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Traces, Unitary Characters and Crossed Products by $\mathbb{Z}$

By

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Abstract

We determine the character group of the infinite unitary group of a unital exact C^*-algebra in terms of K-theory and traces and obtain a description of the infinite unitary group modulo the closure of its commutator subgroup by the same means. The methods are then used to decide when the state space $SK_0(A \times_\alpha \mathbb{Z})$ of the K_0 group of a crossed product by $\mathbb{Z}$ is homeomorphic to $SK_0(A)_{\alpha_*}$ or $T(A)_\alpha$. We also consider the crossed product $A \times_\alpha G$ by a discrete countable abelian group G and give necessary and sufficient conditions for the equality $T(A \times_\alpha G) = T(A)_\alpha$ to hold.

Introduction

In connection with the investigation and classification of the unital C^*-algebras which are inductive limits of finite direct sums of homogeneous or sub-homogeneous C^*-algebras a large new class of simple C^*-algebras have been constructed in [T2], [ET], [V] and [T3]. The main feature of these C^*-algebras, when compared with most more familiar simple C^*-algebras, is that they are *not* topologically spanned by their projections, not even after they have been tensored with a UHF algebra. This property is reflected by the natural affine map $r_A : T(A) \to SK_0(A)$ from the tracial state space $T(A)$ of A to the state space $SK_0(A)$ of $K_0(A)$. Indeed, when A is a simple, exact, separable, unital C^*-algebra, r_A is injective if and only if the span of projections is dense in $A \otimes Q$, where Q can be any (infinite-dimensional) UHF algebra. This follows from [BKR]. The main purpose of this paper is to investigate this phenomenon in more detail and to find out how it can be changed by forming the crossed product corresponding to a $\mathbb{Z}$ -action. It turns out that r_A fails to be injective exactly when the connected component of the unit in the infinite unitary group $U^\infty(A)$ of A admits nontrivial real characters. Thus we are lead to a study of the characters of $U_0^\infty(A)$ and

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$U^{\infty}(A)$ and the results are described in Section 1. It turns out that the character groups can be completely described in terms of K-theory and traces, at least in the case where A is an exact C^*-algebra. In Section 2 we consider a crossed product $A\times_{\alpha}\mathbb{Z}$ of A by an automorphism α and study the relations between the compact convex sets $T(A)$, $SK_0(A)$, $T(A\times_{\alpha}\mathbb{Z})$ and $SK_0(A\times_{\alpha}\mathbb{Z})$. The natural questions become: When is $SK_0(A\times_{\alpha}\mathbb{Z})\simeq SK_0(A)_{\alpha_*}$? When is $SK_0(A\times_{\alpha}\mathbb{Z})\simeq T(A)_{\alpha}$? And when is $T(A\times_{\alpha}\mathbb{Z})\simeq T(A)_{\alpha}$? Our results give answers to these questions. The main tools we employ are those developed by de la Harpe and Skandalis in [dHS] and Pimsner in [P]. In Section 3 we use the de la Harpe-Skandalis determinant to prove that $U_0^{\infty}(A)$ modulo the closure of its commutator subgroup is homeomorphically isomorphic to the quotient $AffT(A)/\overline{\rho(K_0(A))}$, where ρ: $K_0(A)\to AffT(A)$ is the natural map. We also obtain non-stable versions of this and can therefore de-stabilize the results from Sections 1 and 2 for many C^*-algebras. In Section 4 we derive necessary and sufficient conditions for the conclusion $T(A\times_{\alpha}\mathbb{Z})\simeq T(A)_{\alpha}$ to hold. The main tools we employ are those developed by Connes in [C1] and [C2] and by Bedos in [Be]. In fact, we obtain the results for an action of a countable discrete abelian group, not only for $\mathbb{Z}$. Finally, in Section 5, we give a few applications of our results.

§ 1. Characters of the Infinite Unitary Group

Let A be a unital C^*-algebra. Let $\mathcal{T}(A)$ denote the real vector space of bounded selfadjoint trace functionals on A and let $Hom_{ob}(K_0(A),\mathbb{R})$ denote the real vector space of bounded homomorphisms $K_0(A)\to\mathbb{R}$, where 'bounded' is defined relative to the order unit $[1]$, cf. [G]. (Note that $K_0(A)$ is generally only a pre-ordered abelian group with order unit, but the definition of a bounded homomorphism in [G] makes perfect sense also in this case, and the bounded homomorphisms obtained this way are exactly the bounded homomorphisms of the greatest partially ordered quotient of $K_0(A)$, see [G], Proposition 1.1 on page 2.) When we assume that A is exact the natural linear map

$$r_A\colon \mathcal{T}(A)\to Hom_{ob}(K_0(A),\mathbb{R})$$

is surjective because every element of $Hom_{ob}(K_0(A),\mathbb{R})$ is a linear combination of states, see [G], Corollary 7.21 and [BR]. In the following we will occasionally consider r_A as a restriction map and write $\varphi|_{K_0(A)}$ in place of $r_A(\varphi)$. We first identify the kernel of r_A. As we shall see, the main ingredient for that is the de la Harpe-Skandalis determinant as introduced in [dHS].

Let $U^{\infty}(A)$ denote the infinite unitary matrices over A, i.e. $U^{\infty}(A)=\varinjlim U_n(A)$. We shall consider $U^{\infty}(A)$ as a topological group in the inductive limit topology coming from the inclusions $U^n(A)\subset U^{\infty}(A)$. Then $U_0^{\infty}(A)$ is the connected component containing the unit in $U^{\infty}(A)$. Let $Hom_c(U_0^{\infty}(A),\mathbb{R})$ denote the (continuous) real valued characters on $U_0^\infty(A)$. We can then define a map

$$s_A : Hom_c(U_0^\infty(A), \mathbb{R}) \to \mathcal{T}(A)$$

as follows. Let φ be a real character on $U_0^\infty(A)$. Then we set

$$s_A(\varphi)(a) = \varphi(e^{2\pi i a}), \quad a = a^* \in A.$$

Let $DU^\infty(A)$ and $DU_0^\infty(A)$ denote the commutator subgroup of $U^\infty(A)$ and $U_0^\infty(A)$, respectively. (It is well known and easily seen that $DU^\infty(A) = DU_0^\infty(A)$.) The formula

$$e^{ia}e^{ib}e^{-i(a+b)} = \lim_{n\to\infty} e^{ia}e^{ib}(e^{-i\frac{a}{n}}e^{-i\frac{b}{n}})^n \tag{1.1}$$

shows that $e^{ia}e^{ib} = e^{i(a+b)}$ modulo $\overline{DU_0^\infty(A)}$ so that $s_A(\varphi)(a+b) = s_A(\varphi)(a) + s_A(\varphi)(b)$. Thus $s_A(\varphi)$ is a continuous real valued homomorphism on the selfadjoint elements of A and therefore in fact a continuous linear selfadjoint map on A. For any unitary u in A we have that

$$\begin{aligned} s_A(\varphi)(uau^*) = \varphi(ue^{2\pi i a}u^*) &= \varphi\left(\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}\begin{pmatrix} e^{2\pi i a} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u^* & 0 \\ 0 & u \end{pmatrix}\right) \\ &= \varphi(e^{2\pi i a}) = s_A(\varphi)(a), \end{aligned}$$

showing that $s_A(\varphi)$ is a trace. Thus s_A defines a map from $Hom_c(U_0^\infty(A), \mathbb{R})$ to $\mathcal{T}(A)$ which is obviously linear.

Let now α be a $*$-automorphisms of A and consider the corresponding action of $\mathbb{Z}$ on A. α induces an action of $\mathbb{Z}$ on $\mathcal{T}(A)$, $Hom_c(U_0^\infty(A), \mathbb{R})$ and $Hom_{ob}(K_0(A), \mathbb{R})$ in the obvious way and we denote the invariant subspaces by $\mathcal{T}(A)_\alpha$, $Hom_c(U_0^\infty(A), \mathbb{R})_\alpha$ and $Hom_{ob}(K_0(A), \mathbb{R})_{\alpha_*}$, respectively. It is clear that s_A maps $Hom_c(U_0^\infty(A), \mathbb{R})_\alpha$ into $\mathcal{T}(A)_\alpha$ and r_A maps $\mathcal{T}(A)_\alpha$ into $Hom_{ob}(K_0(A), \mathbb{R})_{\alpha_*}$. We then get the following

Lemma 1.1. *Assume that A is exact. The sequence*

$$0 \to Hom_c(U_0^\infty(A), \mathbb{R})_\alpha \xrightarrow{s_A} \mathcal{T}(A)_\alpha \xrightarrow{r_A} Hom_{ob}(K_0(A), \mathbb{R})_{\alpha_*} \to 0$$

is exact.

Proof. The injectivity of s_A follows from the fact that $U_0^\infty(A)$ is generated by the elements of the form e^{ia} for some $a = a^* \in M_\infty(A)$. The surjectivity of r_A is proved as follows. Let $\omega \in Hom_{ab}(K_0(A), \mathbb{R})_{\alpha_*}$. By [BR] we know that $\omega = r_A(\tau)$ for some $\tau \in \mathscr{T}(A)$. Let m be an invariant mean on $\mathbb{Z}$ and define

$$\tau'(a) = \int_{\mathbb{Z}} \tau(\alpha^z(a))dm(z) , \quad a \in A.$$

Then τ' is an α-invariant trace on A such $r_A(\tau') = \omega$. That $im\ s_A \subset ker\ r_A$ follows immediately from the fact that $e^{2\pi i p} = 1$ for all $p = p^* = p^2 \in M_\infty(A)$. Conversely, if $\tau \in \mathscr{T}(A)_\alpha$ vanishes on $K_0(A)$, then the corresponding de la Harpe-Skandalis determinant, Δ_τ, defines a continuous homomorphism $U_0^\infty(A) \to \mathbb{R}$ such that $s_A(\Delta_\tau) = \tau$, see [dHS]. It is clear from the definition of Δ_τ that Δ_τ is α-invariant since τ is. □

Let $\widehat{U_0^\infty(A)}$ denote the characters of $U_0^\infty(A)$, i.e. the continuous homomorphisms $U_0^\infty(A) \to \mathbb{T}$. We shall now show the relation between $Hom_c(U_0^\infty(A), \mathbb{R})$ and $\widehat{U_0^\infty(A)}$. Define homomorphisms

$$e_A : Hom_c(U_0^\infty(A), \mathbb{R}) \to \widehat{U_0^\infty(A)}$$

and

$$\pi_A : \widehat{U_0^\infty(A)} \to Hom_{ob}(K_0(A), \mathbb{Z})$$

as follows. For $\varphi \in Hom_c(U_0^\infty(A), \mathbb{R})$ define $e_A(\varphi)(u) = e^{2\pi i \varphi(u)}$, $u \in U_0^\infty(A)$. To define π_A we identify $K_0(A) = \pi_1(U_0^\infty(A))$ by Bott periodicity. For $\varphi \in \widehat{U_0^\infty(A)}$ we then set $\pi_A(\varphi)$ to be the element of $Hom(K_0(A), \mathbb{Z}) = Hom(\pi_1(U_0^\infty(A)), \pi_1(\mathbb{T}))$, induced by $\varphi : U_0^\infty(A) \to \mathbb{T}$. To see that $\pi_A(\varphi)$ is bounded, define $\varphi_0 : A_{sa} \to \mathbb{T}$ by $\varphi_0(a) = \varphi(e^{2\pi i a})$. Then φ_0 is a continuous homomorphism and since A_{sa} is a topological vector space there is a continuous linear map $\omega_\varphi : A_{sa} \to \mathbb{R}$ such that $\varphi_0(a) = e^{2\pi i \omega_\varphi(a)}$. Since $e^{2\pi i \omega_\varphi(uau^*)} = \varphi(ue^{2\pi i a}u^*) = \varphi(e^{2\pi i a}) = e^{2\pi i \omega_\varphi(a)}$ for all $a = a^* \in A$, we see that ω_φ is a trace. For $p = p^* = p^2 \in M_\infty(A)$ we have that $e^{2\pi i t \omega_\varphi(p)} = \varphi(e^{2\pi i t p})$, $t \in [0, 1]$. It follows that $r_A(\omega_\varphi) = \pi_A(\varphi)$, so that $\pi_A(\varphi)$ is indeed bounded. Actually, we have shown that there is a homomorphism $\kappa_A : \widehat{U_0^\infty(A)} \to \mathscr{T}(A)$, given by $\kappa_A(\varphi) = \omega_\varphi$, such that $\pi_A = r_A \circ \kappa_A$.

Lemma 1.2. *Assume that A is exact. The sequence*

$$0 \to Hom_c(U_0^\infty(A), \mathbb{R}) \xrightarrow{e_A} \widehat{U_0^\infty(A)} \xrightarrow{\pi_A} Hom_{ob}(K_0(A), \mathbb{Z}) \to 0$$

is exact.

Proof. Let $\varphi \in Hom_c(U_0^\infty(A), \mathbb{R})$. If $e_A(\varphi) = 0$ it follows that $\varphi(u) \in \mathbb{Z}$ for all $u \in U_0^\infty(A)$. Since $U_0^\infty(A)$ is connected this is only possible if $\varphi = 0$, proving that

e_A is injective. Furthermore, the divisibility of $Hom_c(U_0^\infty(A), \mathbb{R})$ implies straightforwardly that $im\, e_A \subset ker\, \pi_A$. Let then $\phi \in ker\, \pi_A$. By a well known lifting criterion there is then a continuous map $\phi_0 : U_0^\infty(A) \to \mathbb{R}$ such that $e^{2\pi i \phi_0(\cdot)} = \phi(\cdot)$ and $\phi_0(1) = 0$. Then $\phi_0(uv) - \phi_0(u) - \phi_0(v) \in \mathbb{Z}$ for all $u, v \in U_0^\infty(A)$ and hence the connectedness of $U_0^\infty(A)$ implies that $\phi_0 \in Hom_c(U_0^\infty(A), \mathbb{R})$. Thus $\phi = e_A(\phi_0)$, proving exactness at $\widehat{U_0^\infty(A)}$. Finally, the surjectivity of π_A hinges on the exactness of A : For every $s \in Hom_{ob}(K_0(A), \mathbb{Z})$ there is a trace $\omega \in \mathcal{T}(A)$ with $\omega|_{K_0(A)} = s$. The de la Harpe-Skandalis determinant $\Delta_\omega : U_0^\infty(A) \to \mathbb{T}$ is then a character with $\pi_A(\Delta_\omega) = s$. □

Note that Lemma 1.2 shows that $Hom_c(U_0^\infty(A), \mathbb{R})$ is the largest divisible subgroup of $\widehat{U_0^\infty(A)}$. Since a divisible subgroup is always a direct summand in an abelian group, cf. e.g. [HR], Theorem (A.8), it follows from Lemma 1.2 that

$$\widehat{U_0^\infty(A)} \simeq Hom_c(U_0^\infty(A), \mathbb{R}) \oplus Hom_{ob}(K_0(A), \mathbb{Z}). \tag{1.2}$$

Note also that the short exact sequences of Lemma 1. 2 and Lemma 1. 1, in the case $\alpha = id_A$, fit together via κ_A to form the following commuting diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Hom_c(U_0^\infty(A), \mathbb{R}) & \xrightarrow{e_A} & \widehat{U_0^\infty(A)} & \xrightarrow{\pi_A} & Hom_{ob}(K_0(A), \mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow \| & & \downarrow \kappa_A & & \downarrow \cap & & \\ 0 & \longrightarrow & Hom_c(U_0^\infty(A), \mathbb{R}) & \xrightarrow{S_A} & \mathcal{T}(A) & \xrightarrow{r_A} & Hom_{ob}(K_0(A), \mathbb{R}) & \longrightarrow & 0. \end{array}$$

It follows that we have a short exact sequence

$$0 \to \widehat{U_0^\infty(A)} \to \mathcal{T}(A) \to Hom_{ob}(K_0(A), \mathbb{R})/Hom_{ob}(K_0(A), \mathbb{Z}) \to 0. \tag{1.3}$$

When we want to relate $Hom_c(U_0^\infty(A), \mathbb{R})$ to $Hom_c(U^\infty(A), \mathbb{R})$ and $\widehat{U_0^\infty(A)}$ to $\widehat{U^\infty(A)}$, the discrete group $K_1(A)$ comes into play. Since $K_1(A) = U^\infty(A)/U_0^\infty(A)$ there are natural maps $Hom(K_1(A), \mathbb{R}) \to Hom_c(U^\infty(A), \mathbb{R})$ and $\widehat{K_1(A)} \to \widehat{U_0^\infty(A)}$ obtained by composing a character of $K_1(A)$ with the quotient map $U^\infty(A) \to K_1(A)$ and there are natural maps $Hom_c(U^\infty(A), \mathbb{R}) \to Hom_c(U_0^\infty(A), \mathbb{R})$ and $\widehat{U^\infty(A)} \to \widehat{U_0^\infty(A)}$ obtained by restricting characters of $U^\infty(A)$ to $U_0^\infty(A)$. As a result we get the following

Lemma 1.3. *The sequence*

$$0 \to Hom(K_1(A), \mathbb{R}) \to Hom_c(U^\infty(A), \mathbb{R}) \to Hom_c(U_0^\infty(A), \mathbb{R}) \to 0$$

is exact.

Proof. The only non-trivial assertion is that $Hom_c(U^\infty(A), \mathbb{R}) \to Hom_c(U_0^\infty(A), \mathbb{R})$ is surjective. Let $\varphi \in Hom_c(U_0^\infty(A), \mathbb{R})$. Then φ factors through $U_0^\infty(A)/DU_0^\infty(A)$. Since $DU_0^\infty(A) = DU^\infty(A)$ we have that $U_0^\infty(A)/DU_0^\infty(A) \subset U^\infty(A)/DU^\infty(A)$. Because $\mathbb{R}$ is divisible, φ, regarded as a homomorphism $U_0^\infty(A)/DU_0^\infty(A) \to \mathbb{R}$, admits an extension to $U^\infty(A)/DU^\infty(A)$; i.e. $\varphi: U_0^\infty(A) \to \mathbb{R}$ admits an extension to a homomorphism $U^\infty(A) \to \mathbb{R}$, see e.g. [HR], Theorem (A.7). Since φ is continuous, and $U_0^\infty(A)$ is a neighbourhood of 1, the extension is automatically continuous. □

Lemma 1.4. *The sequence*

$$0 \to \widehat{K_1(A)} \to \widehat{U^\infty(A)} \to \widehat{U_0^\infty(A)} \to 0$$

is exact.

Proof. The proof is the same as for Lemma 1.3 and hence omitted. □

We can now summarize with the following

Theorem 1.5. *Assume that A is exact. There is then a natural exact sequence*

$$0 \to \widehat{K_1(A)} \to \widehat{U^\infty(A)} \to \mathcal{T}(A) \to Hom_{ob}(K_0(A), \mathbb{R})/Hom_{ob}(K_0(A), \mathbb{Z}) \to 0$$

and a factorization

$$\widehat{U^\infty(A)} \simeq \widehat{K_1(A)} \oplus Hom_{ob}(K_0(A), \mathbb{Z}) \oplus Hom_c(U_0^\infty(A), \mathbb{R}).$$

Proof. The exact sequence is obtained by combining (1.3) and Lemma 1.4. To get the factorization observe first that it follows from (1.1) that every element of $U_0^\infty(A)$ is of the form e^{ia} for some $a = a^* \in M_n(A)$, modulo $\overline{DU^\infty(A)}$. Hence $U_0^\infty(A)/\overline{DU^\infty(A)}$ is divisible as a subgroup of $U^\infty(A)/\overline{DU^\infty(A)}$ and therefore a direct summand, see e.g. [HR], Theorem A.8. It can easily be seen that one can choose the projection $U^\infty(A)/\overline{DU^\infty(A)} \to U_0^\infty(A)/\overline{DU^\infty(A)}$ to be continuous. Consequently the extension in Lemma 1.4 splits and hence $\widehat{U^\infty(A)} \simeq \widehat{K_1(A)} \oplus \widehat{U_0^\infty(A)}$. Combine this with (1.2). □

Note that the factorization obtained in Theorem 1.5 is not natural.

§ 2. Crossed Products by $\mathbb{Z}$

Now we consider the crossed product $A \times_\alpha \mathbb{Z}$ and ask when the state space $SK_0(A \times_\alpha \mathbb{Z})$ of the $A \times_\alpha \mathbb{Z}$ is affinely homeomorphic to $SK_0(A)_{\alpha_*}$. The natural inclusion $i : A \to A \times_\alpha \mathbb{Z}$ induces an affine map $Si_* : SK_0(A \times_\alpha \mathbb{Z}) \to SK_0(A)_{\alpha_*}$. This map is always surjective when A is exact. (Every state s of $K_0(A)$ which is α_*-invariant comes from a trace state of A which is α-invariant by Lemma 1.1 and this trace extends in a canonical way to a trace state of $A \times_\alpha \mathbb{Z}$. The corresponding element of $SK_0(A \times_\alpha \mathbb{Z})$ is mapped to s by Si_*.) Therefore the question is only under which conditions Si_* is injective.

By the Pimsner-Voiculescu exact sequence, [PV], Si_* will be injective when $Hom(K_1(A)_{\alpha_*}, \mathbb{R}) = 0$. But this condition is certainly not neccesary, reflecting the fact that there can easily be elements in $Hom(K_1(A)_{\alpha_*}, \mathbb{R})$ which do not extend to a bounded homomorphism of $K_0(A \times_\alpha \mathbb{Z})$. To answer the question we first review a result of Pimsner from [P].

For any trace $\tau \in \mathscr{T}(A \times_\alpha \mathbb{Z})$ there is a homomorphism $\underline{\Delta}^\alpha_\tau : K_1(A)_{\alpha_*} \to \mathbb{R}/\tau(K_0(A))$ such that

$$\underline{\Delta}^\alpha_\tau[u] = \Delta_\tau(u\alpha(u^*)),$$

where $u \in U^\infty(A)$ is an element with $[u] \in K_1(A)_{\alpha_*}$ and $\Delta_\tau : U_0^\infty(A) \to \mathbb{R}/\tau(K_0(A))$ is the de la Harpe-Skandalis determinant associated to τ. Let $q : \mathbb{R} \to \mathbb{R}/\tau(K_0(A))$ be the quotient map. Then

Theorem 2.1. **(Pimsner)** *The sequence*

$$0 \to \tau(K_0(A)) \to \tau(K_0(A) \times_\alpha \mathbb{Z})) \xrightarrow{q} \underline{\Delta}^\alpha_\tau(K_1(A)_{\alpha_*}) \to 0$$

is exact.

This is Theorem 3 in [P] for $n = 1$, except that we do not assume that τ is a state. The proof is the same.

When $\tau \in \mathscr{T}(A)_\alpha$ we denote the canonical extension of τ to $A \times_\alpha \mathbb{Z}$ by $\hat{\tau}$, and call it the *dual trace* of τ. We define a map $h : Hom_c(U_0^\infty(A), \mathbb{R})_\alpha \to Hom_{ob}(K_0(A \times_\alpha \mathbb{Z}), \mathbb{R})$ by $h(\phi) = r_{A \times_\alpha \mathbb{Z}}(\widehat{s_A(\phi)})$ and we let g_A denote the restriction map $Hom_c(U^\infty(A), \mathbb{R}) \to Hom_c(U_0^\infty(A), \mathbb{R})$.

Theorem 2.2. *Assume that A is exact. The sequence*

$$0 \to Hom(K_1(A), \mathbb{R})_{\alpha_*} \to Hom_c(U^\infty(A), \mathbb{R})_\alpha \xrightarrow{g_A} Hom_c(U_0^\infty(A), \mathbb{R})_\alpha \xrightarrow{h} Hom_{ob}(K_0(A \times_\alpha \mathbb{Z}), \mathbb{R}) \to Hom_{ob}(K_0(A), \mathbb{R})_{\alpha_*} \to 0$$

is exact.

Proof. Exactness at $Hom(K_1(A), \mathbb{R})_{\alpha_*}$ and $Hom_c(U^\infty(A), \mathbb{R})_\alpha$ follows from Lemma 1.3.

Exactness at $Hom_c(U_0^\infty(A), \mathbb{R})_\alpha$: Let $\varphi \in Hom_c(U^\infty(A), \mathbb{R})_\alpha$. Then $\tau = s_A \circ g_A(\varphi) \in \mathscr{T}(A)_\alpha$ and

$$\Delta_{\widehat{s_A \circ g_A(\varphi)}}(u\alpha(u^*)) = g_A(\varphi)(u\alpha(u^*)) = \varphi(u\alpha(u^*)) = 0$$

for all $u \in U^\infty(A)$ such that $[u] \in K_1(A)_{\alpha_*}$. So by Theorem 2.1 $\widehat{s_A \circ g_A}(\varphi)(K_0(A \times_\alpha \mathbb{Z})) = s_A \circ g_A(\varphi)(K_0(A))$. The last group is 0 by Lemma 1.1, so we see that $g_A(\varphi) \in \ker h$. Let then $\phi \in Hom_c(U_0^\infty(A), \mathbb{R})_\alpha$ and assume that $h\phi = 0$. This means that $\widehat{s_A(\phi)}(K_0(A \times_\alpha \mathbb{Z})) = \{0\}$. By Lemma 1.1 there is then a $\phi \in Hom_c(U_0^\infty(A \times_\alpha \mathbb{Z}), \mathbb{R})$ such that $s_{A \times_\alpha \mathbb{Z}}(\phi) = \widehat{s_A(\phi)}$. φ extends to an element of $Hom_c(U^\infty(A \times_\alpha \mathbb{Z}), \mathbb{R})$ by Lemma 1.3. (We use here, and in the following, that $A \times_\alpha \mathbb{Z}$ is exact since A is). The restriction of this element to $U^\infty(A)$, considered as a subgroup of $U^\infty(A \times_\alpha \mathbb{Z})$, gives an element $\varphi_0 \in Hom_c(U^\infty(A), \mathbb{R})_\alpha$ such that $g_A(\varphi_0) = \phi$.

Exactness at $Hom_{ob}(K_0(A \times_\alpha \mathbb{Z}), \mathbb{R})$: If $s \in im\ h$, it follows immediately from Lemma 1.1 that $s|_{K_0(A)} = 0$. Conversely assume that $s \in Hom_{ob}(K_0(A \times_\alpha \mathbb{Z}), \mathbb{R})$ and that $s|_{K_0(A)} = 0$. Choose $\tau \in \mathscr{T}(A \times_\alpha \mathbb{Z})$ such that $\tau|_{K_0(A \times_\alpha \mathbb{Z})} = s$. Note that Theorem 2.1 implies that two traces on $A \times_\alpha \mathbb{Z}$ which agree on A must restrict to the same map on $K_0(A \times_\alpha \mathbb{Z})$. Therefore we may assume that $\tau = \hat{\omega}$ for some $\omega \in \mathscr{T}(A)_\alpha$. Since s induces the zero map on $K_0(A)$ we conclude from Lemma 1.1 that $\omega = s_A(\varphi)$ for some $\varphi \in Hom_c(U_0^\infty(A), \mathbb{R})_\alpha$. Then $h(\varphi) = s$.

Exactness at $Hom_{ob}(K_0(A), \mathbb{R})_{\alpha_*}$: Let $\phi \in Hom_{ob}(K_0(A), \mathbb{R})_{\alpha_*}$. By Lemma 1.1 there is an α-invariant trace $\omega \in \mathscr{T}(A)_\alpha$ such that $\omega|_{K_0(A)} = \phi$. Then $r_{A \times_\alpha \mathbb{Z}}(\hat{\omega})|_{K_0(A)} = \phi$. $\square$

Now we consider the question of when the natural map $k(\omega) = r_{A \times_\alpha \mathbb{Z}}(\hat{\omega})$ gives a homeomorphism $T(A)_\alpha \simeq SK_0(A \times_\alpha \mathbb{Z})$. This is answered by the following

Theorem 2.3. *Assume that A is exact. The sequence*

$$0 \to Hom(K_1(A), \mathbb{R})_{\alpha_*} \to Hom_c(U^\infty(A), \mathbb{R})_\alpha \xrightarrow{s_A \circ g_A} \mathscr{T}(A)_\alpha \xrightarrow{k} Hom_{ob}(K_0(A \times_\alpha \mathbb{Z}), \mathbb{R}) \to 0$$

is exact.

Proof. Exactness at $Hom(K_1(A), \mathbb{R})_{\alpha_*}$ follows from Lemma 1.3 as before. Exactness at $Hom_c(U^\infty(A), \mathbb{R})_\alpha$ follows from Theorem 2.2 because s_A is injective.

Exactness at $\mathcal{T}(A)_\alpha$: $k \circ s_A \circ g_A = h \circ g_A = 0$ by Theorem 2.2, so $im\ s_A \circ g_A \subset ker\ k$. If $\varphi \in ker\ k$ it follows that $\varphi \in ker\ r_A$ and hence that $\varphi = s_A(\phi)$ for some $\phi \in Hom_c(U_0^\infty(A), \mathbb{R})_\alpha$ by Lemma 1.1. Then $h(\phi) = 0$ and hence $\phi \in im\ g_A$ by Theorem 2.2. Consequently $\varphi \in im\ s_A \circ g_A$. Exactness at $Hom_{ob}(K_0(A \times_\alpha \mathbb{Z}), \mathbb{R})$ follows from exactness of $A \times_\alpha \mathbb{Z}$ and Theorem 2.1. □

In the next section we show that when the natural map $\pi_1(U^n(A)) \to \pi_1(U^\infty(A)) = K_0(A)$ is surjective and the natural map $\pi_0(U^n(A)) \to \pi_0(U^\infty(A)) = K_1(A)$ an isomorphism, we may replace the infinite unitary groups, $U^\infty(A)$ and $U_0^\infty(A)$, occuring in Theorems 2.2 and 2.3 by $U^n(A)$ and $U_0^n(A)$, respectively.

§ 3. The Infinite Unitary Group Modulo the Closure of its Commutator Subgroup

Above we have used the de la Harpe-Skandalis determinant to relate traces on A to characters of $U_0^\infty(A)$. But the de la Harpe-Skandalis determinant provides direct access to the structure of $U_0^\infty(A)/\overline{DU_0^\infty(A)}$, not only to the characters of this group. Let us review the construction of de la Harpe and Skandalis in a way which allows for a non-stable version.

Let $AffT(A)$ denote the space of continuous affine realvalued functions on $T(A)$. Let $n \in \mathbb{N} \cup \{\infty\}$. For every piecewise smooth path $\eta : [0, 1] \to U_0^n(A)$ such that $\eta(0) = 1$ we define $\Delta_n(\eta) \in AffT(A)$ by

$$\Delta_n(\eta)(\omega) = \frac{1}{2\pi i}\int_0^1 \omega(\eta'(t)\eta(t)^*)dt, \quad \omega \in T(A).$$

The crucial observations are now that

(1) $\Delta_n(\eta)$ depends only on η up to homotopy (with fixed endpoints) and

(2) $\Delta_n(\eta_1\eta_2) = \Delta_n(\eta_1) + \Delta_n(\eta_2)$,

see [dHS], Lemme 3. It follows that Δ_n defines a homomorphism $\Delta_n^0 : \pi_1(U^n(A)) \to AffT(A)$. For $n = \infty$, where $\pi_1(U^\infty(A)) = K_0(A)$ by Bott periodicity, Δ_∞^0 is the canonical map $\rho : K_0(A) \to AffT(A)$, the dual of which is r_A. For each n we can now define a continuous homomorphism

$$\overline{\Delta_n} : U_0^n(A) \to AffT(A)/\overline{\Delta_n^0(\pi_1(U_0^n(A))}$$

by $\overline{\Delta_n}(u) = q(\Delta_n(\eta_u))$, $u \in U_0^n(A)$, where $q : AffT(A) \to AffT(A)/\overline{\Delta_n^0(\pi_1(U^n(A))}$ is the quotient map and η_u is any piecewise smooth path in $U_0^n(A)$ from 1 to u.

Lemma 3.1. *ker* $\overline{\Delta_n} = \overline{DU_0^n(A)}$.

Proof. Let A_0 denote the subset of A_{sa} consisting of elements of the form $x-y$, $x, y \in A_{sa}$ for which there is a sequence $\{c_i\} \subset A$ such that $x = \Sigma_i c_i c_i^*$ and $y = \Sigma_i c_i^* c_i$. By [CP] A_0 is a closed subspace of A_{sa}. Furthermore, we have the following equality for any $a \in A_{sa}$:

$$\text{(3.1)} \qquad sup\{| \omega(a) | : \omega \in T(A)\} = inf\{\| a-x \| : x \in A_0\}.$$

(This equality was only proved when $a \geq 0$ in [CP]. But, as observed by Blackadar in [B], it holds generally. The argument for this is the following: The inequality $\leq$ is trivial, so it suffices to prove the reverse. By the Hahn-Banach theorem there is an element $\varphi \in (A_{sa}/A_0)^*$ of norm 1 such that the right hand side equals $| \varphi(\pi(a)) |$, where $\pi: A_{sa} \to A_{sa}/A_0$ is the quotient map. By 2.7 of [CP] $\varphi \circ \pi$ is a norm 1 trace functional on A_{sa}, showing that the right hand side is $\leq sup\{| \omega(a) | : \omega$ is a norm 1 trace functional on $A_{sa}\}$. But every trace functional on A_{sa} of norm 1 has the form $t\omega_1 - (1-t)\omega_2$, where $t \in [0, 1]$ and $\omega_1, \omega_2 \in T(A)$. Therefore the last quantity equals the left hand side of (3.1).)

$\overline{\Delta_n}$ is a continuous homomorphism so we get the inclusion $\overline{DU_0^n(A)} \subset ker\, \overline{\Delta_n}$ for free. To prove the reversed inclusion we consider only the case $n = \infty$ since the case $n < \infty$ is similar and simpler. Let $u \in ker\, \overline{\Delta_\infty}$. Then $u \in U_0^n(A)$ for some $n \in \mathbb{N}$ and $U_0^n(A) \cap \overline{DU_0^\infty(A)}$ is closed in $U_0^n(A)$. If suffices to find $w \in U_0^n(A) \cap \overline{DU_0^\infty(A)}$ such that $\| u-w \| < 2\pi\epsilon$ for any pre-chose $\epsilon > 0$. Since $u \in ker\, \overline{\Delta_\infty}$ we can find $m \in \mathbb{N}$ and a piecewise smooth path $\eta: [0, 1] \to U_0^n(A)$ such that $\eta(0) = 1$, $\eta(1) = u$ and a piecewise smooth loop γ in $U^m(A)$ such that $\| \Delta_n(\eta) - \Delta_m(\gamma) \| < \epsilon$. Set $\eta_1 = \eta\gamma^*$. Then $\| \Delta_m(\eta_1) \| < \epsilon$. Using (3.1) we can choose $a \in A_{sa}$ such that $\| a \| < \epsilon$ and $\Delta_m(\eta_1)(\omega) = \omega(a)$, $\omega \in T(A)$. Define $\eta_2: [0, 1] \to U_0^m(A)$ by $\eta_2(t) = \eta_1(t)e^{-2\pi ita}$. Then $\| \eta_2(1) - u \| < 2\pi\epsilon$ and $\Delta_m(\eta_2) = 0$. Note that $\eta_2(1) = ue^{-2\pi ia} \in U_0^n(A)$. Thus we can conclude the proof by showing that $\eta_2(1) \in \overline{DU_0^m(A)}$. By [dHS], Lemme 3, (the proof rather than the statement), there are selfadjoint elements $a_1, a_2, \cdots, a_k \in M_m(A)_{sa}$ such that $\eta_2(1) = \Pi_{j=1}^k e^{2\pi ia_j}$ and $\omega(\Sigma_{j=1}^k a_j) = 0$, $\forall \omega \in T(M_m(A))$. By (1.1) $\eta_2(1) = e^{2\pi ix} \bmod \overline{DU_0^m(A)}$ where $x = \Sigma_{j=1}^k a_j$. We assert that $e^{2\pi ix} \in \overline{DU_0^m(A)}$. Since $\omega(x) = 0$ for all $\omega \in T(M_m(A))$ it follows from (3.1) that there is a sequence $\{x_i\} \subset M_m(A)_{sa}$ such that $x_i \to x$ and each x_i is of the form $\Sigma_{j=1}^N b_j b_j^* - b_j^* b_j$ for some $b_1, b_2, \cdots, b_N \in M_m(A)$. To see that $e^{2\pi ix} \in \overline{DU_0^m(A)}$ it therefore suffices to prove that $e^{2\pi i(bb^*-b^*b)} \in \overline{DU_0^m(A)}$ for all $b \in M_m(A)$. Let $\mu \in \mathbb{R}$, $\mu > \| b \|$. Then $w_t = tb + \mu$ is invertible in $M_m(A)$ for all $t \in [0, 1]$ and $u_t = w_t(w_t^* w_t)^{-\frac{1}{2}}$ is a path of unitaries in $U^m(A)$ connecting 1 to u_1. Let $c = b + \mu$ and observe that $bb^* - b^*b = cc^* - c^*c$ and $u_1 c^* c u_1^* = cc^*$. Consequently $e^{2\pi i(bb^*-b^*b)} = e^{2\pi i(cc^*-c^*c)} = e^{2\pi icc^*} u_1^* e^{-2\pi cc^*} u_1 = 0 \bmod \overline{DU_0^m(A)}$. □

Theorem 3.2. *$\overline{\Delta_n}$ gives a homeomorphic group isomorphism*

$$U_0^n(A)/\overline{DU_0^n(A)} \simeq AffT(A)/\overline{\Delta_n^0(\pi_1(U^n(A)))}$$

for every $n \in \mathbb{N} \cup \{\infty\}$. In particular,

$$U_0^\infty(A)/\overline{DU_0^n(A)} \simeq AffT(A)/\overline{\rho(K_0(A))}$$

Proof. Let $a = a^* \in A$ and consider the path $\gamma(t) = e^{2\pi ta}$. Then $\Delta_n(\gamma)(\omega) = \omega(a)$, $\omega \in T(A)$, so $\overline{\Delta_n}$ is clearly surjective for all $n \in \mathbb{N}$. From Lemma 3.1 we get immediately that $\overline{\Delta_n}$ induces a continuous group isomorphism $U_0^n(A)/\overline{DU_0^n(A)} \simeq AffT(A)/\overline{\Delta_n^0(\pi_1(U^n(A)))}$. It therefore suffices to prove that the inverse is also continuous. To see this it suffices to observe that the inverse can be described as follows. For $a \in A_{sa}$ set $\Phi(a) = p(e^{2\pi ia})$ where $p : U_0^n(A) \to U_0^n(A/\overline{DU_0^n(A)}$ is the quotient map. If the map $T(A) \ni \omega \to \omega(a)$ is in $\overline{\Delta_n^0(\pi_1(U^n(A)))}$ we know that $e^{2\pi a} \in \overline{DU_0^n(A)}$ by Lemma 3.1 since $\overline{\Delta_n}(e^{2\pi ia}) = 0$ in this case. This shows that Φ is well-defined as a map $\Phi : AffT(A)/\overline{\Delta_n^0(\pi_1(U^n(A)))} \to U_0^n(A)/\overline{DU_0^n(A)}$. Φ is clearly the inverse of the map induced by $\overline{\Delta_n}$ and its continuity is readily established, also in the case $n = \infty$. □

Corollary 3.3.

$$U^\infty(A)/\overline{DU^\infty(A)} \simeq K_1(A) \oplus AffT(A)/\overline{\rho(K_0(A))}.$$

Proof. As already used in the proof of Theorem 1.5, $U_0^\infty(A)/\overline{DU^\infty(A)}$ is a divisible subgroup of $U^\infty(A)/\overline{DU^\infty(A)}$ and hence a direct summand. □

Note that the inverse Φ of $\overline{\Delta_n}$ constructed in the proof of Theorem 3.2 takes values in $U_0^1(A)$ mod $\overline{DU_0^n(A)}$. Therefore we see that $U_0^n(A)$ is generated, as a group, by $U_0^1(A)$ and $\overline{DU_0^n(A)}$. There must be more direct ways to see this.

The main virtue of Theorem 3.2 in connection with the results of the preceding sections is that it in many cases allows us to replace U_0^∞ in the statements with U_0^n for some finite $n \in \mathbb{N}$. This is because of the following corollary.

Corollary 3.4. *Assume that the natural map $\pi_1(U^n(A)) \to \pi_1(U^\infty(A)) = K_0(A)$ is surjective. Then*

$$U_0^k(A)/\overline{DU_0^k(A)} \simeq U_0^n(A)/\overline{DU_0^n(A)}$$

for all $k \geq n$, *and hence* $Hom_c(U_0^\infty(A), \mathbb{R}) = Hom_c(U_0^n(A), \mathbb{R})$ *and* $\widehat{U_0^\infty(A)} = \widehat{U_0^n(A)}$. *If in addition the natural map* $\pi_0(U^n(A)) \to \pi_0(U^\infty(A)) = K_1(A)$ *is an isomorphism,*

$$\widehat{U^\infty(A)} = \widehat{U^n(A)}.$$

Proof. The first assertion follows immediately from Theorem 3.2. The second follows from the first by comparing the extension in Lemma 1.4 with its non-stable analogue

$$0 \to \widehat{\pi_0(U^n(A))} \to \widehat{U^n(A)} \to \widehat{U^n(A)} \to 0. \ \square$$

Note that the assumptions in Corollary 3.4 are known to hold with $n=1$ for a large class of C^*-algebras, see e.g. [R], [T1]. In fact, it seems to be an open question if this is always the case when A is simple and infinite dimensional.

§ 4. Restricting Traces on $A \times_\alpha G$ to A

Let G be a discrete abelian group and $\alpha: G \to Aut(A)$ an action of G on A. In this section we give a general criterion to decide when the map $R: T(A \times_\alpha G) \to T(A)_\alpha$, obtained by restricting traces on $A \times_\alpha G$ to A, is a homeomorphism. In relation to the subject of the other sections the main interest here lies in the case where $G = \mathbb{Z}$, but the more general case is not more complicated.

It is well known that R is surjective in general and that it can easily fail to be injective. We first prove that R always preserves the extreme boundaries.

Lemma 4.1. *Let* B *be a unital* C^**-algebra and* $A \subset B$ *a unital* C^**-subalgebra. Assume that* B *is generated as a* C^**-algebra by* A *and a family* $\mathscr{U}$ *of unitary* A*-normalizers. Let* $T(A)_i$ *denote the trace states of* A *that are invariant under the action of* Adu *for all* $u \in \mathscr{U}$. *Then the map* $T(B) \to T(A)_i$ *obtained by restricting traces to* A *maps* $\partial_e T(B)$ *into* $\partial_e(T(A)_i)$.

Proof. Let $\omega \in \partial_e T(B)$ and assume that $\omega|_A = t\mu_1 + (1-t)\mu_2$ for some $\mu_1, \mu_2 \in T(A)_i$ and some $t \in]0, 1[$. We must show that $\mu_1 = \mu_2 = \omega|_A$. Let $(\pi_\omega, \mathscr{H}_\omega, \psi_\omega)$ be the GNS-representation of B corresponding to ω. Then ψ_ω is a cyclic trace vector for $\pi_\omega(B)''$ and hence also separating. By the Radon-Nikodym theorem for traces on von Neumann algebras there are unique central elements $z_1, z_2 \in \pi_\omega(A)''$ such that

$$\mu_i(\cdot) = \langle \psi_\omega, z_i \pi_\omega(\cdot) \psi_\omega \rangle, \ \ i = 1, 2.$$

By uniqueness of z_i and invariance of μ_i we see that $\pi_\omega(u) z_i \pi_\omega(u^*) = z_i$, i$=$ 1, 2. Thus z_1 and z_2 are central in $\pi_\omega(B)''$ which is a factor because ω is extremal in $T(B)$. Consequently z_1 and z_2 are scalars and $\mu_1 = \mu_2 = \omega\,|_A$. □

We shall also need the following lemma due to Erik Bedos. In the proof of Proposition 11 of [Be] he established the following

Lemma 4.2. (**Bedos**) *Let $\mathcal{N}$ be a von Neumann algebra and $\beta : G \to Aut(\mathcal{N})$ an action of the discrete abelian group G on $\mathcal{N}$ which admits an invariant normal trace state. If $\mathcal{N} \times_\beta G$ is a factor, the action β must be outer.*

When ω is a state of A we denote the corresponding GNS-representation of A by $(\pi_\omega, \mathscr{H}_\omega, \psi_\omega)$. When ω is α-invariant there is an action $\alpha_\omega : G \to Aut(\pi_\omega(A)'')$ extending α, in the sense that $\alpha_\omega^g(\pi_\omega(a)) = \pi_\omega(\alpha_g(a))$, $a \in A$. The Connes spectrum of α_ω will be denoted by $\Gamma(\alpha_\omega)$, cf. [C1].

Theorem 4.3. *Let (A, G, α) be a C^*-dynamical system with G a countable discrete abelian group and A unital and separable. Then the following four conditions are equivalent*:

(1) *$R : T(A \times_\alpha G) \to T(A)_\alpha$ is a homeomorphism.*
(2) *$\Gamma(\alpha_\omega) = \hat{G}$ for all $\omega \in \partial_e(T(A)_\alpha)$.*
(3) *$\pi_\omega(A)'' \times_{\alpha_\omega} G$ is a factor for all $\omega \in \partial_e(T(A)_\alpha)$.*
(4) *α_ω is properly outer for all $\omega \in \partial_e(T(A)_\alpha)$.*

Proof. Note that α_ω must act centrally ergodically when $\omega \in \partial_e(T(A)_\alpha)$. Therefore (2) and (3) are equivalent by a result of Connes and Takesaki, Corollary 3.4 in [CT]. We prove that (1) ⇒ (3), (3) ⇒ (4) and (4) ⇒ (1). Represent A covariantly in the standard way such that $A \times_\alpha G$ is generated by A and a unitary representation u of G implementing α. It is wellknown that R injective if and only if

$$\mu(au_g) = 0, \;\; a \in A, \;\; g \in G\backslash\{0\}$$

for all $\mu \in T(A \times_\alpha G)$.

(1) ⇒ (3): Let $\omega \in \partial_e(T(A)_\alpha)$. Take $\mu \in T(A \times_\alpha G)$ such that $\mu\,|_A = \omega$. Then μ must be extremal in $T(A \times_\alpha G)$ and hence $\pi_\mu(A \times_\alpha G)''$ is a factor. It is standard to prove that because $< \psi_\mu, \pi_\mu(au_g)\psi_\mu > = 0$, $a \in A$, $g \in G\backslash\{0\}$, we have that $\pi_\mu(A \times_\alpha G)''$ is isomorphic to $\pi_\omega(A)'' \times_{\alpha_\omega} G$ which is therefore a factor.

(3) ⇒ (4): Let $\omega \in \partial_e(T(A)_\alpha)$. Assume there is a non-zero element $h \in G$ such that α_ω^h is not properly outer. There is then a non-zero projection $e \in \pi_\omega(A)''$ such that $\alpha_\omega^h(e) = e$ and such that $\alpha_\omega^h\,|_{e\pi_\omega(A)''e}$ is inner. By a lemma of Borchers, see [Bo], Lemma 5.7, or [GKP], Lemma 8.9.1, it follows that α_ω^h is inner on $c(e)\pi_\omega(A)''$ where $c(e)$ is the central support of e in $\pi_\omega(A)''$. Then $\alpha_\omega^g(c(e))\pi_\omega(A)''$ reduces α_ω^h and the action of α_ω^h on $\alpha_\omega^g(c(e))\pi_\omega(A)''$ is conjugate, via α_ω^g, to its action on

$c(e)\pi_\omega(A)''$ for all $g \in G$. Set

$$p = \bigvee_{g \in G} \alpha_\omega^g(c(e)).$$

It follows easily that α_ω^h is inner on $p\pi_\omega(A)''$. But p is a central α_ω-invariant non-zero projection, and hence p must be 1 since $\omega \in \partial_e(T(A)_\alpha)$. Consequently α_ω^h is inner and hence $\pi_\omega(A)'' \times_{\alpha_\omega} G$ is not a factor by Lemma 4.2.

(4) $\Rightarrow$ (1): It suffices to fix a unitary $w \in A$, a non-zero element $h \in G$, an $\epsilon > 0$ and a trace state $\mu \in \partial_e(T(A \times_\alpha G))$ and prove that $|\mu(wu_h)| \leq \epsilon$. Let $\omega = \mu|_A$ and define $\beta_1 \in Aut(\pi_\omega(A)'')$ by $\beta_1 = Ad\pi_\omega(w) \circ \alpha_\omega^h$ and $\beta \in Aut(\pi_\mu(A)'')$ by $\beta = Ad\pi_\mu(wu_h)$. $\omega \in \partial_e(T(A)_\alpha)$ by Lemma 4.1 and α_ω^h is therefore properly outer by assumption, and hence so is β_1. But β_1 is clearly conjugate to β so we have that β is properly outer on $\pi_\mu(A)''$. Let $\epsilon > 0$ and let f_i, $i \in I$, be a maximal family of orthogonal non-zero projections in $\pi_\mu(A)''$ such that

$$\| f_i \pi_\mu(wu_h) f_i \| \leq \epsilon \quad \forall i \in I.$$

Then $\sum_i f_i = 1$, because if not we can consider $e = 1 - \sum_i f_i \neq 0$. By [C2], 1.2.1, there is then a non-zero projection $f \leq e$ in $\pi_\mu(A)''$ such that $\| f\beta(f) \| \leq \epsilon$ contradicting the maximality. With ϕ_μ the cyclic trace vector for π_μ we can now calculate

$$|\mu(wu_h)| = |\langle \phi_\mu, \pi_\mu(wu_h)\phi_\mu \rangle| = |\sum_i \langle \phi_\mu, f_i \pi_\mu(wu_h)\phi_\mu \rangle| =$$

$$|\sum_i \langle \phi_\mu, f_i \pi_\mu(wu_h) f_i \phi_\mu \rangle| = |\langle \phi_\mu, \sum_i f_i \pi_\mu(wu_h) f_i \phi_\mu \rangle| \leq \epsilon$$

because $\| \sum_i f_i \pi_\mu(wu_h) f_i \| \leq \epsilon$. □

Remark 4.4. Consider the case where A is abelian; i.e. the case when $A = C(X)$ for some compact metrizable Hausdorff space X and $\alpha_g(f) = f \circ \varphi_g$, $f \in C(X)$, for some group φ_g, $g \in G$, of homeomorphisms on X. It follows from [To], Proposition 3.3.9, that the map $R: T(C(X) \times_\alpha G) \to T(C(X))_\alpha$ is a homeomorphism if and only $\{x \in X : \varphi_g(x) = x\} = \emptyset$ for all $g \in G \backslash \{0\}$. This can also be deduced from Theorem 4.3 as follows: Assume first that R is a homeomorphism and take $h \in G \backslash \{0\}$. If $F = \{x \in X : \varphi_h(x) = x\}$ was non-empty we could choose an ergodic Borel probability measure for the action of φ_g, $g \in G$, on F. This measure would then define an ergodic measure μ for the action of φ_g, $g \in G$, on all of X such that α_ω^h would be the trivial automorphism of $\pi_\omega(C(X))''$ when ω denotes the trace on $C(X)$ obtained by integration with respect to μ. Hence $\pi_\omega(C(X))'' \times_{\alpha_\omega} G$ can not be a factor by Lemma 4.2. Since μ is ergodic $\omega \in \partial_e T(C(X))_\alpha)$, so this is a contradiction by Theorem 4.3, (1) $\Rightarrow$ (3). The converse implication holds more generally; we prove in Theorem 4.5 below that R is a homeomorphism when each α_g, $g \in G \backslash \{0\}$, acts without fixed points in $\partial_e T(A)$, also when neither A nor G is abelian. However, this condition is no longer necessary in the non-abelian case as one can see by considering the crossed product $B\times_{\alpha}\mathbb{Z}$ where B is a UHF-algebra and α is a product type action which is outer in the trace representation.

Theorem 4.5. *Let (A, G, α) be a C^*-dynamical system with G an arbitrary discrete group and A unital and separable. Assume that α acts freely on $\partial_e T(A)$, i.e. that $\tau\circ\alpha^g\neq\tau$ for all $\tau\in\partial_e(T(A))$ and for all $g\in G\backslash\{1\}$. It follows that $R: T(A\times_{\alpha,r}G)\rightarrow T(A)_{\alpha}$ is a homeomorphism.*

Proof. We adopt the notation from the proof of Theorem 4.3. To show that

$$\mu(au_g)=0,\ \ \mu\in T(A\times_{\alpha,r}G),\ \ g\in G\backslash\{1\},\ \ a\in A,$$

it suffices to pick a unitary $w\in A$, an $\epsilon>0$, a $g\neq 1$ in G and show that $|\mu(wu_g)|\leq\epsilon$ for all trace states μ on the C^*-algebra B generated by A and u_g. In fact, we may assume that $\mu\in\partial_e T(B)$. Let π_{μ} be the GNS-representaion of B corresponding to μ and set $\beta=Ad\pi_{\mu}(wu_g)\in Aut(\pi_{\mu}(A)'')$. Assume β is not properly outer. To get a contradiction we consider $\omega=\mu|_A$. Let $\beta_1=Ad\pi_{\omega}(w)\circ\alpha_{\omega}^g$. Since β is conjugate to β_1 we have that β_1, and hence also α_{ω}^g, is not properly outer. Thus there is a non-zero α_{ω}^g-invariant projection $e\in\pi_{\omega}(A)''$ such that α_{ω}^g is inner on $e\pi_{\omega}(A)''e$. Since the central support $c(e)$ of e is α_{ω}^g-invariant, we must have that $c(e)=1$ since $\omega\in\partial_e(T(A)_{\alpha^g})$ by Lemma 4.1. The lemma of Borchers, [Bo], Lemma 5.7, or [GKP], Lemma 8.9.1, shows that α_{ω}^g is inner. Thus β is also inner and therefore $\pi_{\mu}(A)''$ must be a factor since $\pi_{\mu}(B)''$ is. Since $\pi_{\mu}(A)''\simeq\pi_{\omega}(A)''$ it follows that the $\pi_{\omega}(A)''$ is also a factor, i.e. $\omega\in\partial_e T(A)$. Since $\omega\circ\alpha_g=\omega$ this conclusion contradicts our assumption. Hence β is properly outer and the inequality $|\mu(wu_g)|\leq\epsilon$ follows as in the proof of Theorem 4.3, (4) $\Rightarrow$ (1). □

§ 5. Applications

It is clear that the results of Sections 2 and 4 answer the questions raised in the introduction, at least in principle. Let us give two general conclusions that are easily obtained from them.

Proposition 5.1. *Let A be a unital exact C^*-algebra and α an automorphism of A. Assume that $r_A: T(A)\rightarrow SK_0(A)$ is a homeomorphism. Then*

$$SK_0(A\times_{\alpha}\mathbb{Z})\simeq SK_0(A)_{\alpha_*}\simeq T(A)_{\alpha}.$$

Proof. We have that $Hom_c(U_0^\infty(A), \mathbb{R}) = 0$ by Lemma 1.1. Consequently the map h in Theorem 2.2 and the map $s_A \circ g_A$ in Theorem 2.3 are both zero. □

This proposition applies when A is exact and is the closed linear span of its projections. But also to any exact C^*-algebras with a unique trace state.

Proposition 5.2. *Let α be an approximately inner automorphism of the exact unital C^*-algebra A. Then*

$$SK_0(A \times_\alpha \mathbb{Z}) \simeq SK_0(A)_{\alpha_*} = SK_0(A).$$

Proof. α acts trivially on $K_1(A)$, $U^\infty(A)/\overline{DU^\infty(A)}$ and $U_0^\infty(A)/\overline{DU_0^\infty(A)}$ since it is approximately inner and hence any character of $K_1(A)$, $U^\infty(A)$ or $U_0^\infty(A)$ is automatically α-invariant. Therefore it follows from Lemma 1.3 that the map h of Theorem 2.2 is zero. □

Thus it often occurs that $SK_0(A \times_\alpha \mathbb{Z}) \simeq SK_0(A)_{\alpha_*}$, so it is natural to ask what such a conclusion means for the position of $K_0(A)$ in $K_0(A \times_\alpha \mathbb{Z})$ under the map i_* induced by the inclusion $i: A \to A \times_\alpha \mathbb{Z}$.

Proposition 5.3. *Assume that $SK_0(A \times_\alpha \mathbb{Z}) \neq \emptyset$. The map $i_* : K_0(A) \to K_0(A \times_\alpha \mathbb{Z})$ induces an affine homeomorphism $SK_0(A \times_\alpha \mathbb{Z}) \simeq SK_0(A)_{\alpha_*}$ if and only if the following condition holds:*
When $y - x \geq [1]$ in $K_0(A \times_\alpha \mathbb{Z})$ there is an $a \in K_0(A)$ and integers $n, m \in N$ such that

$$nx \leq mi_*(a) \leq ny.$$

Proof. Let $\varphi: K_0(A \times_\alpha \mathbb{Z}) \to AffSK_0(A \times_\alpha \mathbb{Z})$ be the state space representation of $K_0(A \times_\alpha \mathbb{Z})$. i_* gives a homeomorphism $SK_0(A \times_\alpha \mathbb{Z}) \simeq SK_0(A)_{\alpha_*}$ if and only if $\varphi \circ i_*(K_0(A))$ has dense span in $AffSK_0(A \times_\alpha \mathbb{Z})$. Assume first that this is the case and let $x, y \in K_0(A \times_\alpha \mathbb{Z})$ such that $y - x \geq [1]$. Then

$$\{f \in AffSK_0(A \times_\alpha \mathbb{Z}) : \varphi(x)(s) < f(s) < \varphi(y)(s),\ s \in SK_0(A \times_\alpha \mathbb{Z})\}$$

is an open non-empty subset of $AffSK_0(A \times_\alpha \mathbb{Z})$ and since $\mathbb{Q}\varphi \circ i_*(K_0(A))$ is dense in $SK_0(A \times_\alpha \mathbb{Z})$, there is an element $a \in K_0(A)$ and $m, k \in \mathbb{N}$ such that $k\varphi(x)(s) < m\varphi \circ i_*(a)(s) < k\varphi(y)(s)$ for all $s \in SK_0(A \times_\alpha \mathbb{Z})$. By [G], Theorem 7.8, this implies that there is a $b \in \mathbb{N}$ such that

$$bkx \leq bmi_*(a) \leq bky.$$

In the opposite direction, if the condition of the statement is satisfied, consider an element $x \in K_0(A \times_\alpha \mathbb{Z})$. For any $k \in \mathbb{N}$, the elements kx and $kx+[1]$ are in a position where the condition applies and hence there are elements $a \in K_0(A)$ and $n, m \in \mathbb{N}$ such that

$$nkx \leq mi_*(a) \leq n(kx+[1]).$$

By applying φ and dividing through by nk we see that there is a rational multiple of $\varphi \circ i_*(a)$ whose norm distance to $\varphi(x)$ is $\leq \frac{1}{k}$. Since the span of $\varphi(K_0(A \times_\alpha \mathbb{Z}))$ is dense in $AffSK_0(A \times_\alpha \mathbb{Z})$, it follows that the same is true for the span of $\varphi \circ i_*(K_0(A))$. □

In the exceptional case where $SK_0(A \times_\alpha \mathbb{Z})$ is empty the same must be true for $SK_0(A)_{\alpha_*}$, at least when A is exact, and every element of $K_0(A \times_\alpha \mathbb{Z})$ is both positive and negative, so there is no problem to consider. It should be noted that in case $K_0(A \times_\alpha \mathbb{Z})$ is a simple pre-ordered abelian group (i.e. every non-zero positive element is an order unit), a property which is automatic when $A \times_\alpha \mathbb{Z}$ is simple, then the condition of Propositon 5.3 is equivalent to the following:

For every $x, y \in K_0(A \times_\alpha \mathbb{Z})$, $x \neq y$, such that $x \leq y$, there are elements $a \in K_0(A)$ and $n, m \in \mathbb{N}$ such that

$$nx \leq mi_*(a) \leq ny.$$

We conclude with an application of Theorem 4.3 and Theorem 4.5. In [Be] E. Bedos studied the question of when a crossed product is simple and has a unique trace state. With the aid of Theorem 4.3 above it is easy to give the following answer to this question in the case of a crossed product by an abelian group:

Theorem 5.4. *Let A be a unital separable C^*-algebra and $\alpha: G \to Aut(A)$ an action of the countable discrete abelian group G on A. Then $A \times_\alpha G$ is simple and has exactly one trace state if and only if A is α-simple, $T(A)_\alpha$ contains exactly one element ω and for this ω we have that the following equivalent conditions hold*:

(1) $\Gamma(\alpha_\omega) = \hat{G}$.

(2) $\pi_\omega(A)'' \times_{\alpha_\omega} G$ *is a factor.*

(3) α_ω *is properly outer.*

Proof. Assume first that $A \times_\alpha G$ is simple with a unique trace state. It is then obvious that A must be α-simple and $T(A)_\alpha$ contain exactly one element, ω. Furthermore, the three conditions on α_ω are satisfied by Theorem 4.3. In the reverse direction observe that the unique element in $T(A)_\alpha$ must be faithful by

α-simplicity of A. Thus the Connes spectrum of α must also be full on the C^*-algebra level, see [GKP], 8.8.9. Then it follows from a result of Olesen and Pedersen that $A \times_\alpha G$ is simple, see [OP] or [GKP], 8.11.12. And $A \times_\alpha G$ has only one trace state by Theorem 4.3. □

One of the themes in [Be] is the construction of examples of simple C^*-algebras with a unique trace state. It is clear that the results obtained in this paper provide new methods for such constructions. With the next example we propose a construction which uses the classification of simple inductive limits of interval algebras, [T2] and [E].

Example 5.5. Let X be compact metric space and φ a uniquely ergodic homeomorphism of X without periodic points. By [T2] there is a simple unital C^*-algebra A which is the inductive limit of algebras of the form $C[0, 1] \otimes M_n$ such that $\partial_e(T(A)) = X$ and $K_0(A)$ is any pre-chosen dense subgroup of $\mathbb{Q}$. By [E] there is an automorphism α of A which induces the given homeomorphism φ on $\partial_e(T(A)) = X$. By combining Theorem 4.5, Theorem 4.3 and Theorem 5.4 we see that the crossed product $A \times_\alpha \mathbb{Z}$ is a simple C^*-algebra with a unique trace state.

Note that we get a simple C^*-algebra even in cases where the crossed product $C(X) \times_\alpha \mathbb{Z}$ is not simple. (Consider, for example, a Denjoy homeomorphism of the circle.) Presently it seems a safe bet that the C^*-algebras we obtain by this method are inductive limits of algebras of the form $C(\mathbb{T}) \otimes M_n$.

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