phi(\lambda R(\lambda)-C)|;\phi\in m_T\}\to 0$ as $\lambda\to\infty$. Also, the positivity of $T(\cdot)$ implies $R(\lambda)\ge \mathbb{0}$ for $\lambda>\omega$. This proves (c)-(e) and (f') for the case $n = 0$.

Now, assume that (c)-(e) and (f') are true for $n=0,1,\cdots,k$. First, we see from $AR(\lambda)=\lambda R(\lambda)-C$ for $\lambda>\omega$ and (b), (e) for $n = k$ that (c) holds for $n = k + 1$ and $A^{k+1}R(\lambda)=\lambda A^kR(\lambda)-A^kC\in B(X)$ is hermitian. Then (f') (with $n = k$) implies that

$$\phi(A^{k+1}R(\lambda))=\phi(\lambda A^kR(\lambda))-\phi(A^kC)=\frac{\lambda\alpha_\phi^k}{\lambda-\alpha_\phi}\phi(C)-\alpha_\phi^k\phi(C)$$

$$=\frac{\alpha_\phi^{k+1}}{\lambda-\alpha_\phi}\phi(C)\ge 0$$

for all $\phi\in m_T$ and $\lambda>\omega$. Hence $A^{k+1}R(\lambda)$ is positive. This proves (e) and the second part of (f') for $n = k + 1$. To prove (d) and the first part of (f') for $n = k + 1$, it suffices to show that $R(C)\subset D(A^{k+1})$ and $\lim_{\lambda\to\infty}\lambda A^{k+1}R(\lambda)$ converges to $A^{k+1}C$ in operator norm.

For every $\phi\in m_T$ and $\lambda,\mu>\omega$

$$|\phi(\lambda A^{k+1}R(\lambda)-\mu A^{k+1}R(\mu))|=|\frac{\lambda\alpha_\phi^{k+1}}{\lambda-\alpha_\phi}\phi(C)-\frac{\mu\alpha_\phi^{k+1}}{\mu-\alpha_\phi}\phi(C)|$$

$$=|\frac{(\mu-\lambda)\alpha_\phi^{k+2}}{(\lambda-\alpha_\phi)(\mu-\alpha_\phi)}\phi(C)|\le\frac{(\mu-\lambda)\omega^{k+2}}{(\lambda-\omega)(\mu-\omega)}\|(C)\|,$$

which converges to 0 uniformly for ϕ in m_T, as $\mu,\lambda\to\infty$. Hence $\lambda A^{k+1}R(\lambda)$ converges in operator norm to a bounded operator E. This with the induction assumption $\|\lambda A^kR(\lambda)-A^kC\|\to 0$ implies $R(A^kC)\subset D(A)$ and $A^{k+1}C=E\in B(X)$ because A is closed. Hence (d) holds for $n = k + 1$. Next, we have for every $\phi\in m_T$

$$\phi(A^{k+1}C)=\lim_{\lambda\to\infty}\phi(\lambda A^{k+1}R(\lambda))=\lim_{\lambda\to\infty}\frac{\lambda\alpha_\phi^{k+1}}{\lambda-\alpha_\phi}\phi(C)=\alpha_\phi^{k+1}\phi(C).$$

This proves the first part of (f') for $n = k + 1$, and the proof is complete.

Theorem 3.3. Let $C\in B(X)$ be an injective operator, and A be a closed operator on X. Then A is the generator of a C-semigroup $T(\cdot)$ that satisfies $T(t)\ge C\ge\mathbb{0}$ and $\|T(t)\|\le Me^{\omega t}$ for all $t\ge 0$ if and only if A has the properties: $C^{-1}AC$

$= A, R(C) \subset D(A^n), A^nC \in B(X), A^nC \geq \mathbf{0}$ and $\|A^nC\| \leq M'\omega^n$ for some $M' \geq \|C\|$ and all $n = 0,1,\cdots$. Moreover, we have $T(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^nC$.

Proof. The necessity follows from Theorem 3.2 To show the sufficiency, define $T(t) := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^nC$ for $t \geq 0$. The hypothesis implies that $T(t) \geq C \geq \mathbf{0}$ and $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Next, we have for $\lambda > \omega$

$$(\lambda - A)\int_0^{\infty} e^{-\lambda t} T(t)dt = (\lambda - A)\sum_{n=0}^{\infty} \int_0^{\infty} e^{-\lambda t} . \frac{t^n}{n!} A^nCdt$$

$$= (\lambda - A)\sum_{n=0}^{\infty} \lambda^{-n-1}A^nC = C.$$

Since $C^{-1}AC = A$, it follows that $T(\cdot)$ is a C-semigroup with generator A (see [10]). This completes the proof.

Corollary 3.4. *Let $C \in B(X)$ be a positive, injective operator, and A be a closed operator on $X = L^p(\mu)(1 < p < \infty)$. Then A is the generator of an exponentially bounded C-semigroup $T(\cdot)$ on X which satisfies $T(t) \geq C \geq \mathbf{0}$ for all $t \geq 0$ if and only if $A \in B(X)$, $A \geq \mathbf{0}$, and $AC = CA$. Under the additional assumption that $\overline{R(C)} = X$, the assertion also holds for the spaces $X = L^1(S, \mu)$ and $X = C_0(\Omega)$ with Ω a locally compact space.*

Proof. Since, as mentioned previously, a hermitian (resp. positive) operator on each of the spaces $C_0(\Omega)$ and $L^p(\mu)$, $1 < p < \infty$, $p \neq 2$, is a multiplication operator by a bounded, real (resp. positive) valued function, the product of hermitian (resp. positive) operators on these spaces is still hermitian (resp. positive). Since the product of two commuting positive operators on a Hilbert space is still positive, the sufficiency part of the corollary follows from the sufficiency part of Theorem 3.3.

Next, we see that, as multiplication operators, positive operators on spaces $X = C_0(\Omega)$, $L^p(\mu)$, $1 \leq p < \infty$, $p \neq 2$, have the property:

$$\mathbf{0} \leq A \leq B \text{ implies } \|Af\| \leq \|Bf\| \text{ for all } f \in X.$$

Commuting operators on a Hilbert space H also have this property. Indeed, if $\mathbf{0} \leq A \leq B$ and $AB = BA$, then $A^{1/2}B = BA^{1/2}$ and $AB^{1/2} = B^{1/2}A$ so that

$$\begin{aligned}\|Ax\|^2 &= \langle A^2x, x\rangle = \langle AA^{1/2}x, A^{1/2}x\rangle \\ &\leq \langle BA^{1/2}x, A^{1/2}x\rangle = \langle ABx, x\rangle \\ &= \langle B^{1/2}AB^{1/2}x, x\rangle = \langle AB^{1/2}x, B^{1/2}x\rangle \\ &\leq \langle BB^{1/2}x, B^{1/2}x\rangle = \|Bx\|^2.\end{aligned}$$

To show the necessity, we see from (b) and (f') of Theorem 3.2 that $0 \le AC \le \omega C$, so that

$$\|ACf\| \le \omega \|Cf\| \text{ for all } f \in X.$$

Hence A is bounded and $\|A\| \le \omega$ if C has dense range. Since injective hermitian operators on reflexive spaces have dense ranges (see the argument in the proof of Theorem 2.5), this is readily true for the case $X = L^p(\mu)(1 < p < \infty)$. Since both C and $CA(= C(C^{-1}AC) = AC)$ are positive operators, for the case $p = 2$, $C^{1/2}$ is an injective positive operator with dense range, and $\langle AC^{1/2}x, C^{1/2}x \rangle = \langle ACx, x \rangle \ge 0$ for all $x \in L^2$. Since A is bounded, this implies that A is positive on L^2. For other cases, there are positive functions h_1 and h_2 such that $Cf = h_1 f$ and $CAf = h_2 f$ for all $f \in X$. The injectivity of C implies that $h_1(s) > 0$ for all $s \in \Omega$ in case $X = C_0(\Omega)$ (resp. a.e. $[\mu]$ in case $X = L^p(\mu)$). It follows that $Af = h_1^{-1} h_2 f$ for all $f \in X$, and so A is positive. This proves the necessity.

Remarks. (i) In particular, Corollary 3.4 asserts that a positive C-semigroup $T(\cdot)$ dominates C on a Hilbert space if and only if its generator A is bounded, positive, and commutes with C. For a simple proof of this assertion for the special case $C = I$, see e.g. [17, Proposition 5.1]. It is worthwhile mentioning that the same assertion holds for positivity preserving C_0-semigroups on Banach lattices (see [14, Proposition 4.8 and Lemma 4.18 on pp. 274–279]).

(ii) Hermitian C-semigroups on the spaces $C_0(\Omega)$ and $L^p(\mu)$, $1 \le p < \infty$, $p \ne 2$, are C-semigroups of multiplication operators. For characterization of abstract multiplication semigroups on Banach lattices, see [14, pp. 287–290] and [16].

§4. Some Examples

In this section we include several examples as illustration of some results in Sections 2 and 3. The first example is a hermitian, contraction C_0-semigroup on ℓ_1.

Example 1. Let $X = \ell_1$ with coordinate vectors $e_1, e_2, \cdots$. Define $T(\cdot) : [0, \infty) \to B(\ell_1)$ by

$$T(t)x = (e^{-nt} x_n) \text{ for } x = (x_n) \in \ell_1 \text{ and } t \ge 0.$$

Let $x \in \ell_1$. If $\varepsilon > 0$ is arbitrary, then there is a positive integer N so that $\sum_{n=N+1}^{\infty} |x_n| < \varepsilon / 2$. Hence

$$\begin{aligned}\|T(t)x - x\| &= \|(e^{-nt}x_n) - (x_n)\| \\ &\le \sum_{n=1}^{N} (1 - e^{-nt})|x_n| + \sum_{n=N+1}^{\infty} (1 + e^{-nt})|x_n| \\ &\le \sum_{n=1}^{N} (1 - e^{-nt})|x_n| + 2(\varepsilon / 2).\end{aligned}$$

Taking lim sup we obtain that $\limsup_{t\to 0^+}\|T(t)x - x\| \le \varepsilon$. Hence $T(\cdot)$ is strongly continuous at $t = 0$. It is easy to see that $T(\cdot)$ is a hermitian (see [4, p. 92]), contraction C_0-semigroup. The infinitesimal generator of $T(\cdot)$ is the operator A defined as

$$\begin{cases} D(A) := \{(x_n) \in \ell_1; (-nx_n) \in \ell_1\}, \\ Ax := (-nx_n) \text{ for } x = (x_n) \in D(A). \end{cases}$$

Since A is unbounded, $T(\cdot)$ is not norm continuous at $t = 0$. By Theorem 2.3 (e), we know that $T(\cdot)$ is norm continuous on $(0, \infty)$. This fact can also be seen from the following estimate:

$$\begin{aligned}\|(T(t) - T(s))x\| &= \sum_{n=1}^{\infty} |e^{-nt} - e^{-ns}\|x_n\| \le \sup_{n\ge 1}(e^{-ns} - e^{-nt})\|x\| \\ &\le \{\sup_{1\le n\le N}(e^{-ns} - e^{-nt}) + e^{-Ns}\}\|x\|, 0 < s \le t, N = 1, 2, \cdots.\end{aligned}$$

Theorem 2.3 (d) asserts that a hermitian integrated C-semigroup $T(\cdot)$ is norm differentiable on $(0, \infty)$ and $T'(\cdot)$ is norm continuous and satisfies $T'(t)T'(s) = T'(t+s)C$ on $(0, \infty)$. The next example shows that $T'(t)$ need not be strongly convergent to C as $t \to 0^+$.

Example 2. Let $T(\cdot)$ be a hermitian C-semigroup on a Banach space X which is not norm continuous at 0 (for instance, the one in Example 1). For each $t \ge 0$ define a linear operator $\mathbb{T}(t)$ on $B(X)$ by $\mathbb{T}(t)S := \int_0^t \mathbb{T}(\tau)S d\tau (S \in B(X))$. Clearly, $\mathbb{T}(\cdot)$ is an integrated L_C-semigroup on $B(X)$, where the operator L_C is the left multiplication by C. Since $V(L_{T(t)}) = V(T(t))$, $\mathbb{T}(\cdot)$ is also hermitian. The norm continuity of $T(\cdot)$ on $(0, \infty)$ implies that $\mathbb{T}(\cdot)$ is norm continuous on $[0, \infty)$ and $\mathbb{T}'(\cdot)$ is norm continuous on $(0, \infty)$. But $\|(\mathbb{T}'(t) - L_C)(I)\| = \|(T(t) - C\| \not\to 0$ as $t \to 0^+$. That is, $\mathbb{T}'(\cdot)$ is not strongly continuous at 0, although it is norm continuous on $(0, \infty)$ and satisfies $\mathbb{T}'(t)\mathbb{T}'(s) = \mathbb{T}'(t+s)C$, t, $s > 0$.

Finally, we exhibit a positive C-semigroup which dominates C, is not exponentially bounded, is uniformly continuous on $[0, \infty)$, and has unbounded generator.

Example 3. Let $X=\ell_1$ and let $C:\ell_1\to\ell_1$ be the operator defined as $Cx=(e^{-n^2}x_n)$ for $x=(x_n)\in\ell_1$. Clearly, C has dense range. Define $T(\cdot):[0,\infty)\to B(\ell_1)$ by

$$T(t)x=(e^{nt-n^2}x_n) \text{ for } x=(x_n)\in\ell_1 \text{ and } t\geq 0.$$

$T(\cdot)$ is a nondegenerate C-semigroup, with generator A defined as

$$\begin{cases} D(A):=\{(x_n)\in\ell_1;(nx_n)\in\ell_1\} \\ Ax:=(nx_n) \text{ for } x=(x_n)\in D(A). \end{cases}$$

Since $T(t)\geq C\geq \mathbf{0}$ for all $t\geq 0$, it follows from Theorem 2.3 (f) that $T(\cdot)$ is norm continuous on $[0,\infty)$. Because of Corollary 3.4, the fact that A is unbounded implies that $T(\cdot)$ is not exponentially bounded. In fact, this is justified by the estimate: $\|T(2n)\|\geq\|T(2n)e_n\|=e^{n^2}$, $n=1,2,\cdots$.

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