

Quadratic Representations of the Canonical Commutation Relations

By

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Abstract

This paper studies a class of representations (called quadratic) of the canonical commutation relations over symplectic spaces of arbitrary dimension, which naturally generalizes coherent and symplectic (i.e. quasifree) representations and which has previously been heuristically employed in the special case of finite degrees of freedom in the physics literature. An explicit characterization of canonical quadratic transformations in terms of a 'standard form' is given, and it is shown that they can be exponentiated to give representations of the Weyl algebra. Necessary and sufficient conditions are presented for the unitary equivalence of these representations with the Fock representation. Possible applications to quantum optics and quantum field theory are briefly indicated.

I. Introduction

Since the canonical commutation relations (henceforth, the CCR) were introduced by Dirac in 1927 as generalizations of Heisenberg's commutation relations in order to discuss radiation theory [6], they have received a great deal of attention. Rigorous mathematical analysis of the CCR for the purposes of quantum field theory began in the 1950's, when it was also realized that in the case of infinitely many degrees of freedom, which is of relevance in quantum statistical mechanics and quantum field theory, there are (uncountably) infinitely many inequivalent irreducible representations of the CCR [8], and that the choice of the proper representation is crucial in any physical application. Various classes of representations of the CCR have been introduced and studied, and in this paper we wish to examine mathematically another class of representations. We

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shall introduce this additional class in a setting that makes clear that it is a natural generalization of two classes that have been intensively studied earlier - the coherent representations and the symplectic representations (also called quasifree representations). It should be mentioned that *special cases* of this class of representations have already been introduced and *heuristically* studied in the physics literature for a *single* degree of freedom [3] [16] and for a *finite* number of degrees of freedom [5]. We give here the first mathematically rigorous treatment of the entire class of quadratic representations, and we do so without restriction on the number of degrees of freedom.

We first give a brief introduction to these classes of representations. Let H be a real test function space with a symplectic form σ defined on it, and let $\{\Phi(f) | f \in H\}$ be a family of densely defined linear operators acting on the complex Hilbert space \mathcal{H} and satisfying

$$[\Phi(f), \Phi(g)] \subset i\sigma(f, g)\mathbf{1}, \quad f, g \in H. \quad (1.1)$$

In other words, $\Phi(f)$ is a representation of the CCR over (H, σ) . The best known such representation is that of Fock (cf. [4] for the first mathematically rigorous discussion of the Fock representation). Representations of the CCR are the basic building blocks in the theory of bosonic systems. The number of physical degrees of freedom of the system determines the dimension of H — in particular, dealing with infinitely many degrees of freedom necessitates handling infinite dimensional spaces H , whence the mathematical problems alluded to above arise. Different, i.e. unitarily inequivalent, representations of the CCR over (H, σ) manifest different physical properties in general. Roughly speaking, the choice of the symplectic space (H, σ) corresponds to a choice of kinematics in the model, whereas the choice of representation corresponds to a choice of dynamics. (For an elaboration of this point, see the texts [7] [25] [9]). A standard technique for constructing such representations is to start with a Fock representation and then perform a canonical transformation, i.e. transformation $\Phi(f) \mapsto \Phi'(f)$, $f \in H$, such that equation (1.1) holds with Φ replaced by Φ' . If the new representation $\Phi'(f)$ is not unitarily equivalent to $\Phi(f)$, then one has the chance of modelling different physics. In this paper, a new class of representations of the CCR, which are not unitarily equivalent to the Fock representation and which therefore open up the opportunity to model new physics, is displayed.

The simplest class of canonical transformations is that of the coherent transformations

$$\Phi(f) \mapsto \Phi'(f) \equiv \Phi(f) + \lambda(f)\mathbf{1},$$

where λ is a real-valued functional on H , which are intimately related with the

coherent states used in quantum optics and elsewhere [12]. Another well-studied class of canonical transformations is that of the symplectic (or quasifree, or Bogoliubov) transformations

$$\Phi(f) \mapsto \Phi'(f) \equiv \Phi(Tf) = \Phi(f) + \Phi((T-1)f),$$

where the operator $T : H \mapsto H$ is symplectic, i.e. satisfies $\sigma(Tf, Tg) = \sigma(f, g)$. Together, the coherent and symplectic transformations constitute the group of inhomogeneous linear canonical transformations, have been intensively studied (see, for example, [1]), and have found application in many branches of theoretical physics. It is known that the coherent representations are unitarily equivalent to the Fock representation if and only if the linear functional λ is bounded [1], and the symplectic transformations are unitarily equivalent to the original Fock representation if and only if the operator $\mathbf{1} - |T| = \mathbf{1} - (T^*T)^{1/2}$ is a Hilbert-Schmidt operator [26].

To avoid technical definitions (which will be given later in the body of this paper) in this introduction, we shall illustrate some of the basic points to be made with a class of easily describable examples. Let $\{q_k, p_k\}_{k=1}^\infty$ be a system of densely defined operators acting on the complex Hilbert space \mathcal{H} and satisfying

$$[q_j, p_k] = i\delta_{jk} \cdot \mathbf{1},$$

and for the standard annihilation and creation operators $a_k \equiv \frac{1}{\sqrt{2}}(q_k + ip_k)$ and $a_k^\dagger \equiv \frac{1}{\sqrt{2}}(q_k - ip_k)$ let there be a vector $\Omega \in \mathcal{H}$ such that $a_k\Omega = 0$ for all $k \in \mathbf{N}$. In other words, we consider the Fock representation of a bosonic system with infinitely many degrees of freedom. In this setting, the coherent, resp. symplectic, canonical transformations can be written as

$$\begin{aligned} (A) \quad & q_k \mapsto q_k, \quad p_k \mapsto p_k + \lambda_k \cdot \mathbf{1}, \\ (B) \quad & q_k \mapsto q_k, \quad p_k \mapsto p_k + \sum_l \lambda_{kl} q_l, \end{aligned}$$

where λ_k, λ_{kl} are real and $\lambda_{kl} = \lambda_{lk}$ for each k, l . Both transformations have the general structure $q_k \mapsto q_k, p_k \mapsto p_k + F_k(q_1, q_2, \dots)$, where the $F_k(q_1, q_2, \dots)$ are real functions of the operators q_1, q_2, \dots . Note that the transformations above are generated by the choice $F_k \equiv \frac{\partial F}{\partial x_k}$, with

$$\begin{aligned} (A) \quad & F(x_1, x_2, \dots) = \sum_k \lambda_k x_k, \\ (B) \quad & F(x_1, x_2, \dots) = \frac{1}{2} \sum_{k,l} \lambda_{kl} : x_k x_l : \equiv \frac{1}{2} \sum_{k,l} \lambda_{kl} \left(x_k x_l - \frac{1}{2} \delta_{kl} \right). \end{aligned}$$

It is a natural next step to choose $F = \frac{1}{3} \sum_{k,l,m} \lambda_{klm} : x_k x_l x_m :$, which would give rise to the transformation [22]

$$q_k \mapsto q_k, \quad p_k \mapsto p_k + \sum_{l,m} \lambda_{klm} : q_l q_m :. \quad (1.2)$$

(In order for this transformation to be canonical, the coefficients $\lambda_{klm} \in \mathbf{R}$ must be totally symmetric in the indices.) We find it natural to call these transformations (and the resulting representations) higher-order¹ transformations (representations) of the CCR.

After defining general quadratic transformations below, those which are canonical will be characterized. It will be shown that for any *canonical* quadratic transformation, there exists a choice of basis in (H, σ) in terms of which the transformation can be brought into the form (1.2)² with $\sum_{l,m} \lambda_{klm}^2 < \infty$. Moreover, it will be proven that any such transformation can be exponentiated to yield a representation of the corresponding Weyl algebra over (H, σ) , and that the transformation produces a representation of the CCR unitarily equivalent with the original Fock representation *if and only if* $\sum_{k,l,m} \lambda_{klm}^2 < \infty$. As we shall see below, this condition can be equivalently expressed in terms of a certain operator being Hilbert-Schmidt.

The paper is organized as follows. After establishing the notation and general setting in Chapter II, quadratic transformations are defined and studied in Chapter III, where the canonical quadratic transformations are explicitly characterized. In Chapter IV it is shown that any canonical quadratic transformation may be exponentiated, and the mentioned necessary and sufficient conditions for their unitary implementability are proven. Finally, further developments, speculations and possible applications are discussed in Chapter V.

Some of the results presented in this paper have been announced in [22].

II. Notation and General Setting

For us the basic starting point is a real (nondegenerate) symplectic space (H, σ) with an associated (regular) Weyl system $(\mathcal{H}, W(f))$ consisting of a complex Hilbert space \mathcal{H} and a mapping $W : H \mapsto \mathcal{U}(\mathcal{H})$ from H into the group $\mathcal{U}(\mathcal{H})$ of unitary operators on \mathcal{H} which satisfies the following axioms [13]:

¹ In the case (1.2), the transformation is called quadratic; for a brief discussion of transformations of degree greater than 2, see Chapter 5 and [22].

² which we call *standard*

$$W(f)W(g) = e^{-i\sigma(f,g)/2}W(f+g), \quad \forall f, g \in H, \tag{2.1}$$

$$W(f)^* = W(-f), \quad \forall f \in H, \tag{2.2}$$

and

$$\mathbf{R} \ni t \mapsto W(tf) \in \mathcal{B}(\mathcal{H}) \text{ is weakly continuous for all } f \in H. \tag{2.3}$$

($\mathcal{B}(\mathcal{H})$ denotes the set of all bounded linear operators $A : \mathcal{H} \mapsto \mathcal{H}$.) Condition (2.3) entails that the map $\mathbf{R} \ni t \mapsto W(tf) \in \mathcal{B}(\mathcal{H})$ is actually strongly continuous, hence by Stone’s Theorem one knows that for each $f \in H$ there exists a self-adjoint operator $\Phi(f)$ on \mathcal{H} such that $W(tf) = e^{it\Phi(f)}$ for all $t \in \mathbf{R}$ and by (2.1) the map $f \mapsto \Phi(f)$ is (real) linear. In fact, there exists a dense domain of vectors $D_w \subset \mathcal{H}$ which is a core of and left invariant by every $\Phi(f)$ [19] [10]; it is on this domain that the linearity just mentioned can be verified. On this domain one also verifies that the generators satisfy the CCR:

$$\Phi(f)\Phi(g) - \Phi(g)\Phi(f) = i\sigma(f, g)\mathbf{1}, \quad \forall f, g \in H. \tag{2.4}$$

We therefore also call the Weyl system $(\mathcal{H}, W(f))$ and its associated generators as above a regular representation of the CCR over (H, σ) .

We shall denote by $\mathcal{A}(H, \sigma)$ the C^* -algebra on \mathcal{H} generated by the operators $\{W(f) \mid f \in H\}$. As the notation indicates, the algebra $\mathcal{A}(H, \sigma)$ does not depend on the choice of representation of the Weyl operators $\{W(f) \mid f \in H\}$ [28] [2, Theorem 5.2.8]. $\mathcal{A}(H, \sigma)$ is a simple, nonseparable C^* -algebra [28] [13]. There are, in fact, many different C^* -algebras one can associate with the CCR (see [11] for a discussion of some of the alternatives), and the one we have chosen is minimal in the sense of set containment [15]; but, for practical purposes the choice is moot, since one is generally interested in a von Neumann algebra which is ‘generated’ by the C^* -algebra, and all the C^* -algebras discussed in [11] (realized concretely on a given representation space) have the same weak closure.

For any real linear map $J : H \mapsto H$ satisfying $\sigma(Jf, Jg) = \sigma(f, g)$, $-\sigma(Jf, f) > 0$ ($f \neq 0$), and $J^2 = -\mathbf{1}$, one can introduce a complex structure on H as follows³ [11]: $(\alpha + i\beta)f \equiv \alpha f + \beta Jf$, for all $f \in H$, and $\alpha, \beta \in \mathbf{R}$. Moreover, $\langle f, g \rangle_{\mathcal{H}} \equiv -\sigma(Jf, g) + i\sigma(f, g)$ defines a scalar product on H such that $(H, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is a complex preHilbert space [11]. If on the other hand one begins with a complex Hilbert space \mathcal{H} with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, then with $\sigma(f, g) = \Im \langle f, g \rangle_{\mathcal{H}}$, (H, σ) is a real symplectic space of the sort with which we began, and with

³ It should be mentioned that such a J does not necessarily exist for arbitrary choice of symplectic space (H, σ) .

$\langle f, g \rangle \equiv \Re \langle f, g \rangle_{\mathcal{H}}$, then $(H, \langle \cdot, \cdot \rangle)$ is a real Hilbert space.

With a choice of a σ -admissible complex structure J on (H, σ) , there is an important representation of $\mathcal{A}(H, \sigma)$ called the Fock representation. This is given as the GNS-representation $\{\mathcal{H}, \pi, \Omega\}$ associated with the state [14] ω_J determined on $\mathcal{A}(H, \sigma)$ by

$$\omega_J(W(f)) \equiv e^{\sigma(Jf, f)/4}. \tag{2.5}$$

Given such a representation, one can define the following ‘annihilation’ and ‘creation’ operators:

$$a(f) \equiv \frac{1}{\sqrt{2}}(\Phi(f) + i\Phi(Jf)), \quad a^\dagger(f) \equiv \frac{1}{\sqrt{2}}(\Phi(f) - i\Phi(Jf)), \tag{2.6}$$

where $\pi(W(tf)) = e^{it\Phi(f)}$. One has then $a(f)\Omega = 0$ for all $f \in H$.

For the purposes of this paper, we shall assume that a choice of σ -admissible complex structure J has been made on (H, σ) , so we have the complex one-particle space \mathcal{H} and a corresponding Fock representation. Since as sets $H = \mathcal{H}$, we shall distinguish notationally the vector f viewed as an element of the real Hilbert space H from the same vector, denoted as \tilde{f} , viewed as an element of the complex Hilbert space \mathcal{H} . If $\{\tilde{e}_k\}_{k \in \mathbf{N}}$ forms an orthonormal basis in \mathcal{H} , then with $Je_k \equiv e_{k'}$, the set $\{e_K\} = \{e_k, e_{k'}\} = \{e_k, Je_k\}$ (in this paper, large-lettered indices K, L, M, \dots are understood to run through the set $\{1, 1', 2, 2', \dots\}$, whereas small-lettered indices k, l, m, \dots run through $\{1, 2, 3, \dots\} \equiv \mathbf{N}$) forms a symplectic orthonormal system in H , in particular

$$\begin{aligned} \sigma(e_k, e_l) = \sigma(e_{k'}, e_{l'}) = 0, \quad \sigma(e_k, e_{l'}) = \delta_{kl}, \\ \langle e_k, e_l \rangle = \langle e_{k'}, e_{l'} \rangle = \delta_{kl}, \quad \langle e_k, e_{l'} \rangle = 0, \end{aligned}$$

where in our notation the last line can be written as $\langle e_K, e_L \rangle = \delta_{KL}$.

With the choice of complex structure made as above, the corresponding Fock state denoted by $\omega_F : \mathcal{A}(H, \sigma) \mapsto \mathbf{C}$ now satisfies

$$\omega_F(W(f)) = e^{-\|f\|^2/4},$$

and the associated GNS-space may be represented by the symmetric Fock space $\mathcal{F}_+(\mathcal{H})$. We recall that the Fock space $\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^\infty \mathcal{H}^n$ ($\mathcal{H}^0 = \mathbf{C}$, \mathcal{H}^n the n -fold tensor product of \mathcal{H} with itself), and that $\mathcal{F}_+(\mathcal{H})$ is the totally symmetric subspace $\bigoplus_{n=0}^\infty P_+ \mathcal{H}^n$ of $\mathcal{F}(\mathcal{H})$, where P_+ is the projection

$$P_+(\tilde{f}_1 \otimes \tilde{f}_2 \otimes \cdots \otimes \tilde{f}_n) = \frac{1}{n!} \sum_{\pi} \tilde{f}_{\pi(1)} \otimes \tilde{f}_{\pi(2)} \otimes \cdots \otimes \tilde{f}_{\pi(n)}$$

($\tilde{f}_i \in \mathcal{H}$, π : permutation on the set $\{1, 2, \dots, n\}$). The projection operator $P_n : \mathcal{F}(\mathcal{H}) \mapsto \mathcal{H}^n$ projects onto the n -particle subspace, and we shall write $\varphi^{(n)} \equiv P_n \varphi$ as the n -particle component of $\varphi \in \mathcal{F}(\mathcal{H})$. The vector $\Omega_0 \equiv (1, 0, 0, \dots) \in \mathcal{F}_+(\mathcal{H})$ is the Fock vacuum. \mathcal{F}_0 is the finite-particle subspace, i.e. the linear span of the ranges of $\{P_n\}_{n \in \mathbf{N}}$. The GNS-representation for the Fock state may be identified with $\{\mathcal{F}_+(\mathcal{H}), \pi_F, \Omega_0\}$.

With $a(\tilde{f})$ the usual annihilation operator in $\mathcal{F}_+(\mathcal{H})$ for $\tilde{f} \in \mathcal{H}$ and the adjoint operator $a(\tilde{f})^*$ the corresponding creation operator, then the linear self-adjoint operator $\Phi_s(\tilde{f}) \equiv \frac{1}{\sqrt{2}} \overline{(a(\tilde{f}) + a(\tilde{f})^*)}$ (the bar denotes the closure of the operator) is called the Segal field operator in $\mathcal{F}_+(\mathcal{H})$. If we also view $\tilde{f} \in \mathcal{H}$ as an element of H , then we have $\Phi_s(\tilde{f}) = \Phi(f)$, after identifying $\mathcal{A}(H, \sigma)$ with $\pi_F(\mathcal{A}(H, \sigma))$ (since they are isomorphic). \mathcal{F}_0 is a core for $\Phi(f)$, and for $\varphi \in \mathcal{F}_0$, $\Phi(f)$ satisfies the bound $\|\Phi(f)\varphi\| \leq \sqrt{2} \sqrt{n_\varphi + 1} \|f\| \|\varphi\|$, where n_φ equals the smallest $n \in \mathbf{N}$ such that $P_N \varphi = 0$ for all $N > n$. If $(f_n)_{n=1}^\infty$ converges in H to f , then for every $\varphi \in \mathcal{F}_0$ the sequence $(\Phi(f_n)\varphi)_{n=1}^\infty$ converges in $\mathcal{F}_+(\mathcal{H})$ to $\Phi(f)\varphi$. To make a notational connection to the discussion in Section I and in keeping with common harmonic oscillator conventions, if $\{e_k\}$ is a symplectic orthonormal basis in H , then the ‘position operator’ and the ‘momentum operator’ corresponding to the k -th degree of freedom are given by

$$q_k \equiv \Phi_s(\tilde{e}_k) = \Phi(e_k), \quad p_k \equiv \Phi_s(i\tilde{e}_k) = \Phi(Je_k) = \Phi(e_{k'})$$

III. Quadratic Transformations

In this chapter we rigorously define what we mean by quadratic transformations and then explicitly characterize those which are canonical, i.e. those which lead to a representation of the CCR. We shall show that any such transformation can be written in the standard form of equation (1.2). In the following, $D(O)$ (resp. $R(O)$) will denote the domain (resp. range) of the operator O .

§3.1. Two-Particle Operators and Generators

We first define and study the two-particle operators and their associated transformations. Let $F \in H \odot H$ have the form $F = \sum_{i=1}^n \alpha_i (f_i \otimes g_i)$, with $\alpha_i \in \mathbf{R}$, f_i, g_i

$\in H$ and $n \in \mathbb{N}$. We define an operator $\Psi(F)$ acting on the domain $\mathcal{F}_0 \subset \mathcal{F}_+(\mathcal{H})$ by

$$\Psi(F)\phi = \Psi\left(\sum_{i=1}^n \alpha_i(f_i \otimes g_i)\right)\phi \equiv \sum_{i=1}^n \alpha_i : \Phi(f_i) \Phi(g_i) : \phi, \quad \phi \in \mathcal{F}_0,$$

where the Wick ordering is taken with respect to the Fock vacuum Ω_0 , i.e.

$$: \Phi(f) \Phi(g) : = \Phi_s(\tilde{f}) \Phi_s(\tilde{g}) - \frac{1}{2} \langle \tilde{f}, \tilde{g} \rangle_{\mathcal{H}} \mathbf{1} = \Phi(f) \Phi(g) - \frac{1}{2} \langle f, g \rangle \mathbf{1} - \frac{i}{2} \sigma(f, g) \mathbf{1}.$$

It is easy to verify that this definition is independent of the special choice of linear representation for F (see [18], pp. 14-15).

Lemma 3.1.1. *For each $\phi \in \mathcal{F}_0$ the mapping $F \mapsto \Psi(F)\phi$ from $H \odot H$ to $\mathcal{F}_+(\mathcal{H})$ is bounded with*

$$\|\Psi(F)\phi\| \leq 2\sqrt{2}\sqrt{(n_\phi+2)(n_\phi+1)}\|F\|\|\phi\|, \tag{3.1.1}$$

for any $\phi \in \mathcal{F}_0$. Hence it may be extended uniquely to a continuous linear map from $H \otimes H (\equiv H^2)$ to $\mathcal{F}_+(\mathcal{H})$. Consequently, for each $F \in H^2$ there exists a linear operator with domain \mathcal{F}_0 , which shall again be denoted by $\Psi(F)$ and which satisfies the same bound (3.1.1).

Proof. 1. With $F = \sum_{i=1}^n \alpha_i(f_i \otimes g_i)$, there is a finite orthonormal and symplectic system $\{e_1, J e_1, \dots, e_m, J e_m\} = \{e_K | K = 1, 1', \dots, m, m'\} = \{e_K\}_{K \in \mathcal{M}}$, with $\mathcal{M} = \{1, 1', \dots, m, m'\}$, $2m \leq n$, such that $F = \sum_{K,L \in \mathcal{M}} \beta_{KL}(e_K \otimes e_L)$. This can be seen by considering the subspace of \mathcal{H} spanned by $\tilde{f}_1, \dots, \tilde{f}_n, \tilde{g}_1, \dots, \tilde{g}_n$ and choosing an orthonormal basis $\{h_1, \dots, h_m\}$ for it. $\{e_i\}$ will denote the corresponding vectors in H , i.e. $\tilde{e}_i = h_i$. Then one has $\tilde{f}_1 = \sum_{i=1}^m \gamma_i \tilde{e}_i$ if and only if $f_1 = \sum_{i=1}^m \Re \gamma_i e_i + \Im \gamma_i J e_i$ (similarly for the other test functions). With F now expressed as $F = \sum_{K,L \in \mathcal{M}} \beta_{KL}(e_K \otimes e_L)$, define $\tilde{F} = \sum_{K,L \in \mathcal{M}} \beta_{KL}(\tilde{e}_K \otimes \tilde{e}_L) \in \mathcal{H}^2$. Then one has $\|\tilde{F}\| \leq \sqrt{2}\|F\|$, which can be verified by direct computation.

2. One then computes:

$$\Psi(F)\phi = \sum_{K,L \in \mathcal{M}} \beta_{KL} \Phi(e_K) \Phi(e_L) : \phi$$

$$\begin{aligned}
 &= \frac{1}{2} \sum_{K,L \in \mathcal{M}} \beta_{KL} \{ a^*(\tilde{e}_K) a^*(\tilde{e}_L) + a(\tilde{e}_K) a(\tilde{e}_L) + a^*(\tilde{e}_K) a(\tilde{e}_L) + a^*(\tilde{e}_L) a(\tilde{e}_K) \} \psi; \\
 \|\Psi(F) \phi\| &\leq \frac{1}{2} \{ \left\| \sum_{K,L \in \mathcal{M}} \beta_{KL} a^*(\tilde{e}_K) a^*(\tilde{e}_L) \phi \right\| + \left\| \sum_{K,L \in \mathcal{M}} \beta_{KL} a(\tilde{e}_K) a(\tilde{e}_L) \phi \right\| \\
 &\quad + \left\| \sum_{K,L \in \mathcal{M}} \beta_{KL} a^*(\tilde{e}_K) a(\tilde{e}_L) \phi \right\| + \left\| \sum_{K,L \in \mathcal{M}} \beta_{KL} a(\tilde{e}_K) a^*(\tilde{e}_L) \phi \right\| \}. \tag{3.1.2}
 \end{aligned}$$

3. Now, in particular, one has

$$\begin{aligned}
 \left\| \sum_{K,L \in \mathcal{M}} \beta_{KL} a^*(\tilde{e}_K) a^*(\tilde{e}_L) \phi \right\|^2 &= \sum_{n=0}^{n_\phi} \left\| \sum_{K,L \in \mathcal{M}} \beta_{KL} a^*(\tilde{e}_K) a^*(\tilde{e}_L) \phi^{(n)} \right\|^2 \\
 &= \sum_{n=0}^{n_\phi} \left\| \sum_{K,L \in \mathcal{M}} \beta_{KL} \sqrt{n+2} \sqrt{n+1} P_+ \tilde{e}_K \otimes \tilde{e}_L \otimes \phi^{(n)} \right\|^2 \\
 &= \sum_{n=0}^{n_\phi} (n+2) (n+1) \|P_+ \tilde{F} \otimes \phi^{(n)}\|^2 \\
 &\leq (n_\phi+2) (n_\phi+1) \|\tilde{F}\|^2 \sum_{n=0}^{n_\phi} \|\phi^{(n)}\|^2 \\
 &\leq 2 (n_\phi+2) (n_\phi+1) \|F\|^2 \|\phi\|^2.
 \end{aligned}$$

Thus, the map $\phi^{(n)} \mapsto a^*(\tilde{e}_K) a^*(\tilde{e}_L) \phi^{(n)}$ is a bounded linear map from $P_+ \mathcal{H}^n$ to $P_+ \mathcal{H}^{(n+2)}$ with norm bounded by $\sqrt{2} \sqrt{(n+2)(n+1)} \|F\|$. The map $\phi^{(n+2)} \mapsto a(\tilde{e}_K) a(\tilde{e}_L) \phi^{(n+2)}$ is (a restriction of) its adjoint and therefore satisfies the bound: $\left\| \sum_{K,L \in \mathcal{M}} \beta_{KL} a(\tilde{e}_K) a(\tilde{e}_L) \phi^{(n+2)} \right\| \leq \sqrt{2} \sqrt{(n+2)(n+1)} \|F\| \|\phi^{(n+2)}\|$, which entails $\left\| \sum_{K,L \in \mathcal{M}} \beta_{KL} a(\tilde{e}_K) a(\tilde{e}_L) \phi \right\|^2 \leq 2 (n_\phi+2) (n_\phi+1) \|F\|^2 \|\phi\|^2$.

4. Similarly, one finds

$$\begin{aligned}
 \sum_{K,L \in \mathcal{M}} \beta_{KL} a^*(\tilde{e}_K) a(\tilde{e}_L) \phi^{(n)} &= \sum_{k=1}^m \sum_{L \in \mathcal{M}} (\beta_{kL} + i\beta_{k'L}) a^*(\tilde{e}_k) a(\tilde{e}_L) \phi^{(n)} \\
 &= \sum_{k=1}^m \sqrt{n} P_+ \tilde{e}_k \otimes [a(\sum_{L \in \mathcal{M}} (\beta_{kL} - i\beta_{k'L}) \tilde{e}_L) \phi^{(n)}].
 \end{aligned}$$

In light of the standard bound $\|a(f) \phi^{(n)}\| \leq \sqrt{n} \|f\| \|\phi^{(n)}\|$, it now follows that

$$\begin{aligned}
 \left\| \sum_{K,L \in \mathcal{M}} \beta_{KL} a^*(\tilde{e}_K) a(\tilde{e}_L) \phi^{(n)} \right\|^2 &\leq n \sum_{k=1}^m \left\| \sum_{L \in \mathcal{M}} (\beta_{kL} + i\beta_{k'L}) \tilde{e}_L \right\|^2 \|\phi^{(n)}\|^2 \\
 &\leq 2n^2 \|F\|^2 \|\phi^{(n)}\|^2.
 \end{aligned}$$

Thus one has the inequality $\left\| \sum_{K,L \in \mathcal{M}} \beta_{KL} a^*(\tilde{e}_K) a(\tilde{e}_L) \phi \right\|^2 \leq 2 (n_\phi+2) (n_\phi+1) \|F\|^2 \|\phi\|^2$.

The same bound is applicable to the remaining term in (3.1.2).

Putting together Steps (2) - (4), one may conclude

$$\|\Psi(F)\psi\| \leq \frac{1}{2} \cdot 4\sqrt{2} \sqrt{(n_\psi+2)(n_\psi+2)} \|F\| \|\psi\|.$$

□

Of course, a consequence of this lemma is that if $\{F_n\} \subset H^2$ converges to $F \in H^2$, then the operators $\Psi(F_n)$ converge strongly to $\Psi(F)$ on \mathcal{F}_0 . This would not be true if in the definition of the two particle operators $\Psi(F)$ normal ordering had not been used (see [18], p.20). Another technically important fact is also implied by the previous lemma.

Corollary 3.1.2. *For any $F \in H^2$, $\Psi(F)$ is a symmetric operator for which the equality $\Psi(F) = \Psi(P_+F) + \Psi(P_-F) = \Psi(P_+F)$ obtains.*

Proof. That $\Psi(F)^* \psi = \Psi(F)\psi$ for $F \in H \odot H$ and $\psi \in \mathcal{F}_0$ is obvious. The continuity property established in Lemma 3.1.1 then entails that the operator $\Psi(F)$ is symmetric on \mathcal{F}_0 for all $F \in H^2$.

Moreover, since $\Psi(f \otimes g) = \Psi(g \otimes f)$, the equality $\Psi(F) = \Psi(P_+F)$ follows for $F \in H \odot H$ and once again by continuity for $F \in H^2$. □

Therefore, with $f \in H$ and $F \in H^2$, we have $(\Phi(f) + \Psi(F))^* \supset \Phi(f) + \Psi(F)^* \supset \Phi(f) + \Psi(F)$, so that the operator $\Phi(f) + \Psi(F)$ is symmetric and closable. If $\Lambda : H \rightarrow P_+H^2$ is a linear, densely defined operator, then we define a quadratic transformation of the field to be

$$\Phi(f) \mapsto \Phi_\Lambda(f) \equiv \overline{\Phi(f) + \Psi(\Lambda f)}, \quad f \in D(\Lambda), \tag{3.1.3}$$

where $\overline{\Phi(f) + \Psi(\Lambda f)}$ denotes the closure of $\Phi(f) + \Psi(\Lambda f)$ on \mathcal{F}_0 . It is easy to verify that $\Phi_\Lambda(f) = \overline{\Phi(f)|_{\mathcal{F}_0} + \Psi(\Lambda f)}$ and $\Phi_\Lambda(f) = \overline{\Phi(f) + \Psi(\Lambda f)}$ (see [18], Satz 3.1). We next shall see that $\Phi_\Lambda(f)$ is self-adjoint.

Proposition 3.1.3. *For every $f \in H$ and $F \in H^2$ the operator $\Phi(f) + \Psi(F)$ is essentially self-adjoint (on \mathcal{F}_0). In particular, for every $f \in D(\Lambda)$ the operator $\Phi_\Lambda(f) \equiv \overline{\Phi(f) + \Psi(\Lambda f)}$ is self-adjoint.*

Proof. Let ψ be a finite particle vector, i.e. let $\psi = (\psi^{(n)})_{n=0}^\infty$, $\psi^{(n)} \in P_+\mathcal{H}^n$, and $\psi^{(n)} = 0$ for all $n > n_\psi$. As a shorthand we write: $\widehat{\Phi} \equiv \Phi(f) + \Psi(F)$ for arbitrary fixed $f \in H$ and $F \in H^2$, as well as $\alpha \equiv \|f\| + 2\|F\|$. Since $\widehat{\Phi}$ maps every finite particle vector onto a vector in \mathcal{F}_0 , one has

$$\psi \in C^\infty(\widehat{\Phi}) \equiv \bigcap_{k=1}^\infty D(\widehat{\Phi}^k).$$

Furthermore, one has

$$\begin{aligned} \|\widehat{\Phi}\phi\| &= \|\Phi(f)\phi + \Psi(F)\phi\| \leq \|\Phi(f)\phi\| + \|\Psi(F)\phi\| \\ &\leq \sqrt{2}\sqrt{n_\phi+1}\|f\|\|\phi\| + 2\sqrt{2}\sqrt{(n_\phi+1)(n_\phi+2)}\|F\|\|\phi\|. \end{aligned}$$

For the first term of the last line see e.g. the proof of Theorem X.41 in [21]; for the second term see the proof of Lemma 3.1.1 above. Therefore one has

$$\|\widehat{\Phi}\phi\| \leq \sqrt{2}\sqrt{(n_\phi+1)(n_\phi+2)}(\|f\|+2\|F\|)\|\phi\| = \sqrt{2}\alpha\sqrt{(n_\phi+1)(n_\phi+2)}\|\phi\|.$$

In light of the fact that the application of $\widehat{\Phi}$ to a finite particle vector increases the highest ‘particle-number’ at most by 2 (since it contains two creation operators), one concludes that

$$\begin{aligned} \|\widehat{\Phi}^n\phi\| &\leq \sqrt{2}\alpha[(n_\phi+2(n-1)+1)(n_\phi+2(n-1)+2)]^{1/2}\|\widehat{\Phi}^{n-1}\phi\| \\ &\leq 2^{n/2}\alpha^n[(n_\phi+1)(n_\phi+2)\cdots(n_\phi+2n)]^{1/2}\|\phi\| \\ &\leq 2^{n/2}\alpha^n\sqrt{(n_\phi+2n)}!\|\phi\|. \end{aligned}$$

In order to obtain a sufficient condition for the convergence of the series $\sum_{n=0}^\infty \frac{1}{n!}\|\widehat{\Phi}^n\phi\|t^n$ ($t > 0$), one can apply the ratio test to its majorant

$\sum_{n=0}^\infty \frac{1}{n!}2^{n/2}\alpha^n t^n \sqrt{(n_\phi+2n)}!\|\phi\|$ to find that

$$\begin{aligned} \frac{(2^{n/2}/n!) \alpha^n t^n \sqrt{(n_\phi+2n)}!}{(2^{(n-1)/2}/(n-1)!) \alpha^{n-1} t^{n-1} \sqrt{(n_\phi+2(n-1))}!} &= \frac{\sqrt{2}}{n} \alpha t \sqrt{(n_\phi+2n-1)(n_\phi+2n)} \\ &\leq \frac{\sqrt{2}}{n} \alpha t (n_\phi+2n) \xrightarrow{n \rightarrow \infty} 2\sqrt{2}\alpha t. \end{aligned}$$

Thus one has $\sum_{n=0}^\infty \frac{1}{n!}\|\widehat{\Phi}^n\phi\|t^n < \infty$ for $t < \frac{1}{2\sqrt{2}\alpha}$, so ϕ is an analytic vector for $\widehat{\Phi}$.

Since therefore all $\phi \in \mathcal{F}_0$ are analytic vectors for $\widehat{\Phi}$, since \mathcal{F}_0 is dense in Fock space and left invariant by $\widehat{\Phi}$, and since $\widehat{\Phi}$ is symmetric, the essential self-adjointness of $\widehat{\Phi}$ follows from Nelson’s Theorem (see e.g. [21], Theorem X.6). □

Remarks. 1. Also the operator $\Psi(F)$ is essentially self-adjoint, as one sees by setting $f=0$. The proof cannot be applied to an N -particle operator $\Psi^{(N)}(F)$, $F \in H^N$ with $N > 2$, since then $\sqrt{(n_\phi+2n)}!$ becomes $\sqrt{(n_\phi+Nn)}!$, and the ratio

test no longer yields convergence.

2. Because of the essential self-adjointness of $\Psi(F)$ and the proven continuity property, one may conclude from standard theorems that if $\{F_n\} \subset H^2$ converges to $F \in H^2$, then the operators $\overline{\Psi(F_n)}$ converge to $\overline{\Psi(F)}$ in the strong resolvent sense, and $\overline{\Psi(F)}$ is the strong graph limit of $\{\overline{\Psi(F_n)}\}$ (see, e.g., [20], Theorems VIII.25 and VIII.26).

§3.2. Canonical Quadratic Transformations

In this section we determine necessary and sufficient conditions on the linear, densely defined map $\Lambda : H \mapsto H^2 \equiv H \otimes H$ such that the quadratic field transformation

$$\Phi(f) \mapsto \Phi_\Lambda(f) \equiv \overline{\Phi(f) + \Psi(\Lambda f)}, \quad f \in D(\Lambda), \tag{3.2.1}$$

is canonical. In light of Proposition 3.1.3, for every $f \in D(\Lambda)$ the transformed field operator $\Phi_\Lambda(f)$ is self-adjoint. We assume Λ to be linear, so that $\Phi_\Lambda(f)$ is linear in f , analogously to $\Phi(f)\varphi$ (for $\varphi \in \mathcal{F}_0$). Moreover, $D(\Lambda)$ should be dense, so that there exists no nonzero $f \perp D(\Lambda)$, i.e. no degree of freedom, for which the transformation (3.2.1) would not be defined. On the other hand, $D(\Lambda) = H$ is not required, as we wish to admit unbounded Λ . Due to Corollary 3.1.2, the range $R(\Lambda)$ of Λ may be taken to lie in P_+H^2 without loss of generality. For convenience, we introduce the following definition.

Definition 3.2.1. *Let \mathcal{L} be the set of all linear, densely defined operators from H to P_+H^2 , and let $\mathcal{L}_{CCR} \subset \mathcal{L}$ be the subset of \mathcal{L} consisting of elements Λ such that the transformation $\Phi(f) \mapsto \Phi_\Lambda(f) = \overline{\Phi(f) + \Psi(\Lambda f)}$, $f \in D(\Lambda)$, is canonical, i.e. one has*

$$[\Phi_\Lambda(f), \Phi_\Lambda(g)] \subset i\sigma(f, g)\mathbf{1}, \tag{3.2.2}$$

whenever $f, g \in D(\Lambda)$.

Note that since \mathcal{F}_0 is an invariant core for $\Phi_\Lambda(f)$, it suffices to verify equation (3.2.2) on the elements of \mathcal{F}_0 . We introduce now a symplectic bilinear form on H^2 .

Definition 3.2.2. *For $F, G \in H^2$ let $\sigma(F, G) \equiv \langle (J \otimes \mathbf{1})F, G \rangle$.*

The map $\sigma : H^2 \times H^2 \mapsto \mathbf{R}$ is bilinear and alternating, since $J \otimes \mathbf{1}$ is unitary

and we have

$$\begin{aligned} \sigma(F, G) &= \langle (J \otimes \mathbf{1})(J \otimes \mathbf{1})F, (J \otimes \mathbf{1})G \rangle \\ &= \langle (-\mathbf{1} \otimes \mathbf{1})F, (J \otimes \mathbf{1})G \rangle = -\langle (J \otimes \mathbf{1})G, F \rangle = -\sigma(G, F). \end{aligned}$$

In addition, this form is nondegenerate, since if $\sigma(F, G) = 0$ for all $F \in H^2$, it follows that

$$0 = \sigma(-(J \otimes \mathbf{1})G, G) = -\langle (J \times \mathbf{1})^2 G, G \rangle = \langle G, G \rangle = \|G\|^2,$$

which entails $G = 0$. We remark that for simple tensors $f_1 \otimes f_2, g_1 \otimes g_2 \in H^2$, one has

$$\sigma(f_1 \otimes f_2, g_1 \otimes g_2) = \langle (Jf_1) \otimes f_2, g_1 \otimes g_2 \rangle = \langle Jf_1, g_1 \rangle \langle f_2, g_2 \rangle = \sigma(f_1, g_1) \langle f_2, g_2 \rangle, \tag{3.2.3}$$

where the final σ naturally is the symplectic form on H .

Definition 3.2.3. For any $f \in H$ and $F \in H^2$, let $b(f, F)$ denote the element of H such that

$$\langle b(f, F), g \rangle = \sigma(f \otimes g, F) \tag{3.2.4}$$

holds for all $g \in H$.

b is a well-defined map from $H \times H^2$ into H , as can be seen as follows. Consider the map

$$g \mapsto \sigma(f \otimes g, F), \quad g \in H, \tag{3.2.5}$$

which is a bounded linear functional on H , since

$$|\sigma(f \otimes g, F)| = |\langle (Jf) \otimes g, F \rangle| \leq \| (Jf) \otimes g \| \| F \| = \| Jf \| \| g \| \| F \| = \| f \| \| F \| \| g \|.$$

Therefore, according to the Riesz representation theorem there exists exactly one vector in H , which we shall denote by $b(f, F)$, so that equation (3.2.4) holds for all $g \in H$. In addition, Riesz' theorem entails that the norm of $b(f, F)$ is equal to that of the map in (3.2.5), hence we have $\|b(f, F)\| \leq \|f\| \|F\|$. Thus b is linear and continuous in each entry, and for $F = f_1 \otimes f_2$ one sees that

$$\langle b(f, f_1 \otimes f_2), g \rangle = \sigma(f \otimes g, f_1 \otimes f_2) = \sigma(f, f_1) \langle g, f_2 \rangle = \langle \sigma(f, f_1) f_2, g \rangle.$$

Since this obtains for all $g \in H$, one has

$$b(f, f_1 \otimes f_2) = \sigma(f, f_1) f_2. \quad (3.2.6)$$

Thus, we could have defined b via linear and then continuous extension of (3.2.6). We define next a related map B .

Definition 3.2.4. For any $F, G \in H^2$, let $B(F, G)$ denote that element of H^2 for which $b(h, B(F, G)) = b(b(h, F), G)$ holds for all $h \in H$.

Lemma 3.2.5. B is a well-defined bounded linear map from $H^2 \times H^2$ into H^2 .

Proof. 1. Let $\{e_s\}$ be an orthonormal basis in H and fix $F, G \in H^2$. Then $\{Je_s\}$ is also an orthonormal basis, and one has

$$F = \sum_{r,s} \alpha_{rs} (Je_r) \otimes (Je_s) = \sum_r (Je_r) \otimes f_r,$$

with $\alpha_{rs} \in \mathbf{R}$ and $f_r \equiv \sum_s \alpha_{rs} Je_s$. Define furthermore the vectors $B_s \in H^2$, $B_s \equiv (Je_s) \otimes b(b(e_s, F), G)$, which are mutually orthogonal and whose norms satisfy

$$\begin{aligned} \|B_s\| &= \|b(b(e_s, F), G)\| \leq \|b(e_s, F)\| \|G\| \\ &= \|b(e_s, \sum_r (Je_r) \otimes f_r)\| \|G\| = \sum_r \sigma(e_s, Je_r) f_r \|G\| \\ &= \|\sum_r \langle Je_s, Je_r \rangle f_r\| \|G\| = \|f_s\| \|G\|. \end{aligned}$$

(The continuity of b was used.) Therefore,

$$\sum_s \|B_s\|^2 \leq \|G\|^2 \sum_s \|f_s\|^2 = \|G\|^2 \|F\|^2 < \infty,$$

and $\sum_s B_s$ is a well-defined vector in H^2 . But for this vector, one has

$$\begin{aligned} b(h, \sum_s B_s) &= \sum_s b(h, B_s) = \sum_s \sigma(h, Je_s) b(b(e_s, F), G) \\ &= b(b(\sum_s \langle Jh, Je_s \rangle e_s, F), G) = b(b(h, F), G). \end{aligned}$$

2. Let B_1 and B_2 be two vectors in H^2 such that $b(h, B_1) = b(h, B_2) = b(b(h, F), G) \forall h \in H$ for given $F, G \in H^2$. Then $b(h, B_1 - B_2) = 0$ and $\langle b(h, B_1 - B_2), g \rangle = \sigma(h \otimes g, B_1 - B_2) = 0$ for all $g, h \in H$. Since, however, $\sigma(\cdot, \cdot)$ is linear and continuous in the first entry, one has therefore $\sigma(K, B_1 - B_2) = 0$ for all $K \in H^2$. But by the nondegeneracy of σ , it follows that $B_1 = B_2$.

3. From step (1) one obtains the bound $\|B(F, G)\| \leq \|F\| \|G\|$ and thus the continuity of B . □

Note that from the above arguments, if $\{e_s\}$ is an orthonormal basis in H and $f_1, f_2, g_1, g_2 \in H$, then

$$\begin{aligned} B(f_1 \otimes f_2, g_1 \otimes g_2) &= \sum_s (Je_s) \otimes b(e_s, f_1 \otimes f_2, g_1 \otimes g_2) \\ &= \sum_s \sigma(e_s, f_1) \sigma(f_2, g_1) (Je_s) \otimes g_2 \\ &= \sigma(f_2, g_1) \sum_s \langle Je_s, f_1 \rangle (Je_s) \otimes g_2 = \sigma(f_2, g_1) f_1 \otimes g_2, \end{aligned}$$

in analogy to (3.2.6).

We have introduced the maps b and B , because they appear in the computation of the commutators of transformed fields, as we see in the next theorem.

Proposition 3.2.6. *For every $f \in H, F, G \in P_+H^2$, and $\varphi \in \mathcal{F}_0$ one has*

$$[\Phi(f), \Psi(G)]\varphi = 2i\Phi(b(f, G))\varphi, \tag{3.2.7}$$

$$[\Psi(F), \Psi(G)]\varphi = 4i\Psi(P_+B(F, G))\varphi + 2i\sigma(F, G)\varphi. \tag{3.2.8}$$

It therefore follows that for any $\Lambda \in \mathcal{L}$, one has

$$\begin{aligned} [\Phi_\Lambda(f), \Phi_\Lambda(g)]\varphi &= i\sigma(f, g)\varphi + 2i\Phi(b(f, \Lambda g) - b(g, \Lambda f))\varphi \\ &\quad + 4i\Psi(P_+B(\Lambda f, \Lambda g))\varphi + 2i\sigma(\Lambda f, \Lambda g)\varphi. \end{aligned} \tag{3.2.9}$$

Proof. To begin, let G be of the form $G = P_+(g_1 \otimes g_2)$. Then one computes

$$\begin{aligned} [\Phi(f), \Psi(P_+(g_1 \otimes g_2))]\varphi &= [\Phi(f), \Psi(g_1 \otimes g_2)]\varphi = [\Phi(f), \cdot : \Phi(g_1) \Phi(g_2) :]\varphi \\ &= [\Phi(f), \Phi(g_1) \Phi(g_2)]\varphi \\ &= [\Phi(f), \Phi(g_1)]\Phi(g_2)\varphi + \Phi(g_1)[\Phi(f), \Phi(g_2)]\varphi \\ &= i\sigma(f, g_1)\Phi(g_2)\varphi + i\sigma(f, g_2)\Phi(g_1)\varphi \\ &= i\Phi(b(f, g_1 \otimes g_2) + b(f, g_2 \otimes g_1))\varphi \\ &= 2i\Phi(b(f, P_+(g_1 \otimes g_2)))\varphi. \end{aligned} \tag{3.2.10}$$

This equation holds as well for the dense linear span of such simple tensor products and hence for all $G \in P_+H^2$, as soon as it is seen that both sides of the equation are continuous in G . First, the map $G \mapsto 2i\Phi(b(f, G))\varphi$ is continuous, since both maps $G \mapsto b(f, G)$ and $h \mapsto \Phi(h)\varphi$ are continuous. The map $G \mapsto [\Phi(f), \Psi(G)]\varphi$ is also bounded:

$$\begin{aligned} \|\Phi(f), \Psi(G)\| \varphi &\leq \|\Phi(f) \Psi(G)\| \varphi + \|\Psi(G) \Phi(f)\| \varphi \\ &\leq \sqrt{2} \sqrt{n_\varphi + 3} \|f\| \|\Psi(G)\| \varphi + 2\sqrt{2} \sqrt{(n_\varphi + 2)(n_\varphi + 3)} \|G\| \|\Phi(f)\| \varphi \\ &\leq 8\sqrt{(n_\varphi + 1)(n_\varphi + 2)(n_\varphi + 3)} \|f\| \|G\| \|\varphi\| \end{aligned}$$

(see (3.1.1)).

With $\{e_s\}$ an orthonormal basis in H , $F = \sum_{r,s} \alpha_{rs} e_r \otimes e_s$, $G = \sum_{p,q} \gamma_{pq} e_p \otimes e_q$, $\alpha_{rs} = \alpha_{sr}$, $\gamma_{pq} = \gamma_{qp} \in \mathbf{R}$ and $\sigma_{rs} \equiv \sigma(e_r, e_s)$, one has

$$\begin{aligned} [\Psi(F), \Psi(G)] \varphi &= \sum_{p,q} \gamma_{pq} [\Psi(F), \Phi(e_p) \Phi(e_q)] \varphi \\ &= \sum_{p,q} \gamma_{pq} \{ \Phi(e_p) [\Psi(F), \Phi(e_q)] + [\Psi(F), \Phi(e_p)] \Phi(e_q) \} \varphi \\ &= -2i \sum_{p,q} \gamma_{pq} \{ \Phi(e_p) \Phi(b(e_q, F)) + \Phi(b(e_p, F)) \Phi(e_q) \} \varphi \\ &= -2i \sum_{p,q} \gamma_{pq} \{ : \Phi(e_p) \Phi(b(e_q, F)) : + \frac{1}{2} \langle e_p, b(e_q, F) \rangle \\ &\quad + \frac{i}{2} \sigma(e_p, b(e_q, F)) + : \Phi(b(e_p, F)) \Phi(e_q) : \\ &\quad + \frac{1}{2} \langle b(e_p, F), e_q \rangle + \frac{i}{2} \sigma(b(e_p, F), e_q) \} \varphi \\ &= -2i \sum_{p,q} \gamma_{pq} \{ \Psi(\sum_{r,s} \alpha_{rs} (\sigma_{qr} e_p \otimes e_s + \sigma_{pr} e_s \otimes e_q)) \\ &\quad + \frac{1}{2} \sigma(e_q \otimes e_p + e_p \otimes e_q, F) + \frac{i}{2} (\sigma(e_p, b(e_q, F)) \\ &\quad - \sigma(e_q, b(e_p, F))) \} \varphi. \end{aligned}$$

The last term vanishes, because it is antisymmetric and γ_{pq} is symmetric in p and q , so with

$$\begin{aligned} P_+ B(F, G) &= \sum_{p,q} \gamma_{pq} \sum_{r,s} \alpha_{rs} \sigma_{sp} P_+(e_r \otimes e_q) \\ &= \frac{1}{2} \left(\sum_{p,q} \gamma_{pq} \sum_{r,s} \alpha_{rs} \sigma_{rp} e_s \otimes e_q + \sum_{p,q} \gamma_{pq} \sum_{r,s} \alpha_{rs} \sigma_{rq} e_p \otimes e_s \right) \end{aligned}$$

and

$$\sigma(F, G) = \sum_{p,q} \gamma_{pq} \sigma(F, e_p \otimes e_q) = \sum_{p,q} \gamma_{pq} \sigma(F, e_q \otimes e_p),$$

one obtains the right side of equation (3.2.8). □

Remark. Note that the first two equations in the Proposition are valid only for symmetric $F, G \in H^2$.

A first result characterizing canonical quadratic transformations can now

be given.

Proposition 3.2.7. *Let $\Lambda \in \mathcal{L}$. The transformation $\Phi(f) \mapsto \Phi_\Lambda(f)$, $f \in D(\Lambda)$, is canonical if and only if, for all $f, g \in D(\Lambda)$, one has*

$$b(f, \Lambda g) = b(g, \Lambda f) \tag{3.2.11}$$

and

$$P_+B(\Lambda f, \Lambda g) = 0. \tag{3.2.12}$$

Proof. 1. (\Leftarrow) If equations (3.2.11) and (3.2.12) are satisfied, then by (3.2.9) the transformation is canonical, whenever $\sigma(\Lambda f, \Lambda g) = 0$ holds. This, then, will be proven. Choose an orthonormal basis $\{e_s\}$ in H and write

$$\Lambda f = \sum_{r,s} \alpha_{rs} e_r \otimes e_s = \sum_r e_r \otimes f_r = \sum_r f_r \otimes e_r$$

($\alpha_{rs} = \alpha_{sr} \in \mathbf{R}$, $f_r \equiv \sum_s \alpha_{rs} e_s$) and correspondingly $\Lambda g = \sum_s g_s \otimes e_s$. Then one sees that

$$0 = P_+B(\Lambda f, \Lambda g) = P_+ \sum_{r,s} B(e_r \otimes f_r, g_s \otimes e_s) = \sum_{r,s} \sigma(f_r, g_s) P_+(e_r \otimes e_s),$$

in particular, $\sigma(f_r, g_r) = 0$ for every r , and thus

$$\sigma(\Lambda f, \Lambda g) = \sum_{r,s} \sigma(f_r \otimes e_r, g_s \otimes e_s) = \sum_{r,s} \sigma(f_r, g_s) \langle e_r, e_s \rangle = \sum_r \sigma(f_r, g_r) = 0.$$

2. (\Rightarrow) On the other hand, if $\Phi(f) \mapsto \Phi_\Lambda(f)$ is canonical, then by equation (3.2.9) one has

$$2\Psi(K) \varphi + \Phi(k) \varphi + \sigma(\Lambda f, \Lambda g) \varphi = 0 \quad \forall \varphi \in \mathcal{F}_0, \tag{3.2.13}$$

where the quantity $P_+B(\Lambda f, \Lambda g)$ has been designated by K and $b(f, \Lambda g) - b(g, \Lambda f)$ by k . One must therefore show that $K=0$ and $k=0$. With this goal in mind, by (3.2.13) one has

$$\begin{aligned} 0 &= [\Phi(h_1), [\Phi(h_2), 2\Psi(K) + \Phi(k) + \sigma(\Lambda f, \Lambda g) \mathbf{1}]] \varphi \\ &= 2[\Phi(h_1), [\Phi(h_2), \Psi(K)]] \varphi = 4i[\Phi(h_1), \Phi(b(h_2, K))] \varphi \end{aligned}$$

for all $h_1, h_2 \in H$, $\varphi \in \mathcal{F}_0$, where in the last step equation (3.2.7) was used. This leads to

$$0 = -4\sigma(h_1, b(h_2, K))\varphi = -4\sigma(h_2 \otimes (Jh_1), K)\varphi = -4\langle (Jh_2) \otimes (Jh_1), K \rangle \varphi,$$

so that $K=0$. Equation (3.2.13) then entails that $\Phi(k)\varphi + \sigma(\Lambda f, \Lambda g)\varphi = 0$, from which it follows that $k=0$. \square

Remark 3.2.8. We next note that if $F = \sum_r f_r \otimes e_r = \sum_r e_r \otimes f_r \in P_+H^2$ and $G = \sum_s g_s \otimes e_s = \sum_s e_s \otimes g_s \in P_+H^2$, then

$$\begin{aligned} 2P_+B(F, G) &= \sum \sigma(f_r, g_s) (e_r \otimes e_s + e_s \otimes e_r) \\ &= \sum_{r,s} (\sigma(f_r, g_s) e_r \otimes e_s - \sigma(g_s, f_r) e_s \otimes e_r) = B(F, G) - B(G, F); \end{aligned}$$

thus one has $P_+B(\Lambda f, \Lambda g) = 0$ if and only if $B(\Lambda f, \Lambda g) = B(\Lambda g, \Lambda f)$.

A variant of Proposition 3.2.7 will now be presented, which demonstrates that if the quadratic transformation is canonical, then not only the symmetric part of $B(\Lambda f, \Lambda g)$ must be zero, but, in fact, also the antisymmetric part vanishes.

Proposition 3.2.9. *Let $\Lambda \in \mathcal{L}$. The transformation $\Phi(f) \mapsto \Phi_\Lambda(f)$, $f \in D(\Lambda)$, is canonical if and only if, for all $f, g \in D(\Lambda)$, one has*

$$b(f, \Lambda g) = b(g, \Lambda f) \quad (3.2.14)$$

and

$$B(\Lambda f, \Lambda g) = 0. \quad (3.2.15)$$

Proof. Given the preceding proposition, one may assume that (3.2.14) and (3.2.12) hold. Using the equality

$$\langle B(F, G), (Jh) \otimes (Jk) \rangle = \sigma(b(h, F), b(k, G))$$

(for general $F, G \in P_+H^2$, $h, k \in H$) and Remark 3.2.8, one finds for $f, g, h, k \in D(\Lambda)$:

$$\begin{aligned} \langle B(\Lambda f, \Lambda g), (Jh) \otimes (Jk) \rangle &= \sigma(b(h, \Lambda f), b(k, \Lambda g)) = -\sigma(b(k, \Lambda g), b(h, \Lambda f)) \\ &= -\langle B(\Lambda g, \Lambda f), (Jk) \otimes (Jh) \rangle = -\langle B(\Lambda f, \Lambda g), (Jk) \otimes (Jh) \rangle \\ &= -\sigma(b(k, \Lambda f), b(h, \Lambda g)) = -\sigma(b(f, \Lambda k), b(g, \Lambda h)) \\ &= -\langle B(\Lambda k, \Lambda h), (Jf) \otimes (Jg) \rangle = -\langle B(\Lambda h, \Lambda k), (Jf) \otimes (Jg) \rangle \\ &= -\sigma(b(f, \Lambda h), b(g, \Lambda k)) = -\sigma(b(h, \Lambda f), b(k, \Lambda g)) \end{aligned}$$

$$= -\langle B(Af, Ag), (Jh) \otimes (Jk) \rangle.$$

Since $D(\Lambda)$ is dense in H , the set of vectors of the form $(Jh) \otimes (Jk)$ is total in H^2 ; hence it follows that $B(Af, Ag) = 0$. \square

We next reformulate condition (3.2.15) in a more symplectic geometrical context.

Lemma 3.2.10. *Let $\Lambda \in \mathcal{L}$. The condition (3.2.15) holds exactly when there exists a closed isotropic subspace $M \subset H$ (i.e. $\sigma(m, n) = 0, \forall m, n \in M$) such that $R(\Lambda) \subset M \otimes M$.*

Proof. Since it has been shown that the range of Λ consists of symmetric vectors, the relation $R(\Lambda) \subset M \otimes M$ may be replaced by $R(\Lambda) \subset P_+(M \otimes M)$.

1. (\Leftarrow) Let $f, g \in D(\Lambda)$ and $\{n_i\}$ be an orthonormal basis in M , which by hypothesis must satisfy $\sigma(n_i, n_j) = 0$. Since by assumption the vectors Af and Ag are elements of $M \otimes M$, one may write $Af = \sum_i f_i \otimes n_i$ and $Ag = \sum_j g_j \otimes n_j$, with $f_i, g_j \in M$. Employing the continuity of B , one obtains

$$B(Af, Ag) = \sum_i \sum_j B(f_i \otimes n_i, g_j \otimes n_j) = \sum_i \sum_j \sigma(n_i, n_j) f_i \otimes g_j = 0.$$

2. (\Rightarrow) In this direction the isotropic subspace will be explicitly constructed. If $\{e_s\}$ is an orthonormal basis in H and one writes $F \in R(\Lambda)$ in the form $F = \sum_s f_s \otimes e_s$, then one obtains the vectors $f_1, f_2, f_3, \dots \in H$. This leads to a map from an element $F \in R(\Lambda)$ to a set $\{f_s\}$ of vectors. The range of this map, as F runs through $R(\Lambda)$, will be called M_0 . The closure of the linear span of M_0 will be called M . This will be shown to be the desired subspace.

Let f and g be chosen from $D(\Lambda)$; then one has $Af = \sum_r f_r \otimes e_r = \sum_r e_r \otimes f_r$ (since Af is symmetric), $Ag = \sum_s g_s \otimes e_s$, and $f_r, g_s \in M_0$. If

$$0 = B(Af, Ag) = \sum_r \sum_s \sigma(f_r, g_s) e_r \otimes e_s,$$

then $\sigma(f_r, g_s) = 0$ and $\sigma(f, g) = 0$ hold for all $f, g \in M_0$. Since the set M_0 is total in M , it therefore follows that $\sigma(m, n) = 0$ for all $m, n \in M$.

On the other hand, if $F \in R(\Lambda)$, then also $F \in M \otimes H$, and therefore one may write $F = \sum_i m_i \otimes \widehat{f}_i$, with an orthonormal basis $\{m_i\}$ in M and $\widehat{f}_i \in H$. For an arbitrary element n_j of this basis and an $h \in H$, one then has

$$\langle F, n_j \otimes h \rangle = \sum_i \langle m_i, n_j \rangle \langle \widehat{f}_i, h \rangle = \langle \widehat{f}_j, h \rangle.$$

But F is symmetric, so one also has

$$\langle F, n_j \otimes h \rangle = \langle F, h \otimes n_j \rangle = \sum_i \langle n_i, h \rangle \langle \widehat{f}_i, n_j \rangle.$$

Choosing h from the orthogonal space M^\perp to M , one concludes $\langle \widehat{f}_j, h \rangle = 0, \forall h \in M^\perp$, in other words one has $\widehat{f}_j \in M^{\perp\perp} = \overline{M} = M$. But since then all \widehat{f}_i in the sum $F = \sum_i n_i \otimes \widehat{f}_i$ belong to M , one clearly has $F \in M \otimes M$, which completes the proof. \square

We summarize the main results of this section up to this point in the following theorem.

Theorem 3.2.11. *The transformation $\Phi(f) \mapsto \Phi_\Lambda(f), f \in D(\Lambda)$ with $\Lambda \in \mathcal{L}$ is canonical (i.e., $\Lambda \in \mathcal{L}_{CCR}$) if and only if $b(f, \Lambda g) = b(g, \Lambda f)$ for all $f, g \in D(\Lambda)$ and $R(\Lambda) \subset M \otimes M$ for some closed, isotropic subspace M of H .*

§3.3. The Standard Form of Canonical Quadratic Transformations

In this section we shall prove that if $\Lambda \in \mathcal{L}_{CCR}$, then there exists a suitable basis in H in terms of which the map Λ has a particularly convenient form. In fact, we shall be able to explicitly characterize the elements of \mathcal{L}_{CCR} in this manner. This technical point is crucial in the arguments of the following sections.

Proposition 3.3.1. *For any $\Lambda \in \mathcal{L}_{CCR}$ there exists a symplectic orthonormal basis $\{e_k, e_{k'}\} = \{e_K : K=1, 1', 2, 2', \dots\}$ in H and real numbers $\lambda_{ijk}, i, j, k=1, 2, 3, \dots$, which are completely symmetric in the three indices, and satisfy*

$$\sum_{i,j} \lambda_{ijk}^2 < \infty$$

for every k , such that the operator Λ may be expressed as:

$$\Lambda f = \sum_{i,j} \sum_k \lambda_{ijk} \sigma(e_k, f) e_i \otimes e_j, \tag{3.3.1}$$

for all $f \in D(\Lambda)$.

Proof. 1. From Theorem 3.2.11 it is known that there exists a closed isotropic subspace M of H with $R(\Lambda) \subseteq M \otimes M$. Let $\{m_i\}$ be an orthonormal basis in M and consider the corresponding vectors \widetilde{m}_i in \mathcal{H} . Since one has $\langle \widetilde{m}_i, \widetilde{m}_j \rangle_{\mathcal{H}} =$

$\langle m_i, m_j \rangle + i\sigma(m_i, m_j) = \langle m_i, m_j \rangle = \delta_{ij}$, these vectors form an orthonormal system, which can be completed in the case that it is not already complete in \mathcal{H} . From the discussion in Section 2, the resulting orthonormal basis $\{\tilde{n}_k\}$ in \mathcal{H} can be transformed into a symplectic orthonormal basis $\{n_k, Jn_k\} = \{n_k, n_{k'}\} = \{n_K\}$ in H . In the subset $\{n_k\}$ of this system are contained the original vectors m_i . Calling $N \subset H$ the closed subspace spanned by $\{n_k\}$, one has by construction: $M \subset N$, $R(\Lambda) \subset N \otimes N$ and $\sigma(m, n) = 0$, for any $m, n \in N$. One may therefore replace M by N , which has the hypothesized properties of M and which, moreover, has the advantage of being a maximal isotropic subspace of H . Note that one has $JN = N^\perp$ and $N \oplus (JN) = H$.

Next let $f \in D(\Lambda)$, $g \in H$ and $n \in N \cap D(\Lambda)$. Since $\Lambda f \in N \otimes N$, it may be written as $\Lambda f = \sum_i f_i \otimes g_i$, with $f_i, g_i \in N$. Thus one has $\sigma(n \otimes g, \Lambda f) = \sum_i \sigma(n, f_i) \langle g, g_i \rangle = 0$. Using (3.2.14) one finds then that

$$\sigma(f \otimes g, \Lambda n) = \langle b(f, \Lambda n), g \rangle = \langle b(n, \Lambda f), g \rangle = \sigma(n \otimes g, \Lambda f) = 0,$$

which entails $\Lambda n = 0$. One may therefore extend Λ to an operator $\widehat{\Lambda}$ by defining:

$$\begin{aligned} D(\widehat{\Lambda}) &\equiv D(\Lambda) + N = \{f \in H \mid f = l + n, l \in D(\Lambda), n \in N\}, \\ \widehat{\Lambda}f &= \widehat{\Lambda}(l + n) \equiv \Lambda l, \end{aligned} \tag{3.3.2}$$

where $f = l + n \in D(\widehat{\Lambda})$, $l \in D(\Lambda)$, and $n \in N$. Since the decomposition $f = l + n$ need not be unique, it must be checked that the definition (3.3.2) makes sense. Therefore, consider two decompositions $f = l_1 + n_1, f = l_2 + n_2$ of $f \in D(\widehat{\Lambda})$ ($l_1, l_2 \in D(\Lambda)$, $n_1, n_2 \in N$). Since $n_1 - n_2 = l_2 - l_1$, one has $n_1 - n_2 \in N \cap D(\Lambda)$; hence $\Lambda(n_1 - n_2) = 0$, as has just been shown above. Thus one has $\Lambda l_2 = \Lambda(n_1 - n_2 + l_1) = \Lambda l_1$ and $\widehat{\Lambda}(l_2 + n_2) = \Lambda l_2 = \Lambda l_1 = \widehat{\Lambda}(l_1 + n_1)$. It is straightforward to verify that $D(\widehat{\Lambda})$ is a linear subspace of H , and that $\widehat{\Lambda}$ is linear. Moreover, one also has $R(\widehat{\Lambda}) = R(\Lambda)$ and $\widehat{\Lambda} \in \mathcal{L}$.

If $\Lambda \in \mathcal{L}_{CCR}$, then also for its extension one has $\widehat{\Lambda} \in \mathcal{L}_{CCR}$, as can be seen: Since the condition $R(\widehat{\Lambda}) = R(\Lambda) \subset N \otimes N$ has already been verified, one needs only check (3.2.14). But for arbitrary $f_1 = l_1 + n_1, f_2 = l_2 + n_2 \in D(\widehat{\Lambda})$, one computes

$$b(f_1, \widehat{\Lambda}f_2) = b(l_1 + n_1, \Lambda l_2) = b(l_1, \Lambda l_2) = b(l_2, \Lambda l_1) = b(f_2, \widehat{\Lambda}f_1).$$

2. Since $\widehat{\Lambda}f = \Lambda f$, for any $f \in D(\Lambda)$, it will suffice to show that the theorem holds for the extended operator. This will be technically easier. Let P be the orthogonal projection onto the subspace N , so that $P^\perp = I - P$ is the orthogonal projection onto $N^\perp = JN$. The set $P^\perp D(\widehat{\Lambda})$ is dense in N^\perp and is a separable pre-

Hilbert space; therefore it contains an orthonormal basis $\{e_k\}$, which is also an orthonormal basis for $P^\perp D(\widehat{\Lambda}) = N^\perp$. Hence $\{-Je_k\}$ is an orthonormal basis in $-JN^\perp = -JJN = N$, and if one sets $-Je_k = e_k$, then $Je_k = e_k$ and $\{e_k, e_k\} = \{e_K\}$ is a symplectic orthonormal basis in H contained in $D(\widehat{\Lambda})$: the containment $\{e_k\} \subset N \subseteq D(\widehat{\Lambda})$ is clear. Now, if f is an arbitrary element in $D(\widehat{\Lambda})$, then $Pf \in N \subset D(\widehat{\Lambda})$ and $P^\perp f = f - Pf \in D(\widehat{\Lambda})$. In other words, $P^\perp D(\widehat{\Lambda}) \subset D(\widehat{\Lambda})$. However, as $\{e_k\}$ is by construction a subset of $P^\perp D(\widehat{\Lambda})$, one also has $\{e_k\} \subset D(\widehat{\Lambda})$.

3. It shall be seen that the constructed basis $\{e_k\}$ and the numbers defined by $\lambda_{ijk} \equiv \langle e_i \otimes e_j, \widehat{\Lambda}Je_k \rangle$ satisfy the conclusions of the proposition. First, since $\widehat{\Lambda}Je_k \in P_+H^2$, it follows that $\lambda_{ijk} = \lambda_{jik}$. Moreover, one has

$$\lambda_{ijk} = -\sigma((Je_i) \otimes e_j, \widehat{\Lambda}Je_k) = -\langle b(Je_i, \widehat{\Lambda}Je_k), e_j \rangle = -\langle b(Je_k, \widehat{\Lambda}Je_i), e_j \rangle = \lambda_{kji}.$$

This implies that λ_{ijk} is invariant under exchange of the first with the second, as well as the first with the third index, so it is completely symmetric in the three indices.

Next it is noted that

$$\begin{aligned} \infty > \|\widehat{\Lambda}Je_k\|^2 &= \left\| \sum_{I,J} \langle e_I \otimes e_J, \widehat{\Lambda}Je_k \rangle e_I \otimes e_J \right\|^2 \\ &= \sum_{I,J} \langle e_I \otimes e_J, \widehat{\Lambda}Je_k \rangle^2 \geq \sum_{i,j} \langle e_i \otimes e_j, \widehat{\Lambda}Je_k \rangle^2 = \sum_{i,j} \lambda_{ijk}^2, \end{aligned}$$

for every k , since $Je_k \in D(\widehat{\Lambda})$. Further, for $f \in D(\widehat{\Lambda})$ one has

$$\widehat{\Lambda}f = \sum_{I,J} \langle e_I \otimes e_J, \widehat{\Lambda}f \rangle e_I \otimes e_J = \sum_{I,J} \langle e_I \otimes e_J, \widehat{\Lambda} \sum_K \langle Je_K, f \rangle Je_K \rangle e_I \otimes e_J.$$

Employing once again (3.2.14), one sees that for any e_I, e_J the linear map

$$f \mapsto \langle e_I \otimes e_J, \widehat{\Lambda}f \rangle = -\langle b(Je_I, \widehat{\Lambda}f), e_J \rangle = -\langle b(f, \widehat{\Lambda}Je_I), e_J \rangle$$

is bounded, since $\|b(f, F)\| \leq \|f\| \|F\|$. Thus, one may pull the infinite sum through the scalar product above to obtain

$$\widehat{\Lambda}f = \sum_{I,J} \sum_K \langle Je_K, f \rangle \langle e_I \otimes e_J, \widehat{\Lambda}Je_K \rangle e_I \otimes e_J.$$

Since on the one hand, $\widehat{\Lambda}Je_K \in N \otimes N$, and on the other hand, $\widehat{\Lambda}JN^\perp = \widehat{\Lambda}N = \{0\}$, it follows that the only scalar products satisfying $\langle e_I \otimes e_J, \widehat{\Lambda}Je_K \rangle \neq 0$ are those for which $e_I, e_J, e_K \in N$, in other words, the vectors must come from $\{e_k\}$. Thus, one

has finally

$$\widehat{\Lambda}f = \sum_{i,j} \sum_k \langle J e_k, f \rangle \langle e_i \otimes e_j, \widehat{\Lambda} J e_k \rangle e_i \otimes e_j = \sum_{i,j} \sum_k \sigma(e_k, f) \lambda_{ijk} e_i \otimes e_j.$$

□

The next step in our program to explicitly characterize the elements of \mathcal{L}_{CCR} is the following proposition, which is in some sense the converse of the preceding result.

Proposition 3.3.2. *Let $\{e_k\}$ be an orthonormal system in H with $\sigma(e_k, e_l) = 0$ for all k, l , and let λ_{ijk} be real numbers which are completely symmetric in the three indices satisfying $\sum_{i,j} \lambda_{ijk}^2 < \infty$ for every k . Then the operator $\Lambda : H \rightarrow H^2$ defined by*

$$D(\Lambda) \equiv \{f \in H \mid \sum_{i,j} \left(\sum_k \lambda_{ijk} \sigma(e_k, f) \right)^2 < \infty\},$$

$$\Lambda f \equiv \sum_{i,j} \sum_k \lambda_{ijk} \sigma(e_k, f) e_i \otimes e_j,$$

for $f \in D(\Lambda)$ is contained in \mathcal{L}_{CCR} , i.e. the quadratic transformation $\Phi(f) \mapsto \Phi_\Lambda(f)$, $f \in D(\Lambda)$, is canonical.

Proof. 1. It shall be assumed that $\{e_k\}$ is an infinite orthonormal system, and that the indices i, j, k run from 1 to infinity, since the finite case is relatively trivial. The first step is to show that for every i, j and $f \in H$ the series $\sum_k \lambda_{ijk} \sigma(e_k, f)$ is absolutely convergent. Since $\{e_k\}$ is also an orthonormal system in H , Bessel's inequality yields $\|f\|^2 \geq \sum_k \langle J e_k, f \rangle^2 = \sum_k \sigma(e_k, f)^2$, for $f \in H$. Thus the sequence $\varphi \equiv \{|\sigma(e_1, f)|, |\sigma(e_2, f)|, |\sigma(e_3, f)|, \dots\}$ belongs to $\ell_2(\mathbf{R})$ and $\|\varphi\|_{\ell_2}^2 \leq \|f\|^2$. In addition, the sequence $\lambda_{ij} \equiv \{|\lambda_{ij1}|, |\lambda_{ij2}|, |\lambda_{ij3}|, \dots\} \in \ell_2(\mathbf{R})$, since $\sum_{i,j} \lambda_{kij}^2 < \infty$ entails $\sum_j \lambda_{kij}^2 < \infty$. In the Hilbert space $\ell_2(\mathbf{R})$ the Cauchy-Schwarz inequality implies:

$$\langle \lambda_{ij}, \varphi \rangle_{\ell_2} = \sum_k |\lambda_{ijk}| |\sigma(e_k, f)| \leq \|\lambda_{ij}\|_{\ell_2} \|\varphi\|_{\ell_2} < \infty, \tag{3.3.3}$$

so that $\sum_k \lambda_{ijk} \sigma(e_k, f)$ is absolutely convergent.

2. The next step is to show that $\Lambda \in \mathcal{L}$. The linearity of Λ is clear, once it is seen that $f, g \in D(\Lambda)$ implies that $f+g \in D(\Lambda)$. But, in fact,

$$\begin{aligned} \sum_{i,j} \left(\sum_k \lambda_{ijk} \sigma(e_k, f+g) \right)^2 &= \sum_{i,j} \left(\sum_k \lambda_{ijk} \sigma(e_k, f) + \sum_k \lambda_{ijk} \sigma(e_k, g) \right)^2 \\ &\leq 2 \sum_{i,j} \left\{ \left(\sum_k \lambda_{ijk} \sigma(e_k, f) \right)^2 + \left(\sum_k \lambda_{ijk} \sigma(e_k, g) \right)^2 \right\} \\ &= 2 \sum_{i,j} \left(\sum_k \lambda_{ijk} \sigma(e_k, f) \right)^2 + 2 \sum_{i,j} \left(\sum_k \lambda_{ijk} \sigma(e_k, g) \right)^2 < \infty, \end{aligned}$$

where the trivial fact $(\alpha + \beta)^2 \leq (\alpha + \beta)^2 + (\alpha - \beta)^2 = 2(\alpha^2 + \beta^2)$ for real numbers was employed.

From Step (1) of the proof of Proposition 3.3.1, it is already known that there exists a symplectic orthonormal basis $\{n_i, n_{i'}\} = \{n_L\}$ of H with $\{n_i\} \supseteq \{e_k\}$. For any element n_L of this basis the quantity $\sigma(e_k, n_L)$ is nonzero for at most one choice of k , and since $\sum_{i,j} \lambda_{ijk}^2 < \infty$, it must be the case that $n_L \in D(\Lambda)$. Hence, the linear operator Λ is defined on the linear hull of $\{n_K\}$ and consequently is densely defined. The relation $R(\Lambda) \subseteq P_+H^2$ is trivially fulfilled, because of the symmetry of λ_{ijk} . It has therefore been proven that $\Lambda \in \mathcal{L}$.

3. Since the sufficient condition of Lemma 3.2.10 is clearly satisfied, it suffices to demonstrate that equation (3.2.14) holds in order to conclude that $\Lambda \in \mathcal{L}_{CCR}$. For $f, g \in D(\Lambda)$ one verifies that

$$\begin{aligned} b(f, \Lambda g) &= b(f, \sum_{i,j,k} \lambda_{ijk} \sigma(e_k, g) e_i \otimes e_j) = \sum_{i,j,k} \lambda_{ijk} \sigma(e_k, g) b(f, e_i \otimes e_j) \\ &= \sum_j \sum_i \sum_k \lambda_{ijk} \sigma(e_k, g) \sigma(f, e_i) e_j = \sum_j \sum_k \sum_i \lambda_{ijk} \sigma(e_i, g) \sigma(f, e_k) e_j \end{aligned}$$

and

$$b(g, \Lambda f) = \sum_j \sum_i \sum_k \lambda_{ijk} \sigma(e_k, f) \sigma(g, e_i) e_j.$$

These two expressions are equal, if the sums \sum_i and \sum_k can be exchanged, and that will be the case if the double sum

$$\sum_i \sum_k \lambda_{ijk} \sigma(e_k, f) \sigma(g, e_i)$$

is absolutely convergent. To demonstrate that this is the case here, consider the sequences φ and λ_{ij} from the first part of this proof and also the sequences

$$\psi \equiv (|\sigma(e_1, g)|, |\sigma(e_2, g)|, |\sigma(e_3, g)|, \dots), \quad \lambda_j \equiv (\|\lambda_{1j}\|_{\ell_2}, \|\lambda_{2j}\|_{\ell_2}, \|\lambda_{3j}\|_{\ell_2}, \dots);$$

ψ is an element of ℓ_2 for the same reason that φ is, and also λ_j belongs to ℓ_2 , since $\sum_i \|\lambda_{ij}\|_{\ell_2}^2 = \sum_i \sum_k \lambda_{ijk}^2 < \infty$. Therefore, one has

$$\infty > \|\varphi\|_{\ell_2} \langle \lambda_j, \psi \rangle_{\ell_2} = \|\varphi\|_{\ell_2} \sum_i \|\lambda_{ij}\|_{\ell_2} |\sigma(e_i, g)| \geq \sum_i \sum_k |\lambda_{ijk} \sigma(e_k, f) \sigma(e_i, g)|,$$

where equation (3.3.3) was used in the last step. This demonstrates the desired absolute convergence, and $b(f, \Lambda g) = b(g, \Lambda f)$ follows. □

We next shall show that each $\Lambda \in \mathcal{L}_{CCR}$ has a maximal extension $\Lambda_{\max} \in \mathcal{L}_{CCR}$. To this end we make the following definition.

Definition 3.3.3 For a given $\Lambda \in \mathcal{L}_{CCR}$ determine the quantities $\{e_K\}$ and λ_{ijk} as in Proposition 3.3.1, and then define with these an operator via Proposition 3.3.2. The resulting operator will be denoted by Λ_{\max} .

The fact emerged in the proof of Proposition 3.3.1 that the basis $\{e_K\}$ and the numbers λ_{ijk} are not necessarily determined uniquely by Λ . It shall be made clear that the operator Λ_{\max} is independent of this choice of $\{e_K\}$ and λ_{ijk} . An immediate consequence of Proposition 3.3.2 is that Λ_{\max} is an element of \mathcal{L}_{CCR} ; moreover, for any $f \in D(\Lambda)$ we have

$$\|A_f\|^2 = \sum_{i,j} (\sum_k \lambda_{ijk} \sigma(e_k, f))^2 < \infty,$$

so that $f \in D(\Lambda_{\max})$, and Λ_{\max} is indeed an extension of Λ . We next see that Λ_{\max} is a maximal extension of Λ in the class \mathcal{L}_{CCR} .

Proposition 3.3.4. If $\Lambda' \in \mathcal{L}_{CCR}$ is an extension of Λ , then $\Lambda' \subset \Lambda_{\max}$. Furthermore, Λ_{\max} is a closed operator.

Proof. 1. First consider the case $\Lambda' \supset \Lambda_{\max}$. If $f \in D(\Lambda')$, then set $\Lambda'f = \sum_I e_I \otimes f_I$, where $\{e_I\}$ is the symplectic orthonormal basis from which Λ and Λ_{\max} are constructed, and the vectors $f_I \in H$ are obtained from

$$b(Je_I, \Lambda'f) = \sum_J \sigma(Je_I, e_J) f_J = - \sum_J \delta_{IJ} f_J = -f_I.$$

Since $Je_I \in D(\Lambda_{\max})$, by assumption one also has $Je_I \in D(\Lambda')$, so that

$$\begin{aligned} f_I &= -b(Je_I, \Lambda'f) = -b(f, \Lambda'Je_I) = -b(f, \Lambda_{\max} Je_I) \\ &= -b\left(f, \sum_{m,n} \sum_k \lambda_{mnk} \sigma(e_k, Je_I) e_m \otimes e_n\right) = - \sum_{m,n} \sum_k \lambda_{mnk} \delta_{kI} \sigma(f, e_m) e_n. \end{aligned}$$

Therefore, $f_{I'} = 0$ and $f_I = \sum_n \sum_m \lambda_{mni} \sigma(e_m, f) e_n$, which entails

$$\|A'f\|^2 = \sum_I \|f_I\|^2 = \sum_i \sum_n \sum_m (\sum \lambda_{mni} \sigma(e_m, f))^2.$$

Since this quantity must be finite, it follows that $f \in D(\Lambda_{\max})$.

2. In general, one cannot assume that $D(\Lambda)$ or $D(\Lambda')$ contain a symplectic orthonormal basis, but, as has been argued before, there exists an orthonormal basis $\{k_s\}$ for the scalar product in H such that $Jk_s \in D(\Lambda) \subset D(\Lambda')$. Using *this*

basis we write as above

$$\begin{aligned} \Lambda_{\max} f &= \sum_s k_s \otimes l_{\max, s}(f) = \sum_s l_{\max, s}(f) \otimes k_s, \\ l_{\max, s}(f) &= -b(Jk_s, \Lambda_{\max} f), \quad f \in D(\Lambda_{\max}) \end{aligned}$$

and

$$\Lambda' g = \sum_s k_s \otimes l'_s(g) = \sum_s l'_s(g) \otimes k_s, \quad l'_s(g) = -b(k_s, \Lambda' g), \quad g \in D(\Lambda').$$

Since $f \in D(\Lambda)$ entails $\Lambda_{\max} f = \Lambda f = \Lambda' f$, $l_{\max, s}(\cdot)$ and $l'_s(\cdot)$ coincide on $D(\Lambda)$. Because $l_{\max, s}(f) = -b(f, \Lambda_{\max} Jk_s)$ and $l'_s(g) = -b(g, \Lambda' Jk_s)$, the linear maps $f \mapsto l_{\max, s}(f)$ and $g \mapsto l'_s(g)$ are continuous. Thus, since the continuous extension of a linear, continuous, densely defined operator is unique, $l_{\max, s}(\cdot)$ and $l'_s(\cdot)$ are restrictions of the same continuous map $l_s(\cdot)$ defined on all of H .

If $f \in D(\Lambda_{\max})$ and $g \in D(\Lambda')$, then the sum $\sum_s k_s \otimes l_s(f+g)$ converges strongly, since by construction $\sum_s k_s \otimes l_s(f)$ and $\sum_s k_s \otimes l_s(g)$ converge. One may therefore define an operator $\tilde{\Lambda}$ by

$$\begin{aligned} \tilde{\Lambda} h &\equiv \sum_s k_s \otimes l_s(h) = \sum_s l_s(h) \otimes k_s, \\ h \in D(\tilde{\Lambda}) &\equiv D(\Lambda_{\max}) + D(\Lambda') = \{f+g \mid f \in D(\Lambda_{\max}), g \in D(\Lambda')\}. \end{aligned}$$

One clearly has $\tilde{\Lambda} \supset \Lambda_{\max}$ and $\tilde{\Lambda} \supset \Lambda'$.

It will next be shown that $\tilde{\Lambda} \in \mathcal{L}_{CCR}$. First it is noted that since $\tilde{\Lambda}(h) = \tilde{\Lambda}(f+g) = \Lambda_{\max} f + \Lambda' g \in P_+ H^2$, one has $\tilde{\Lambda} \in \mathcal{L}$. With $f, g \in D(\tilde{\Lambda})$ one has

$$B(\tilde{\Lambda} f, \tilde{\Lambda} g) = \sum_{r,s} \sigma(l_r(f), l_s(g)) k_r \otimes k_s$$

and

$$b(f, \tilde{\Lambda} g) = \sum_s \sigma(f, l_s(g)) k_s.$$

Choosing two sequences (f_m) and (g_n) from $D(\Lambda)$, which converge to f , resp. $g \in D(\tilde{\Lambda})$, it follows from $B(\Lambda f_m, \Lambda g_n) = 0$ that $\sigma(l_r(f_m), l_s(g_n)) = 0$ and from $b(f_m, \Lambda g_n) = b(g_n, \Lambda f_m)$ that $\sigma(f_m, l_s(g_n)) = \sigma(g_n, l_s(f_m))$, for every r, s, m and n . Thus,

$$\begin{aligned} \sigma(l_r(f), l_s(g)) &= \sigma(l_r(\lim_m f_m), l_s(g)) = \sigma(\lim_m l_r(f_m), l_s(g)) \\ &= \lim_m \sigma(l_r(f_m), l_s(g)) = \lim_m \lim_n \sigma(l_r(f_m), l_s(g_n)) = 0 \end{aligned}$$

entails $B(\tilde{\Lambda}f, \tilde{\Lambda}g) = 0$, and

$$\sigma(f, l_s(g)) = \lim_m \lim_n \sigma(f_m, l_s(g_n)) = \lim_m \lim_n \sigma(g_n, l_s(f_m)) = \sigma(g, l_s(f))$$

entails $b(f, \tilde{\Lambda}g) = b(g, \tilde{\Lambda}f)$. Hence one has $\tilde{\Lambda} \in \mathcal{L}_{CCR}$. From the first step of this proof, it then follows from $\tilde{\Lambda} \supset \Lambda_{\max}$ the equality $\tilde{\Lambda} = \Lambda_{\max}$, so that $\Lambda' \subset \tilde{\Lambda} = \Lambda_{\max}$.

3. Finally, one turns to the closedness of Λ_{\max} . To this end, one considers Λ_{\max} in the form

$$D(\Lambda_{\max}) = \{f \in H \mid \sum_i \|h_i(f)\|^2 < \infty\}, \quad \Lambda_{\max}f = \sum_i e_i \otimes h_i(f)$$

with the continuous functions $h_i(f) = -b(j_{e_i}, \Lambda_{\max}f) = \sum_j \sum_k \lambda_{ijk} \sigma(e_k, f) e_j$. Let (f_n) be a sequence in $D(\Lambda_{\max})$ which converges to $f \in H$ so that $(\Lambda_{\max}f_n)$ converges to $F \in P_+H^2$. Set $F = \sum_I e_I \otimes g_I$, so that $g_i = \lim_n h_i(f_n)$ and $g_{i'} = 0$, since

$$\Lambda_{\max}f_n = \sum_i e_i \otimes h_i(f_n) \xrightarrow{n \rightarrow \infty} F = \sum_I e_I \otimes g_I.$$

But because of the continuity of $h_i(\cdot)$, $h_i(f_n)$ converges to $h_i(f)$. Therefore, $g_i = h_i(f)$, so $F = \sum_i e_i \otimes h_i(f)$, which implies $\sum_i \|h_i(f)\|^2 = \|F\|^2$ and thus also $f \in D(\Lambda_{\max})$ and $\Lambda_{\max}f = F$, concluding the proof. \square

From the proof of this last proposition we see that for every $\Lambda \in \mathcal{L}_{CCR}$ there is a unique maximal extension $\Lambda_{\max} \in \mathcal{L}_{CCR}$. Moreover, each such Λ is closable, since it possesses a closed extension (namely Λ_{\max}), and, of course, $\overline{\Lambda} \subset \overline{\Lambda_{\max}} = \Lambda_{\max}$. It may be the case that $\tilde{\Lambda} = \Lambda_{\max}$, but this has not been determined.

Since Λ_{\max} uniquely exists, it is henceforth admissible to consider Λ as being defined on the symplectic orthonormal basis $\{e_K\}$, where it takes the values $\Lambda e_k = 0$ and $\Lambda e_{k'} = \sum_{i,j} \lambda_{ijk} e_i \otimes e_j$. We can now finally state what we mean by the standard form of a canonical quadratic transformation $\Phi(f) \mapsto \Phi_\Lambda(f)$ in terms of the field operators $\Phi(e_K)$, where $\{e_K\}$ is a symplectic orthonormal basis in which Λ takes the form (3.3.1).

$$q_k \equiv \Phi(e_k) \mapsto \Phi_\Lambda(e_k) = \overline{\Phi(e_k) + \Psi(\Lambda e_k)} = q_k,$$

and

$$\begin{aligned} p_k \equiv \Phi(e_{k'}) \mapsto \Phi_\Lambda(e_{k'}) &= \overline{\Phi(e_{k'}) + \Psi(\Lambda e_{k'})} \\ &= p_k + \overline{\Psi(\sum_{i,j} \lambda_{ijk} e_i \otimes e_j)} \end{aligned}$$

$$\begin{aligned} &= \overline{p_k + \sum_{i,j} \lambda_{ijk} \Psi(e_i \otimes e_j)} \\ &= \overline{p_k + \sum_{i,j} \lambda_{ijk} (: q_i q_j : |_{\mathcal{F}_0})}. \end{aligned}$$

In short, the standard form of a canonical quadratic transformation is

$$q_k \mapsto q_k, \quad p_k \mapsto \overline{p_k + \sum_{i,j} \lambda_{ijk} (: q_i q_j : |_{\mathcal{F}_0})}. \tag{3.3.4}$$

The above results demonstrate that every canonical quadratic transformation can be written in this standard form, so that the apparently special case introduced in [22] was already perfectly general.

IV. Unitary Equivalence Characterized

In the previous section, we have characterized the canonical quadratic transformations, but in general there is certainly no guarantee that a representation of the CCR can be exponentiated to obtain a representation of the corresponding Weyl algebra. The first section of this chapter is dedicated to the proof that, in fact, representations of the CCR which are obtained from the Fock representation via canonical quadratic transformations can be exponentiated. Then we present the proof of the main theorem of this paper, which characterizes those quadratic representations which are unitarily equivalent to the Fock representation. In particular, we shall prove the following theorem.

Theorem 4.1. *Let $\Lambda \in \mathcal{L}_{CCR}$ and let $\Phi_\Lambda(f)$ be the corresponding representation of the CCR. $\Phi_\Lambda(f)$ is unitarily equivalent to the Fock representation $\Phi(f)$ if and only if the map $\Lambda : H \mapsto H \otimes H$ is Hilbert-Schmidt. Moreover, the corresponding Weyl representation $W_\Lambda(f)$ is unitarily equivalent to the Fock representation $W(f)$ if and only if the map $\Lambda : H \mapsto H \otimes H$ is Hilbert-Schmidt.*

§4.1. Quadratic Transformations and Weyl Algebras

First we shall prove that any canonical quadratic representation can be exponentiated to obtain a representation of the Weyl algebra.

Theorem 4.1.1. *Let $\Lambda \in \mathcal{L}_{CCR}$ and define the unitary operators $W_\Lambda(f)$, $f \in D(\Lambda)$, on $\mathcal{F}_+(\mathcal{H})$ by $W_\Lambda(f) \equiv e^{i\Phi_\Lambda(f)}$. Then for any $f, g \in D(\Lambda)$,*

$$(a) \ W_\Lambda(-f) = W_\Lambda(f)^* \text{ and } (b) \ W_\Lambda(f) W_\Lambda(g) = e^{-i\sigma(f,g)/2} W_\Lambda(f+g).$$

Proof. 1. The unitarity of $W_\Lambda(f)$ is an immediate consequence of Proposition 3.1.3, and assertion (a) is clear: $W_\Lambda(-f) = e^{i\Phi_\Lambda(-f)} = e^{-i\Phi_\Lambda(f)} = W_\Lambda(f)^*$. Thus, only assertion (b) is nontrivial. The proof of (b) uses analytic vectors and the exponential series for $e^{i\Phi_\Lambda(f)}$. An analogous proof for the Segal field Φ instead of Φ_Λ is in [21] (see Theorem X.41).

If $f=0$ or $g=0$, then the Weyl relations (b) obtain trivially. Hence, it shall be assumed that $f \neq 0$ and $g \neq 0$ are chosen from $D(\Lambda)$. Define the functional $\alpha \in D(\Lambda)'$ by $\alpha(h) \equiv \|h\| + 2\|\Lambda h\|$, and let k be a natural number so that $k > 4\sqrt{2} \max\{\alpha(f), \alpha(g)\}$ and $0 < \beta \equiv \frac{1}{k} < \frac{1}{4\sqrt{2} \max\{\alpha(f), \alpha(g)\}}$. Finally, let $\varphi = (\varphi^{(0)}, \varphi^{(1)}, \dots, \varphi^{(n_\varphi)}, 0, \dots)$ be a finite-particle vector.

From Proposition 3.1.3 one knows that the series $\sum_{m=0}^\infty \frac{t^m}{m!} \|\Phi_\Lambda(\beta f + \beta g)^m \varphi\|$ converges for all t such that $0 \leq t < 1/2\sqrt{2} \alpha(\beta f + \beta g)$ and that φ is an analytic vector for $\Phi_\Lambda(\beta f + \beta g)$. Thus one has

$$e^{i\Phi_\Lambda(\beta f + \beta g)} \varphi = \sum_{m=0}^\infty \frac{i^m}{m!} \Phi_\Lambda(\beta f + \beta g)^m \varphi,$$

since β has been chosen to satisfy $|i| < \frac{1}{2\sqrt{2} \alpha(\beta f + \beta g)}$. Therefore, one may write

$$\begin{aligned} e^{-i\sigma(\beta f, \beta g)/2} W_\Lambda(\beta f + \beta g) \varphi &= e^{-i\sigma(\beta f, \beta g)/2} \sum_{m=0}^\infty \frac{i^m}{m!} \Phi_\Lambda(\beta f + \beta g)^m \varphi \\ &= \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{i^{m+n}}{n!m!} \left(-\frac{1}{2}\right)^n \beta^{2n+m} \sigma(f, g)^n (\Phi_\Lambda(f) + \Phi_\Lambda(g))^m \varphi. \end{aligned} \tag{4.1.1}$$

On the other hand, φ is also an analytic vector of $\Phi_\Lambda(\beta g)$, and

$$e^{i\Phi_\Lambda(\beta g)} \varphi = \sum_{m=0}^\infty \frac{i^m}{m!} \Phi_\Lambda(\beta g)^m \varphi.$$

From the proof of Proposition 3.1.3 it is already established that

$$\|\Phi_\Lambda(f)^n \varphi\| \leq 2^{n/2} \alpha(f)^n [(n_\varphi + 1)(n_\varphi + 2) \cdots (n_\varphi + 2n)]^{1/2} \|\varphi\|,$$

where $\alpha(f) \equiv \|f\| + 2\|\Lambda f\|$. Thus one obtains

$$\begin{aligned} \|\Phi_\Lambda(f)^n \Phi_\Lambda(g)^m \varphi\| &\leq 2^{n/2} \alpha(f)^n [(n_\varphi + 2m + 1) \cdots (n_\varphi + 2m + 2n)]^{1/2} \|\Phi_\Lambda(g)^m \varphi\| \\ &\leq 2^{n/2} \alpha(f)^n 2^{m/2} \alpha(g)^m [(n_\varphi + 1) \cdots (n_\varphi + 2m)] \end{aligned}$$

$$\begin{aligned} & \cdot (n_\varphi + 2m + 1) \cdots (n_\varphi + 2m + 2n)]^{1/2} \|\varphi\| \\ & \leq 2^{(n+m)/2} \alpha(f)^n \alpha(g)^m \sqrt{(n_\varphi + 2m + 2n)!} \|\varphi\| \\ & \leq 2^{(n+m)/2} \alpha(f)^n \alpha(g)^m (n_\varphi + 4n)!^{1/4} (n_\varphi + 4m)!^{1/4} \|\varphi\|. \end{aligned}$$

The last inequality follows from the straightforward estimate $(n + m)! \leq \sqrt{(2n)!(2m)!}$. Hence one may bound $\sum_{n,m=0}^\infty \frac{t^n}{n!m!} \|\Phi_\Lambda(\beta f)^n \Phi_\Lambda(\beta g)^m \varphi\|$ from above by

$$\left(\sum_{n=0}^\infty \frac{2^{n/2} t^n}{n!} \alpha(\beta f)^n (n_\varphi + 4n)!^{1/4} \right) \left(\sum_{m=0}^\infty \frac{2^{m/2}}{m!} \alpha(\beta g)^m (n_\varphi + 4m)!^{1/4} \right) \|\varphi\|.$$

Applying again the ratio test to, e.g., the second of the series in this product, one obtains

$$\begin{aligned} & \frac{(2^{m/2}/m!) \alpha(\beta g)^m (n_\varphi + 4m)!^{1/4}}{(2^{(m-1)/2}/(m-1)!) \alpha(\beta g)^{m-1} (n_\varphi + 4m - 4)!^{1/4}} \\ & = \sqrt{2} \alpha(\beta g) \frac{1}{m} [(n_\varphi + 4m - 3) \cdots (n_\varphi + 4m)]^{1/4} \\ & \leq \sqrt{2} \alpha(\beta g) \frac{1}{m} (n_\varphi + 4m) \xrightarrow{m \rightarrow \infty} 4\sqrt{2} \alpha(\beta g). \end{aligned}$$

After a similar argument for the first of the series in the product, one may thereby conclude that the double series (4.1.1) converges whenever $4\sqrt{2} \beta \alpha(f) t < 1$ and $4\sqrt{2} \beta \alpha(g) < 1$. The latter condition is satisfied by the specified choice of β . Thus one may conclude that (4.1.1) converges if $t < \frac{1}{4\sqrt{2} \beta \alpha(f)}$, where also the case $t=1$ is included. Hence for arbitrary n ,

$$\sum_{m=0}^\infty \frac{1}{m!} \|\Phi_\Lambda(\beta f)^n \Phi_\Lambda(\beta g)^m \varphi\| < \infty,$$

from which one may conclude that $\sum_{m=0}^\infty \frac{i^m}{m!} \Phi_\Lambda(\beta f)^n \Phi_\Lambda(\beta g)^m \varphi$ converges. Since $\Phi_\Lambda(\beta f)^n$ is self-adjoint and therefore closed, it follows from this convergence that of $\sum_{m=0}^\infty \frac{i^m}{m!} \Phi_\Lambda(\beta g)^m \varphi$, which entails that $W_\Lambda(\beta g) \varphi$ is in the domain of definition of $\Phi_\Lambda(\beta f)^n$ and also that

$$\Phi_\Lambda(\beta f)^n W_\Lambda(\beta g) \varphi = \sum_{m=0}^\infty \frac{i^m}{m!} \Phi_\Lambda(\beta f)^n \Phi_\Lambda(\beta g)^m \varphi.$$

The above estimates imply that $W_\Lambda(\beta g)\varphi$ is in fact an analytic vector for $\Phi_\Lambda(\beta f)$, and one may use the exponential series again, i.e.

$$e^{i\Phi_\Lambda(\beta f)}e^{i\Phi_\Lambda(\beta g)}\varphi = \sum_{n,m=0}^{\infty} \frac{i^{n+m}}{n!m!}\beta^{n+m}\Phi_\Lambda(f)^n\Phi_\Lambda(g)^m\varphi. \tag{4.1.2}$$

Since the double series (4.1.1) and (4.1.2) are both *absolutely* convergent, one can arbitrarily re-order the series without changing the sum with the aim to compare terms of the same degree in β . Then, in the same manner as one establishes the validity of the Baker-Hausdorff formula ($e^Ae^B = e^{[A,B]^{1/2}}e^{A+B}$ for $[A, B] \in \mathbf{C}$), one verifies the equality of (4.1.1) and (4.1.2). For any $\varphi \in \mathcal{F}_0$ one has proven the validity of the equation $W_\Lambda(\beta f)W_\Lambda(\beta g)\varphi = e^{-\frac{i}{2}\sigma(\beta f, \beta g)}W_\Lambda(\beta f + \beta g)\varphi$. Since the Weyl operators are unitary, this relation is also fulfilled for all other vectors in Fock space.

2. In order to eliminate the quantity β , one writes

$$W_\Lambda(f) = W_\Lambda(k\beta f) = e^{ik\Phi_\Lambda(\beta f)} = W_\Lambda(\beta f)^k \text{ and } W_\Lambda(f)W_\Lambda(g) = W_\Lambda(\beta f)^kW_\Lambda(\beta g)^k.$$

But

$$\begin{aligned} W_\Lambda(\beta f)W_\Lambda(\beta g) &= e^{-i\sigma(\beta f, \beta g)/2}W_\Lambda(\beta f + \beta g) \\ &= e^{-i\sigma(\beta f, \beta g)}e^{-i\sigma(\beta g, \beta f)/2}W_\Lambda(\beta g + \beta f) \\ &= e^{-i\sigma(\beta f, \beta g)}W_\Lambda(\beta g)W_\Lambda(\beta f). \end{aligned}$$

Thus the product $W_\Lambda(f)W_\Lambda(g)$ can be rewritten:

$$W_\Lambda(f)W_\Lambda(g) = W_\Lambda(\beta f)^{k-1}e^{-i\sigma(\beta f, \beta g)}W_\Lambda(\beta g)W_\Lambda(\beta f)W_\Lambda(\beta g)^{k-1}.$$

By repeating this commutation one can manipulate $W_\Lambda(f)W_\Lambda(g)$ until only the products $W_\Lambda(\beta f)W_\Lambda(\beta g)$ arise. Every such commutation yields a factor $e^{-i\sigma(\beta f, \beta g)}$, and the required number of commutations is $(k-1) + (k-2) + \dots + (k - (k-1)) = \frac{1}{2}k(k-1)$. In this manner one obtains the desired result:

$$\begin{aligned} W_\Lambda(f)W_\Lambda(g) &= e^{-i\sigma(\beta f, \beta g) \cdot \frac{1}{2}k(k-1)}(W_\Lambda(\beta f)W_\Lambda(\beta g))^k \\ &= e^{-\frac{i}{2}k(k-1)\sigma(\beta f, \beta g)}e^{-\frac{i}{2}k\sigma(\beta f, \beta g)}W_\Lambda(\beta f + \beta g)^k \\ &= e^{-\frac{i}{2}k^2\sigma(\beta f, \beta g)}W_\Lambda(k(\beta f + \beta g)) = e^{-\frac{i}{2}\sigma(f, g)}W_\Lambda(f+g). \quad \square \end{aligned}$$

Consider the C^* -subalgebra $\mathcal{A}(D(\Lambda))$ of the CCR-algebra $\mathcal{A}(H, \sigma)$ generated by the set $\{W(f) \in \mathcal{A}(H, \sigma) \mid f \in D(\Lambda)\}$. If Λ is bounded, then $D(\Lambda) = H$ and

$\mathcal{A}(D(\Lambda)) = \mathcal{A}(H, \sigma)$. But if Λ is unbounded, we may consider it maximally defined by replacing it with Λ_{\max} . The resulting algebra $\mathcal{A}(D(\Lambda))$ is then ([2], Theorem 5.2.9) a proper subalgebra of $\mathcal{A}(H, \sigma)$. By the uniqueness of the CCR-algebra over a preHilbert space, the algebra $\mathcal{A}(D(\Lambda))$ is just the CCR-algebra over $(D(\Lambda), \sigma)$. On the other hand, the C^* -algebra $\mathcal{A}_\Lambda(D(\Lambda))$ generated by the operators $W_\Lambda(f) = e^{i\Phi_\Lambda(f)}$, $f \in D(\Lambda)$ on Fock space is, by Theorem 4.1.1 and [2], Theorem 5.2.8, isomorphic to $\mathcal{A}(D(\Lambda))$, i.e. there exists a unique C^* -isomorphism π_Λ from $\mathcal{A}(D(\Lambda))$ onto $\mathcal{A}_\Lambda(D(\Lambda))$, such that $\pi_\Lambda(W(f)) = W_\Lambda(f)$ for all $f \in D(\Lambda)$.

Corollary 4.1.2. *For any $\Lambda \in \mathcal{L}_{CCR}$, π_Λ is a faithful, regular, irreducible representation of the CCR-algebra $\mathcal{A}(D(\Lambda))$.*

Proof. Since the other assertions are obvious, only the proof of the claim of irreducibility will be sketched. If $A \in \mathcal{B}(\mathcal{F}_+(\mathcal{H}))$ commutes with all elements of $\mathcal{A}_\Lambda(D(\Lambda))$, then it commutes with every $\Phi_\Lambda(f)$. Since q_k and p_k can be expressed in terms of the $\Phi_\Lambda(e_j)$ (which is clear from the standard form (3.3.4)), A commutes also with all q_k and p_k . Since the Fock representation is irreducible, A must be trivial. □

We comment that the representation π_Λ is not quasifree in general. In fact, since $\langle \Omega_0, \Phi_\Lambda(f) \Omega_0 \rangle = 0$, the truncated three-point-function equals

$$\langle \Omega_0, \Phi_\Lambda(f_1) \Phi_\Lambda(f_2) \Phi_\Lambda(f_3) \Omega_0 \rangle$$

and the truncated four-point-function is

$$\begin{aligned} & \langle \Omega_0, \Phi_\Lambda(f_1) \Phi_\Lambda(f_2) \Phi_\Lambda(f_3) \Phi_\Lambda(f_4) \Omega_0 \rangle \\ & - \langle \Omega_0, \Phi_\Lambda(f_1) \Phi_\Lambda(f_2) \Omega_0 \rangle \langle \Omega_0, \Phi_\Lambda(f_3) \Phi_\Lambda(f_4) \Omega_0 \rangle \\ & - \langle \Omega_0, \Phi_\Lambda(f_1) \Phi_\Lambda(f_3) \Omega_0 \rangle \langle \Omega_0, \Phi_\Lambda(f_2) \Phi_\Lambda(f_4) \Omega_0 \rangle \\ & - \langle \Omega_0, \Phi_\Lambda(f_1) \Phi_\Lambda(f_4) \Omega_0 \rangle \langle \Omega_0, \Phi_\Lambda(f_2) \Phi_\Lambda(f_3) \Omega_0 \rangle. \end{aligned}$$

If one chooses $\Lambda f \equiv \lambda \sigma(e, f) e \otimes e$ with $\lambda \in \mathbf{R} \setminus \{0\}$ and $e \in H$ a unit vector, and sets $f_1 = f_2 = e$, $f_3 = f_4 = Je$, then one finds that neither of these truncated functions vanishes. Of course, that does not exclude the possibility that π_Λ is unitarily equivalent to a quasifree representation. But note that since quadratic representations are irreducible and since pure quasifree states are Fock states [14], π_Λ is unitarily equivalent to a quasifree representation *if and only if* it is unitarily equivalent to a Fock representation (though, of course, not necessarily the original Fock representation). It appears that, in fact, a quadratic representation is unitarily equivalent to a quasifree representation if and only if both are unitarily equivalent to the original Fock representation [23]. Moreover, π_Λ

is unitarily equivalent to a *coherent* representation if and only if once again both are unitarily equivalent to the original Fock representation [23]. These results will be proven elsewhere.

§4.2. Unitary Equivalence Implies Λ Hilbert-Schmidt

Though, as has just been shown, the algebras $\mathcal{A}(D(\Lambda))$ and $\mathcal{A}_\Lambda(D(\Lambda))$ are isomorphic, what is of primary physical concern is whether they are unitarily equivalent. (Unfortunately, it is a common mistake in the physics literature to overlook the difference between these two notions.) To specify exactly what we shall mean below, we shall say that the transformation $\Phi(f) \rightarrow \Phi_\Lambda(f)$ is unitarily implementable (or the two representations $\Phi(f)$ and $\Phi_\Lambda(f)$ are unitarily equivalent), when there exists a unitary operator $U : \mathcal{F}_+(\mathcal{H}) \mapsto \mathcal{F}_+(\mathcal{H})$ such that $U\Phi(f)U^*\psi = \Phi_\Lambda(f)\psi$ for all $f \in D(\Lambda)$ and all $\psi \in D(\Phi_\Lambda(f))$. (Recall that we have defined the operators $\Phi(f)$ and $\Phi_\Lambda(f)$ to be closed and have shown that they are, in fact, self-adjoint.) This entails, in particular, that $U^*D(\Phi_\Lambda(f)) \subset D(\Phi(f))$, for all $f \in D(\Lambda)$, and these domains of definition can indeed depend upon the choice of f . Note that by Theorem 4.1.1, $\Phi(f) \rightarrow \Phi_\Lambda(f)$ is unitarily implementable if and only if $UW(f)U^* = W_\Lambda(f)$ for all $f \in D(\Lambda)$. First we consider the proof of the following lemma.

Lemma 4.2.1. *Let $\Lambda \in \mathcal{L}_{CCR}$. If the canonical transformation $\Phi(f) \rightarrow \Phi_\Lambda(f)$ is unitarily implementable, then Λ is bounded.*

Proof. 1. Let U be the unitary such that $U\Phi(f)U^* = \Phi_\Lambda(f)$ for all $f \in D(\Lambda)$. Thus the set $U\mathcal{F}_0$ is dense in $\mathcal{F}_+(\mathcal{H})$ and is contained in the domain of definition of $\Phi_\Lambda(f)$. Using the closedness of $\Phi_\Lambda(f)$, let $\psi \in U\mathcal{F}_0 \subset D(\Phi_\Lambda(f))$ and $\{\psi_k\} \subset \mathcal{F}_0$ converge to ψ in such a manner that $\Phi_\Lambda(f)\psi_k \rightarrow \Phi_\Lambda(f)\psi$. Then, since $\Psi(\Lambda f)$ is quadratic in the creation and annihilation operators, it is clear that

$$P_n\Phi_\Lambda(f)\psi = \lim_{k \rightarrow \infty} P_n\Phi_\Lambda(f)\psi_k = \lim_{k \rightarrow \infty} P_n\Phi_\Lambda(f)(P_{n-2} + \dots + P_{n+2})\psi_k.$$

But since $P_n\Phi_\Lambda(f)$ is closed, it follows that $P_n\Phi_\Lambda(f)\psi = P_n\Phi_\Lambda(f)(P_{n-2} + \dots + P_{n+2})\psi$. Arguing similarly on the components $\Phi(f)$ and $\Phi(\Lambda f)$ and using the comments following equation (3.1.3), one obtains

$$\begin{aligned} P_nU\Phi(f)U^*\psi &= P_n\Phi_\Lambda(f)\psi = P_n(\Phi(f) + \Psi(\Lambda f))(P_{n-2} + \dots + P_{n+2})\psi \\ &= P_n\Phi(f)(P_{n-1} + P_{n+1})\psi + P_n\Psi(\Lambda f)(P_{n-2} + P_n + P_{n+2})\psi, \end{aligned}$$

in particular, for $n=2$:

$$P_2 U \Phi(f) U^* \phi - P_2 \Phi(f) (P_1 + P_3) \phi = P_2 \Psi(\Lambda f) (P_0 + P_2 + P_4) \phi.$$

But since $U^* \phi \in \mathcal{F}_0$ and $(P_1 + P_3) \phi \in \mathcal{F}_0$, the maps $f \mapsto \Phi(f) U^* \phi$ and $f \mapsto \Phi(f) (P_1 + P_3) \phi$ are continuous, and thus so are the maps $f \mapsto P_2 U \Phi(f) U^* \phi$ and $f \mapsto P_2 \Phi(f) (P_1 + P_3) \phi$. Hence it follows that the map $f \mapsto P_2 \Psi(\Lambda f) (P_0 + P_2 + P_4) \phi$ from $D(\Lambda)$ to $\mathcal{F}_+(H)$ is continuous.

2. Consider now the expression $P_2 \Psi(\Lambda f) (P_0 + P_2 + P_4) \phi$. One has

$$\begin{aligned} \|P_2 \Psi(\Lambda f) (P_0 + P_2 + P_4) \phi\| &= \|P_2 \Psi(\Lambda f) (P_0 + P_2 + P_4) \phi\| + \|-P_2 \Psi(\Lambda f) (P_2 + P_4) \phi\| \\ &\quad - \|P_2 \Psi(\Lambda f) (P_2 + P_4) \phi\| \\ &\geq \|P_2 \Psi(\Lambda f) (P_0 + P_2 + P_4) \phi - P_2 \Psi(\Lambda f) (P_2 + P_4) \phi\| \\ &\quad - \|P_2 \Psi(\Lambda f) (P_2 + P_4) \phi\| \\ &= \|P_2 \Psi(\Lambda f) P_0 \phi\| - \|P_2 \Psi(\Lambda f) (P_2 + P_4) \phi\|. \end{aligned} \tag{4.2.1}$$

Now, of course, $P_0 \phi = (\phi^{(0)}, 0, 0, \dots) = \phi^{(0)} \Omega_0$, where $\phi^{(0)} \in \mathbb{C}$. Since $\Lambda \in \mathcal{L}_{CCR}$, Theorem 3.3.1 yields $\Lambda = \sum_{i,j} \sum_k \lambda_{ijk} \sigma(e_k, \cdot) e_i \otimes e_j$, and one finds

$$\begin{aligned} P_2 \Psi(\Lambda f) P_0 \phi &= \phi^{(0)} P_2 \Psi \left(\sum_{i,j} \sum_k \lambda_{ijk} \sigma(e_k, f) e_i \otimes e_j \right) \Omega_0 \\ &= \phi^{(0)} \sum_{i,j} \sum_k \lambda_{ijk} \sigma(e_k, f) P_2 : \Phi(e_i) \Phi(e_j) : \Omega_0 \\ &= \frac{1}{2} \phi^{(0)} \sum_{i,j} \sum_k \lambda_{ijk} \sigma(e_k, f) a(\tilde{e}_i) * a(\tilde{e}_j) * \Omega_0 \\ &= \frac{1}{2} \frac{\sqrt{2}}{2} \phi^{(0)} \sum_{i,j} \sum_k \lambda_{ijk} \sigma(e_k, f) (0, 0, \tilde{e}_i \otimes \tilde{e}_j + \tilde{e}_j \otimes \tilde{e}_i, 0, \dots) \\ &= \frac{1}{\sqrt{2}} \phi^{(0)} \left(0, 0, \sum_{i,j} \sum_k \lambda_{ijk} \sigma(e_k, f) \tilde{e}_i \otimes \tilde{e}_j, 0, 0, \dots \right). \end{aligned}$$

Since $\{\tilde{e}_i\}$ is an orthonormal basis in \mathcal{H} , one finds for the norm squared:

$$\|P_2 \Psi(\Lambda f) P_0 \phi\|^2 = \frac{1}{2} |\phi^{(0)}|^2 \sum_{i,j} \left(\sum_k \lambda_{ijk} \sigma(e_k, f) \right)^2 = \frac{1}{2} |\phi^{(0)}|^2 \|\Lambda f\|^2.$$

Up to this point ϕ was an arbitrary vector from $U\mathcal{F}_0$, which is dense in Fock space. Thus choose ϕ such that $\|\phi - \Omega_0\| < \varepsilon$; then one has

$$\varepsilon^2 > \|\phi - \Omega_0\|^2 = \|(\phi^{(0)} - 1, \phi^{(1)}, \phi^{(2)}, \dots)\|^2 = |\phi^{(0)} - 1|^2 + \sum_{n=1}^{\infty} \|\phi^{(n)}\|^2, \tag{4.2.2}$$

which implies $\varepsilon > |1 - \phi^{(0)}| \geq 1 - |\phi^{(0)}|$, yielding $|\phi^{(0)}| > 1 - \varepsilon$. Thus one finds that for this choice of ϕ , one has $\|P_2 \Psi(\Lambda f) P_0 \phi\| \geq \frac{1 - \varepsilon}{\sqrt{2}} \|\Lambda f\|$.

Now to the second term in (4.2.1). In this case one has

$$\begin{aligned} \|P_2 \Psi(\Lambda f) (P_2 + P_4) \phi\| &\leq \| \Psi(\Lambda f) (P_2 + P_4) \phi \| \\ &\leq \| \Psi(\Lambda f) (P_2) \phi \| + \| \Psi(\Lambda f) P_4 \phi \| \\ &\leq 2\sqrt{2}\sqrt{3 \cdot 4} \|\Lambda f\| \|\phi^{(2)}\| + 2\sqrt{2}\sqrt{5 \cdot 6} \|\Lambda f\| \|\phi^{(4)}\| \end{aligned}$$

(see (3.1.1)). From (4.2.2) one extracts the bounds $\|\phi^{(2)}\| < \varepsilon$ and $\|\phi^{(4)}\| < \varepsilon$ for the specified choice of ϕ , so that $\|P_2 \Psi(\Lambda f) (P_2 + P_4) \phi\| \leq 2\sqrt{2} (\sqrt{3 \cdot 4} + \sqrt{5 \cdot 6}) \varepsilon \|\Lambda f\| \leq 2\sqrt{2} (\sqrt{4 \cdot 4} + \sqrt{6 \cdot 6}) \varepsilon \|\Lambda f\| = 20\sqrt{2} \varepsilon \|\Lambda f\|$.

With these two results, one can now continue (4.2.1):

$$\|P_2 \Psi(\Lambda f) (P_0 + P_2 + P_4) \phi\| \geq \left(\frac{1 - \varepsilon}{\sqrt{2}} - 20\sqrt{2} \varepsilon \right) \|\Lambda f\| = \beta \|\Lambda f\|,$$

with $\beta \equiv \frac{1 - \varepsilon}{\sqrt{2}} - 20\sqrt{2} \varepsilon$. Choosing $\varepsilon < \frac{1}{41}$, one finds $\beta > 0$. Therefore, for arbitrary $f \in D(\Lambda)$ one has found that

$$\|\Lambda f\| \leq \frac{1}{\beta} \|P_2 \Psi(\Lambda f) (P_0 + P_2 + P_4) \phi\|.$$

Since it has already been demonstrated that the map $f \mapsto P_2 \Psi(\Lambda f) (P_0 + P_2 + P_4) \phi$ is bounded, this implies that the operator Λ is bounded, completing the proof. \square

This lemma is then used in the proof of the next proposition.

Proposition 4.2.2. *Let $\Lambda \in \mathcal{L}_{CCR}$. If the canonical transformation $\Phi(f) \rightarrow \Phi_\Lambda(f)$ is unitarily implementable, then Λ is Hilbert-Schmidt.*

Proof. 1. By Lemma 4.2.1 it may be assumed that Λ is bounded and everywhere defined. Certainly Λ_{\max} is bounded, so $\bar{\Lambda} \subset \Lambda_{\max}$ is everywhere defined. But it is straightforward to show that $U\Phi(f)U^* = \Phi_\Lambda(f)$, for all $f \in D(\Lambda)$, entails $U\Phi(f)U^* = \Phi_{\bar{\Lambda}}(f)$, for all $f \in D(\bar{\Lambda}) = H$, since it has already been remarked that $f_n \rightarrow f$ in H implies that the self-adjoint $\Phi_\Lambda(f)$ is the strong graph limit of the self-adjoint $\Phi_\Lambda(f_n)$ (see [18], pp. 62-63 for further details).

It may therefore be assumed that $U\Phi(f)U^* = \Phi_\Lambda(f)$ for all $f \in H$, Λ is bounded and U is unitary. Choose a symplectic orthonormal basis $\{e_k, e_{k'}\}$ in H , in which Λ takes on its standard form, and consider

$$\begin{aligned} \Omega &\equiv U\Omega_0, \\ a_k &\equiv a(\tilde{e}_k) = \frac{1}{\sqrt{2}} (\Phi(e_k) + i\Phi(Je_k)) = \frac{1}{\sqrt{2}} (q_k + ip_k), \end{aligned}$$

$$Ua_k U^* = \frac{1}{\sqrt{2}} (\Phi_\Lambda(e_k) + i\Phi_\Lambda(Je_k)) = \frac{1}{\sqrt{2}} (q_k + i\overline{p_k + i \sum_{l,m} \lambda_{klm} : q_l q_m : |_{\mathcal{F}_\sigma}}),$$

where (3.3.4) has been used. With $\Omega \equiv U\Omega_0$, one has $Ua_k U^* \Omega = 0$, so that (for $n \geq 2$)

$$\begin{aligned} 0 &= P_n Ua_k U^* \Omega = P_n Ua_k U^* (P_{n-2} + \dots + P_{n+2}) \Omega \\ &= P_n \left(a_k + \frac{i}{2\sqrt{2}} \sum_{l,m} \lambda_{klm} (a_l^* a_m^* + 2a_l^* a_m + a_l a_m) \right) (P_{n-2} + \dots + P_{n+2}) \Omega \\ &= a_k P_{n+1} \Omega + \frac{i}{2\sqrt{2}} \sum_{l,m} \lambda_{klm} (a_l^* a_m^* P_{n-2} \Omega + 2a_l^* a_m P_n \Omega + a_l a_m P_{n+2} \Omega). \end{aligned} \tag{4.2.3}$$

Expanding with respect to the basis $\{\tilde{e}_i\}$, one writes $P_m \Omega = (0, \dots, 0, \Omega^{(m)}, 0, \dots)$ and $\Omega^{(m)} = \sum_{i_1, \dots, i_m} \omega_{i_1, \dots, i_m}^{(m)} \tilde{e}_{i_1} \otimes \dots \otimes \tilde{e}_{i_m}$, with $m \in \mathbb{N}$ and $\omega_{i_1, \dots, i_m}^{(m)} \in \mathbb{C}$, totally symmetric in the m indices. Setting this into (4.2.3) and isolating a particular component $\tilde{e}_{i_1} \otimes \dots \otimes \tilde{e}_{i_n}$, one finds for every $n \geq 2$,

$$\begin{aligned} \sqrt{n+1} \omega_{ki_1, \dots, i_n}^{(n+1)} + \frac{i}{2\sqrt{2}} \left\{ \frac{1}{\sqrt{n(n-1)}} \sum_{\substack{\nu, \mu=1 \\ \nu+\mu=n}}^n \lambda_{ki_\nu i_\mu} \omega_{i_1, \dots, \hat{i}_\nu, \dots, \hat{i}_\mu, \dots, i_n}^{(n-2)} \right. \\ \left. + 2 \sum_{m=1}^{\infty} \sum_{\nu=1}^n \lambda_{kmi_\nu} \omega_{mi_1, \dots, \hat{i}_\nu, \dots, i_n}^{(m)} + \sqrt{(n+2)(n+1)} \sum_{l,m=1}^{\infty} \lambda_{klm} \omega_{lm i_1, \dots, i_n}^{(n+2)} \right\} = 0, \end{aligned} \tag{4.2.4}$$

where the symbol $\hat{}$ over an index signifies that it should be left out of the sum.

2. For any $m \in \mathbb{N}$, one has $\|\Omega^{(m)}\|^2 = \sum_{i_1, \dots, i_m} |\omega_{i_1, \dots, i_m}^{(m)}|^2 \leq 1$ and, in particular, $\sum_{i_1, \dots, i_n} |\omega_{ki_1, \dots, i_n}^{(n+1)}|^2 < \infty$, for an arbitrary fixed index k . Moreover, since $\sum_{i_\nu, i_\mu} \lambda_{ki_\nu i_\mu}^2 < \infty$ (see Prop. 3.3.1), one also has

$$\begin{aligned} \infty &> \sum_{i_\nu, i_\mu} \lambda_{ki_\nu i_\mu}^2 \sum_{i_\sigma} |\omega_{i_1, \dots, \hat{i}_\nu, \dots, \hat{i}_\mu, \dots, i_\sigma}^{(n-2)}|^2 = \sum_{i_\nu, i_\mu, i_\sigma} |\lambda_{ki_\nu i_\sigma} \omega_{i_1, \dots, \hat{i}_\nu, \dots, \hat{i}_\mu, \dots, i_\sigma}^{(n-2)}|^2 \\ &= \sum_{i_\nu, i_\mu, i_\sigma} |\lambda_{ki_\nu i_\sigma} \omega_{i_1, \dots, \hat{i}_\nu, \dots, \hat{i}_\mu, \dots, i_\sigma}^{(n-2)}|^2. \end{aligned}$$

To study the term $\sum_{m=1}^{\infty} \lambda_{kmi_\nu} \omega_{mi_1, \dots, \hat{i}_\nu, \dots, i_n}^{(m)}$ in (4.2.4) choose a vector $\sum_{m, i_1, \dots, \hat{i}_\nu, \dots, i_n} \Re \omega_{mi_1, \dots, \hat{i}_\nu, \dots, i_n}^{(m)} e_m \otimes \dots \otimes e_{i_n}$ from $H^{\otimes n}$ (the n -fold direct product of H with itself) and apply to it the operator $(\Lambda J) \otimes I^{\otimes n}$, which is continuous and defined everywhere. This results in a vector from $H^{\otimes n+1}$:

$$\begin{aligned} (\Lambda J \otimes I^{\otimes n}) \sum_{m, i_1, \dots, \hat{i}_\nu, \dots, i_n} \Re \omega_{mi_1, \dots, \hat{i}_\nu, \dots, i_n}^{(m)} e_m \otimes \dots \otimes e_{i_n} \\ = \sum_{m, i_1, \dots, \hat{i}_\nu, \dots, i_n} \Re \omega_{mi_1, \dots, \hat{i}_\nu, \dots, i_n}^{(m)} (\Lambda J e_m) \otimes \dots \otimes e_{i_n} \end{aligned}$$

$$= \sum_{m, i_1, \dots, i_n} \Re \omega_{m i_1 \dots i_n}^{(n)} \left(\sum_{k, i} \lambda_{k i m} e_k \otimes e_i \right) \otimes \dots \otimes e_{i_n}.$$

The square norm of this vector equals $\sum_{k, i, i_1, \dots, i_n} (\sum_m \lambda_{k i m} \Re \omega_{m i_1 \dots i_n}^{(n)})^2$ and must be finite. The same considerations employing the imaginary part of $\omega_{m i_1 \dots i_n}^{(n)}$ yield $\sum_{k, i, i_1, \dots, i_n} (\sum_m \lambda_{k i m} \Im \omega_{m i_1 \dots i_n}^{(n)})^2 < \infty$, and together they lead to

$$\begin{aligned} \infty &> \sum_{k, i, i_1, \dots, i_n} \left(\sum_m \lambda_{k i m} \Re \omega_{m i_1 \dots i_n}^{(n)} \right)^2 + \sum_{k, i, i_1, \dots, i_n} \left(\sum_m \lambda_{k i m} \Im \omega_{m i_1 \dots i_n}^{(n)} \right)^2 \\ &= \sum_{k, i, i_1, \dots, i_n} \left\{ \left(\Re \sum_m \lambda_{k i m} \omega_{m i_1 \dots i_n}^{(n)} \right)^2 + \left(\Im \sum_m \lambda_{k i m} \omega_{m i_1 \dots i_n}^{(n)} \right)^2 \right\} \\ &= \sum_{k, i, i_1, \dots, i_n} \left| \sum_m \lambda_{k i m} \omega_{m i_1 \dots i_n}^{(n)} \right|^2. \end{aligned}$$

For the last term in (4.2.4) the operator $(\Lambda_j)^*$ is useful. With $\Lambda_j e_k = \sum_{l, m} \lambda_{k l m} e_l \otimes e_m$, one has $(\Lambda_j)^* e_l \otimes e_m = \sum_k \lambda_{k l m} e_k$, and since $\Lambda_j : H \mapsto H \otimes H$ is bounded, its adjoint $(\Lambda_j)^* : H \otimes H \mapsto H$ and $(\Lambda_j)^* \otimes I^{\otimes n-1} : (H \otimes H) \otimes H^{\otimes n-1} \mapsto H^{\otimes n}$ are bounded (and everywhere defined). Thus one has

$$\begin{aligned} (\Lambda_j)^* \otimes I^{\otimes n-1} &\sum_{l, m, i, i_1, \dots, i_{n-1}} \Re \omega_{l m i_1 \dots i_{n-1}}^{(n+2)} e_l \otimes e_m \otimes e_{i_1} \otimes \dots \otimes e_{i_{n-1}} \\ &= \sum_{l, m, i_1, \dots, i_{n-1}} \Re \omega_{l m i_1 \dots i_{n-1}}^{(n+2)} \left(\sum_k \lambda_{k l m} e_k \right) \otimes e_{i_1} \otimes \dots \otimes e_{i_{n-1}}, \end{aligned}$$

with the square norm satisfying $\sum_{k, i_1, \dots, i_n} (\sum_{l, m} \lambda_{k l m} \Re \omega_{l m i_1 \dots i_n}^{(n+2)})^2 < \infty$. With this and the corresponding bound for the imaginary part, one finds as above that

$$\sum_{k, i_1, \dots, i_n} \left| \sum_{l, m} \lambda_{k l m} \omega_{l m i_1 \dots i_n}^{(n+2)} \right|^2 < \infty.$$

3. Pick $\nu, \mu \in \{1, \dots, n\}$ with $\nu \neq \mu$, and write $i = i_\nu, j = i_\mu$ and relabel $\{i_1, \dots, i_\nu, \dots, i_\mu, \dots, i_n\}$ by $\{i_3, \dots, i_n\}$. One has seen that all the terms in (4.2.4) except possibly $\lambda_{k i j} \omega_{i_3 \dots i_n}^{(n-2)}$ are square summable over k, i and j , so that all other quantities entering into (4.2.4) and indexed by k, i and j can be viewed as vectors in $\ell_2 \otimes \ell_2 \otimes \ell_2$. Therefore solving (4.2.4) for $\lambda_{k i j} \omega_{i_3 \dots i_n}^{(n-2)}$, it follows that $(\lambda_{k i j} \omega_{i_3 \dots i_n}^{(n-2)})_{k, i, j=1, 2, \dots}$ must be an element of $\ell_2 \otimes \ell_2 \otimes \ell_2$, i.e.,

$$\sum_{k, i, j} |\lambda_{k i j} \omega_{i_3 \dots i_n}^{(n-2)}|^2 = |\omega_{i_3 \dots i_n}^{(n-2)}|^2 \sum_{k, i, j} \lambda_{k i j}^2 < \infty,$$

so that $\sum_{k, i, j} \lambda_{k i j}^2 < \infty$ (one can always find a nonzero $\omega_{i_3 \dots i_n}^{(n-2)}$ for some n , since $\Omega \neq 0$). With this the proof is completed, since

$$\|A\|_2^2 = \sum_K \|Ae_K\|^2 = \sum_{K, J, I} \langle e_I \otimes e_J, Ae_K \rangle^2 = \sum_{k, i, j} \lambda_{kij}^2,$$

so that $\|A\|_2 < \infty$. □

§4.3. Λ Hilbert-Schmidt Implies Unitary Equivalence

We next wish to prove the converse to Proposition 4.2.2, i.e. that $A \in \mathcal{L}_{CCR}$ Hilbert-Schmidt entails that the canonical transformation $\Phi(f) \mapsto \Phi_A(f)$ is unitarily implementable. The previously demonstrated standard form of the elements of \mathcal{L}_{CCR} will be crucial here, since it will enable us to use Q -space techniques to facilitate the explicit construction of the desired unitary U . Once the quadratic transformation has been written in standard form (3.3.1), the unitary implementing the transformation should be (at least formally):

$$U = \exp\left(-\frac{i}{3} \sum_{i, j, k} \lambda_{ijk} : \Phi(e_i) \Phi(e_j) \Phi(e_k) : \right), \tag{4.3.1}$$

as can be verified with the help of the Baker-Campbell-Hausdorff formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \dots$$

However, proving that if A is Hilbert-Schmidt, then (4.3.1) actually defines a unitary operator intertwining the two representations is not trivial.

Let $A \in \mathcal{L}_{CCR}$ be given and then written in standard form, with $\{\lambda_{klm}\}_{k, l, m \in \mathbb{N}}$ the resulting sequence of real numbers totally symmetric with respect to permutations of the indices k, l, m , and $\sum_{k, l, m=1}^{\infty} \lambda_{klm}^2 < \infty$, and let $\{e_k, e'_k\} = \{e_K | K=1, 1', 2, 2', \dots\} \subset H$ be the resulting symplectic orthonormal basis. Moreover, let $x = (x_1, x_2, \dots)$ be a point in $Q \equiv \times_{k=1}^{\infty} \mathbb{R}$, and Σ be the σ -algebra generated by the cylinder sets of Q with Lebesgue measurable base. Then $\mu = \otimes_{k=1}^{\infty} \mu_k$, where each μ_k is the Gaussian measure $d\mu_k = \pi^{-\frac{1}{2}} e^{-x_k^2} dx_k$, is a probability measure on (Q, Σ) .

It is well-known that there exists a unitary map S of $\mathcal{F}_+(\mathcal{H})$ onto $L^2(Q, d\mu)$ such that [24] [27] [21]

$$S\Omega_0 = 1 \text{ and } SP_+(\tilde{e}_{k_1} \otimes \tilde{e}_{k_2} \otimes \dots \otimes \tilde{e}_{k_r}) = (r!)^{-\frac{1}{2}} (\sqrt{2})^r : x_{k_1} x_{k_2} \dots x_{k_r} : ,$$

$$S\Phi(e_k)S^{-1} \equiv q_k = (\text{multiplication by } x_k, \text{ and}$$

$$S\Phi(Je_k)S^{-1} \equiv p_k = \frac{1}{i} \frac{\partial}{\partial x_k} + ix_k,$$

where the operator equations are understood to hold on the dense set $S\mathcal{F}_0$. We shall occasionally drop the symbol S , when the identification is clear.

Let $\underline{k} = \{k_1, k_2, \dots, k_r\}$ be a multiple index in $\mathbf{N}^{(r)} = \times_{j=1}^r \mathbf{N}$, $\{\lambda_{\underline{k}}\}_{\underline{k} \in \mathbf{N}^{(r)}}$ a sequence of real numbers, totally symmetric in \underline{k} , $\sum_{\underline{k}} \lambda_{\underline{k}}^2 < \infty$, and $I_n = \{\underline{k} \mid \max\{k_1, k_2, \dots, k_r\} \leq n\}$. Then

$$f_n^{(r)} = \sum_{\underline{k} \in I_n} \lambda_{\underline{k}} e_{k_1} \otimes e_{k_2} \otimes \dots \otimes e_{k_r} \in H^{(r)},$$

with $\|f_n^{(r)}\|^2 = \sum_{\underline{k} \in I_n} \lambda_{\underline{k}}^2$. Since with $m < n$, we have $\|f_n^{(r)} - f_m^{(r)}\|^2 = \sum_{\underline{k} \in I_n \setminus I_m} \lambda_{\underline{k}}^2$ and $\sum_{\underline{k} \in I_n} \lambda_{\underline{k}}^2$ converges with $n \rightarrow \infty$, it follows that

$$f_n^{(r)} \rightarrow f^{(r)} = \sum_{\underline{k}} \lambda_{\underline{k}} e_{k_1} \otimes e_{k_2} \otimes \dots \otimes e_{k_r} \text{ in } H^{(r)}.$$

We want to study the sequence of operators

$$A(f_n^{(r)}) = \sum_{\underline{k} \in I_n} \lambda_{\underline{k}} : \Phi(e_{k_1}) \Phi(e_{k_2}) \dots \Phi(e_{k_r}) : ,$$

the Q -space realization of which is given by $A_n = A(f_n^{(r)}) = \sum_{\underline{k} \in I_n} \lambda_{\underline{k}} : x_{k_1} x_{k_2} \dots x_{k_r} : .$ Note that since it is a polynomial, $A_n = A_n(x)$ is in $L^2(Q, d\mu)$ with

$$\begin{aligned} \|A_n\|^2 &= 2^{-r} r! \sum_{\underline{k} \in I_n} \lambda_{\underline{k}}^2 = 2^{-r} r! \|f_n^{(r)}\|^2, \\ \text{and } \|A_n - A_m\|^2 &= 2^{-r} r! \|f_n^{(r)} - f_m^{(r)}\|^2 \end{aligned} \tag{4.3.2}$$

(cf. the proof of Lemma I.18 of [27]). Therefore, $A_n(x)$ converges in $L^2(Q, d\mu)$, and we shall call the a.e.-defined limit $A(x) = \sum_{\underline{k}} \lambda_{\underline{k}} : x_{k_1} x_{k_2} \dots x_{k_r} : ,$ which up to a factor of $\sqrt{r!} (\sqrt{2})^{-r}$ corresponds to $f^{(r)}$.

The advantage of the Q -space formulation is that all functions of $\{\Phi(e_k), k = 1, 2, \dots\}$ become multiplication operators on $L^2(Q, d\mu)$. $A_n(x)$ and $A(x)$ are measurable, real-valued functions on Q which are finite a.e. $[\mu]$. So with $D(A) \equiv \{\varphi \mid A\varphi \in L^2(Q, d\mu)\}$ (similarly for $D(A_n)$), $(A_n\varphi)(x) = A_n(x)\varphi(x)$ and $(A\varphi)(x) = A(x)\varphi(x)$ are self-adjoint operators (cf. [20], VIII. 3 Proposition 1). Thus, for every $f^{(r)} \in P_+H^{(r)}$, $A(f^{(r)})$ represents a well-defined self-adjoint multiplication operator.

We collect a number of useful technical results concerning these operators.

- Lemma 4.3.1.** 1. $L^\infty(Q, d\mu) \subset L^2(Q, d\mu)$ is a common core for all A_n, A .
 2. As operators on $L^2(Q, d\mu)$, $A_n \rightarrow A$ in the strong resolvent sense.
 3. On $L^\infty(Q, d\mu)$ the operator family $A(f^{(r)})$ is strongly continuous in $f^{(r)} \in P_+H^{(r)}$.
 4. Let f be a bounded continuous function on \mathbf{R} . Then $f(A_n)\varphi \rightarrow f(A)\varphi$ for all $\varphi \in L^2(Q, d\mu)$, and especially, $e^{itA_n} \rightarrow e^{itA}$ strongly for each $t \in \mathbf{R}$. Moreover, A is the strong graph limit of A_n .

Proof. 1. Of course, $L^\infty(Q, d\mu)$ is dense in $L^2(Q, d\mu)$. For any vector $\varphi \in L^\infty(Q, d\mu)$, one has $\|A_n\varphi\| \leq \|\varphi\|_\infty \|A_n\| < \infty$ and $\|A\varphi\| \leq \|\varphi\|_\infty \|A\| < \infty$, showing that $L^\infty(Q, d\mu) \subset D(A_n)$ and $L^\infty(Q, d\mu) \subset D(A)$, respectively. A_n, A are multiplication by real-valued functions. Hence, for all $t \in \mathbf{R}$, one has $e^{itA_n} : L^\infty(Q, d\mu) \rightarrow L^\infty(Q, d\mu)$ and $e^{itA} : L^\infty(Q, d\mu) \rightarrow L^\infty(Q, d\mu)$. Assertion (1) now follows from [20], Theorem VIII.11.

2. For $\varphi \in L^\infty(Q, d\mu)$ one has $\|A_n\varphi - A\varphi\| \leq \|\varphi\|_\infty \|A_n - A\| \xrightarrow{n \rightarrow \infty} 0$. Hence, $A_n\varphi \rightarrow A\varphi$ for each $\varphi \in L^\infty(Q, d\mu)$, which according to [20], Theorem VIII.25, implies $A_n \rightarrow A$ in the strong resolvent sense.

3. For $\varphi \in L^\infty(Q, d\mu)$ one has from Step (2) and equation (4.3.2)

$$\|A(f^{(r)})\varphi - A(g^{(r)})\varphi\|^2 \leq 2^{-r} r! \|\varphi\|_\infty^2 \|f^{(r)} - g^{(r)}\|^2.$$

4. Assertion (4) follows from the previous steps and standard theorems (see, e.g. [20], Theorems VIII.20, VIII.21 and VIII.26). □

Remark. Assertion 4 means in particular that if $f_n = \sum_{k=1}^n \alpha_k e_k \xrightarrow{n \rightarrow \infty} f \in H$ then $\Phi(f)$ is the strong graph limit of $\Phi(f_n)$. Since the role of p_k and q_k as multiplication operators could have been interchanged, it is also true that if $g_n = \sum_{k=1}^n \beta_k e_k \xrightarrow{n \rightarrow \infty} g \in H$ then $\Phi(g)$ is the strong graph limit of $\Phi(g_n)$.

In the next lemma we want to show that $\mathcal{F}_0 \subset D(A(f^{(r)}))$ for all $f^{(r)} \in H^{(r)}$, that in contrast to $L^\infty(Q, d\mu)$, \mathcal{F}_0 is invariant under $A(f^{(r)})$ and that the analogue of Lemma 4.3.1, Assertion 3, also holds on \mathcal{F}_0 . Note that on \mathcal{F}_0 , we have $\Phi(e_k) = \frac{1}{\sqrt{2}}(a^*(\tilde{e}_k) + a(\tilde{e}_k))$. Inserting $\frac{1}{\sqrt{2}}(a^*(\tilde{e}_k) + a(\tilde{e}_k))$ for each $\Phi(e_k)$ in $A(f_n^{(r)}) = \sum_{k \in I_n} \lambda_k : \Phi(e_{k_1}) \Phi(e_{k_2}) \cdots \Phi(e_{k_r}) :$, we observe that $A(f_n^{(r)})$ is a finite sum of finite products of $a^*(\tilde{e}_{k_i}), a(\tilde{e}_{k_i})$ under which \mathcal{F}_0 is invariant. Hence, $\mathcal{F}_0 \subset D(A_n)$ for all n .

In order to express $A(f_n^{(r)})$ in terms of $a^*(\tilde{e}_{k_i}), a(\tilde{e}_{k_i})$ we define for $0 \leq j \leq r$

$$B_j(f_n^{(r)}) = \sum_{\underline{k} \in I_n} \lambda_{\underline{k}} a^*(\tilde{e}_{k_1+1}) a^*(\tilde{e}_{k_1+2}) \cdots a^*(\tilde{e}_{k_r}) a(\tilde{e}_{k_1}) a(\tilde{e}_{k_1-1}) \cdots a(\tilde{e}_{k_1}).$$

Since with $\underline{k} \in I_n$ we sum over a symmetric index set and since $\lambda_{\underline{k}}$ is totally symmetric in \underline{k} , $B_j(f_n^{(r)})$ does not depend on the specific ordering of the indices $\{k_1, k_2, \dots, k_r\}$. Taking into account this symmetry we see that on \mathcal{F}_0 the equality

$$A(f_n^{(r)}) = (\sqrt{2})^{-r} \sum_{j=0}^r \binom{r}{j} B_j(f_n^{(r)}) \tag{4.3.3}$$

holds. Let us denote by $H_{s,0}^{(r)}$ the subset of $P_+H^{(r)}$ which consists of finite symmetrized linear combinations of the basis $e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_r}$. Hence, $f_n^{(r)}$ is a typical example of a vector in $H_{s,0}^{(r)}$.

Lemma 4.3.2. *Let $f^{(r)} \in P_+H^{(r)}$. Then*

1. $\mathcal{F}_0 \subset D(A(f^{(r)}))$,
2. \mathcal{F}_0 is invariant under $A(f^{(r)})$,
3. $A(f^{(r)})$ is strongly continuous in $f^{(r)}$ on \mathcal{F}_0 .

Proof. By the preceding discussion, assertions (1) and (2) are obvious for $f^{(r)} \in H_{s,0}^{(r)}$. Let $\varphi = \{\varphi^{(0)}, \varphi^{(1)}, \varphi^{(2)}, \dots\} \in \mathcal{F}_0$ and $\varphi^{(s)} = 0$ for $s > n_\varphi$. By a natural embedding $\varphi^{(s)} = \{0, \dots, 0, \varphi^{(s)}, 0, \dots\}$. Consider the expansion $\varphi^{(s)} = \sum_{\underline{i}} \varphi_{\underline{i}}^{(s)} \tilde{e}_{i_1} \otimes \tilde{e}_{i_2} \otimes \cdots \otimes \tilde{e}_{i_s}$ with $\varphi_{\underline{i}}^{(s)} \in \mathbb{C}$, totally symmetric in $\underline{i} = \{i_1, \dots, i_s\}$. Then, for $j \leq s \leq n_\varphi$,

$$\begin{aligned} B_j(f_n^{(r)}) \varphi^{(s)} &= P_+ \sum_{\underline{k} \in I_n} \sum_{\underline{i}} \lambda_{\underline{k}} \varphi_{\underline{i}}^{(s)} n(s, j, r) \delta_{k_1 i_1} \delta_{k_2 i_2} \cdots \delta_{k_j i_j} \tilde{e}_{k_{j+1}} \otimes \cdots \otimes \tilde{e}_{k_r} \otimes \tilde{e}_{i_{j+1}} \otimes \cdots \otimes \tilde{e}_{i_s} \\ &= P_+ \sum_{\underline{k} \in I_n} \sum_{\underline{i} \in I_n} \lambda_{\underline{k}} \varphi_{k_1 \dots k_{j+1} \dots i_s} n(s, j, r) \tilde{e}_{k_{j+1}} \otimes \cdots \otimes \tilde{e}_{k_r} \otimes \tilde{e}_{i_{j+1}} \otimes \cdots \otimes \tilde{e}_{i_s}, \end{aligned}$$

where $n(s, j, r) = \sqrt{s(s-1) \cdots (s-j+1)} \sqrt{(s-j+1)(s-j+2) \cdots (s-j+r-j)}$. Taking into account that P_+ is a projection operator, using the Schwarz inequality and $|n(s, j, r)|^2 \leq s^j (s+r)^{r-j} \leq (s+r)^r$, one gets

$$\|B_j(f_n^{(r)}) \varphi^{(s)}\|^2 \leq (s+r)^r \|f_n^{(r)}\|^2 \|\varphi^{(s)}\|^2. \tag{4.3.4}$$

Note that $B_j(f_n^{(r)}) \varphi^{(s)} = 0$ for $j > s$, so the inequality (4.3.4) holds also in this case. Since $s \leq n_\varphi$, $\varphi = \sum_{s=0}^{n_\varphi} \varphi^{(s)}$ and $\|\varphi^{(s)}\|^2 \leq \|\varphi\|^2$, $\|B_j(f_n^{(r)}) \varphi\| \leq (n_\varphi + 1) (n_\varphi + r)^{\frac{r}{2}} \|f_n^{(r)}\| \|\varphi\|$ by the triangle inequality. Inserting $A(f_n^{(r)})$ in the form given by equation (4.3.3) we arrive at

$$\|A(f_n^{(r)})\varphi\| \leq (\sqrt{2})^r (n_\varphi + r)^{\frac{r}{2}+1} \|f_n^{(r)}\| \|\varphi\|. \tag{4.3.5}$$

Consequently, $\|A(g^{(r)})\varphi - A(h^{(r)})\varphi\| = \|A(g^{(r)} - h^{(r)})\varphi\| \leq (\sqrt{2})^r (n_\varphi + r)^{\frac{r}{2}+1} \|g^{(r)} - h^{(r)}\| \|\varphi\|$ for all $g^{(r)}, h^{(r)} \in H_{s,0}^{(r)}$, proving assertion (3) for $f^{(r)} \in H_{s,0}^{(r)}$. Choosing $f_n^{(r)}, f_m^{(r)}$ for $g^{(r)}, h^{(r)}$ in (4.3.5), one sees that $A(f_n^{(r)})\varphi$ converges, and, since $A(f^{(r)})$ is the strong graph limit of $A(f_n^{(r)})$, $\varphi \in D(A(f^{(r)}))$ and $A(f_n^{(r)})\varphi \rightarrow A(f^{(r)})\varphi$. This proves assertion (1) and together with (4.3.5) also assertion (3). Note that for all n , $A(f_n^{(r)})$ creates at most r particles. Thus $(A(f_n^{(r)})\varphi)^{(s)} = 0$ for $s > n_\varphi + r$, which is then true also for the limit $A(f^{(r)})\varphi$, i.e. $A(f^{(r)})\varphi \in \mathcal{F}_0$. \square

We now come back to the standard form of Λ and decompose $f \in H$ as $f^{re} + f^{im}$, $f^{re} = \sum_k \alpha_k e_k$, $f^{im} = \sum_k \beta_k J e_k$, so that $\Lambda f = \sum_{l,m} \sum_k \lambda_{klm} \beta_k e_l \otimes e_m$. Note that $\|\Lambda f\|^2 = \sum_{l,m} (\sum_k \lambda_{klm} \beta_k)^2 \leq \sum_{k,l,m} \lambda_{klm}^2 \|f\|^2 < \infty$ showing $\Lambda f \in P_+ H^{(2)}$ for all $f \in H$.

In Proposition 3.1.3 above it has been shown that $\Phi_\Lambda(f) = \Phi(f) + \Phi(\Lambda f)$ is essentially self-adjoint on \mathcal{F}_0 . On \mathcal{F}_0 we have $\Phi_\Lambda(f^{re} + f^{im}) = \Phi_\Lambda(f^{re}) + \Phi_\Lambda(f^{im}) = \Phi(f^{re}) + \Phi(f^{im}) + \Psi(\Lambda f^{im})$ with $\Psi(\Lambda f^{im}) = \sum_{l,m} \sum_k \lambda_{klm} \beta_k \cdot \Phi(e_l) \Phi(e_m)$. It shall be shown that on \mathcal{F}_0 which is a core for $\Phi_\Lambda(f)$, one has

$$\begin{aligned} \Phi_\Lambda(f^{re}) &= \Phi(f^{re}) = U\Phi(f^{re})U^* \text{ and} \\ \Phi_\Lambda(f^{im}) &= \Phi(f^{im}) + \Psi(\Lambda f^{im}) = U\Phi(f^{im})U^*, \end{aligned}$$

with the unitary operator $U = e^{-\frac{i}{3}A(f^{(3)})}$, where $f^{(3)} = \sum_{k,l,m} \lambda_{klm} e_k \otimes e_l \otimes e_m$ and $A(f^{(3)}) = \sum_{k,l,m} \lambda_{klm} : \Phi(e_k) \Phi(e_l) \Phi(e_m) :$.

In $L^2(Q, d\mu)$, $A(f^{(3)})$ is multiplication by $A(x) = \sum_{k,l,m} \lambda_{klm} : x_k x_l x_m :$ and U^* is multiplication by $e^{\frac{i}{3}A(x)}$. Similarly, $\Phi(f^{re})$, is multiplication by $\Phi_{fre}(x) = \sum_k \alpha_k x_k$ with $D(\Phi(f^{re})) = \{\varphi \mid \Phi(f^{re})\varphi \in L^2(Q, d\mu)\}$. Therefore, $D(\Phi(f^{re}))$ is left invariant by U^* , and, obviously, $U\Phi(f^{re})U^*\varphi = \Phi(f^{re})\varphi$ for all $\varphi \in D(\Phi(f^{re})) \supset \mathcal{F}_0$.

Next consider $\Phi_\Lambda(Je_i) = \Phi(Je_i) + \Psi(\Lambda Je_i) = \Phi(Je_i) + \sum_{l,m} \lambda_{ilm} : \Phi(e_l) \Phi(e_m) :$, which on $S\mathcal{F}_0 \subset L^2(Q, d\mu)$ takes the form

$$\Phi_\Lambda(Je_i) = p_i + \sum_{l,m} \lambda_{ilm} : x_l x_m : \quad \text{with} \quad p_i = \frac{1}{i} \frac{\partial}{\partial x_i} + i x_i.$$

Lemma 4.3.3. $A(x) = \sum_{k,l,m} \lambda_{klm} : x_k x_l x_m :$ is differentiable by every x_i and $\frac{\partial A}{\partial x_i}(x) = 3 \sum_{l,m} \lambda_{ilm} : x_l x_m :$.

Proof. Writing $A(x)$ in the form

$$\begin{aligned} A(x) &= \lambda_{iii} : x_i^3 : + 3 \sum_{l \neq i} \lambda_{iil} : x_i^2 x_l : + 3 \sum_{l, m \neq i} \lambda_{ilm} : x_i x_l x_m : + \sum_{k, l, m \neq i} \lambda_{klm} : x_k x_l x_m : \\ &= \lambda_{iii} : x_i^3 : + 3 : x_i^2 : \sum_{l \neq i} \lambda_{iil} : x_l : + 3 : x_i : \sum_{l, m \neq i} \lambda_{ilm} : x_l x_m : + \sum_{k, l, m \neq i} \lambda_{klm} : x_k x_l x_m : \end{aligned}$$

one sees that

$$\begin{aligned} \frac{\partial A}{\partial x_i}(x) &= 3 \left(\lambda_{iii} : x_i x_i : + 2 \sum_{l \neq i} \lambda_{iil} : x_i x_l : + \sum_{l, m \neq i} \lambda_{ilm} : x_l x_m : \right) \\ &= 3 \sum_{l, m} \lambda_{ilm} : x_l x_m : . \end{aligned}$$

The convergence of the respective infinite sums is the same as discussed following (4.3.2). □

Since $\sum_{i,m} \lambda_{ilm} e_l \otimes e_m \in P_+ H^{(2)}$, we see by Lemma 4.3.2 that $U^* \mathcal{F}_0 \subset D(p_i)$, because for $\varphi \in \mathcal{F}_0$, $p_i U^* \varphi = \left(\frac{1}{i} \frac{\partial}{\partial x_i} + i x_i \right) e^{\frac{i}{3} A(x)} \varphi(x) = e^{\frac{i}{3} A(x)} \left(\frac{1}{i} \frac{\partial}{\partial x_i} + i x_i \right) + \frac{1}{3} \frac{\partial A}{\partial x_i}(x) \varphi(x)$. Hence

$$p_i U^* \varphi = \left(p_i + \sum_{l,m} \lambda_{ilm} : x_l x_m : \right) \varphi, \tag{4.3.6}$$

i.e. $U \Phi(Je_i) U^* \varphi = \Phi_\Lambda(Je_i) \varphi$ for all $\varphi \in \mathcal{F}_0$. Equation (4.3.6) also implies $U \Phi(f^{im}) U^* = \Phi_\Lambda(f^{im})$ on \mathcal{F}_0 for those f^{im} which are finite linear combinations of $\{J e_k\}_{k=1}^\infty$. Now let $f_n^{im} = \sum_{k=1}^n \beta_k J e_k \xrightarrow{n \rightarrow \infty} f^{im} \in H$. Then, with $\varphi \in \mathcal{F}_0$,

$$\begin{aligned} \Phi(f_n^{im}) U^* \varphi &= U^* \Phi_\Lambda(f_n^{im}) \varphi = U^* \left(\Phi(f_n^{im}) + \sum_{l,m,k} \lambda_{klm} \beta_k : \Phi(e_l) \Phi(e_m) : \right) \varphi \\ &= U^* \left(\Phi(f_n^{im}) \varphi + A(\Lambda f_n^{im}) \varphi \right). \end{aligned} \tag{4.3.7}$$

Since Λ is bounded, $\Lambda f_n^{im} \rightarrow \Lambda f^{im} \in P_+ H^{(2)}$ and by Lemma 4.3.2, $A(\Lambda f_n^{im}) \varphi \rightarrow A(\Lambda f^{im}) \varphi$. Taking limits on both sides of equation (4.3.7) yields, according to the remark after Lemma 4.3.1, $U \Phi(f^{im}) U^* \varphi = \Phi_\Lambda(f^{im}) \varphi$ for all $\varphi \in \mathcal{F}_0$.

We summarize the results of this section as

Proposition 4.3.4. *Let $\Phi(f) \mapsto \Phi_\Lambda(f)$ be a quadratic transformation of the CCR. If $\Lambda \in \mathcal{L}_{CCR}$ is Hilbert-Schmidt, then $\Phi_\Lambda(f)$ is unitarily equivalent to the Fock representation $\Phi(f)$.*

It is clear then that Theorem 4.1 is a consequence of Theorem 4.1.1, Corollary

4.1.2, and Propositions 4.2.2 and 4.3.4.

V. Additional Comments and Developments

In this chapter we collect a number of relevant comments on further developments and possible applications of quadratic transformations.

Heuristic Comments on General Quadratic Canonical Transformations

Combining the purely quadratic transformations we have rigorously studied above with the well-known inhomogeneous linear transformations of [1], we obtain the general (formal) quadratic transformation of $\Phi: \Phi(f) \mapsto \Phi_{\text{gen}}(f)$, where

$$\begin{aligned} \Phi_{\text{gen}}(f) \equiv & \sum_{k=1}^{\infty} (\alpha_k(f) a_k^* + \tilde{\alpha}_k(f) a_k) \\ & + \sum_{k,l=1}^{\infty} (\alpha_{kl}(f) a_k^* a_l^* + \tilde{\alpha}_{kl}(f) a_k a_l + 2\beta_{kl}(f) a_k^* a_l) + \ell(f), \end{aligned} \tag{5.1}$$

with $f \in H$, $\alpha_k, \tilde{\alpha}_k, \alpha_{kl}, \tilde{\alpha}_{kl}, \beta_{kl}, \ell$ (complex-valued) linear functionals on H and $a_k = a(\tilde{e}_k)$, $a_k^* = a(\tilde{e}_k)^*$ annihilator and creator for the k -th degree of freedom ($\{\tilde{e}_k\}$ is an orthonormal basis in \mathcal{H}). A term of the form $a_k a_l^*$ can be absorbed into the $a_k^* a_l$ - and ℓ -terms using the commutation relations. $\Phi_{\text{gen}}(f)$ should be symmetric, whence it follows that $\tilde{\alpha}_k(f) = \overline{\alpha_k(f)}$, $\tilde{\alpha}_{kl}(f) = \overline{\alpha_{kl}(f)}$, and $\beta_{kl}(f) = \overline{\beta_{lk}(f)}$, $\ell(f) \in \mathbb{R}$. Since a_k^* commutes with a_l^* , one may assume without loss of generality that $\alpha_{kl}(f) = \alpha_{lk}(f)$.

The ‘linear’ part of $\Phi_{\text{gen}}(f)$ can be written as

$$\sum_{k=1}^{\infty} (\alpha_k(f) a_k^* + \overline{\alpha_k(f)} a_k) = \Phi(Tf),$$

with $T: H \mapsto H$ defined by

$$Tf \equiv \sqrt{2} \sum_{k=1}^{\infty} \{\Re \alpha_k(f) e_k + \Im \alpha_k(f) e_{k'}\}, \tag{5.2}$$

or equivalently $\widetilde{T}f = \sqrt{2} \sum_k \alpha_k(f) \tilde{e}_k$. Defining, in addition, the operator $\Gamma: H \mapsto P_+(H \otimes H)$ by

$$\Gamma f = \sum_{K,L} \gamma_{KL}(f) e_K \otimes e_L, \quad \gamma_{KL}(f) \in \mathbb{R}, \quad \gamma_{KL}(f) = \gamma_{LK}(f),$$

with $\gamma_{kl}(f) \equiv \Re(\alpha_{kl}(f) + \beta_{kl}(f))$, $\gamma_{k'l'}(f) \equiv \Re(-\alpha_{kl}(f) + \beta_{kl}(f))$, $\gamma_{kl'}(f) \equiv \Im(\alpha_{kl}(f) - \beta_{kl}(f))$, and $\gamma_{k'l}(f) \equiv \Im(\alpha_{kl}(f) + \beta_{kl}(f))$ and setting it into the two-particle operator $\Psi(\cdot)$, we obtain

$$\begin{aligned} \Psi(I_f) &= \frac{1}{2} \sum_{K,L} \gamma_{KL}(f) \{ a(\tilde{e}_K)^* a(\tilde{e}_L)^* + a(\tilde{e}_K) a(\tilde{e}_L) + 2a(\tilde{e}_K)^* a(\tilde{e}_L) \} \\ &= \frac{1}{2} \sum_{k,l} \{ (\gamma_{kl}(f) - \gamma_{k'l'}(f) + i\gamma_{kl'}(f) + i\gamma_{k'l}(f)) a_k^* a_l^* \\ &\quad + (\gamma_{kl}(f) - \gamma_{k'l'}(f) - i\gamma_{kl'}(f) - i\gamma_{k'l}(f)) a_k a_l \\ &\quad + 2(\gamma_{kl}(f) + \gamma_{k'l'}(f) - i\gamma_{kl'}(f) + i\gamma_{k'l}(f)) a_k^* a_l \}, \end{aligned}$$

which is exactly the quadratic part of $\Phi_{\text{gen}}(f)$, since $\gamma_{kl}(f) - \gamma_{k'l'}(f) + i\gamma_{kl'}(f) + i\gamma_{k'l}(f) = 2\alpha_{kl}(f)$, and $\gamma_{kl}(f) + \gamma_{k'l'}(f) - i\gamma_{kl'}(f) + i\gamma_{k'l}(f) = 2\beta_{kl}(f)$. Hence we have

$$\Phi_{\text{gen}}(f) = \Phi(Tf) + \Psi(I_f) + \ell(f), \tag{5.3}$$

so that, at least formally, the two-particle operators $\Psi(\cdot)$ we have been working with are sufficiently general to reproduce the transformations (5.1).

As in the proofs in Section 3.2, we see that if the transformation $\Phi(f) \mapsto \Phi_{\text{gen}}(f)$ is to be canonical, then the operator T must be symplectic. In particular, one has

$$\begin{aligned} [\Phi_{\text{gen}}(f), \Phi_{\text{gen}}(g)] &= i\sigma(Tf, Tg) + 2i\Phi(b(Tf, \Gamma g) - b(Tg, \Gamma f)) \\ &\quad + 4i\Psi(P_{+B}(\Gamma f, \Gamma g)) + 2i\sigma(\Gamma f, \Gamma g). \end{aligned}$$

Thus, as before, requiring $[\Phi(h_1), [\Phi(h_2), [\Phi_{\text{gen}}(f), \Phi_{\text{gen}}(g)]]] = [\Phi(h_1), [\Phi(h_2), i\sigma(f, g)I]] = 0$ for arbitrary $h_1, h_2 \in H$, we find $P_{+B}(\Gamma f, \Gamma g) = 0$, which implies $\sigma(\Gamma f, \Gamma g) = 0$. Therefore, we conclude that $\sigma(Tf, Tg) = \sigma(f, g)$, i.e. T is symplectic.

Now, since T is symplectic and $\sigma(\cdot, \cdot)$ is nondegenerate, T is injective, and, assuming that T^{-1} is defined on all of H (in infinitely many dimensions, this is not always assured), we can write $\Phi_{\text{gen}}(T^{-1}f) = \Phi(f) + \Psi(\Gamma T^{-1}f) + \ell(T^{-1}f)$, and $[\Phi_{\text{gen}}(T^{-1}f), \Phi_{\text{gen}}(T^{-1}g)] = [\Phi(f) + \Psi(\Gamma T^{-1}f), \Phi(g) + \Psi(\Gamma T^{-1}g)]$. Since T^{-1} is also symplectic, it follows that

$$[\Phi_{\text{gen}}(T^{-1}f), \Phi_{\text{gen}}(T^{-1}g)] = i\sigma(f, g),$$

so that the transformation $\Phi(f) \mapsto \Phi(f) + \Psi(\Gamma T^{-1}f)$ must be canonical. Thus, the resulting conditions on $\Gamma T^{-1} \equiv \Lambda$ (i.e. $\Gamma = \Lambda T$) are $\Lambda \in \mathcal{L}_{CCR}$ (i.e., see Theorem 3.2.11 and Proposition 3.3.1). To summarize, we have seen (at least

formally) that necessary and sufficient conditions for the general quadratic transformation (5.1) to be canonical are that the operator T defined in (5.2) is symplectic and together with the operator Γ defined in (5.3) must satisfy $\Gamma T^{-1} \in \mathcal{L}_{CCR}$.

We compare this result with the special case $\dim \mathcal{H} = 1$ handled previously in the literature [3], where a general quadratic canonical transformation was understood as

$$\begin{aligned} q &\mapsto Q = a_1(q + yp)^2 + aq + bp + a_4, \\ p &\mapsto P = a_1x(q + yp)^2 + cq + dp + b_4 \end{aligned} \tag{5.4}$$

with $a_1, a_4, a, b, c, d, x, y \in \mathbf{R}$ satisfying the conditions

$$ad - bc = 1, \quad y = \frac{bx - d}{ax - c}.$$

In fact, this is subsumed in our class of transformations; explicitly, with H spanned by the symplectic orthonormal basis e and $Je \equiv e'$, then our $\Phi_{\text{gen}}(e)$ becomes their Q and $\Phi_{\text{gen}}(e')$ becomes P , if we set $\ell(e) = a_4, \ell(e') = b_4, Te = ae + be', Te' = ce + de'$, and $\Gamma = \Lambda T$, with $\Lambda = \sigma(l, \cdot) l \otimes l, l = \lambda e + \lambda' e', \lambda = a_1^{1/3}(ax - c)^{1/3}$, and $\lambda' = a_1^{1/3} \frac{bx - d}{(ax - c)^{2/3}}$. T is symplectic, since $\sigma(Te, Te') = ad - bc = 1$, and also $\Lambda \in \mathcal{L}_{CCR}$, since after pulling out the factor $\|\ell\|^3$, it is in standard form.

Quadratic Transformations and Automorphisms of the Weyl Algebra

It is known [13] that the coherent transformation $\Phi(f) \mapsto \Phi(f) + \lambda(f)\mathbf{1}, f \in D(\lambda)$, induces an automorphism on the C^* -algebra $\mathcal{A}(D(\lambda))$ via $W(f) \mapsto e^{i\lambda(f)}W(f)$. The same is true on the algebra $\mathcal{A}(D(T))$ for the symplectic transformation $\Phi(f) \mapsto \Phi(Tf), f \in D(T)$, via $W(f) \mapsto W(Tf)$, whenever $R(T) = D(T)$. But the canonical quadratic transformations do not induce such automorphisms, in other words, in general there exists no automorphism α on $\mathcal{A}(D(\Lambda))$, such that $\pi_F(\alpha(W(f))) = e^{i\Phi_\Lambda(f)}$ (recall that π_F denotes the Fock representation, so that $\pi_F(W(f)) = e^{i\Phi(f)}$), for all $f \in D(\Lambda)$. This is due to the fact that in general the operator $e^{i\Phi_\Lambda(f)}$ simply does not lie in the norm closure of the set $\{e^{i\Phi(f)} | f \in H\}$. This assertion is demonstrated in [18] for the simplest nontrivial example — H two-dimensional with symplectic orthonormal basis $\{e, Je\}$ and $\Lambda f = \lambda \sigma(e, f)e \otimes e, \lambda > 0$.

Transformations of Order Higher Than Quadratic

It is, of course, natural to consider higher-order transformations which go beyond the quadratic [22]. Define an r -particle operator $\Psi^{(r)}$ by

$$\Psi^{(r)}(F)\varphi \equiv \sum_{i=1}^n \alpha_i : \Phi(f_{1i}) \Phi(f_{2i}) \cdots \Phi(f_{ri}) : \varphi, \quad \varphi \in F_0 \tag{5.5}$$

for $F = \sum_{i=1}^n \alpha_i (f_{1i} \otimes f_{2i} \otimes \cdots \otimes f_{ri})$ ($r, n \in \mathbf{N}, \alpha_i \in \mathbf{R}, f_{ki} \in H$). As with the two-particle operator Ψ , one finds that the map $F \mapsto \Psi^{(r)}(F)\varphi$ is bounded when $\varphi \in \mathcal{F}_0$, so that again $\Psi^{(r)}(F)\varphi$ can be continuously extended to all of H^r . Because of the normal ordering in (5.5), $\Psi^{(r)}(F) = \Psi^{(r)}(P_+F)$ once again, and $\Psi^{(r)}(F)$ is a symmetric operator. Thus a canonical transformation of order r could be defined mathematically as $\Phi(f) \mapsto \overline{\Phi(f) + \Psi^{(r)}(\Lambda f)}$, $f \in D(\Lambda)$, where Λ is a densely defined linear operator from H to P_+H^r such that the transformation is canonical.

However, the techniques of this paper break down at two important points for $r > 2$. It turns out that the norm of the map $F \mapsto \Psi^{(r)}(F)\varphi$ with $\varphi \in \mathcal{F}_0$ involves a factor $(n_\varphi + r)^{r/2}$, which for $r > 2$ prevents the finite-particle vectors from being analytic vectors of $\Psi^{(r)}(F)$. It is therefore not evident that $\overline{\Phi(f) + \Psi^{(r)}(F)}$ is self-adjoint, nor that the exponentiation of such a transformation would result in a representation of the corresponding Weyl algebra. In fact, showing that an operator like $\Psi^{(r)}(F)$ is essentially self-adjoint for $r > 2$ is an old and difficult problem in quantum field theory. One possibility to deal with these difficult operator-theoretic questions is to go to the Q -space formalism, as in Section 4.3. But in order to do that usefully, it would be necessary to use the additional information $\Lambda \in \mathcal{L}_{CCR}^{(r)}$ to bring the operator Λ into a suitable standard form, which leads to the second sticking point.

Here the problem is that if Λ is to be brought into standard form (we write only the case $r=3$)

$$\Lambda f = \sum_{i,j,k,l} \lambda_{ijkl} \sigma(n_l, f) n_i \otimes n_j \otimes n_k,$$

with a symplectic orthonormal basis $\{n_k\}$ in H , then because of $R(\Lambda) \subseteq N \otimes N \otimes N$, $\sigma(m, n) = 0 \forall m, n \in \mathbf{N}$ it would be necessary for the entire vector

$$\sum_{p,q,s,u,v,w} \alpha_{pqst} \beta_{uvvw} \sigma_{su} e_p \otimes e_q \otimes e_v \otimes e_w$$

to vanish, and not just its symmetric component. We did not succeed in deriving

this condition from the requirement that the transformation be canonical (see [18] for further details).

It should also be pointed out that the commutator of an r - and an s -particle operator yields a sum consisting of an $(r+s-2)$ -, an $(r+s-4)$ -, an $(r+s-6)$ -particle operator, and so forth. Thus the operators $\mathbf{1}$, $\Phi(f)$ ($f \in H$), and $\Psi^{(2)}(F)$ ($F \in P_+H^2$) form a Lie algebra, whereas a Lie algebra which contains a cubic (or higher) operator also contains all higher-order operators. Thus, both algebraically and analytically, the quadratic case has a special status.

Some Applications of Canonical Quadratic Transformations

As previously indicated in the introduction, an important application of quadratic transformations will be to generate non-Fock representations in which one can more suitably model and compute quantities of physical interest. This is exemplified heuristically in [17] and especially in [16]. Such transformations should also be of use in constructing exactly soluble models and have applications in the field of special functions (see [5] and the references given there).

Next consider the following formal Hamilton operator:

$$\begin{aligned} H &= \sum_{k=1}^{\infty} \left\{ \frac{\omega_k}{2} (p_k^2 + q_k^2 - 1) + \sqrt{2} \mu_k (p_k : q_k^2 : + : q_k^2 : p_k) + 4\nu_k (: q_k^2 :)^2 \right\} \\ &= \sum_{k=1}^{\infty} \left\{ \omega_k a_k^* a_k + i \mu_k (a_k^{*3} + a_k^{*2} a_k - a_k^* a_k^2 - a_k^3) + \nu_k (a_k^{*2} + 2a_k^* a_k + a_k^2)^2 \right\} \end{aligned}$$

where $\omega_k, \mu_k, \nu_k \in \mathbb{R}$ and $\omega_k, \nu_k > 0$. The canonical quadratic transformation

$$q_k \mapsto q_k' = q_k, \quad p_k \mapsto p_k' = p_k + \lambda_k : q_k^2 : \quad (5.6)$$

yields

$$\begin{aligned} H \mapsto H' &= \sum_{k=1}^{\infty} \left\{ \omega_k a_k^* a_k + i \left(\mu_k + \frac{\omega_k \lambda_k}{2\sqrt{2}} \right) (a_k^{*3} + a_k^{*2} a_k - a_k^* a_k^2 - a_k^3) \right. \\ &\quad \left. + \left(\nu_k + \frac{\mu_k \lambda_k}{\sqrt{2}} + \frac{\omega_k}{8} \lambda_k^2 \right) (a_k^{*2} + 2a_k^* a_k + a_k^2)^2 \right\}, \end{aligned}$$

which with respect to H merely has a different coupling constant. But then with the choice $\lambda_k = -2\sqrt{2} \frac{\mu_k}{\omega_k}$, the transformed Hamiltonian becomes purely quartic:

$$H' = \sum_{k=1}^{\infty} \left\{ \omega_k a_k^* a_k + \left(\nu_k - \frac{\mu_k^2}{\omega_k} \right) (a_k^{*2} + 2a_k^* a_k + a_k^2)^2 \right\}.$$

This operator H' , for suitable choice of the energies and coupling constants ω_k , μ_k , ν_k is well-defined in Fock space and bounded from below, whereas the original Hamiltonian may not be. And if, in fact, the transformation (5.6) is not unitarily implementable, i.e. when $\sum_{k=1}^{\infty} \lambda_k^2$ diverges, but H' is well-defined on Fock space, then H is not a well-defined operator on Fock space. If, on the other hand, the quadratic transformation is unitarily implementable, then the original operator H is well-defined on Fock space and bounded from below (and the physics of the system described by H may just as well be described by H'). Thus, such quadratic transformations may be used to transfer information from one representation to another.

Finally, we mention that the Q -space formulation of quantum field theory implies that every result on the unitary equivalence (resp. nonequivalence) of a given representation of the CCR with the Fock representation yields a corresponding result of the equivalence (resp. nonequivalence) of measures on infinite-dimensional spaces. However, we shall not take more space to spell out these results here. For further applications, see Chapter 9 of [18].

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