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Norm Additivity Conditions for Normal Linear Functionals on von Neumann Algebras

By

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§1. Introduction

Let M be a von Neumann algebra and let φ and ψ be bounded linear functionals on M. We then have the norm inequality $\|\varphi+\psi\| \leq \|\varphi\|+\|\psi\|$. On the other hand, it is well-known that if φ and ψ are positive, then $\|\varphi+\psi\| = \|\varphi\|+\|\psi\|$. In general, however, such an equality does not necessarily holds if both φ and ψ are not positive. The purpose of this paper is to investigate when the norm equality $\|\varphi+\psi\| = \|\varphi\|+\|\psi\|$ holds for given normal linear functionals φ and ψ. Then the fact to play an essential role is the following:

Let M be a von Neumann algebra and let φ be a normal linear functional on M. Then we have

$$\varphi(\cdot) = |\varphi|(v\,\cdot), \quad |\varphi|(\cdot) = \varphi(v'\cdot), \quad \textit{and} \quad \|\varphi\| = \||\varphi|\|$$

for all partial isometries v' in M satisfying that $\varphi(v') = \|\varphi\|$, where $|\varphi|$ denotes the absolute value of φ (c.f. [1, Lemma 2.3]).

In connection with this fact, we may expect that the set of those elements x, in the unit ball of M, with $\varphi(x) = \|\varphi\|$ has nice information on norms and absolute values of normal linear functionals on M. In fact, by employing such a set, we will give necessary and sufficient conditions for a pair of normal linear functionals φ and ψ to satisfy that $\|\varphi+\psi\| = \|\varphi\|+\|\psi\|$ or that $|\varphi+\psi| = |\varphi|+|\psi|$ (Theorem 2.1).

§2. Results

Let M be a von Neumann algebra and let φ be a normal linear functional on M. By the *polar decomposition* of φ, we mean the following expression:

$$\varphi(\cdot) = |\varphi|(v\,\cdot) \quad \text{and} \quad |\varphi|(\cdot) = \varphi(v'\cdot)$$

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for some partial isometry v in M and a uniquê positive linear functional $|\varphi|$ on M which satisfies that

$$\|\varphi\| = \||\varphi|\| \text{ and } |\varphi(x)|^2 \le \|\varphi\| |\varphi|(x^* x) \qquad (*)$$

for all x in M (cf. [2, 3.6.7], [3, 1.14.4], or [4, III. 4.2]). Note that this condition (*) ensures the uniqueness of $|\varphi|$ (cf. [2, 3.6.7] or [4, III. 4.6]). More precisely, if a positive linear functional ψ satisfies that

$$\|\psi\| = \|\varphi\| \text{ and } |\varphi(x)|^2 \le \|\varphi\| \psi(x^* x),$$

then $\psi = |\varphi|$.[1] In general, there is some freedom for the choice of v, as is seen from [1, Lemma 2.3]. However, if vv^* is exactly equal to the support projection $s(|\varphi|)$ of $|\varphi|$, i.e., the smallest of all projections p such that $|\varphi|(p\cdot) = |\varphi|(\cdot)$, then v is uniquely determined (cf. [3, 1.14.4], [4, III. 4.6]).

Now we set

$$M_\varphi = \{x \in M \mid \|x\| \le 1, \ \varphi(x) = \|\varphi\|\},$$

which is a non-empty and weakly compact face of the unit ball of M. Hence M_φ contains a partial isometry (cf. [2, 1.4.7], [3, 1.6.5], [4, I.10.2]).

Positive linear functionals φ and ψ on a C^*-algebra always satisfy that $\|\varphi + \psi\| = \|\varphi\| + \|\psi\|$ and $\varphi + \psi$ is positive. These are generalized as follows.

Theorem 2.1. *Let M be a von Neumann algebra and let φ and ψ be normal linear functionals on M. Then the following conditions are equivalent*:

(1) *$M_\varphi \cap M_\psi$ is not empty.*

(2) $|\varphi + \psi| = |\varphi| + |\psi|$.

(3) $\|\varphi + \psi\| = \|\varphi\| + \|\psi\|$.

(4) $M_\varphi \cap M_\psi = M_{\varphi+\psi}$.

Proof. We first show that $M_\varphi \cap M_\psi \subset M_{\varphi+\psi}$. Without loss of generality, we can assume that $M_\varphi \cap M_\psi$ is not empty. For any x in $M_\varphi \cap M_\psi$, we have

$$\|\varphi\| + \|\psi\| = \varphi(x) + \psi(x) = (\varphi + \psi)(x) \le \|\varphi + \psi\|.$$

Since this mean that $(\varphi + \psi)(x) = \|\varphi + \psi\|$, we see that $x \in M_{\varphi+\psi}$.

$(1) \Rightarrow (2)$. Let v be a partial isometry in $M_\varphi \cap M_\psi$. Since $v \in M_{\varphi+\psi}$, it follows from [1, Lemma 2.3] that

$$|\varphi + \psi|(\cdot) = (\varphi + \psi)(v\,\cdot) = \varphi(v\,\cdot) + \psi(v\,\cdot) = |\varphi|(\cdot) + |\psi|(\cdot).$$

$(2) \Rightarrow (3)$. Since $|\varphi + \psi|$, $|\varphi|$ and $|\psi|$ are positive linear functionals, we have

$$\begin{aligned} \|\varphi + \psi\| &= \||\varphi + \psi|\| = |\varphi + \psi|(1) = (|\varphi| + |\psi|)(1) \\ &= |\varphi|(1) + |\psi|(1) = \||\varphi|\| + \||\psi|\| = \|\varphi\| + \|\psi\|. \end{aligned}$$

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[1] As the definition of a polar decomposition, we adopt the (right) polar decomposition mentioned in [2, 3.6.7] and our absolute value $|\varphi|$ means $|\varphi^*|$ in the sense of the (left) polar decomposition mentioned in [3, 1.14.4] and [4, III. §4].

(3) $\Rightarrow$ (4). We have only to show that $M_{\varphi+\psi} \subset M_\varphi \cap M_\psi$. Take any x from $M_{\varphi+\psi}$. We then have

$$\varphi(x)+\psi(x)=(\varphi+\psi)(x)=\|\varphi+\psi\|=\|\varphi\|+\|\psi\|.$$

Now denote by $\Re z$ the real part of a complex number z and by $\Im z$ the imaginary part of z, respectively. Since $\|\varphi\|+\|\psi\|$ is a real number, it follows from the above equality that

$$\Re\varphi(x)+\Re\psi(x)=\Re(\varphi(x)+\psi(x))=\|\varphi\|+\|\psi\|.$$

Here remark that

$$\|\omega\| \geq |\omega(x)| \geq |\Re\omega(x)| \qquad (**)$$

for every bounded linear functional ω on M. We thus have

$$0 \leq \|\varphi\|-\Re\varphi(x)=\Re\psi(x)-\|\psi\| \leq 0.$$

Hence we conclude that $\|\varphi\|=\Re\varphi(x)$ and $\|\psi\|=\Re\psi(x)$. This and the inequality $(**)$ show that $\Im\varphi(x)=0$, i.e., $\|\varphi\|=\varphi(x)$. Similarly we obtain that $\|\psi\|=\psi(x)$. We thus see that $x \in M_\varphi \cap M_\psi$.

(4) $\Rightarrow$ (1). Since $M_{\varphi+\psi}$ is not empty, this implication is clear. Q.E.D.

Here recall that positive linear functionals φ and ψ on a C^*-algebra are said to be *orthogonal* if $\|\varphi-\psi\|=\|\varphi\|+\|\psi\|$.

Corollary 2.2. *Let M be a von Neumann algebra and let φ and ψ be positive normal linear functionals on M. Then the following conditions are equivalent*:

(1) *φ and ψ are orthogonal.*

(2) *$M_\varphi \cap M_{-\psi}$ is not empty.*

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