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On Boundedness and $\| \cdot \|_p$ Continuity of Second Quantisation

By

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Abstract

In this short note, we determine precisely which operators have the property that their (full, symmetric or antisymmetric) second quantisation is an operator which is bounded or belongs to one of the various Schatten ideals; we also note that in 'the interior' of the natural domain, the second quantisation is a continuous map.

§1. Introduction

Operators A for which the second quantised operator $\Gamma(A)$ belongs to the Hilbert-Schmidt class are known to be of central importance in the construction of white noise functionals - see [**H**]. The problem discussed in this short note was brought to our attention by [**O**], where use is made of the fact (explicitly proved in [**H**]) that if $A \in \mathscr{C}_1(\mathscr{H})$ and if $\|A\|_1 < 1$, then $\|\Gamma(A)\|_1 \leq \frac{1}{(1-\|A\|_1)}$. In fact this estimate for the 'symmetric' second quantisation becomes an equality for the 'full' second quantisation. Thus, in the notation discussed below, $\Gamma_f(A)$ is of trace class if and only if $\|A\|_1 < 1$.

The two reasons for this short note are: (1) it shows that, when restricted to the symmetric (or Boson) Fock space, or to the antisymmetric (or Fermion) Fock space, the second quantisation of many more operators turn out to be of trace class; and (2) the proof of the continuity assertion established here does not seem to be entirely straightforward.

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We give a brief outline of the paper. We consider all three kinds of second quantisation $A \to \Gamma_\varepsilon(A)$, $\varepsilon \in \{f,s,a\}$, where the subscripts stand for 'full', 'symmetric' and 'antisymmetric' respectively. (See §2 for the notation.) We are primarily concerned with the 'natural domains of definition' and continuity properties of the map Γ_ε. Specifically, we identify precisely which operators A have the property that Γ_ε is bounded (see Theorem 3.3(a)) or belongs to the Schatten class $\mathscr{C}_p$, $p \in [1, \infty]$ (see Theorem 3.10(a)), and then determine the points of continuity of Γ_ε in this natural domain (see Theorem 3.3(b) for the bounded case, and Theorem 3.10(b) for the case of the Schatten classes).

§2. Notation

The symbol $\mathscr{H}$ will always denote a complex Hilbert space, and the symbols $\mathscr{L}(\mathscr{H})$ and $\mathscr{C}_\infty(\mathscr{H})$ will denote the algebras of all bounded and compact operators on $\mathscr{H}$ respectively, and the norm on $\mathscr{L}(\mathscr{H})$ will be denoted by $\| \ \|_\infty$. For $1 \leq p < \infty$, we use the symbol $\mathscr{C}_p(\mathscr{H})$ for the ideal of operators belonging to the Schatten class, and the symbol $\| \ \|_p$ to denote the Schatten p-norm.

If $A \in \mathscr{L}(\mathscr{H})$, recall that the expressions 'essential spectrum' and 'essential norm' of A, denoted by $\sigma_{ess}(A)$ and $\|A\|_{ess}$, respectively, denote the spectrum and norm of the image of A in $\mathscr{L}(\mathscr{H})/\mathscr{C}_\infty(\mathscr{H}))$ (the so-called Calkin algebra).

We shall find it convenient to work with the following obvious generalisation of the usual notion of singular values of a compact operator. (For a detailed exposition of these *generalised s-numbers* - as they are called in the context of general semifinite von Neumann algebras - see [**FK**].) If $A \in \mathscr{L}(\mathscr{H})$, we define the (clearly non-increasing) sequence $\{s_n(A)\}_{n=1}^\infty$ (of generalised singular values) by

$$s_n(A) = \sup_{\dim \mathscr{M} = n} \ \min_{x \in \mathscr{M}, \|x\|=1} \|Ax\|.$$

Using the symbol $|A|$ for the unique positive square root of A^*A, it is easy to see that there are two possibilities:

(i) $s_n(A)$ belongs to the 'essential spectrum' $\sigma_{ess}(|A|)$; in this case, $s_{n+1}(A) = s_n(A)$; and

(ii) $s_n(A) \notin \sigma_{ess}(|A|)$; in this case, $s_n(A)$ is an eigenvalue of $|A|$ of finite multiplicity, say m, and there are two cases depending on the number $l = \# \{k \leq n : s_k = s_n\}$, viz., (a) if $l < m$, then $s_{n+1}(A) = s_n(A)$; and (b) if $l = m$, then $s_{n+1}(A)$ is nothing but the norm of the restriction of $|A|$ to the spectral subspace corresponding to $[0, s_n(|A|))$.

We shall have more than one occasion to use the obvious fact that

$$\inf_n s_n(A) = \lim_n s_n(A) = \|A\|_{ess}. \tag{2.1}$$

For a positive integer n, let $\otimes^n\mathscr{H}$ denote the (Hilbert space) tensor product of n copies of $\mathscr{H}$. Let $S_n \ni \sigma \mapsto U_\sigma$ denote the natural unitary representation of the symmetric group S_n on $\otimes^n\mathscr{H}$. We shall write $\mathscr{H}^{(f)n} = \otimes^n\mathscr{H}$ to stress that this is the *full* tensor space; this is because we shall also be working with the following subspaces of $\mathscr{H}^{(f)n}$, the so-called *symmetric* and *anti-symmetric* tensor powers:

$$\mathscr{H}^{(s)n} = \{x \in \mathscr{H}^{(f)n} : U_\sigma x = x \ \forall \ \sigma \in S_n\}$$
$$\mathscr{H}^{(a)n} = \{x \in \mathscr{H}^{(f)n} : U_\sigma x = (-1)^\sigma x \ \forall \ \sigma \in S_n\}$$

where, of course, $\sigma \mapsto (-1)^\sigma$ denotes the alternating character on S_n.

In the sequel, we shall consistently use the symbol ε to stand for one of the letters f, s or a. Thus, for instance, if $\varepsilon \in \{f,s,a\}$ and if $A \in \mathscr{L}(\mathscr{H})$, we shall use the symbol $A^{(\varepsilon)n}$ to denote the restriction of $\otimes^n A$ to the invariant subspace $\mathscr{H}^{(\varepsilon)n}$ of $\otimes^n\mathscr{H}$. Similarly, we use the symbol $\Gamma_\varepsilon(\mathscr{H}) = \oplus_{n=0}^\infty \mathscr{H}^{(\varepsilon)n}$, with the convention that $\mathscr{H}^{(\varepsilon)0} = \mathbf{C} \ \forall \ \varepsilon \in \{f,s,a\}$. Finally, for any $\varepsilon \in \{f,s,a\}$ and $A \in \mathscr{L}(\mathscr{H})$, we shall define $\Gamma_\varepsilon(A)$ to be the (in general unbounded) operator $\Gamma_\varepsilon(A) = \oplus_{n=0}^\infty A^{(\varepsilon)n}$, with the natural domain given by $\{\oplus_{n=0}^\infty x_n \in \Gamma_\varepsilon(\mathscr{H}) : \sum_{n=0}^\infty \|A^{(\varepsilon)n} x_n\|^2 < \infty\}$. (We adopt the usual convention that $A^{(\varepsilon)0} = id_{\mathbf{C}}$.)

It is, of course, common knowledge that if $C \in \mathscr{L}(\mathscr{H})$ is a 'contraction' (i.e., $\|C\|_\infty \leq 1$), then so also is $\Gamma_f(C)$ (and hence also $\Gamma_\varepsilon(C)$, $\varepsilon \in \{s,a\}$). We shall also have occasion to use the obvious fact that for arbitrary $A \in \mathscr{L}(\mathscr{H})$ and contraction C as above, we have: $\Gamma_\varepsilon(CA) = \Gamma_\varepsilon(C)\Gamma_\varepsilon(A)$.

§3. The Text

We begin with a simple lemma whose obvious proof we omit.

Lemma 3.1. *Let $A \in \mathscr{L}(\mathscr{H})$, $n \geq 1$. Then*

$$\|A^{(\varepsilon)n}\|_\infty = \begin{cases} s_1(A)^n = \|A\|_\infty^n & \text{if} \quad \varepsilon = f \text{ or } \varepsilon = s \\ s_1(A) \ s_2(A) \cdots s_n(A) & \text{if} \quad \varepsilon = a. \end{cases}$$

Corollary 3.2.

$$A \in \mathscr{C}_\infty(\mathscr{H}) \Rightarrow \Gamma_a(A) \in \mathscr{C}_\infty(\Gamma_a(\mathscr{H})) .$$

Proof. Pick an integer n such that $s_{n+1}(A) < \frac{1}{2}$. It follows that $s_1(A)s_2(A)$

$\cdots s_{n+l}(A) \leq s_1(A) s_2(A) \cdots s_n(A) \left(\frac{1}{2}\right)^l$. Thus, $\|A^{(a)m}\|_\infty \to 0$. Since the compactness of A implies that of each $A^{(a)m}$, the corollary follows. □

The following theorem identifies the 'natural domain of definition', from the point of view of bounded operators, of the map Γ_ε, as well as the points in that domain where the map is continuous.

Theorem 3.3. *Let* $A \in \mathscr{L}(\mathscr{H})$.

(a) *Then* $\Gamma_\varepsilon(A)$ *is a bounded operator if and only if* $A \in \mathscr{D}_\varepsilon$, *where*

$$\mathscr{D}_\varepsilon = \begin{cases} \{T \in \mathscr{L}(\mathscr{H}) : \|T\|_\infty \leq 1\} & \text{if} \quad \varepsilon \in \{f, s\} \\ \{T = C + K_1 : C \in \mathscr{D}_f,\ K_1 \in \mathscr{C}_1(\mathscr{H})\} & \text{if} \quad \varepsilon = a. \end{cases}$$

(b) *If* $A \in \mathscr{D}_\varepsilon$, *then the map* $\Gamma_\varepsilon : \mathscr{D}_\varepsilon \to \mathscr{L}(\Gamma_\varepsilon(\mathscr{H}))$ *is continuous at* A (*with both domain and range metrised by* $\| \quad \|_\infty$) *if and only if*

$$\begin{aligned} &\|A\|_\infty < 1 \quad \text{if} \quad \varepsilon \in \{f, s\} \\ &\|A\|_{ess} < 1 \quad \text{if} \quad \varepsilon = a. \end{aligned}$$

Proof. (a) For $\varepsilon \in \{f, s\}$, this is an immediate consequence of Lemma 3.1. So suppose $\varepsilon = $ a.

To start with, if $A = U|A|$ is the polar decomposition of A, we also have $|A| = U^*A$; since U is a partial isometry (so that $\|U\| = \|U^*\| \leq 1$), it follows that A satisfies either of the conditions asserted to be equivalent if and only if $|A|$ does. Thus, we may assume without loss of generality that $A \geq 0$; also, we shall simply write s_n for $s_n(A)$.

Notice, by Lemma 3.1, that $\Gamma_a(A)$ is a bounded operator if and only if sup_n $s_1 s_2 \cdots s_n < \infty$. We consider three cases.

Case 1. $inf_n\ s_n > 1$.

In this case, the sequence $\{s_1 s_2 \cdots s_n\}_{n=1}^\infty$ clearly diverges to infinity and so $\Gamma_a(A)$ is unbounded. Also, the identity 2.1 shows that, in this case, no compact (let alone a trace-class) perturbation of A can be a contraction.

Case 2. $\exists\, n$ *such that* $s_n \leq 1$.

In this case, it is clear that $sup_m \{s_1 s_2 \cdots s_m\} \in \{s_1 s_2 \cdots s_m : 1 \leq m \leq n\}$, and so $\Gamma_a(A)$ is bounded. On the other hand, if $P = 1_{(s_n, \infty)}(A)$ denotes the indicated spectral projection of A, then $A = A(1-P) + AP$ is a decomposition of A as the sum of a contraction and a finite-rank operator.

Case 3. $inf_n\ s_n=1$ *and* $s_n>1\ \forall n$.

The hypothesis implies that $\{s_n\}$ is a decreasing sequence which strictly decreases to 1. This implies that there exists an orthonormal sequence $\{x_n\}$ such that $Ax_n=s_nx_n\ \forall n$.

Write $s_n=1+r_n$. Then the boundedness of the sequence $\{s_1\ s_2\cdots s_n\}$ (which is a strictly increasing sequence under the hypothesis of this case) is equivalent to the convergence of the infinite product $\Pi_{n=1}^{\infty}(1+r_n)$, which is equivalent to the convergence of the infinite series $\sum_{n=1}^{\infty}r_n$.

Suppose now that $\Gamma_a(A)$ is bounded; thus, $\sum_{n=1}^{\infty}r_n<\infty$. Define the operator K_1 by $K_1x=\sum_{n=1}^{\infty}r_n\langle x,x_n\rangle x_n$. It is easily seen that $A=(A-K_1)+K_1$ is a decomposition of A as the sum of a contraction and a trace-class operator.

Suppose, conversely, that $A=C+K$ is a decomposition of A as the sum of a contraction and a trace-class operator. In case 3 and with the preceding notation, if we set $P_nx=\sum_{i=1}^{n}\langle x,x_i\rangle x_i$, note that

$$\begin{aligned}\|K\|_1 &\geq |Tr(KP_n)|\\ &\geq |Tr(AP_n)|-|Tr(CP_n)|\\ &=\sum_{i=1}^{n}s_i-|Tr(CP_n)|\\ &\geq \sum_{i=1}^{n}s_i-n\\ &=\sum_{i=1}^{n}r_i\,.\end{aligned}$$

(In the penultimate step, we used the fact that P_n is a projection of rank n and C is a contraction, so that $|Tr(CP_n)|\leq\|C\|_\infty\|P_n\|_1\leq n$.) This shows that the series $(\sum_{n=1}^{\infty}r_n)$ and consequently the product $(\Pi_{n=1}^{\infty}s_n)$ converges, thereby establishing the boundedness of $\Gamma_a(A)$, and completing the proof of the part (a) of the proposition.

(b) First consider the case $\varepsilon\in\{f,s\}$. Begin by noting that if $A,B\in\mathscr{L}(\mathscr{H})$, and if $\gamma=\max\{\|A\|_\infty,\|B\|_\infty\}$, then we have the following inequality,

$$\|\otimes^nA-\otimes^nB\|_\infty\leq n\gamma^{n-1}\|A-B\|_\infty\ \forall n\,. \tag{3.2}$$

Fix $\|A\|_\infty<1$, and choose $0<\delta<1-\|A\|_\infty$. If $\|B-A\|_\infty<\delta$, then we have $\gamma<1$ (in the notation of inequality 3.2) and hence $\kappa=sup_n n\gamma^{n-1}<\infty$, and we see that $\|\Gamma_f(A)-\Gamma_f(B)\|_\infty\leq\kappa\|A-B\|_\infty<\kappa\delta$. This clearly implies that also $\|\Gamma_s(A)-\Gamma_s(B)\|_\infty<\kappa\delta$. This shows that Γ_ε is continuous at A, for $\varepsilon\in\{f,s\}$.

Suppose conversely that $\|A\|_\infty=1$. Then notice that, for any $\varepsilon\in\{f,s\}$ and $t\in$

$[0, 1)$, we have

$$\|\Gamma_\varepsilon(tA) - \Gamma_\varepsilon(A)\|_\infty = sup_n(1-t^n) = 1\ ,$$

and hence $\Gamma_\varepsilon(tA)$ does not tend to $\Gamma_\varepsilon(A)$ in $\|\ \ \|_\infty$ as t increases to 1.

Now consider the case $\varepsilon = a$. Begin by noting that the 'max-min' definition of s_n clearly implies that

$$|s_n(A) - s_n(B)| \leq \|A - B\|_\infty \ \forall A, B \in \mathcal{L}(\mathcal{H})\ . \tag{3.3}$$

Suppose now that $A \in \mathcal{D}_a$ and that $\|A\|_{ess} < 1$; and suppose η is any prescribed positive number, which we assume without loss of generality to be less than 1. The hypothesis and the identity 2.1 imply - as in the proof of Corollary 3.2 - that $lim_{n\to\infty}(s_1(A)\,s_2(A)\cdots s_n(A)) = 0$; so we can pick an integer N such that $s_1(A)\,s_2(A)\cdots s_N(A) < \frac{\eta}{3}$. This implies (since $\eta < 1$ and since the sequence $\{s_n(A)\}$ is non-increasing) that $s_n(A) < 1\ \forall n \geq N$, and consequently that

$$\|A^{(a)n}\|_\infty = s_1(A)\,s_2(A)\cdots s_n(A) < \frac{\eta}{3}\ \forall n \geq N. \tag{3.4}$$

Next, since the mapping $T \mapsto T^{(a)n}$ is norm-continuous - see the inequality 3.2, for instance - we can find $\delta_n > 0$ such that

$$\|B - A\|_\infty < \delta_n \Rightarrow \|B^{(a)n} - A^{(a)n}\|_\infty < \frac{\eta}{3}\ . \tag{3.5}$$

Set $\delta = min\left\{\delta_1, \cdots, \delta_N, \frac{\eta}{3}\right\}$, and suppose $B \in \mathcal{D}_a$ and $\|B - A\|_\infty < \delta$. Then, we have $\|B^{(a)n} - A^{(a)n}\|_\infty < \frac{\eta}{3}$ for $1 \leq n \leq N$, and in particular, by the inequality 3.4, we see that $s_1(B)\,s_2(B)\cdots s_N(B) = \|B^{(a)N}\|_\infty \leq \frac{2\eta}{3} < 1$; this implies, as before that $s_n(B) < 1$ $\forall n \geq N$, and hence that $\|B^{(a)n}\|_\infty < \frac{2\eta}{3}\ \forall n \geq N$. So, if $n \geq N$, we see that $\|B^{(a)n} - A^{(a)n}\|_\infty \leq \|B^{(a)n}\|_\infty + \|A^{(a)n}\|_\infty < \eta$. Hence, for the arbitrarily prescribed $\eta > 0$, we have exhibited a $\delta > 0$ such that $\|\Gamma_a(B) - \Gamma_a(A)\|_\infty < \eta$ whenever $B \in \mathcal{D}_a$ and $\|B - A\|_\infty < \delta$.

Conversely, suppose $A \in \mathcal{D}_a$ and $\|A\|_{ess} \geq 1$. Then, we see from the identity 2.1 that $s_n(A) \geq 1\ \forall n$; hence, for any $t \in [0,1)$, we have

$$\|\Gamma_a(tA) - \Gamma_a(A)\|_\infty = sup_n(1-t^n)\,s_1(A)\cdots s_n(A) \geq 1\ ,$$

and again $\Gamma_a(tA)$ does not tend to $\Gamma_a(A)$ in $\| \quad \|_\infty$ as t increases to 1. □

Remark 3.4. *It should be clear that $\mathcal{D}_\varepsilon$ is closed under products and adjoints and that $\Gamma_\varepsilon(AB)=\Gamma_\varepsilon(A)\Gamma_\varepsilon(B)$ and $\Gamma_\varepsilon(A^*)=\Gamma_\varepsilon(A)^*$, whenever $A, B\in\mathcal{D}_\varepsilon$.*

We now define what will be shown later (in Theorem 3.10(a)) to be the 'natural domain of definition', from the point of view of the Schatten ideals, of the map Γ_ε.

Definition 3.5. *For $1\leq p\leq\infty$, define*

$$\mathbf{D}_p^{(\varepsilon)}=\begin{cases} \{A\in\mathcal{C}_p(\mathcal{H}): \|A\|_p<1\} & \text{if } \varepsilon=f \\ \{A\in\mathcal{C}_p(\mathcal{H}): \|A\|_\infty<1\} & \text{if } \varepsilon=s \\ \mathcal{C}_p(\mathcal{H}) & \text{if } \varepsilon=a. \end{cases}$$

Lemma 3.6. *If $A\in\mathbf{D}_1^{(\varepsilon)}$, then $\Gamma_\varepsilon(A)\in\mathcal{C}_1(\Gamma_\varepsilon(\mathcal{H}))$, and*

$$\|\Gamma_\varepsilon(A)\|_1=\begin{cases} \dfrac{1}{(1-\|A\|_1)} & \text{if } \varepsilon=f \\ \prod_{n=1}^{\infty}\left(\dfrac{1}{1-s_n(A)}\right) & \text{if } \varepsilon=s \\ \prod_{n=1}^{\infty}(1+s_n(A)) & \text{if } \varepsilon=a. \end{cases}$$

Proof. If we write $\lambda_n=s_n(A)$, the definitions imply that an enumeration of the eigenvalues (except possibly for 0) of $|\Gamma_\varepsilon(A)|$, counted according to multiplicity, is given by

$$\begin{array}{ll} \{\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} : n\geq 0,\ 1\leq i_1, i_2,\cdots, i_n<\infty\} & \text{if } \varepsilon=f, \\ \{\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} : n\geq 0,\ 1\leq i_1\leq i_2\leq\cdots\leq i_n<\infty\} & \text{if } \varepsilon=s, \\ \{\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} : n\geq 0,\ 1\leq i_1<i_2<\cdots<i_n<\infty\} & \text{if } \varepsilon=a. \end{array}$$

It follows that

$$\|\Gamma_\varepsilon(A)\|_1 = Tr\Gamma_\varepsilon(|A|)$$

$$= \begin{cases} \sum_{n=0}^{\infty}\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} & \text{if } \varepsilon=f \\ \sum_{n=0}^{\infty}\sum_{i_1=1}^{\infty}\sum_{i_2=i_1}^{\infty}\cdots\sum_{i_n=i_{n-1}}^{\infty}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} & \text{if } \varepsilon=s \\ \sum_{n=0}^{\infty}\sum_{i_1=1}^{\infty}\sum_{i_2=i_1+1}^{\infty}\cdots\sum_{i_n=i_{n-1}+1}^{\infty}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} & \text{if } \varepsilon=a \end{cases}$$

$$= \begin{cases} \frac{1}{(1-\|A\|_1)} & \text{if } \varepsilon=f \\ \prod_{n=1}^{\infty}\left(\frac{1}{1-s_n(A)}\right) & \text{if } \varepsilon=s \\ \prod_{n=1}^{\infty}(1+s_n(A)) & \text{if } \varepsilon=a. \end{cases}$$

□

Remark 3.7. (1) *The same arguments show that, for any* $A\in\mathbf{D}_1^{(\varepsilon)}$, *we have*

$$Tr\Gamma_\varepsilon(A) = \begin{cases} \frac{1}{(1-Tr\,A)} & \text{if } \varepsilon=f \\ \prod_{n=1}^{\infty}\left(\frac{1}{1-\lambda_n}\right) & \text{if } \varepsilon=s \\ \prod_{n=1}^{\infty}(1+\lambda_n) & \text{if } \varepsilon=a \end{cases}$$

where $\{\lambda_n(A): n=1, 2, \cdots\}$ *is an enumeration of the non-zero eigenvalues of* A *counted according to algebraic multiplicity. Notice that in the case of symmetric Fock space (i.e.,* $\varepsilon=s$*), the right side is nothing but the Fredholm determinant of* $(1-A)^{-1}$; *since* $\Gamma_s(A)$ *may be viewed as a 'quantised' Neumann series, it might be interesting to find a more 'functorial' proof of the validity of this identity.*

(2) *If* $A\in\mathscr{C}_\infty(\mathscr{H})$, *then clearly* $A^{(a)n}\in\mathbf{C}_\infty(\mathscr{H}^{(a)n})$ $\forall\, n$ *and since the compactness of* A *implies that the sequence* $\{s_n(A)\}_{n=1}^{\infty}$, *and consequently also the sequence* $\{s_1(A)\,s_2(A)\cdots s_n(A)\}_{n=1}^{\infty}$, *converges to zero, it follows that* $\Gamma_a(A)\in\mathbf{C}_\infty(\Gamma(\mathscr{H}))$. *It is even more clear that if* $A\in\mathbf{D}_\infty^{(\varepsilon)}$, $\varepsilon\in\{f,s\}$, *then* $\Gamma_\varepsilon(A)$ *is compact.*

(3) *Notice that the last step in the proof of Lemma* 3.6 *needs the assumption that* $A\in\mathbf{D}_1^{(\varepsilon)}$. *Suppose now that* $\varepsilon=s$. *Then, even if* P *is a rank one projection, it is clear that* $\Gamma_s(P)$ *is an infinite-rank projection and hence not compact; but, for any* $t\in(0,1)$, *the lemma ensures that* $\Gamma_s(tP)$ *is of trace class; since* $\Gamma_s(P)$ *is not compact, it cannot be the case that* $\lim_{t\to1-0}\|\Gamma_s(tP)-\Gamma_s(P)\|_\infty=0$; *thus, the map* Γ_s *(resp.,* Γ_f*) is* **not** *continuous as a map from the set of (not necessarily strict) contractions on* $\mathscr{H}$

into the set of contractions on $\Gamma_s(\mathcal{H})$ *(resp.,* $\Gamma_s(\mathcal{H})$*) - with respect to the* $\| \ \|_\infty$*-metric on domain and range. On the other hand, it is true - see Theorem* 3.10(b), *for instance - that* $\|\Gamma_a(tP)-\Gamma_a(P)\|_\infty \to 0$.

Corollary 3.8. *If* $A \in \mathbf{D}_p^{(\varepsilon)}$*, where* $1 \leq p < \infty$*, then* $\Gamma_\varepsilon(A) \in \mathscr{C}_p(\Gamma_\varepsilon(\mathcal{H}))$*, and*

$$\|\Gamma_\varepsilon(A)\|_p^p = \begin{cases} \dfrac{1}{(1-\|A\|_p^p)} & \text{if } \varepsilon = f \\ \displaystyle\prod_{n=1}^{\infty}\left(\frac{1}{1-s_n(A)^p}\right) & \text{if } \varepsilon = s \\ \displaystyle\prod_{n=1}^{\infty}(1+s_n(A)^p) & \text{if } \varepsilon = a. \end{cases}$$

Proof. This is an immediate consequence of Lemma 3.6, since $\|T\|_p^p = \||T|^p\|_1$ and $|\Gamma_\varepsilon(A)|^p = \Gamma_\varepsilon(|A|^p)$. □

Lemma 3.9. *For any* $p \in [1,\infty)$ *and* $\varepsilon \in \{f,s,a\}$*, the map*

$$\mathbf{D}_p^{(\varepsilon)} \ni A \overset{\phi}{\mapsto} \|\Gamma_\varepsilon(A)\|_p$$

is a continuous map from $\mathbf{D}_p^{(\varepsilon)}$ *(equipped with the* $\| \ \|_p$*-metric) into* $\mathbf{R}$ *(with the usual topology).*

Proof. The desired assertion, at least when $\varepsilon = f$, is an immediate consequence of Corollary 3.8.

Suppose now that $\varepsilon = s$. Consider the following sets and maps:

$$X_1 = \mathbf{D}_p^{(s)},\ X_2 = \mathbf{D}_p^{(s)}{}_+ = \{T \in \mathbf{D}_p^{(s)} : T \geq 0\},\ X_1 \ni A \overset{mod}{\mapsto} |A| \in X_2$$

$$X_3 = \{(\lambda_n)_{n=1}^\infty \in \ell_p : 0 \leq \lambda_n < 1\ \forall n\},\ X_2 \ni T \overset{s}{\mapsto} (s_n(T))_{n=1}^\infty \in X_3$$

$$X_4 = \{(\lambda_n)_{n=1}^\infty \in \ell_1 : 0 \leq \lambda_n < 1\ \forall n\},\ X_3 \ni (\lambda_n) \overset{\psi}{\mapsto} (\lambda_n^p) \in X_4$$

$$X_4 \ni (\alpha_n) \overset{d}{\mapsto} \prod_{n=1}^{\infty}(1-\alpha_n) \in \mathbf{R}.$$

(The sets X_1, X_2, X_3 and X_4 are viewed as metric subspaces of the normed spaces $\mathscr{C}_p(\mathcal{H})$, $\mathscr{C}_p(\mathcal{H})$, ℓ_p and ℓ_1 respectively.)

In terms of these maps, it is clear from Corollary 3.8 that the map ϕ of this Lemma admits the factorisation

$$\phi(A) = [d \circ \psi \circ s \circ mod(A)]^{-\frac{1}{p}}.$$

On the other hand that, it is known - see [D] and [K] - that, for $1 \leq p < \infty$, the map $A \mapsto |A|$ is a continuous map from $\mathscr{C}_p(\mathscr{H})$ into itself; it is also known that - see [B] -

$$\|s(A) - s(B)\|_{\ell_p} \leq \|A - B\|_p \ \forall \text{ self-adjoint } A, B \in \mathscr{C}_p,$$

and that - see [GK] - the Fredholm determinant, and consequently our map d, is continuous. The map ψ is also easily seen to be continuous; one proof uses the following two simple facts (and the dominated convergence theorem): (a) a sequence in a metric space converges to a limit, say x, if and only if every subsequence of that sequence has a further subsequence which converges to x; and (b) if a sequence $\{f_n\}$ converges, to f, say, in L^p (of some measure space), then there is a subsequence, say $\{g_n\}$, of $\{f_n\}$, and an element, say g, in L^p, such that $|g_n| \leq g$ a.e.

Finally consider the case $\varepsilon = a$. Now consider the following spaces and maps:

$$Y_1 = \mathscr{C}_p(\mathscr{H}),\ Y_2 = \mathscr{C}_p(\mathscr{H})_+,\ Y_1 \ni A \overset{mod}{\mapsto} |A| \in Y_2$$

$$Y_3 = (\ell_p)_+,\ Y_2 \ni A \overset{s}{\mapsto} (s_n(A))_{n=1}^{\infty}$$

$$Y_4 = \ell_1,\ Y_3 \ni (s_n) \overset{\psi}{\mapsto} (s_n^p) \in Y_4$$

$$Y_4 \ni (a_n) \overset{d}{\mapsto} \prod_{n=1}^{\infty} (1 + a_n)\ .$$

In terms of these maps, we find, from Corollary 3.8 that

$$\phi(A) = (d \circ \psi \circ s \circ mod(A))^{\frac{1}{p}}$$

and deduce as before that the map ϕ is continuous. □

The next theorem is the 'Schatten-class' counterpart of Theorem 3.3; notice that in this case, the map is continuous wherever defined.

Theorem 3.10. *Let $1 \leq p \leq \infty$, $\varepsilon \in \{f,s,a\}$ and $A \in \mathscr{L}(\mathscr{H})$. Then,*

(a) *$\Gamma_\varepsilon(A) \in \mathscr{C}_p(\Gamma_\varepsilon(\mathscr{H})) \Longleftrightarrow A \in \mathbf{D}_p^{(\varepsilon)}$; and*

(b) *Γ_ε defines a continuous map from $\mathbf{D}_p^{(\varepsilon)}$ into $\mathscr{C}_p(\Gamma_\varepsilon(\mathscr{H}))$ (with respect to the $\| \ \|_p$-metric on domain and range).*

Proof. (a) If $p \in [1, \infty)$, this is an immediate consequence of Corollary 3.8 and the fact that A is a direct summand of $\Gamma_\varepsilon(A)$.

For $p=\infty$, suppose $A\in\mathscr{C}_\infty(\mathscr{H})$ and $\|A\|_\infty<1$. Then $A^{(f)n}$ is compact for each n and $\|A^{(f)n}\|_\infty\to 0$, and so $\Gamma_f(A)$, and hence also $\Gamma_s(A)$, is compact. Conversely, if $\Gamma_s(A)$ is compact (which is the case if $\Gamma_f(A)$ is compact), then A is clearly compact (since it is a direct summand of $\Gamma_s(A)$), and also $\|A\|_\infty\leq 1$ since $\|\Gamma_s(A)\|_\infty<\infty$; but if $\|A\|_\infty$ were not strictly less than 1, then 1 would be an eigenvalue of $|A|$, hence 1 would be an eigenvalue of infinite multiplicity of $\Gamma_s(|A|)$, which would contradict the assumed compactness of $\Gamma_s(A)$. This takes care of $\varepsilon\in\{f,s\}$.

The assertion, when $p=\infty$ and $\varepsilon=a$, follows from Remark 3.7(2) (and the fact that A is a direct summand of $\Gamma_\varepsilon(A)$).

(b) The case $p=\infty$ is an immediate consequence of Theorem 3.3 - since $\|K\|_{ess}=0\ \forall K\in\mathscr{C}_\infty(\mathscr{H})$. So assume $1\leq p<\infty$.

Suppose $A_n\to A$ in $\mathbb{D}_p^{(\varepsilon)}$. To start with, this is easily seen to imply that the sequence $\{\Gamma_\varepsilon(A_n)\}$ converges to $\Gamma_\varepsilon(A)$ in the weak operator topology.

On the other hand, if $\mathscr{K}$ is any Hilbert space, and if a sequence $\{T_n\}$ is in $\mathscr{C}_p(\mathscr{K})$ and $T\in\mathscr{C}_p(\mathscr{K})$ where $1\leq p<\infty$, then $\{T_n\}$ converges to T in the weak*-topology (in the Banach space sense) if and only if it converges to T in the weak operator topology, because $\mathscr{C}_p(\mathscr{K})$ coincides with the dual Banach space of $\mathscr{C}_q(\mathscr{K})$ ($p^{-1}+q^{-1}=1$, $1<q\leq\infty$) and the finite-rank operators are dense in $\mathscr{C}_q(\mathscr{K})$. It is known that if $\{T_n\}\subset\mathscr{C}_p(\mathscr{K})$ converges to an operator $T\in\mathscr{C}_p(\mathscr{K})$ in the weak*-topology and $\|T_n\|_p\to\|T\|_p$ where $1\leq p<\infty$, then $\|T_n-T\|_p\to 0$. For $p=1$, this is proved in [**LM**]; for $1<p<\infty$, it is a fact - see [**M**] - that $\mathscr{C}_p(\mathscr{K})$ is a uniformly convex Banach space, and it is known - see [**S**] - that if X is a uniformly convex Banach space, and if a sequence $\{x_n\}$ converges weakly to x in X, then a necessary and sufficient condition for $\{x_n\}$ to converge in the norm to x is that $\|x_n\|\to\|x\|$. Now an appeal to Lemma 3.9 completes the proof. □

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