Publ. RIMS, Kyoto Univ.
32 (1996), 85-116

CQ*-Algebras: Structure Properties

By

Fabio BAGARELLO* and Camillo TRAPANI**

Abstract

Some structure properties of CQ*-algebras are investigated. The usual multiplication of a quasi *-algebra is generalized by introducing a weak- and strong product. The *-semisemplicity is defined via a suitable family of positive sesquilinear forms and some consequences of this notion are derived. The basic elements of a functional calculus on these partial algebraic structures are discussed.

§ 1. Introduction

Quasi *-algebras were introduced by Lassner [1, 2] with the purpose of providing a reasonable mathematical environment where properly dealing with the thermodynamical limit of certain quantum statistical problems which did not fit into the set-up of the algebraic formulation of quantum theories developed by Haag and Kastler [3].

In [4] we begun a systematic analysis of a special class of quasi *-algebras, the so called CQ*-algebras, taking particular care for those mathematical aspects which are more relevant for applications.

A CQ*-algebra is, roughly speaking, a complete normed quasi *-algebra $\mathscr{A}$ containing two dense C*-algebras $R\mathscr{A}$ and $L\mathscr{A}$ (each one with respect to its own norm and its own involution) mapped one into the other by the involution of $\mathscr{A}$. Typical examples of this structure are provided by the family of bounded operators in a scale of Hilbert spaces, in the non-commutative case [4], and by L^p-spaces on (locally) compact Hausdorff measure spaces [5], in the commutative case.

A CQ*-algebra can also be viewed as a *partial *-algebra* [6]-[10] whose lattice of multipliers consists only of four elements $\{\mathscr{A}, R\mathscr{A}, L\mathscr{A}, \mathscr{A}_0\}$, where $\mathscr{A}_0 = R\mathscr{A} \cap L\mathscr{A}$. The first point of interest for us is to investigate the possibility of refining the lattice of multipliers. This is done by introducing two different notions of multipli-

Communicated by H. Araki, March 14, 1995. Revised July 21, 1995.
1991 Mathematics Subject Classification: 46H25, 46L05
* Dipartimento di Matematica e Applicazioni, Facoltà d' Ingegneria, Università di Palermo, I-90128 Palermo, Italy
** Istituto di Fisica, Università di Palermo, I-90123 Palermo, Italy

cation. The *strong* multiplication is defined invoking the familiar notion of *closable* linear map whereas the *weak* one is obtained via an appropriately defined family of sesquilinear forms. In both cases they mimick the notion of strong and weak multiplication discussed in [6], [7] and [9] for closable operators. All this is done in view of extending to CQ*-algebras the well-known functional calculus for C*-algebras: the first thing we need for this purpose is in fact to have at hand the largest possible set of invertible elements.

As a matter of fact, both the strong and the weak multiplication are well-behaved in the case the CQ*-algebra under consideration is **-semisimple*. We will introduce this concept by generalizing one of the possible equivalent characterizations of the *-semisemplicity for C*-algebras, the one in terms of the Gel'fand seminorms.

-Semisimple CQ-algebras share with C*-algebras a lot of topological properties, described by several different norms which is possible to introduce on them. These norms coincide for C*-algebras and we show that this fact is indeed characteristic of C*-algebras.

§ 2. Preliminaries and Examples

Throughout the paper we will extensively use the notion of partial *-algebra [6, 9].

A partial *-algebra is a vector space $\mathscr{A}$ with involution $A \to A^*$ [i.e. $(A+\lambda B)^* = A^*+\bar{\lambda}B^*$; $A = A^{**}$] and a subset $\Gamma \subset \mathscr{A} \times \mathscr{A}$ such that (i) $(A, B) \in \Gamma$ implies $(B^*, A^*) \in \Gamma$; (ii) (A, B) and $(A, C) \in \Gamma$ imply $(A, B+\lambda C) \in \Gamma$; and (iii) if $(A, B) \in \Gamma$, then there exists an element $AB \in \mathscr{A}$ and for this multiplication the distributive property holds in the following sense: if $(A, B) \in \Gamma$ and $(A, C) \in \Gamma$ then

$$AB+AC = A(B+C)$$

Furthermore $(AB)^* = B^*A^*$.

The product is not required to be associative.

The partial *-algebra $\mathscr{A}$ is said to have a unit if there exists an element $\mathbb{I}$ (necessarily unique) such that $\mathbb{I}^* = \mathbb{I}$, $(\mathbb{I}, A) \in \Gamma$, $\mathbb{I}A = A\mathbb{I} = A$, $\forall A \in \mathscr{A}$.

If $(A, B) \in \Gamma$ then we say that A is a left multiplier of B [and write $A \in L(B)$] or B is a right multiplier of A [$B \in R(A)$]. For $\mathscr{S} \subset \mathscr{A}$ we put $L\mathscr{S} = \bigcap_{A \in \mathscr{S}} L(A)$; the set $R\mathscr{S}$ is defined in analogous way. The set $M\mathscr{S} = L\mathscr{S} \cap R\mathscr{S}$ is called the set of universal multipliers of $\mathscr{S}$.

Following Lassner [1], [2], we call quasi *-algebras a special family of partial *-algebras, namely, those for which the set $M\mathscr{A}$ of universal multipliers is a *-algebra. We give the complete definition for reader's convenience.

Let $\mathscr{A}$ be a linear space and $\mathscr{A}_0$ a *-algebra contained in $\mathscr{A}$. We say that $\mathscr{A}$ is a quasi *-algebra with distinguished *-algebra $\mathscr{A}_0$ (or, simply, over $\mathscr{A}_0$) if (i) the right and left multiplications of an element of $\mathscr{A}$ and an element of $\mathscr{A}_0$ are always defined and linear; and (ii) an involution * (which extends the involution of $\mathscr{A}_0$) is defined in $\mathscr{A}$ with the property $(AB)^* = B^*A^*$ whenever the multiplication is defined.

A quasi *-algebra $(\mathscr{A}, \mathscr{A}_0)$ is said to have a unit $\mathbb{I}$ if there exists an element $\mathbb{I} \in \mathscr{A}_0$ such that $A\mathbb{I} = \mathbb{I}A = A, \ \forall A \in \mathscr{A}$.

A quasi *-algebra $(\mathscr{A}, \mathscr{A}_0)$ is said to be a topological quasi *-algebra if in $\mathscr{A}$ is defined a locally convex topology ξ such that (a) the involution is continuous and the multiplications are separately continuous; and (b) $\mathscr{A}_0$ is dense in $\mathscr{A}[\xi]$.

Following [11], if $(\mathscr{A}[\xi], \mathscr{A}_0)$ is a topological quasi *-algebra, by ξ_0 we will denote the weakest locally convex topology on $\mathscr{A}_0$ such that for every bounded set $\mathscr{M} \in \mathscr{A}[\xi]$ the family of maps $B \to AB, B \to BA$; $A \in \mathscr{M}$ from $\mathscr{A}_0\ [\xi_0]$ into $\mathscr{A}\ [\xi]$ is equicontinuous.
In this case $\mathscr{A}_0\ [\xi_0]$ is a locally convex *-algebra. The topology ξ_0 will be called the *reduced topology* of ξ.

In [4] we considered a special class of quasi *-algebras, called CQ*-algebras, which arise as completions of C *-algebras. Let us begin with a purely algebraic definition.

Definition 2.1. *A rigged quasi *-algebra $\mathscr{A}$ is a partial *-algebra for which there exist two vector subspaces $\mathscr{A}_\flat$ and $\mathscr{A}_\sharp$ such that*

(i) $(\mathscr{A}_\flat)^* = \mathscr{A}_\sharp$

(ii) $\Gamma = \{(A, B) \in \mathscr{A} \times \mathscr{A} : A \in \mathscr{A}_\sharp$ *or* $B \in \mathscr{A}_\flat\}$

(iii) *both* $\mathscr{A}_\flat$ *and* $\mathscr{A}_\sharp$ *are algebras with respect to the partial multiplication* $(A, B) \in \Gamma \to AB \in \mathscr{A}$ *defined in* $\mathscr{A}$

The multiplication $(A, B) \in \Gamma \to AB \in \mathscr{A}$ *is supposed to be (weakly) semi-associative; i.e.* $(AB)C = A(BC)\ \forall A \in \mathscr{A}$ *and* $\forall B, C \in \mathscr{A}_\flat$,

Definition 2.2. *A rigged quasi *-algebra* $\{\mathscr{A}, *, \mathscr{A}_\flat, \flat\}$ *is called a CQ*-algebra if*

(i) $\mathscr{A}$ *is a Banach space under the norm and* $\| A^* \| = \| A \|\ \forall A \in \mathscr{A}$

(ii) $\mathscr{A}_\flat$ *is a C *-algebra with respect to the norm* $\| \ \|_\flat$ *and to the involution* $\flat$

(iii) $\mathscr{A}_\sharp$ *carries the norm* $\| \ \|_\sharp$, *defined by* $\| A \|_\sharp = \| A^* \|_\flat$ *(thus the involution * is an isometric anti-isomorphisms of* $\mathscr{A}_\flat$ *onto* $\mathscr{A}_\sharp$*) and* $A^{\sharp *} = A^{*\flat}\ \forall A \in \mathscr{A}_\sharp$

(iv) $\| B \|_\flat = \sup_{\| A \| \leq 1} \| AB \|$

(v) $\mathscr{A}_0 = \mathscr{A}_\flat \cap \mathscr{A}_\sharp$ *is* $\| \ \|_\flat$*-dense in* $\mathscr{A}_\flat$ *and* $\mathscr{A}_\flat$ *is* $\| \ \|$*-dense in* $\mathscr{A}$

Throughout the paper we will always assume that $\mathscr{A}$ has a unit $\mathbb{I} \in \mathscr{A}_0$. Moreover, by (ii) of Definition 2.1, $\mathscr{A}_\flat$ (resp., $\mathscr{A}_\sharp$) concides with the set $R\mathscr{A}$ (resp., $L\mathscr{A}$) of the right (resp., left) multipliers of $\mathscr{A}$. For this reason, we will often write $R\mathscr{A}$ instead of $\mathscr{A}_\flat$, etc.

Example 2.3. Operators on scales of Hilbert spaces.—Let $\mathscr{H}$ be a Hilbert space with scalar product $\langle .,. \rangle$ and λ $(.,.)$ a positive sesquilinear closed form defined on a dense domain $\mathscr{D}_\lambda \subset \mathscr{H}$. Then $\mathscr{D}_\lambda$ becomes a Hilbert space, that we denote by $\mathscr{H}_\lambda$, with respect to the scalar product

$$\langle f, g\rangle_\lambda = \langle f, g\rangle + \lambda(f, g) \tag{1}$$

Let $\mathscr{H}_{\bar\lambda}$ be the Hilbert space of conjugate linear forms on $\mathscr{H}_\lambda$.

This is the canonical way to get a scale of Hilbert spaces ([14], VIII.6)

$$\mathscr{H}_\lambda \xrightarrow{i} \mathscr{H} \xrightarrow{j} \mathscr{H}_{\bar\lambda} \tag{2}$$

where i and j are continuous embeddings with dense range. In fact, the identity map i embeds $\mathscr{H}_\lambda$ in $\mathscr{H}$ and the map $j: \psi \in \mathscr{H} \to j(\psi) \in \mathscr{H}_{\bar\lambda}$, where $j(\psi)(\phi) = \langle \phi, \psi \rangle$, $\forall \phi \in \mathscr{H}_\lambda$ is a linear imbedding of $\mathscr{H}$ into $\mathscr{H}_{\bar\lambda}$. Indentifying $\mathscr{H}_\lambda$ and $\mathscr{H}$ with their respective images in $\mathscr{H}_{\bar\lambda}$ we can read (2) as a chain of topological inclusions

$$\mathscr{H}_\lambda \subset \mathscr{H} \subset \mathscr{H}_{\bar\lambda}$$

The representation theorem for sesquilinear forms implies ([15], Ch. VI, Sect. 2) the existence of a selfadjoint positive operator H such that $D((1+H)^{1/2}) = \mathscr{D}_\lambda = \mathscr{H}_\lambda \subseteq \mathscr{H}$ and

$$\langle f,g\rangle_\lambda = \langle (1+H)^{1/2} f, (1+H)^{1/2} g\rangle \qquad \forall f,g \in \mathscr{D}_\lambda \tag{3}$$

The operator $R = (1+H)^{1/2}$ has a bounded inverse R^{-1} which maps $\mathscr{H}$ into $\mathscr{H}_\lambda$. As a result, we can write:

$$\langle f, g\rangle_\lambda = \langle Rf, Rg\rangle = \langle Uf, Ug\rangle_\lambda \qquad \forall f, g \in \mathscr{H}_\lambda$$

Here U is the operator from $\mathscr{H}_\lambda$ onto $\mathscr{H}_{\bar\lambda}$ whose existence is ensured by the Riesz lemma.

Let $\mathscr{B}(\mathscr{H}_\lambda, \mathscr{H}_{\bar\lambda})$ be the Banach space of bounded operators from $\mathscr{H}_\lambda$ into $\mathscr{H}_{\bar\lambda}$ and let us denote with $\| A \|_{\lambda\bar\lambda}$ the natural norm of $A \in \mathscr{B}(\mathscr{H}_\lambda, \mathscr{H}_{\bar\lambda})$.

In $\mathscr{B}(\mathscr{H}_\lambda, \mathscr{H}_{\bar\lambda})$ we can introduce an involution in the following way: to each element $A \in \mathscr{B}(\mathscr{H}_\lambda, \mathscr{H}_{\bar\lambda})$ we associate the linear map A^* from $\mathscr{H}_\lambda$ into $\mathscr{H}_{\bar\lambda}$ defined by the equation

$$\langle A^* f, g\rangle = \overline{\langle Ag, f\rangle} \qquad \forall f, g \in \mathscr{H}_\lambda$$

As can be easily proved $A^* \in \mathscr{B}(\mathscr{H}_\lambda, \mathscr{H}_{\bar\lambda})$ and $\| A^* \|_{\lambda\bar\lambda} = \| A \|_{\lambda\bar\lambda}$ $\forall A \in \mathscr{B}(\mathscr{H}_\lambda, \mathscr{H}_{\bar\lambda})$.

Let $\mathscr{B}(\mathscr{H}_\lambda)$ denotes the C^*-algebra of bounded operators on $\mathscr{H}_\lambda$ (the usual involution of $\mathscr{B}(\mathscr{H}_\lambda)$ will be denoted here as $\flat$) and $\mathscr{B}(\mathscr{H}_{\bar\lambda})$ the C^*-algebra of bounded operators on $\mathscr{H}_{\bar\lambda}$ (the natural involution of $\mathscr{B}(\mathscr{H}_{\bar\lambda})$ is denoted as $\sharp$). Then, both $\mathscr{B}(\mathscr{H}_\lambda)$ and $\mathscr{B}(\mathscr{H}_{\bar\lambda})$ are contained in $\mathscr{B}(\mathscr{H}_\lambda, \mathscr{H}_{\bar\lambda})$ and $A \in \mathscr{B}(\mathscr{H}_\lambda)$ if, and only if, $A^* \in \mathscr{B}(\mathscr{H}_{\bar\lambda})$. Moreover $B^{\flat *} = B^{*\sharp}$ $\forall B \in \mathscr{B}(\mathscr{H}_\lambda)$.

Defining the algebraic operations in the natural way, it is quite easy to show that $(\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda}), *, \mathcal{B}(\mathcal{H}_\lambda), \flat)$ is a rigged quasi *-algebra. The distinguished *-algebra of $\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$ is

$$\mathcal{B}^+(\mathcal{H}_\lambda) = \{A \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda}) \ : \ A, A^* \in \mathcal{B}(\mathcal{H}_\lambda)\}$$

Actually, $(\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda}), *, \mathcal{B}(\mathcal{H}_\lambda), \flat)$ is a *CQ* *-algebra if $\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$ and $\mathcal{B}(\mathcal{H}_\lambda)$ carry their natural norms. In fact $\mathcal{B}^+(\mathcal{H}_\lambda)$ is dense in $\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$ and the other requirements of Definition 2.2 are also fulfilled. For the details see [4, Example 3.3]

Example 2.4. Hilbert algebras — Let $\mathcal{A}_0$ be an achieved left Hilbert algebra with identity e and involution $\sharp$ and let $\mathcal{H}$ the Hilbert space obtained by completing $\mathcal{A}_0$ with respect to its own scalar product. Then, as is known [16, Ch.10], the commutant $\mathcal{A}_0'$ of $\mathcal{A}_0$ is an achieved right Hilbert algebra in $\mathcal{H}$ with (the same) identity and involution $\flat$. The involution in $\mathcal{H}$ is defined by the modular conjugation operator J. For shortness we put $\mathcal{H}_\flat = \mathcal{A}_0'$ and $\mathcal{H}_\sharp = \mathcal{A}_0$. It is easy to check that $(\mathcal{H}, J, \mathcal{H}_\flat, \flat)$ is a rigged quasi *-algebra in the sense of Definition 2.1.

As for the norms, one defines for $\eta \in \mathcal{H}_\flat$,

$$\| \eta \|_\flat \equiv \| \pi_0'(\eta) \| = \sup_{\|\xi\| \le 1} \| \xi \eta \|$$

where $\pi_0'(\eta)$ denotes the regular *-representation of $\mathcal{A}_0'$ in $\mathcal{B}(\mathcal{H})$. We also define $\| \xi \|_\sharp = \| J\xi \|_\flat, \ \ \forall \xi \in \mathcal{H}_\sharp$.

Is $(\mathcal{H}, J, \mathcal{H}_\flat, \flat)$ a CQ*-algebra? First of all, we observe that conditions (i) and (iv) of Definition 2.2 are obviously fulfilled, whereas condition (iii) follows from the known equality $(J\xi)^\flat = J\xi^\sharp, \ \ \forall \xi \in \mathcal{H}_\sharp$. As for (ii), the C*-property for the norm $\| \ \|_\flat$ is easily obtained from the fact that π_0' is a *-representation of $\mathcal{H}_\flat$ into $\mathcal{B}(\mathcal{H})$. To show the completeness of $\mathcal{H}_\flat = \mathcal{A}_0'$ one has to take into account the equality:

$$\mathcal{A}_0' = \{\eta \in D(S^*) : \pi_0'(\eta) \text{ is bounded}\}$$

where S is, as usual, the closure of the operator S_0 defined on the dense domain $\mathcal{A}_0^2$ by $\eta \in \mathcal{A}_0^2 \mapsto \eta^\sharp \in \mathcal{H}$.

Now, if $\{\eta_k\}$ is a $\| \ \|_\flat$-Cauchy sequence in $\mathcal{H}_\flat$, since $e \in \mathcal{A}_0'$, one can find an element $\eta \in \mathcal{H}$ such that η_k converges to η with respect to the Hilbert norm; moreover since, as is known, for each $\eta \in \mathcal{A}_0'$, $\eta^\flat = S^*\eta$, the sequence $\{S^*\eta_k\}$ is also convergent. Therefore $\eta \in D(S^*)$. The fact that $\pi_0'(\eta)$ is bounded follows easily from the norm completeness of $\mathcal{B}(\mathcal{H})$.

To conclude that $(\mathcal{H}, J, \mathcal{H}_\flat, \flat)$ is a CQ*-algebra, we should prove the density of $\mathcal{H}_\flat \cap \mathcal{H}_\sharp$ in $\mathcal{H}_\flat$ with respect to $\| \ \|_\flat$. We do not have a definite result in this direction; however in [16, Sect. 10.19] it is shown that the set

$$\{f_r(\Delta) f_s(\Delta^{-1})\eta : \eta \in \mathcal{H}_\flat, \, r, s > 0\},$$

where $f_m(x) = \exp(-mx)$ and Δ is the modular operator, is contained in $\mathcal{H}_\flat \cap \mathcal{H}_\sharp$. This set is, in a sense, quite rich; indeed, a simple application of the spectral theorem for the operator Δ and of the Lebesgue dominated convergence theorem shows that $f_r(\Delta)f_s(\Delta^{-1})\eta$ converges to η with respect to the Hilbert norm, for each $\eta \in \mathscr{H}_\flat$. We leave a deeper analysis of these points to a further paper.

Extesions of the notion of (left) Hilbert algebra in the framework of partial *-algebras have been studied by Inoue in [17].

The general structure of CQ*-algebras is simplified a lot for the so-called proper CQ*-algebras.

Definition 2.5. *A CQ*-algebra* $\{\mathscr{A}, *, R\mathscr{A}, \flat\}$ *is called proper if* $R\mathscr{A} = L\mathscr{A}$ *and if* $A^\flat = A^\sharp$, $\forall A \in R\mathscr{A}$

In [4] it is proved that from the above definition it follows that

(i) $\| A \|_\sharp = \| A \|_\flat \quad \forall A \in R\mathscr{A}$;

(ii) all the abelian CQ*-algebras (i.e. $R\mathscr{A} = L\mathscr{A}$ and $AB = BA\ \forall A \in \mathscr{A}, B \in R\mathscr{A}$) are proper.

In [4] we have also proved the following constructive Proposition:

Proposition 2.6. *Let* $\mathscr{C}$ *be a C *-algebra with norm* $\| \ \|_1$ *and involution* *. *Let* $\| \ \|$ *be another norm on* $\mathscr{C}$, *weaker than* $\| \ \|_1$ *and such that*

(i) $\| A \| = \| A^* \| \quad \forall A \in \mathscr{C}$;

(ii) $\| AB \| \leq \| A \| \, \| B \|_1 \quad \forall A, B \in \mathscr{C}$.

Then the completion $\tilde{\mathscr{C}}$ *of* $\mathscr{C}$, *with its natural norm, is a proper CQ*-algebra over* $\mathscr{C}$, *with* $* = \flat$.

We will now give some examples of proper CQ*-algebras.

Example 2.7. L_ρ-spaces.—

Let μ be a measure in a non-empty point set X. Let M^+ be the collection of all the μ-measurable functions on X. We assume that to each $f \in M^+$ it corresponds a number $\rho(f) \in [0, \infty]$ such that:

i) $\rho(f) = 0$ iff $f = 0$ a.e. in X;

ii) $\rho(f_1 + f_2) \leq \rho(f_1) + \rho(f_2)$;

iii) $\rho(af) = a\rho(f)\ \forall a \in R_+$;

iv) let $f_n \in M^+$ and $f_n \uparrow f$ a.e. in X. Then $\rho(f_n) \uparrow \rho(f)$.

Following [13] we call ρ a *function norm*. Let us define

$$L_\rho \equiv \{f \in M^+ : \rho(f) < \infty\}.$$

With this definition it has been proved in [13] that the space L_ρ is a Banach space, that is it is complete, with respect to the norm $\| f \| \equiv \rho(| f |)$.

Some L_ρ spaces generate examples of abelian CQ*-algebras.

(A) Let (X, μ) be a measure space with μ a regular Borel measure on the compact Hausdorff space X. As usual, we denote by $L^p(X, d\mu)$ the Banach space of all (equivalence classes of) measurable functions $f: X \to \mathbb{C}$ such that

$$\| f \|_p \equiv \left(\int_X | f |^p \, d\mu \right)^{1/p} < \infty.$$

On $L^p(X)$ we consider the natural involution $f \in L^p(X) \mapsto f^* \in L^p(X)$ with $f^*(X) = \overline{f(x)}$. Clearly L^p is an L_ρ space (with $\| \, \|_p \equiv \| \, \|$).
We denote with $C(X)$ the C *-algebra of continuous functions defined on X. The pair $(L^p(X, \mu), C(X))$ provides the basic commutative example of topological quasi *-algebra.
It turns also out that $(L^p(X, \mu), C(X))$ is a proper abelian CQ*-algebra, for any $p \geq 1$, since the p-norm satisfies all the conditions of Proposition 2.6. These spaces have been analyzed with a certain care in [5].

(B) Let X be a compact Hausdorff space and $M = \{\mu_\alpha, \alpha \in I\}$ a family of Borel measures on X, for which there exists a constant $C > 0$ such that $\mu_\alpha(X) \leq C \;\; \forall \alpha \in I$. Let $\| \, \|_{p, \alpha}$ be the norm on $L^p(X, \mu_\alpha)$. Of course each norm is related to a particular function norm $\rho_{p, \alpha}(f)$. Let us define, for $\phi \in C(X)$

$$\| \phi \|_{p, I} \equiv \sup_{\alpha \in I} \| \phi \|_{p, \alpha}.$$

In [13] it is shown that the map $\rho_{p, I}$ related to this norm still satisfies all the requirements of a function norm so that the completion of $C(X)$ with respect to $\| \, \|_{p, I}$, $L^p_I(X, M)$, is a Banach space. Furthermore $L^p_I(X, M)$ is contained in the intersection of all the $L^p(X, \mu_\alpha)$ spaces.
Moreover, in the hypothesis above, it is easy to prove that $\| \, \|_{p, I}$ also satisfies the conditions of the Proposition 2.6. Therefore $(L^p_I(X, M), C(X))$ is an abelian proper CQ*-algebra.

(C) Let X, M and $\rho_{p, \alpha}$ be as above. For a sequence $\{a_n\}$ of positive constants, we define

$$\rho_p(f) \equiv \sum_{n \in I} a_n \rho(f)_{p, n}.$$

Then the space $L_p(X, M)$ (the completion of $C(X)$ with respect to the norm $\| \, \|_p$ generated by ρ_p) is a Banach space which, if the sequence $\{a_n\}$ is summable, contains the space $L^p_I(X, M)$ of the previous example. Again, $(L_p(X, M), C(X))$ is an abelian proper CQ*-algebra.

Example 2.8. Non-commutative L^p-spaces.—Let $\mathscr{A}_0$ be a Hilbert algebra with unit e, π_0 the left regular representation of $\mathscr{A}_0$ in its norm-completion $\mathscr{H}$ and $\mathfrak{U}(\mathscr{A}_0)$ the left von Neumann algebra of $\mathscr{A}_0$. Let us denote by τ_0 the natural trace on $\mathfrak{U}(\mathscr{A}_0)^+$ (the positive cone of $\mathfrak{U}(\mathscr{A}_0)$). If T is a *measurable* operator in Segal's sense [18] and $T \geq 0$ one defines (we refer in the following to [19, Sect. 3] for definitions and theorems)

$$\mu(T) = \sup\{\tau_0\overline{(\pi_0(\xi))};\ 0 \le \pi_0(\xi) \le T,\ \xi \in (\mathscr{A}_0)_b^2\}$$

where $(\mathscr{A}_0)_b^2 = \{x \in \mathscr{H}: \overline{\pi_0(x)} \in \mathscr{B}(\mathscr{H})\}$. Let $L^p(\tau_0)$, $1 \le p < \infty$ be the space of all measurable operators T such that $\mu(|T|^p) < \infty$. Then, $L^p(\tau_0)$ is a Banach space with respect to the norm $\| T \|_p = \mu(| T |^p)^{1/p}$. $L^\infty(\tau_0)$ is identified with $\mathfrak{U}(\mathscr{A}_0)$ with its own norm. Since,

$$\| T \|_p = \| T^* \|_p, \quad \forall T \in L^p(\tau_0)$$

and

$$\| TS \|_p \le \| T \|_p \| S \|_\infty, \quad \forall T \in L^p(\tau_0),\ S \in \mathfrak{U}(\mathscr{A}_0)$$

and $\mathfrak{U}(\mathscr{A}_0)$ is dense in $L^p(\tau_0)$, applying Proposition 2.6, we get that $L^p(\tau_0)$ is a (non-abelian) proper CQ*-algebra over $\mathfrak{U}(\mathscr{A}_0)$.

Example 2.9. Let $\mathscr{A}_0$ be a C *-algebra (with unit $\mathbb{I}$) with respect to the norm $\| \|_0$ and the involution *. Let Φ be a linear map of $\mathscr{A}_0$ into itself with $\Phi(A^*) = \Phi(A)^*$, $\forall A \in \mathscr{A}_0$. Suppose that the following inequality is fulfilled, for all $A, B \in \mathscr{A}_0$

$$\| \Phi(AB) \|_0 \le \| \Phi(A) \|_0 \| B \|_0 \tag{4}$$

Let us assume that $\| \Phi(\mathbb{I}) \|_0 = 1$ and define a new norm on $\mathscr{A}_0$ by

$$\| A \| \equiv \| \Phi(A) \|_0.$$

It is easy to verify that this norm satisfies the condition of Proposition 2.6. Therefore, the $\| \|$-completion $\mathscr{A}$ of $\mathscr{A}_0$ is a proper CQ*-algebra over $\mathscr{A}_0$.
Of course, the inequality (4) automatically holds if Φ is a *-homomorphism, [12]. However in this case the two norms coincide, as always when $\| \|$ is a Banach algebra norm on $\mathscr{A}_0$.

§ 3. The Weak- and Strong-Multiplication

In this Section we will focus our attention on the problem of refining the multiplication in a CQ*-algebra (in the sense of obtaining a richer lattice of multipliers). This is already a significant question in some very simple situations. It is clear, for instance, that in $(L^p(X), C(X))$ the multiplication is defined not only between elements of $\mathscr{A} = L^p(X)$ and elements belonging to $\mathscr{A}_0 = C(X)$: indeed, any essentially bounded discontinuous functions with support in X can be multiplied with any function of $L^p(X)$ and the result is again in $L^p(X)$. We will show here that it is possible to introduce in a CQ* -algebra two different multiplications, both extending the usual one, and we discuss some of their properties. The first one, called the *strong* multiplication and indicated with $\bullet$, is obtained via a *closure* procedure. The second one, called *weak* multiplication, $\circ$, is defined via a suitable family of sesquilinear forms.

Let $(\mathscr{A}, *, R\mathscr{A}, \flat)$ be an arbitrary CQ*-algebra. Given $A \in \mathscr{A}$ we consider the linear map $L_A : B \in R\mathscr{A} \mapsto AB \in \mathscr{A}$. Since $\| AB \| \leq \| A \| \, \| B \|_\flat$, L_A is continuous from $R\mathscr{A}(\| \, \|_\flat)$ into $\mathscr{A}(\| \, \|)$, while, in general, it is not continuous from $R\mathscr{A}(\| \, \|)$ into $\mathscr{A}(\| \, \|)$.

Definition 3.1. *We say that $A \in \mathscr{A}$ is closable to the right if L_A is closable as a map from $R\mathscr{A}$ into $\mathscr{A}$.*

The closability of L_A means that $\forall \{B_n\} \subset R\mathscr{A}$ such that $B_n \xrightarrow{\| \|} 0$ in $\mathscr{A}$ and $AB_n \xrightarrow{\| \|} Y \in \mathscr{A}$ then $Y = 0$.

If $A \in \mathscr{A}$ is closable to the right we define the domain of its closure

$$D(\overline{L_A}) \equiv \{B \in \mathscr{A} : \exists \{B_n\} \subset R\mathscr{A} : B_n \xrightarrow{\| \|} B \text{ and such that } AB_n \text{ is } \| \, \| \text{-converging}\}$$

and, for $B \in D(\overline{L_A})$

$$\overline{L_A}(B) \equiv \| \, \| - \lim_{n \to \infty} AB_n \tag{5}$$

Since $D(\overline{L_A}) \supseteq \mathscr{A}_0$ then this set is dense in $\mathscr{A}$.

Of course, in the same way, $\forall A \in \mathscr{A}$ one can consider a right multiplication map R_A defined by $R_A : B \in L\mathscr{A} \to BA \in \mathscr{A}$.

The domain of the closure of R_A is now the set

$$D(\overline{R_A}) \equiv \{B \in \mathscr{A} : \exists \{B_n\} \subset L\mathscr{A} : B_n \xrightarrow{\| \|} B \text{ and such that } B_nA \text{ is } \| \, \| \text{-converging}\}$$

and, for $B \in D(\overline{R_A})$

$$\overline{R_A}(B) \equiv \| \, \| - \lim_{n \to \infty} B_n A \tag{6}$$

The right and left multiplications are linked with each other by the following

Lemma 3.2. *Given $A \in \mathscr{A}$, R_A is closable if, and only if, L_{A^*} is closable. Moreover $D(\overline{R_A})^* = D(\overline{L_{A^*}})$*

It is useful to remark that any element $\mathscr{A}$ in $R\mathscr{A}$ $(L\mathscr{A})$ is closable to the right (left) and that $D(\overline{L_A}) = \mathscr{A}$ $(D(\overline{R_A}) = \mathscr{A})$.

If L_A is closable and $B \in D(\overline{L_A})$ and if, at the same time, R_B is also closable and $A \in D(\overline{R_B})$, then one would expect that the equality $\overline{L_A}(B) = \overline{R_B}(A)$ holds. This is, indeed, true and will be proved in Proposition 3.18, making use of a weaker notion of multiplication.

Definition 3.3. *A CQ*-algebra $(\mathscr{A}, *, R\mathscr{A}, \flat)$ is said to be fully-closable if L_A is closable $\forall A \in \mathscr{A}$.*

Due to Lemma 3.2 a CQ*-algebra $(\mathscr{A}, *, R\mathscr{A}, \flat)$ is fully-closable if, and only if R_A is closable $\forall A \in \mathscr{A}$.

The fully-closability of a CQ*-algebra seems a very strong requirement. We are going to discuss some equivalent condition and also to discuss an example.

Let $\mathscr{A}'$ denote the dual Banach space of $\mathscr{A}$. For $\eta \in \mathscr{A}'$ and $A \in \mathscr{A}$ we put $\eta_A^L(B) \equiv \eta(AB)$, $\forall B \in R\mathscr{A}$. We observe that η_A^L is continuous on $R\mathscr{A}(\| \|_\flat)$.

Proposition 3.4. *Let $A \in \mathscr{A}$. The map L_A is closable if, and only if, the set*

$$F_A \equiv \{\eta \in \mathscr{A}' : \ \eta_A^L \text{ is continuous on } R\mathscr{A}(\| \|)\}$$

is $\sigma(\mathscr{A}', \mathscr{A})$-dense in $\mathscr{A}'$.

Proof. This is nothing but an application of a well known theorem on the existence of closed extensions of linear maps [20, Ch. 7, Sect. 36.3] □

Example 3.5. We now use the above proposition to prove that the CQ*-algebra $(L^p(X, d\mu), C(X))$, with $2 \leq p < \infty$, discussed in Example 2.7, is fully-closable.

We call p' the index conjugate to p, $(p^{-1} + p'^{-1} = 1)$. The dual of the space $L^p(X, d\mu)$ is therefore $L^{p'}(X, d\mu)$. For $f \in L^p(X, d\mu)$ and $g \in L^{p'}(X, d\mu)$ we put $g_f(\phi) \equiv \int_X f\phi g \, d\mu$, $\phi \in C(X)$.

The functional g_f is continuous on $C(X)$ with respect to the norm $\| \|_p$ if, and only if, $fg \in L^{p'}(X, d\mu)$. Therefore the set F_f of Proposition 3.4 is

$$F_f \equiv \{g \in L^{p'}(X, d\mu) : fg \in L^{p'}(X, d\mu)\}$$

If $p \geq 2$, this set is $\| \|$-dense in $L^{p'}(X, d\mu)$ since it contains $C(X)$. A fortiori F_f is $\sigma(L^{p'}, L^p)$-dense in $L^{p'}$.

We conclude that the CQ*-algebras $(L^p(X, d\mu), C(X))$ for any $p \geq 2$ are fully-closable.

The same conclusion can be obtained also from the very first definition of fully-closability. As a matter of fact, we showed in [5] that the same statement holds for $p > 1$ without limitation on X and, if $\mu(X) < \infty$, also for $p = 1$.

We introduce now right approximate identities of a CQ*-algebra . This notion extends the concept of approximate identities of a C *-algebra and their existence will imply our CQ*-algebra to be fully-closable.

Definition 3.6. *Let $(\mathscr{A}, *, R\mathscr{A}, \flat)$ be a CQ*-algebra. A right approximate identity is a net $\{E_\alpha\}$ of elements of $R\mathscr{A}$ such that*

i) *E_α is a bilateral approximate identity of $R\mathscr{A}$;*

ii) $\lim_\alpha \| AE_\alpha - A \| = 0 \ \forall A \in \mathscr{A}$.

We further say that the right approximate identity $\{E_\alpha\}$ is regularizing if $AE_\alpha \in R\mathscr{A} \ \forall A \in \mathscr{A}$.

The difinition of left approximate identity is an obvious modification of the previous one.

Remark 3.7. (1)—If $\{E_\alpha\}$ is a right approximate identity, then $\{E_\alpha^*\}$ is a left approximate identity.
(2)—If $\{E_\alpha\}$ is a regularizing left approximate identity, then it is easy to show by a simple limit argument that, for each α, $E_\alpha(AB) = (E_\alpha A)B \quad \forall A \in \mathscr{A}, B \in R\mathscr{A}$.

Proposition 3.8. *Any CQ*-algebra has a right approximate identity (and then also a left approximate identity).*

Proof. $R\mathscr{A}$ is a C-algebra, then it has an increasing approximate identity bounded by 1 (i.e., $\| E_\alpha \|_\flat \le 1$). Let $A \in \mathscr{A}$ and $\{A_n\} \subset R\mathscr{A}$ be a sequence $\| \, \|$-converging to $\mathscr{A}$; then we get*

$$\| A - AE_\alpha \| \le \| A - A_n \| + \| A_n - A_n E_\alpha \| + \| A_n E_\alpha - AE_\alpha \|$$
$$\le \| A - A_n \| + \| A_n \| \, \| \mathbb{I} - E_\alpha \|_\flat + \| A - A_n \| \, \| E_\alpha \|_\flat \to 0$$

since $\| \mathbb{I} - E_\alpha \|_\flat \to 0$ *and* $\| E_\alpha \|_\flat \le 1$. □

By means of regularizing approximate identities one can give a sufficient condition for a CQ*-algebra to be fully closable.

Proposition 3.9. *If a CQ*-algebra $(\mathscr{A}, *, R\mathscr{A}, \flat)$ has a left regularizing approximate identity, then $\mathscr{A}$ is fully closable.*

Proof. Let $\{E_\alpha\}$ be a regularizing left approximate identity of $\mathscr{A}$ and let us define $\eta_\alpha(X) \equiv \eta(E_\alpha X)$, for $\eta \in \mathscr{A}'$ and $\forall X \in \mathscr{A}$. Due to the continuity of η, we have $\eta_\alpha(X) \to \eta(X)$, $\forall X \in \mathscr{A}$. Now, if $A \in \mathscr{A}$ and $B \in R\mathscr{A}$, we define $\eta^L_{\alpha, A}(B) \equiv \eta_\alpha(AB) = \eta(E_\alpha(AB)) = \eta((E_\alpha A)B)$. Then,

$$| \eta^L_{\alpha, A}(B) | = | \eta_\alpha(AB) | \le \| E_\alpha A \|_\# \| B \|.$$

This implies that $\eta_\alpha \in F_A$, $\forall A \in \mathscr{A}$. Therefore F_A is $\sigma(\mathscr{A}', \mathscr{A})$-dense in $\mathscr{A}'$, $\forall A \in \mathscr{A}$. □

We now introduce a different multiplication, which we call *weak*.

Definition 3.10. *Let $(\mathscr{A}, *, R\mathscr{A}, \flat)$ be a CQ*-algebra. We denote as $\mathscr{S}(\mathscr{A})$ the set of sesquilinear forms Ω on $\mathscr{A} \times \mathscr{A}$ with the following properties:*

(i) $\Omega(A, A) \ge 0 \;\; \forall A \in \mathscr{A}$;
(ii) $\Omega(AB, C) = \Omega(B, A^*C) \; \forall A \in \mathscr{A}, \forall B, C \in R\mathscr{A}$;
(iii) $| \Omega(A, B) | \le \| A \| \, \| B \| \quad \forall A, B \in \mathscr{A}$.

Given $\Omega \in \mathcal{S}(\mathcal{A})$ we define the following positive sesquilinear form on $\mathcal{A} \times \mathcal{A}$:

$$\Omega^*(X, Y) \equiv \Omega(Y^*, X^*) \ \forall X, Y \in \mathcal{A}. \tag{7}$$

Then Ω^* satisfies conditions (i) and (iii) of Def. 3.10, while condition (ii) should be substituted with the following one:

(ii′) $\Omega^*(BA, C) = \Omega^*(B, CA^*) \, \forall A \in \mathcal{A}, \forall B, C \in L\mathcal{A}$.

If we call $\mathcal{S}(\mathcal{A})^*$ the set of sesquilinear forms on $\mathcal{A} \times \mathcal{A}$ satisfying (i), (ii′) and (iii), it is easy to prove that Ω belongs to $\mathcal{S}(\mathcal{A})$ if, and only if, Ω^* belongs to $\mathcal{S}(\mathcal{A})^*$.

Moreover, if $\Omega \in \mathcal{S}(\mathcal{A})$ and $B \in R\mathcal{A}$, with $\| B \|_\flat \leq 1$, we set

$$\Omega_B(X, Y) \equiv \Omega(XB, YB) \ \forall X, Y \in \mathcal{A}. \tag{8}$$

It is easy to prove that Ω_B still belongs to $\mathcal{S}(\mathcal{A})$.

Remark 3.11. —It is well known that to any bounded sesquilinear form Ω on $\mathcal{A} \times \mathcal{A}$ it corresponds a continuous linear map $T_\Omega \in \mathcal{B}(\mathcal{A}, \mathcal{A}')$, defined by the formula

$$\langle A, T_\Omega(B) \rangle \equiv \Omega(A, B) \qquad \forall A, B \in \mathcal{A},$$

where $\mathcal{A}'$ is the conjugate dual of $\mathcal{A}$ with respect to the form $\langle .,. \rangle$.

In particular, if Ω belongs to $\mathcal{S}(\mathcal{A})$, the corresponding T_Ω satisfies the following properties:

(i) $\langle A, T_\Omega(A) \rangle \geq 0, \ \forall A \in \mathcal{A}$;

(ii) $\langle AB, T_\Omega(C) \rangle = \langle B, T_\Omega(A^*C) \rangle, \ \forall A \in \mathcal{A}, \forall B, C \in R\mathcal{A}$;

(iii) $\| T_\Omega \| \leq 1$ as an operator from $\mathcal{A}$ into $\mathcal{A}'$.

The next Proposition shows that normalized elements of $\mathcal{S}(\mathcal{A})$ give rise to states, in the usual sense, on $R\mathcal{A}$.

Proposition 3.12. *Let $\Omega \in \mathcal{S}(\mathcal{A})$ with $\Omega(\mathbb{I}, \mathbb{I}) = 1$, then the linear functional ω_Ω on $R\mathcal{A}$ defined by*

$$\omega_\Omega(X) = \Omega(X, \mathbb{I}), \ X \in R\mathcal{A}$$

is positive in $R\mathcal{A}$; i.e. $\omega_\Omega(X^\flat X) = \Omega(X, X^{\flat *}) \geq 0, \ \forall X \in R\mathcal{A}$.

Proof. By the Schwarz inequality we get

$$| \omega_\Omega(X) | = | \Omega(X, \mathbb{I}) | \leq \Omega(X, X)^{1/2} \Omega(\mathbb{I}, \mathbb{I})^{1/2} \leq \| X \|_\flat.$$

Therefore $\| \omega_\Omega \| \leq 1$; on the other hand,

$$\| \omega_\Omega \| \geq \omega_\Omega(\mathbb{I}) = \Omega(\mathbb{I}, \mathbb{I}) = 1.$$

Thus $\| \omega_\Omega \| = \omega_\Omega(\mathbb{I})$. Hence ω_Ω is positive. □

Remark 3.13. Since ω_Ω is positive, one has

$$\omega_\Omega(X^\flat) = \overline{\omega_\Omega(X)} = \Omega(\mathbb{I}, X)$$

and therefore $\Omega(X^\flat, \mathbb{I}) = \Omega(X^*, \mathbb{I})$, $\forall X \in R\mathscr{A}$. From this it follows easily that

$$\Omega(X^* B^{\flat *}, C) = \Omega(X^\flat B, C^{\flat *}), \quad \forall X, B, C \in R\mathscr{A}.$$

From now on we will study certain 'weak' properties related to a convenient family of sesquilinear forms on $\mathscr{A} \times \mathscr{A}$. Of course both $\mathscr{S}(\mathscr{A})$ and $\mathscr{S}(\mathscr{A})^*$ are good candidates, as well as their intersection. However, since we have in mind essentially vector states, which may satisfy (ii) but not (ii′), we will consider only the family $\mathscr{S}(\mathscr{A})$. There is no other reason for this 'symmetry breaking'.

Let $(\mathscr{A}, *, R\mathscr{A}, \flat)$ be an arbitrary CQ*-algebra and let $X, Y \in \mathscr{A}$.

Definition 3.14. *We say that X (Y) is a weak left (right) multiplier of Y (X), if there exists a unique element $Z \in \mathscr{A}$ such that*

$$\Omega(YB, X^*C) = \Omega(ZB, C) \ \forall \Omega \in \mathscr{S}(\mathscr{A}), \ \forall B, C \in R\mathscr{A}. \tag{9}$$

In this case we write $Z = X \circ Y$ and $X \in L_w(Y)$ or $Y \in R_w(X)$.

It is straightforward to prove that, if the usual product is defined, then this coincides with the weak one. More explicitly, if $X \in \mathscr{A}$, $C \in R\mathscr{A}$, then $X \in L_w(C)$ and $X \circ C = XC$.

At this point, let us now define the following subset $\Gamma_w \subset \mathscr{A} \times \mathscr{A}$

$$\Gamma_w \equiv \{(X, Y) \in \mathscr{A} \times \mathscr{A} : \exists ! Z \in \mathscr{A} : \Omega(YB_1, X^*B_2) = \Omega(ZB_1, B_2) \\ \forall \Omega \in \mathscr{S}(\mathscr{A}), \ \forall B_1, B_2 \in R\mathscr{A}\} \tag{10}$$

As usual we put, for $X \in \mathscr{A}$, $R_w(X) = \{Y \in \mathscr{A} : (X, Y) \in \Gamma_w\}$ and we define in similar way $L_w(X)$.

Remark 3.15. It is clear that if $\mathscr{A}$ satisfies the following condition:

$$\Omega(XB, C) = 0 \ \forall \Omega \in \mathscr{S}(\mathscr{A}) \ ; \ \forall B, C \in R\mathscr{A} \text{ implies } X = 0$$

or the equivalent one

$$\Omega(X, X) = 0 \ \forall \Omega \in \mathscr{S}(\mathscr{A}) \text{ implies } X = 0$$

then the element Z in Definition 3.14, if it exists, is necessarily unique.

We will come back to these two conditions in the next Section.

Proposition 3.16. *Let $(\mathscr{A}, *, R\mathscr{A}, \flat)$ be a CQ*-algebra. Then $(\mathscr{A}, \Gamma_w, \circ)$ is a partial *-algebra and $R\mathscr{A} \subset R_w(\mathscr{A})$.*

This result easily follows from the definition of partial *-algebra, see [9]. This partial *-algebra is, in general, non-associative.

Proposition 3.17. *If X is closable to the right and if $Y \in D(\overline{L_X})$ then $X \in L_w(Y)$ and $X \circ Y = \overline{L_X}(Y)$.*

Proof. Indeed, if $Y \in D(\overline{L_X})$, then there exists a sequence $\{Y_n\} \subset R\mathscr{A}$ such that $Y_n \to Y$ and $XY_n \to Z$ in the norm of $\mathscr{A}$. Then, for $\Omega \in \mathscr{S}(\mathscr{A})$ and $B_1, B_2 \in R\mathscr{A}$ we get

$$|\Omega((XY_n - Z)B_1, B_2)| \leq \| XY_n - Z \| \, \| B_1 \|_b \| B_2 \|_b .$$

It follows that $\Omega((XY_n)B_1, B_2) \to \Omega(ZB_1, B_2) \; \forall \Omega \in \mathscr{S}(\mathscr{A}), \; \forall B_1, B_2 \in R\mathscr{A}$. Therefore $\Omega(YB_1, X^*B_2) = \lim_{n\to\infty} \Omega(Y_n B_1, X^* B_2) = \lim_{n\to\infty} \Omega(XY_n B_1, B_2) = \Omega(ZB_1, B_2)$ □

Making use of the above Proposition, and of the analogous statement for the left-closability, we can deduce the following statement, which easily follows from the uniqueness of the product $\circ$.

Proposition 3.18. *Let X be closable to the right and $Y \in D(\overline{L_X})$. Let furthermore Y be closable to the left and $X \in D(\overline{R_Y})$. Then $\overline{R_Y}(X) = \overline{L_X}(Y) = X \circ Y$.*

At this point we can define the *strong* product in the following way

Definition 3.19. *In the hypotheses of Proposition 3.18 we define $X \bullet Y \equiv \overline{R_Y}(X) = \overline{L_X}(Y)$.*

An obvious consequence is that the weak and the strong products coincide whenever they are both defined.

Remark 3.20. $(L^p(X, \mu), C_0(X))$ is a very simple instance where the strong and weak multiplication coincide, [5].

It is a natural question to ask whether $\mathscr{A}$ is a partial *-algebra with respect to the strong multiplication too. More precisely, if $\mathscr{A}$ is fully closable, then we can define

$$\Gamma_s \equiv \{(X, Y) \in \mathscr{A} \times \mathscr{A} \colon Y \in D(\overline{L_X}) \text{ and } X \in D(\overline{R_Y})\} \tag{11}$$

(In analogy with the case of the weak-multiplication, we define, for $X \in \mathscr{A}$, $R_s(X) = \{Y \in \mathscr{A} \colon (X, Y) \in \Gamma_s\}$, etc.)

Is then $(\mathscr{A}, \Gamma_s, \bullet)$ a partial *-algebra? The answer is, in general, negative because of the possible lack of the distributivity. This unpleasant feature depends on the following fact: if $(A, B) \in \Gamma_s$ and $(A, C) \in \Gamma_s$ then certainly $B + C \in D(\overline{L_A})$; but on the other side, we only get $A \in D(\overline{R_B}) \cap D(\overline{R_C})$ and this is, in general, different from $D(\overline{R_{B+C}})$ (this is the same pathology discussed in [6, Add./Err.] and has the same topological motivations.) The conditions $(X, Y) \in \Gamma_s \Leftrightarrow (Y^*, X^*) \in \Gamma_s$ and $(X \bullet Y)^* = Y^* \bullet X^*$ are, on the contrary, always fulfilled.

The definition of strong multiplication allows also to give the following weaker form of the associative law in $(\mathscr{A}, \Gamma_w, \circ)$:

Proposition 3.21. *If* $A \in D(\overline{R_B})$, $(A \bullet B, C) \in \Gamma_w$ *and* $(B, C) \in \Gamma_w$ *then* $(A, B \circ C) \in \Gamma_w$ *and* $(A \bullet B) \circ C = A \circ (B \circ C)$.

Proof. First we recall that $A \in D(\overline{R_B})$ if, and only if, $A^* \in D(\overline{L_{B^*}})$. Therefore there exists a sequence $\{R_n\} \subset L\mathscr{A}$: $R_n \to A^*$ and $B^* R_n$ is convergent in $\mathscr{A}$.

If $\Omega \in \mathscr{S}(\mathscr{A})$, $S_1, S_2 \in R\mathscr{A}$ we get

$$\begin{aligned}\Omega((A \bullet B) \circ C)S_1, S_2) &= \Omega((A \circ B) \circ C)S_1, S_2) \\ &= \Omega(CS_1, (A \circ B)^* S_2) = \Omega(CS_1, B^* \circ A^* S_2) \\ &= \lim_{n \to \infty} \Omega(CS_1, B^* R_n S_2) = \lim_{n \to \infty} \Omega((B \circ C)S_1, R_n S_2) \\ &= \Omega((B \circ C)S_1, A^* S_2) = \Omega(A \circ (B \circ C)S_1, S_2)\end{aligned}$$

□

§ 4. *-Semisimple CQ*-algebras

Lemma 4.1. *Let* $(\mathscr{A}, *, R\mathscr{A}, \flat)$ *be a CQ*-algebra. Let us consider the following three sets*

$$\mathfrak{R}_1 = \{X \in \mathscr{A}: \Omega(X, X) = 0 \ \ \forall \Omega \in \mathscr{S}(\mathscr{A})\}$$
$$\mathfrak{R}_2 = \{X \in \mathscr{A}: \Omega(XB, C) = 0 \ \ \forall \Omega \in \mathscr{S}(\mathscr{A}) ; \ \forall B, C \in R\mathscr{A}\}$$
$$\mathfrak{R}_3 = \{X \in \mathscr{A}: \Omega(XB, XC) = 0 \ \ \forall \Omega \in \mathscr{S}(\mathscr{A}) ; \ \forall B, C \in R\mathscr{A}\}$$

Then $\mathfrak{R}_1 = \mathfrak{R}_2 = \mathfrak{R}_3 \equiv \mathfrak{R}^{(*)}$.

The set $\mathfrak{R}^{(*)}$ *is called the* <u>*-radical</u> *of* $\mathscr{A}$.

Proof. The inclusion $\mathfrak{R}_3 \subseteq \mathfrak{R}_2$ follows immediately from the Schwarz inequality.

Next we will show that $\mathfrak{R}_2 \subseteq \mathfrak{R}_1$. Let $X \in \mathscr{A}$, and $\{X_n\}$ be a sequence of elements in $R\mathscr{A}$ $\|\ \|$-converging to X. If $\Omega \in \mathscr{S}(\mathscr{A})$, then $\Omega(X, X_n) = 0 \forall n$, and the $\|\ \|$-continuity of the sesquilinear form Ω, implies that $\Omega(X, X) = 0$.

Finally, the inclusion $\mathfrak{R}_1 \subseteq \mathfrak{R}_3$ follows easily from the Schwarz inequality making use of the form Ω_B defined in (8). □

Lemma 4.2. *The* *-*radical* $\mathfrak{R}^{(*)}$ *of a CQ*-algebra* $(\mathscr{A}, *, R\mathscr{A}, \flat)$ *has the following properties*:

(i) $\mathfrak{R}^{(*)}$ *is a linear subspace of* $\mathscr{A}$

(ii) *If* $X \in \mathfrak{R}^{(*)}$ *then* $X^* \in \mathfrak{R}^{(*)}$
(iii) $\mathfrak{R}^{(*)} \cap R\mathscr{A}$ *is a right ideal of* $R\mathscr{A}$
(iv) *If* $X \in L\mathscr{A}$, $Y \in \mathfrak{R}^{(*)}$ *and* $Z \in R\mathscr{A}$ *then* $XY \in \mathfrak{R}^{(*)}$ *and* $YZ \in \mathfrak{R}^{(*)}$

The proof is straightforward.

Remark 4.3. The property (iii) of Lemma 4.2 has a *left* counterpart: $\mathfrak{R}^{(*)} \cap L\mathscr{A}$ is a left ideal of $L\mathscr{A}$ and, analogously, $\mathfrak{R}^{(*)} \cap \mathscr{A}_0$ is a *-ideal of $\mathscr{A}_0$.

It is rather natural to call *quasi *-ideal* a subset of $\mathscr{A}$ which has the properties (i)-(iv) of Lemma 4.2.

Definition 4.4. *We call *-semisimple any CQ*-algebra* $(\mathscr{A}, *, R\mathscr{A}, \flat)$ *such that* $\mathfrak{R}^{(*)} = \{0\}$.

For a *-semisimple CQ*-algebra $(\mathscr{A}, *, R\mathscr{A}, \flat)$ the set

$$\mathfrak{R}_4 = \{X \in \mathscr{A}: X^* \circ X \text{ is well-defined and } X^* \circ X = 0\}$$

coincides with the *-radical and, therefore, reduces to $\{0\}$. We will be mainly concerned with such CQ*-algebras. The reason is that the *-semisimplicity turns out to be a structure property which simplifies the (heavy) general framework developed in [4]. Moreover many interesting examples of CQ*-algebras are indeed *-semisimple as we shall see later.

We recall that the notion of *-semisimplicity in the ordinary Banach *-algebras theory can be formulated in terms similar to those used here. Our set-up is indeed an extension of the Gel'fand description of the *-semisimplicity. Actually, if $\mathscr{A}_0$ is a Banach *-algebra then $\mathscr{A}_0$ is *-semisimple if, when $A \in \mathscr{A}_0$ is such that $\omega(A^*A) = 0$ for all ω in the set $\mathscr{P}(\mathscr{A}_0)$ of all positive functionals with norm less or equal to 1, $A = 0$ results.

If $\mathscr{A}_0$ is *-semisimple then the Gelfand seminorm

$$\| A \|_*^2 \equiv \sup_{\omega \in \mathscr{P}(\mathscr{A}_0)} \omega(A^*A) \tag{12}$$

is actually a norm which satisfies the C*-property.

Given a CQ*-algebra $(\mathscr{A}, *, R\mathscr{A}, \flat)$ and a sesquilinear form $\Omega \in \mathscr{S}(\mathscr{A})$, we define the positive linear functional $\omega_\Omega(A) \equiv \Omega(A, \mathbb{I})$ where A is taken in $\mathscr{A}_0$, [4]. It is easy to see that any such ω_Ω belongs to $\mathscr{P}(\mathscr{A}_0)$.

Proposition 4.5. *Let* $(\mathscr{A}, *, R\mathscr{A}, \flat)$ *be a CQ*-algebra. If* $\mathscr{A}$ *is *-semisimple then* $\mathscr{A}_0$ *is *-semisimple.*

Proof. Let $\omega(A^*A) = 0 \ \ \forall \omega \in \mathscr{P}(\mathscr{A}_0)$. Then, in particular, we will have $\omega_\Omega(A^*A) = 0 \, \forall \Omega \in \mathscr{S}(\mathscr{A})$. This implies that, for all such Ω, $\Omega(A, A) = 0$, that is

$A = 0$. □

Definition 4.6. *Given a *-semisimple CQ*-algebra* $(\mathscr{A}, *, R\mathscr{A}, \flat)$ *we define*

$$\| X \|_{\alpha}^{2} \equiv \sup_{\Omega \in \mathscr{S}(\mathscr{A})} \Omega(X, X). \tag{13}$$

and

$$\| X \|_{(\alpha)} = \max\{\| X \|_{\alpha}, \| X^* \|_{\alpha}\} \tag{14}$$

We say that the CQ-algebra* $(\mathscr{A}, *, R\mathscr{A}, \flat)$ *is* *regular* *if* $\| X \| = \| X \|_{\alpha}$ $\forall X \in \mathscr{A}$.

The Gelfand seminorm (12) can be compared with the other norms, $\| \ \|$ and $\| \ \|_0$ which enter in our structure. Actually, it is easy to prove the following

Corollary 4.7. *Let* $(\mathscr{A}, *, R\mathscr{A}, \flat)$ *be a regular CQ*-algebra. Then* $\| A \| \leq \| A \|_* \leq \| A \|_0$, $\forall A \in \mathscr{A}_0$.

Due to the definition of $\mathscr{S}(\mathscr{A})$, the inequality $\| X \|_{\alpha} \leq \| X \|$, $\forall X \in \mathscr{A}$ holds. The regular CQ*-algebras also satisfy the converse inequality. Moreover it is easily seen that if $\mathscr{A}$ is regular then we also have $\| A \| = \| A \|_{(\alpha)}$, $\forall A \in \mathscr{A}$.

Let us now give some examples of *-semisimple CQ*-algebras.

Example 4.8. We start considering an abelian example, that is $(L^p(X, \mu), C(X))$, where (X, μ) is a measure space with X a compact Hausdorff space and $p \geq 2$. We know from [4, 5] that $(L^p(X, \mu), C(X))$ is an abelian proper CQ*-algebra with $\flat = \sharp = *$. We will first show that for all $f \in L^p(X, d\mu)$ there exists a sesquilinear form $\Omega_f \in \mathscr{S}(L^p(X))$ such that $\Omega_f(f, f) = \| f \|_p^2$. In the following we fix for simplicity $X = [0, 1]$.

Given $f \in L^p(X, d\mu)$, we define for all $\psi, \phi \in L^p(X, d\mu)$ a sesquilinear form Ω_f as

$$\Omega_f(\psi, \phi) \equiv \| f \|_p^{2-p} \int_0^1 \psi(x)\overline{\phi(x)} \, | f(x) |^{p-2} \, dx$$

It is easy to verify the first two conditions of Definition 3.10. The last condition requires twice the use of the Hölder inequality

$$| \Omega_f(\psi, \phi) | \leq \| f \|_p^{2-p} \| \psi \|_p \| \phi \, | f |^{p-2} \|_{p'} \leq \| \psi \|_p \| \phi \|_p.$$

Therefore $\Omega_f \in \mathscr{S}(\mathscr{A})$. Moreover, from the definition itself, $\Omega_f(f, f) = \| f \|_p^2$.

This fact immediately implies the *-semisemplicity of $(L^p(X, d\mu), C(X, d\mu))$. As for the regularity, we already know that $\| f \|_{\alpha} \leq \| f \|_p$ $\forall f \in L^p(X, d\mu)$. To prove the regularity of the CQ*-algebra we have to prove the converse inequality.

This easily follows from the above property; in fact we have $\|f\|_\alpha \equiv \sup_{\Omega \in \mathcal{S}(\mathcal{A})} \Omega(f, f) \geq \Omega_f(f, f) = \|f\|_p$.

For $p < 2$, the *-semisemplicity fails. More details can be found in [5].

Example 4.9. The second example is again proper but not abelian. Let $(\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar{\lambda}}), *, \mathcal{B}(\mathcal{H}_\lambda), \flat)$ be the CQ*-algebra considered in [4, Example 3.3]. Let $\mathcal{B}^+(\mathcal{H}_\lambda) = \mathcal{B}(\mathcal{H}_\lambda) \cap \mathcal{B}(\mathcal{H}_{\bar{\lambda}})$ and set

$$\mathcal{B}^+_\gamma(\mathcal{H}_\lambda) = \{X \in \mathcal{B}^+(\mathcal{H}_\lambda) : \lambda(Xf, g) = \lambda(f, X^*g) \ \forall f, g \in \mathcal{H}_\lambda\}$$

i.e., the set of elements of $\mathcal{B}^+(\mathcal{H}_\lambda)$ *commuting* with λ. It is readily checked that

$$\mathcal{B}^+_\gamma(\mathcal{H}_\lambda) = \{X \in \mathcal{B}^+(\mathcal{H}_\lambda) : X^* = X^\flat\}.$$

It turns out that $\mathcal{B}^+_\gamma(\mathcal{H}_\lambda)$ is a C *-algebra (with respect to the norm of $\mathcal{B}(\mathcal{H}_\lambda)$) and then the $\| \ \|_{\lambda, \bar{\lambda}}$-closure $\mathcal{B}_\gamma(\mathcal{H}_\lambda, \mathcal{H}_{\bar{\lambda}})$ of $\mathcal{B}^+_\gamma(\mathcal{H}_\lambda)$ in $\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar{\lambda}})$ is a proper CQ*-algebra with $\flat = *$ (Proposition 2.6).
Let us now prove that $\mathcal{B}_\gamma(\mathcal{H}_\lambda, \mathcal{H}_{\bar{\lambda}})$ is *-semisimple.
Let $f \in \mathcal{H}_\lambda$ with $\|f\|_\lambda = 1$. For $X, Y \in \mathcal{B}_\gamma(\mathcal{H}_\lambda, \mathcal{H}_{\bar{\lambda}})$ set

$$\Omega_f(X, Y) = \langle Xf, Yf \rangle_{\bar{\lambda}}.$$

Then it is easy to check that $\Omega_f \in \mathcal{S}(\mathcal{B}_\gamma(\mathcal{H}_\lambda, \mathcal{H}_{\bar{\lambda}}))$ (condition (ii) of Definition 3.10 follows from a simple limit argument). It is clear that if $\Omega_f(X, X) = \|Xf\|^2_{\bar{\lambda}} = 0$ $\forall f \in \mathcal{H}_\lambda$, then $X = 0$. This proves our claim.

The next proposition shows that the *-semisemplicity has relevant consequences also for the algebraic structure of $\mathcal{A}$.

Proposition 4.10. *If a CQ*-algebra $(\mathcal{A}, *, R\mathcal{A}, \flat)$ is *-semisimple, then $\mathcal{A}$ is fully-closable.*

Proof. Let $A \in \mathcal{A}$ and $\{C_n\} \subset R\mathcal{A}$ a sequence $\| \ \|$-converging to zero and such that $\| \ \| - \lim_{n \to \infty} AC_n = Y$. Then, if $\Omega \in \mathcal{S}(\mathcal{A})$ and $B_1, B_2 \in R\mathcal{A}$, we get

$$|\Omega(YB_1, B_2)| \leq |\Omega((Y - AC_n)B_1, B_2)| + |\Omega(C_nB_1, A^*B_2)|$$
$$\leq \|Y - AC_n\| \, \|B_1\|_\flat \|B_2\|_\flat + \|C_n\| \, \|B_1\|_\flat \|A^*B_2\|_\flat \to 0$$

Therefore $\Omega(YB_1, B_2) = 0$, $\forall \Omega \in \mathcal{S}(\mathcal{A})$, $\forall B_1, B_2 \in R\mathcal{A}$.

The *-semisemplicity of $\mathcal{A}$ and Lemma 4.1, imply $Y = 0$. This proves the statement. □

It is worth mentioning, at this point, that for *-semisimple abelian CQ*-algebras a generalization of the well-known Gel'fand theorem on the representation of an abelian C *-algebra as a C *-algebra of functions can be proved, supporting the idea that the notion of *-semisimplicity is the right one in order to get significant structure properties, [5].

It is sometimes convenient to consider also a stronger notion (equivalent to *-semisemplicity for proper CQ*-algebras). Let $\mathscr{S}_0(\mathscr{A})$ denote the subset of $\mathscr{S}(\mathscr{A})$ consisting of those elements Ω satisfying also the following additional condition:

(15) $$\Omega(X^*B, C) = \Omega(X^\flat B, C), \quad \forall X, B, C \in R\mathscr{A}.$$

We define *strongly *-semisimple* a CQ*-algebra $(\mathscr{A}, *, R\mathscr{A}, \flat)$ such that the following condition holds:

(16) $$\text{If } \Omega(X, X) = 0, \quad \forall \Omega \in \mathscr{S}_0(\mathscr{A}) \text{ then } X = 0.$$

It turns out that any strongly *-semisimple CQ*-algebra $(\mathscr{A}, *, R\mathscr{A}, \flat)$ is proper and $* = \flat$ (this follows immediately from the fact that the *strong *-radical* can be characterized in analogy to $\mathfrak{R}_2$ in Lemma 4.1) and conversely.

Further, we mention the fact that sesquilinear forms of $\mathscr{S}(\mathscr{A})$ satisfying (15) drastically simplify the GNS-construction discussed in [4]. Indeed, if $(\mathscr{A}, *, R\mathscr{A}, \flat)$ is a CQ*-algebra and $\Omega \in \mathscr{S}_0(\mathscr{A})$, then one also has:

$$\Omega(B, B) = \Omega(B, B^{\flat *}), \quad \forall B \in R\mathscr{A}$$

and this equality makes easier to check the conditions given in [4] for the general case. Let us now sketch the construction. Let $\mathscr{K} = \{A \in \mathscr{A}: \Omega(A, A) = 0\}$. Let us consider the linear space $\mathscr{A}/\mathscr{K}$; an element of this set will be denoted as $\lambda_\Omega(A)$, $A \in \mathscr{A}$. Clearly, $\mathscr{A}/\mathscr{K} = \lambda_\Omega(\mathscr{A})$ is a pre-Hilbert space with respect to the scalar product $(\lambda_\Omega(A), \lambda_\Omega(B)) = \Omega(A, B), A, B \in \mathscr{A}$. We denote by $\mathscr{H}_\Omega$ the Hilbert space obtained by the completion of $\lambda_\Omega(\mathscr{A})$. Then Ω is *invariant* in the sense of [10]. This means, in this case, that Ω satisfies condition (ii) of Definition 3.10 and that $\lambda_\Omega(R\mathscr{A})$ is dense in $\mathscr{H}_\Omega$. Indeed, let $\lambda_\Omega(A) \in \lambda_\Omega(\mathscr{A})$ and let $\{A_n\}$ be a sequence in $R\mathscr{A}$ converging to A in the norm of $\mathscr{A}$. Then from the inequality

$$\Omega(A - A_n, A - A_n) \leq \| A - A_n \|^2$$

it follows that $\lambda_\Omega(A_n) \to \lambda_\Omega(A)$ in $\mathscr{H}_\Omega$.
If we put

$$\pi_\Omega(A)\lambda_\Omega(B) = \lambda_\Omega(AB) \quad B \in R\mathscr{A},$$

then $\pi_\Omega(A)$ is a well-defined closable operator with domain $\lambda_\Omega(R\mathscr{A})$ in $\mathscr{H}_\Omega$. More precisely it is an element of the partial O *-algebra $\mathscr{L}^+(\lambda_\Omega(R\mathscr{A}), \mathscr{H}_\Omega)$ [9, 10]. The map $A \mapsto \pi_\Omega(A)$ is a *-representation of partial *-algebras in the sense of [10].
We define now the following set:

$$\mathscr{D}_\Omega = \left\{ A \in \mathscr{A}: \sup_{B \in R\mathscr{A}, \Omega(B, B) \neq 0} \frac{\Omega(AB, AB)}{\Omega(B, B)} < \infty \right\}$$

then

(i) $\mathscr{D}_\Omega$ is a linear space;

(ii) $\mathscr{D}_\Omega \supset R\mathscr{A}$;

(iii) if $A \in \mathscr{D}_\Omega$ and $B \in R\mathscr{A}$, then $AB \in \mathscr{D}_\Omega$

If $\mathscr{D}_\Omega = \mathscr{A}$ then Ω is *admissible* in the sense of [4].

From the definition itself, it follows easily that $\pi_\Omega(\mathscr{D}_\Omega) \subseteq \mathscr{B}(\mathscr{H}_\Omega)$, i.e., each element of $\mathscr{D}_\Omega$ is represented by a bounded operator in Hilbert space.

§ 5. Norms on a *-semisimple CQ*-algebra

As shown in [4], the topological structure of a CQ*-algebra $(\mathscr{A}, *, R\mathscr{A}, \flat)$ is described in terms of four, generally different, norms: $\| \|, \| \|_0, \| \|_\flat, \| \|_\#$.

In this Section we introduce more inequivalent norms which are very useful to investigate some structure properties of the *-semisimple CQ*-algebras. We already defined the norm $\| \|_\alpha$ in (13). We now define

$$\| X \|_\beta \equiv \sup\{| \Omega(XB, B) |; \Omega \in \mathscr{S}(\mathscr{A}), B \in R\mathscr{A}, \| B \|_\flat \leq 1\} \tag{17}$$

Lemma 4.1 and the fact that $\mathscr{A}$ is *-semisimple ensure that $\| \|_\beta$ is really a norm. Moreover, making use of the polarization identity, it is possible to prove that it generates the same topology as the one defined by the following norm

$$\| X \|_{\beta'} \equiv \sup\{| \Omega(XB_1, B_2) |; \Omega \in \mathscr{S}(\mathscr{A}), \| B_1 \|_\flat, \| B_2 \|_\flat \leq 1\}.$$

Here B_1 and B_2 are both taken in $R\mathscr{A}$.

Let us remind that if Ω belongs to $\mathscr{S}(\mathscr{A})$ and $B \in R\mathscr{A}$, with $\| B \|_\flat \leq 1$, then the form Ω_B still belongs to $\mathscr{S}(\mathscr{A})$. The following equivalent definition can therefore be given:

$$\| X \|_\alpha^2 = \sup_{\Omega \in \mathscr{S}(\mathscr{A}), \| B \|_\flat \leq 1} \Omega(XB, XB) \tag{18}$$

and

$$\| X \|_\beta \equiv \sup_{\Omega \in \mathscr{S}(\mathscr{A})} | \Omega(X, \mathbb{I}) | \tag{19}$$

Furthermore, it is easy to prove that $\| X \|_\beta = \| X^* \|_\beta$ and $\| X \|_{(\alpha)} = \| X^* \|_{(\alpha)}$ $\forall X \in \mathscr{A}$. Moreover we can prove that

$$\| X \|_\beta \leq \| X \|_\alpha \quad \forall X \in \mathscr{A}.$$

A short remark is in order: the above inequality holds if $\|\mathbb{I}\| = 1$. This does not hold in general, as it is easy to see looking at the abelian CQ*-algebra $(L^p(X, d\mu), C(X))$, with $1 < \mu(X) < \infty$. However, in this simple case, we can consider the equivalent (in the sense of the norm) CQ*-algebra $(L^p(X, d\nu), C(X))$, where $\nu(E) \equiv \frac{\mu(E)}{\mu(X)}$, $\forall E \in X$. It is easy to see that in this space now $\|\mathbb{I}\| = 1$. We will always assume that $\|\mathbb{I}\| = 1$, even for non abelian CQ*-algebras.

With this in mind, it is easy to prove the following inequality:

(20) $$\| X \|_\beta \leq \| X \|_{(\alpha)} \leq \| X \|, \quad \forall X \in \mathscr{A}$$

where the last inequality follows from the definition of the $\| \|_\alpha$ and from the property (iii) of the set $\mathscr{S}(\mathscr{A})$.

Two interesting properties of the norm (17) are given by the following

Proposition 5.1. *Let $(\mathscr{A}, *, R\mathscr{A}, \flat)$ be a *-semisimple CQ*-algebra. If $X \circ Y$ is well-defined for a certain pair X, $Y \in \mathscr{A}$, then*

(21) $$\| X \circ Y \|_\beta \leq \| X^* \|_\alpha \| Y \|_\alpha \leq \| X \|_{(\alpha)} \| Y \|_{(\alpha)} \leq \| X \| \, \| Y \|$$

Moreover, if $X^* \circ X$ is defined for a certain $X \in \mathscr{A}$ then

(22) $$\| X \|_\alpha^2 = \| X^* \circ Y \|_\beta \leq \| X^* \circ X \|_\alpha$$

The inequality (21) easily follows from the definition of the weak multiplication, the Schwarz inequality for the positive sesquilinear forms and from the definition of the α-norm. The second statement is again a consequence of the above ingredients and of (21).

The next proposition shows that if the *extreme* norms coincide then the structure is, say, trivialized.

Proposition 5.2. *Given a *-semisimple CQ*-algebra $(\mathscr{A}, *, R\mathscr{A}, \flat)$. If $\| X \|_\beta = \| X \|$, $\forall X \in \mathscr{A}$, then $\mathscr{A}$ is a C*-algebra with respect to the multiplication $\circ$.*

Proof. We start proving that in the above hypothesis the set $R_w(X)$ is $\| \|$-closed $\forall X \in \mathscr{A}$. Let $\{Y_n\} \subseteq R_w(X)$, with Y_n converging in $\| \|$ to a certain Y. To prove that $Y \in R_w(X)$ we start observing that, if $\Omega \in \mathscr{S}(\mathscr{A})$, $B_1, B_2 \in R\mathscr{A}$ then $\Omega(YB_1, X^*B_2) = \| \| - \lim_{n\to\infty} \Omega((X \circ Y_n)B_1, B_2)$.

Moreover, if $\| X \|_\beta = \| X \|$, using Proposition 5.1 we deduce that

$$\| X \circ (Y_n - Y_m) \| = \| X \circ (Y_n - Y_m) \|_\beta \leq \| X^* \|_\alpha \| (Y_n - Y_m) \|_\alpha$$
$$= \| X \| \, \| (Y_n - Y_m) \| \to 0.$$

This implies that $\{X \circ Y_n\}$ is a $\| \|$-Cauchy sequence in $\mathscr{A}$. Therefore there exists a $Z \in \mathscr{A}$ which is the $\| \|$ limit of this sequence.

From the previous step we conclude that

$$\Omega(YB_1, X^* B_2) = \Omega(ZB_1, B_2) \quad \forall \Omega \in \mathscr{S}(\mathscr{A}), \ \forall B_1, B_2 \in R\mathscr{A}.$$

This implies that $Y \in R_w(X)$, so that the set $R_w(X)$ is closed.

We now notice that $R_w(\mathscr{A}) \equiv \cap_{X \in \mathscr{A}} R_w(X)$ is $\| \|$-closed in $\mathscr{A}$; but it is also dense since $R\mathscr{A} \subset R_w(\mathscr{A})$. Therefore $R_w(\mathscr{A}) \equiv \mathscr{A}$. We conclude therefore that $\mathscr{A}$ is an algebra. From (21), which in the hypothesis of the Proposition reads $\| X \circ Y \| \leq \| X \| \, \| Y \|$, we know that $\mathscr{A}$ is a Banach algebra. Furthermore here (22), $\| X^* \circ X \| = \| X \|^2$, gives the C*-property for the elements of $\mathscr{A}$. □

Remark 5.3. Of course, by (20), if $\| X \|_\beta = \| X \|$ then also $\| X \|_\alpha = \| X \|$. We will now define two more norms, $\| X \|_R$ and $\| X \|_L$ and two subsets of $\mathscr{A}$, $\mathscr{A}_R$ and $\mathscr{A}_L$ where they are respectively finite. As we will see in Section 6, they will play a relevant role in the study of some spectral properties of a CQ*-algebra.

For $A \in \mathscr{A}$, we put

$$\| A \|_R^2 \equiv \sup \left\{ \frac{\Omega(AB, AB)}{\Omega(B, B)}; \Omega \in \mathscr{S}(\mathscr{A}), B \in R\mathscr{A}, \Omega(B, B) \neq 0 \right\} \tag{23}$$

$$\| A \|_L \equiv \| A^* \|_R \tag{24}$$

These norms are not necessarily finite for arbitrary $A \in \mathscr{A}$. We introduce therefore the following non empty subsets of $\mathscr{A}$:

$$\mathscr{A}_R \equiv \{A \in \mathscr{A}: \| A \|_R < \infty\}$$

and

$$\mathscr{A}_L \equiv \{A \in \mathscr{A}: \| A \|_L < \infty\}.$$

It is easy to prove that $\| \, \|_R$ and $\| \, \|_L$ are really norms and also that they can be expressed in equivalent forms

$$\| A \|_R^2 = \sup_{\Omega \in \mathscr{S}(\mathscr{A})} \frac{\Omega(A, A)}{\Omega(\mathbb{I}, \mathbb{I})}, \tag{25}$$

$$\| A \|_L^2 \equiv \sup_{\Omega \in \mathscr{S}(\mathscr{A}), B \in L\mathscr{A}: \Omega^*(B, B) \neq 0} \frac{\Omega^*(BA, BA)}{\Omega^*(B, B)} \equiv \sup_{\Omega \in \mathscr{S}(\mathscr{A})} \frac{\Omega^*(A, A)}{\Omega^*(\mathbb{I}, \mathbb{I})} \tag{26}$$

The above norms satisfy the following inequalities:

$$\| A \|_R \geq \| A \|_\alpha \ \forall A \in \mathscr{A}_R \tag{27}$$

and

$$\| A \|_L \geq \| A^* \|_\alpha \ \forall A \in \mathscr{A}_L, \tag{28}$$

which are used to prove the following

Proposition 5.4. *Both $\mathscr{A}_R$ and $\mathscr{A}_L$ are linear normed spaces containing $\mathscr{A}_0$ as a subspace. Moreover $A \in \mathscr{A}_R$ if, and only if, $\mathscr{A}^* \in \mathscr{A}_L$. Finally, if $\mathscr{A}$ is regular, then both $\mathscr{A}_R$ and $\mathscr{A}_L$ are Banach spaces.*

Proof. We start proving that $\mathscr{A}_0$ belongs to both $\mathscr{A}_R$ and $\mathscr{A}_L$. This follows from the fact that the weak product of $X \in \mathscr{A}_0$ with X^* exists (together with all its powers) and coincides with the usual product. Therefore the following estimate, obtained using k times the Schwarz inequality and property (iii) of the set $\mathscr{S}(\mathscr{A})$, holds true:

$$\Omega(X, X) \leq \| (X^*X)^{2^{k-1}} \|^{1/2^{k-1}} \Omega(\mathbb{I}, \mathbb{I})^{1/2+1/4+\ldots+1/2^k}$$

Taking the limit $k \to \infty$ and recalling that $\| B \| \leq \| B \|_0$, see [4], we get $\Omega(X, X) \leq \Omega(\mathbb{I}, \mathbb{I}) \, \| X^*X \|_0$ and therefore

$$\| X \|_R \leq \| X \|_0 < \infty \tag{29}$$

This inequality shows that if X belongs to $\mathscr{A}_0$ then $X \in \mathscr{A}_R$. Being $\mathscr{A}_0$ closed with respect to the involution $*$, X also belongs to $\mathscr{A}_L$.

In order to prove that, if $\mathscr{A}$ is regular, $\mathscr{A}_R$ and $\mathscr{A}_L$ are Banach spaces, we only have to show the completeness of, say, $\mathscr{A}_R$. Let us consider a sequence $\{A_n\} \subset \mathscr{A}_R$ which is $\| \, \|_R$-Cauchy. We need to verify that it is also $\| \, \|_R$-converging to an element $B \in \mathscr{A}_R$.

Inequality (27) for regular algebras becomes $\| X \|_R \geq \| X \| \quad \forall X \in \mathscr{A}_R$. Therefore, if $\{A_n\}$ is $\| \, \|_R$-Cauchy it is also $\| \, \|$-Cauchy. Using the $\| \, \|$-completeness of $\mathscr{A}$ we conclude that there exists an element $B \in \mathscr{A}$ which is the $\| \, \|$-limit of A_n.

To prove that B belongs to $\mathscr{A}_R$ we observe that

$$\| B \|_R^2 = \sup_{\Omega \in \mathscr{S}(\mathscr{A})} \frac{\Omega(B, B)}{\Omega(\mathbb{I}, \mathbb{I})} = \sup_{\Omega \in \mathscr{S}(\mathscr{A})} \lim_{n \to \infty} \frac{\Omega(A_n, A_n)}{\Omega(\mathbb{I}, \mathbb{I})} = \lim_{n \to \infty} \sup_{\Omega \in \mathscr{S}(\mathscr{A})} \frac{\Omega(A_n, A_n)}{\Omega(\mathbb{I}, \mathbb{I})}$$

so that $\| B \|_R = \lim_{n \to \infty} \| A_n \|_R$. This limit is finite since, being $\{A_n\}$ $\| \, \|_R$-Cauchy, then the sequence $\{\| A_n \|_R\}$ is convergent. It is worthwhile to observe that the interchange of lim and sup above is possible due to the uniformity of $\Omega(A_n, A_n)/\Omega(\mathbb{I}, \mathbb{I})$ in n.

Finally, using the uniqueness of the limit, we also prove that $B = \| \, \|_R - \lim_{n \to \infty} A_n$.

A completely analogous proof can be set on to prove completeness of $\mathscr{A}_L$. □

In principle, we are not sure that $\mathscr{A}_0$ is really a proper subset of, say, $\mathscr{A}_R$. The following proposition, however, implies the proper nature of this inclusion.

We need first to introduce the notion of *weak length* of an element, see [10, 21] :

Definition 5.5. *We say that $X \in \mathscr{A}$ has weak length N if all the weak product $X^{(k)} \circ X^{(l)}$, with $k+l = n$, $n = 1, 2, \ldots N$ exist and coincide for fixed n. In this case we write $l_w(X) = N$.*

Proposition 5.6. *Let $(\mathscr{A}, *, R\mathscr{A}, \flat)$ be a *-semisimple CQ*-algebra and $X \in \mathscr{A}$ be such that:*

(i) *$X^* \circ X$ is well-defined;*

(ii) *$l_w(X^* \circ X) = \infty$;*

(iii) $\lim \inf_{k \to \infty} \| (X^* \circ X)^{2^k} \|^{1/2^k} < \infty$.

Then $X \in \mathscr{A}_R$. In particular, if $X = X^*$ then $X \in \mathscr{A}_R \cap \mathscr{A}_L$

Remark 5.7. Obviously condition (iii) is satisfied if there exists a positive constant M such that $\| (X^* \circ X)^{2^k} \| < M^{2^k}$, $\forall k \in \mathbb{N}$. It is also worth remarking that if X^* satisfies the assumptions of Proposition 5.6 then $X \in \mathscr{A}_L$ and this implies the second part of the statement. Furthermore, for a self-adjoint X the conditions (i) and (ii) can be replaced with the unique requirement $l_w(X) = \infty$.

Instead of giving the proof, which is very similar to the one of Proposition 5.4, we observe that any element of $\mathscr{A}_0$ satisfies the conditions of the above Proposition, but these are not the only ones. In $(L^p[0, 1], d\mu), C[0, 1])$ any step function $s(x)$ defined on $[0, 1]$ is in $L^p([0, 1], d\mu)$ but not in $C([0, 1])$. It is immediate to verify that $s(x)$ satisfies the above hypothesis. Therefore, in general, $\mathscr{A}_0$ is properly included in $\mathscr{A}_R$ and $\mathscr{A}_L$.

The set $\mathscr{A}_L$ contains, as we shall see in a while, the Banach algebra $\mathscr{A}_\lambda$ of the λ-bounded elements:

Definition 5.8. *Let $(\mathscr{A}, *, R\mathscr{A}, \flat)$ be a *-semisimple CQ*-algebra and $X \in \mathscr{A}$. We say that X is λ-bounded if $D(\overline{L_X}) = \mathscr{A}$.*

The terminology is motivated by the fact that, in this case, the map

$$A \in \mathscr{A} \mapsto \overline{L_X}(A) = X \bullet A \in \mathscr{A}$$

is an everywhere defined and closed, therefore bounded, linear map of $\mathscr{A}$ into itself; therefore, there exists $C > 0$ such that

$$\| X \bullet A \| \leq C \| A \|.$$

We put

$$\| X \|_{\hat{\#}} = \sup_{\| A \| \leq 1} \| X \bullet A \|. \tag{30}$$

The fact that $\| X \|_{\hat{\#}} = \| X \|_{\#}$ for $X \in L\mathscr{A}$ motivates the notation we used. Of course, we can consider as well the set $\mathscr{A}_\rho$ of all ρ-bounded elements which are defined analogously to λ-bounded elements. All the statements concerning the set $\mathscr{A}_\lambda$ have an obvious 'right' counterpart which we will not give explicitly.

Proposition 5.9. *The set $\mathscr{A}_\lambda$ of all the λ-bounded elements is a Banach algebra with respect to the strong multiplication $\bullet$ and the norm $\| \, \|_{\hat{\#}}$. If $X \in \mathscr{A}_\lambda$ then $X^* \in \mathscr{A}_\rho$. Moreover $L\mathscr{A} \subseteq \mathscr{A}_\lambda \subseteq \mathscr{A}_L$.*

Proof. It is easy to show that if $X, Y \in \mathscr{A}_\lambda$ and $\mu \in \mathbb{C}$ then $X+Y$, μX and $X \bullet Y$ all belong to $\mathscr{A}_\lambda$. Then $\mathscr{A}_\lambda$ is an algebra, since it is isomorphic to a subalgebra of the algebra $\mathscr{L}(\mathscr{A})$ of bounded operators in the Banach space $\mathscr{A}$. We will show now that $\mathscr{A}_\lambda$ is, in fact, isomorphic to a closed subalgebra of $\mathscr{L}(\mathscr{A})$. First notice

that for $X \in \mathscr{A}_\lambda$, $\| X \|_{\sharp}^{\natural}$ coincides with the norm of $\bar{L}_X$ as a bounded operator in $\mathscr{A}$. Let now $\{X_n\}$ be a sequence in $\mathscr{A}_\lambda$ such that $\bar{L}_{X_n}$ converges to $L \in \mathscr{L}(\mathscr{A})$ with respect to the natural norm of $\mathscr{L}(\mathscr{A})$.
Since

$$\| (X_n - X_m) \bullet A \| \leq \| X_n - X_m \|_{\sharp}^{\natural} \| A \|,$$

taking $\mathscr{A} = \mathbb{I}$, there exists $X \in \mathscr{A}$ such that $\| X_n - X \| \to 0$ and for each $A \in \mathscr{A}$, there exists $Y_A \in \mathscr{A}$ such that $\| X_n \bullet A - Y_A \| \to 0$. By Proposition 4.10 the right multiplication by A is a closed linear map in $\mathscr{A}$, then it follows that $X \in D(\bar{R}_A)$ and $X_n \bullet A \to X \bullet A = \bar{L}_X(A)$ in the norm of $\mathscr{A}$. This implies that $\bar{L}_X = L$ and so X is λ-bounded.
If $X \in \mathscr{A}_\lambda$, it is immediate to prove that the map $A \in \mathscr{A} \mapsto A \bullet X^* \in \mathscr{A}$ is everywhere defined.
The inclusion $L\mathscr{A} \subseteq \mathscr{A}_\lambda$ is obvious, whereas the second inclusion can be deduced from Proposition 5.6, taking into account the above identification of $\mathscr{A}_\lambda$ (and of $\mathscr{A}_\rho$) with a subalgebra of $\mathscr{L}(\mathscr{A})$ and the inequality (30). □

The λ-bounded elements will be useful in deriving some aspects of the functional calculus in a *-semisimple CQ*-algebra. This will be discussed to some extent in the next Section.

§ 6. Basics for a Functional Calculus

Proposition 5.6 suggests that the norms $\| \|_R$ and $\| \|_L$ should play a role similar to that of the spectral radius in the theory of C *-algebras.

In this Section we will deepen this question which is apparently closely linked to the possibility of generalizing to CQ*-algebras some aspects of the functional calculus for C *-algebras.

First we need to introduce the notion of inverse of an element of a CQ*-algebra.

The main problem which arises when one tries to define the inverse of an element in a partial *-algebra consists in its non-uniqueness. This fact depends on the possible lack of associativity in a partial *-algebra.

However, Proposition 3.21 provides a possible way to overcome the problem.

Definition 6.1. *An element $X \in \mathscr{A}$ has a (strong-) inverse in $\mathscr{A}$ if there exists $X^{-1} \in R_s(X) \cap L_s(X)$ such that*

$$X \bullet X^{-1} = X^{-1} \bullet X = \mathbb{I}$$

Due to Proposition 3.21, the inverse, when it exists, is unique.

Remark 6.2. In spite of the fact that the weak-multiplication makes of any

CQ*-algebra, a partial *-algebra, the inverse will be always considered in the above *strong* sense because of its uniqueness. For this reason, in what follows, we will systematically omit (as far as no ambiguity arises) the adjective *strong* speaking of the inverse.

It is worth mentioning that for some elements of $\mathscr{A}$ the inverse may exists in an even stronger sense: this is defined coming back to the original lattice of multipliers $\{\mathscr{A}, R\mathscr{A}, L\mathscr{A}, \mathscr{A}_0\}$. We do not enter in the details because this definition is of little use.

We list in the next proposition, without proving them, some elementary properties of the inverse.

Proposition 6.3. *Let X, Y be invertible elements of $\mathscr{A}$. Then*
(i) $(X^{-1})^{-1} = X$;
(ii) *If* $X \in L_w(Y)$ *and* $Y^{-1} \in L_w(X^{-1})$ *then* $(X \circ Y)^{-1}$ *exists and* $(X \circ Y)^{-1} = Y^{-1} \circ X^{-1}$;
(iii) $(X^*)^{-1} = (X^{-1})^*$

Definition 6.4. *Let $X \in \mathscr{A}$. The domain of regularity $\Delta(X)$ of X is the following subset of $\mathbb{C}$*

$$\Delta(X) = \{z \in \mathbb{C} : (X-z)^{-1} \text{ exists in } \mathscr{A}\}$$

The resolvent $\rho(X)$ of X is the largest open subset of $\Delta(X)$ where the function $z \to f(z) = (X-z)^{-1}$ is analytic with respect to the norm of $\mathscr{A}$.
The set $\sigma(X) = \mathbb{C} \setminus \rho(X)$ is called the spectrum of X.

In general, $\rho(X) \subset \Delta(X)$, in contrast with the Banach algebra case.

Example 6.5. In $L^2(0, 1)$, let us consider the function $u(x) = x^{\frac{1}{4}}$ which is continuous in $[0, 1]$. It is readily seen that the spectrum of u in the C *-algebra $C[0, 1]$ is exactly the closed interval $[0, 1]$. Since $u^{-1}(x) = x^{-\frac{1}{4}}$ is in $L^2(0, 1)$, then $0 \in \Delta(u)$. Nevertheless, $0 \notin \rho(u)$. Indeed, setting $f(z) = (u-z)^{-1}$, we have $f'(0) = x^{-\frac{1}{2}} \notin L^2(0, 1)$. In conclusion, $\rho(u) = \Delta(u) \setminus \{0\}$.

As is clear, the function $f(z) = (X-z)^{-1}$ has the power series expansion $f(z) = \Sigma_{n=0}^{\infty} T_n (z-z')^n$ throughout the largest open disk with center z' contained in $\rho(X)$, for each $z' \in \rho(X)$. The coefficients T_n (which belong to $\mathscr{A}$) are given by

$$T_n = \frac{1}{2\pi i} \int_C \frac{f(w)dw}{(w-z')^{n+1}} = \frac{f^{(n)}(z')}{n!} \tag{31}$$

for each closed curve C surrounding z', with $C \subset \rho(X)$ (the integral is, clearly, understood to converge with respect to the norm of $\mathscr{A}$).

Lemma 6.6. *Let $z, z' \in \Delta(X)$; then $(X-z)^{-1} \circ (X-z')^{-1}$ is well defined and*

(32) $$(X-z)^{-1}-(X-z')^{-1} = (z-z')(X-z)^{-1}\circ(X-z')^{-1}$$

and therefore,

$$(X-z)^{-1}\circ(X-z')^{-1} = (X-z')^{-1}\circ(X-z)^{-1}, \quad \forall z, z' \in \Delta(X)$$

Proof. The strong product $(X-z)\bullet(X-z')^{-1}$ is, as is easily seen, well defined and one has

$$(X-z)\bullet(X-z')^{-1} = ((X-z')-(z-z'))\bullet(X-z')^{-1} = \mathbb{I}-(z-z')(X-z')^{-1}$$

where we made use of the distributivity of the weak multiplication.
Now, using Proposition 3.21 we get $(X-z)^{-1}\circ((X-z)\bullet(X-z')^{-1}) = (X-z')^{-1}$ and thus

$$(X-z')^{-1} = (X-z)^{-1}\circ(\mathbb{I}-(z-z')(X-z')^{-1})$$

So if $\Omega\in\mathcal{S}(\mathcal{A})$, $B_1, B_2\in R(\mathcal{A})$ we have

$$\begin{aligned}&\Omega((X-z')^{-1}B_1, B_2)\\ &\quad=\Omega((\mathbb{I}-(z-z')(X-z')^{-1})B_1, (X^*-\bar{z})^{-1}B_2)\\ &\quad=\Omega(B_1, (X^*-\bar{z})^{-1}B_2)-(z-z')\Omega((X-z')^{-1}B_1, (X^*-\bar{z})^{-1}B_2)\end{aligned}$$

This implies that $(X-z)^{-1}\circ(X-z')^{-1}$ is well defined and

$$(X-z)^{-1}-(X-z')^{-1} = (z-z')(X-z)^{-1}\circ(X-z')^{-1}$$

□

The first statement of the next Proposition is concerned with a very elementary aspect of the functional calculus. However in the framework of partial *-algebras it needs a non-trivial proof which puts in evidence how far partial *-algebras are from the ordinary *-algebras.

Proposition 6.7. *Let $X\in\mathcal{A}$ with $\rho(X)\neq\emptyset$. Then the following statements hold:*
(i) *If $z\in\rho(X)$, all weak powers $(X-z)^{-n}$ exist in $\mathcal{A}$ and, setting $f(z) = (X-z)^{-1}$, one has:*

(33) $$f^{(n)}(z) = n!(X-z)^{-(n+1)}$$

(ii) *If $X = X^*$ and $\sigma(X)\neq\emptyset$, then $(X-z)^{-1}\in\mathcal{A}_R\cap\mathcal{A}_L$, $\forall z\in\rho(X)\cap\mathbb{R}$.*

Proof. (i) We proceed by induction on n. Let $n = 1$. The function $f(z)$ is, clearly $\|\ \|$-continuous; so, if $\Omega\in\mathcal{S}(\mathcal{A})$; $B_1, B_2\in R\mathcal{A}$ making use of Lemma 6.6, we have

$$\lim_{z'\to z}\Omega\left(\frac{f(z')-f(z)}{z'-z}B_1, B_2\right) = \lim_{z'\to z}\Omega((X-z')^{-1}B_1, (X^*-\bar{z})^{-1}B_2)$$

But

$$|\Omega(((X-z')^{-1}-(X-z)^{-1})B_1, (X^*-\bar{z})^{-1}B_2)|$$
$$\leq \|(X-z')^{-1}-(X-z)^{-1}\| \, \|B_1\|_\flat \|(X^*-\bar{z})^{-1}\| \, \|B_2\|_\flat .$$

And therefore $\Omega(f'(z)B_1, B_2) = \Omega((X-z)^{-1}B_1, (X^*-\bar{z})^{-1}B_2)$.
Let $n \in \mathbb{N}$ and assume that $\forall r \leq n$ $(X-z)^{-r}$ exists and that

$$f^{(r-1)}(z) = (r-1)!(X-z)^{-r}$$

Then, for any $k \geq 1$ we have

$$\Omega(f^{(n)}(z)B_1, B_2) = \lim_{z' \to z} \Omega\left(\frac{f^{(n-1)}(z')-f^{(n-1)}(z)}{z'-z} B_1, B_2\right)$$
$$= (n-1)! \lim_{z' \to z} \Omega\left(\frac{(X-z')^{-n}-(X-z)^{-n}}{z'-z} B_1, B_2\right)$$
$$= (n-1)! \lim_{z' \to z} \Omega\left((X-z')^{-(n-k)}B_1, \frac{(X^*-\overline{z'})^{-k}-(X^*-\bar{z})^{-k}}{z'-z} B_2\right)$$
$$+(n-1)! \lim_{z' \to z} \Omega\left(\frac{(X-z')^{-(n-k)}-(X-z)^{-(n-k)}}{z'-z} B_1, (X^*-\bar{z})^{-k}B_2\right)$$
$$= (n-1)!\, k\Omega((X-z)^{-(n-k)}B_1, (X^*-\bar{z})^{-(k+1)}B_2)$$
$$+(n-1)!\,(n-k)\Omega((X-z)^{-(n-k+1)}B_1, (X^*-\bar{z})^{-k}B_2)$$

This implies that $(X-z)^{-k} \circ (X-z)^{-(n-k+1)}$ is well-defined if, and only if, $(X-z)^{-(k+1)} \circ (X-z)^{-(n-k)}$ is well-defined. We will now show that if $(X-z)^{-n}$ is well-defined then $(X-z)^{-k} \circ (X-z)^{-l}$ exists for any k, l such that $k+l = n+1$, hence the weak-power $(X-z)^{-(n+1)}$ is well-defined. Indeed, by the hypothesis of induction, we get

$$(X-z)^{-k} = \frac{f^{(k-1)}(z)}{(k-1)!} = \frac{1}{2\pi i}\int_{|\xi-z|=r_1} \frac{f(\xi)d\xi}{(\xi-z)^k}, \quad 1 \leq k \leq n-1$$

and thus, applying the calculus of residues, we obtain for $\Omega \in \mathcal{S}(\mathcal{A})$, $B_1, B_2 \in R\mathcal{A}$

$$\Omega\left(\frac{1}{2\pi i}\int_{|\eta-z|=r_2} \frac{f(\eta)d\eta}{(\eta-z)^l} B_1, -\frac{1}{2\pi i}\int_{|\xi-z|=r_1} \frac{f^*(\xi)d\bar{\xi}}{(\bar{\xi}-\bar{z})^k} B_2\right)$$
$$= -\frac{1}{4\pi^2}\Omega\left(\int_{|\xi-z|=r_1}\int_{|\eta-z|=r_2} \frac{f(\xi)\circ f(\eta)}{(\xi-z)^k(\eta-z)^l}\, d\xi\, d\eta\, B_1, B_2\right)$$
$$= -\frac{1}{4\pi^2}\Omega\left(\int_{|\xi-z|=r_1}\int_{|\eta-z|=r_2} \frac{f(\xi)-f(\eta)}{(\xi-\eta)(\xi-z)^k(\eta-z)^l}\, d\xi\, d\eta\, B_1, B_2\right)$$
$$= \frac{1}{2\pi i}\Omega\left(\int_{|\xi-z|=r_1} \frac{f(\xi)d\xi}{(\xi-z)^{k+l}} B_1, B_2\right) = \Omega(f^{(n)}(z)B_1, B_2)$$

where we also made use of Lemma 6.6. Therefore, since the right hand side exists due to the analiticity of $f(z)$, $(X-z)^{-k}\circ(X-z)^{-l}$ exists for any k, l such that $k+l=n+1$; from this it follows

(34) $$f^{(n)}(z)=(n-1)!k(X-z)^{-(k+1)}\circ(X-z)^{-(n-k)} \\ +(n-1)!(n-k)(X-z)^{-k}\circ(X-z)^{-(n-k+1)}$$

and then

(35) $$f^{(n)}(z)=n!(X-z)^{-(n+1)}$$

(ii) Since $\sigma(X)\neq\emptyset$, for $z\in\rho(X)$ and $r\leq d(z,\partial\Omega)\neq 0$, we get the inequality

$$\begin{aligned}\|(X-z)^{-k}\| &= \left\|\frac{1}{2\pi i}\int_{|\xi-z|=r}\frac{f(\xi)d\xi}{(\xi-z)^k}\right\| \\ &\leq \frac{1}{2\pi}\int_{|\xi-z|=r}\frac{\|(X-\xi)^{-1}\|\,|d\xi|}{|(\xi-z)^k|} \\ &\leq \frac{1}{2\pi}\frac{1}{r^k}\int_{|\xi-z|=r}\|(X-\xi)^{-1}\|\,|d\xi|.\end{aligned}$$

If $z\in\mathbb{R}$ then Proposition 5.6 can be applied. □

Remark 6.8. Notice that the analiticity of $f(z)=(X-z)^{-1}, z\in\rho(X)$ implies the existence of $(X-z)^{-n}$, $\forall n\in\mathbb{N}$ but not the existence of $(X-z)^n$ for $n>1$. As an example, let us consider the CQ*-algebra $(L^2(X,\mu), C(X))$ where $X=[0,1]$ and μ is the Lebesgue measure on X. The function $v(x)=x^{-\frac{1}{4}}$ is in $L^2(X,\mu)$; obviously, $0\in\Delta(v)$ since $v^{-1}(x)=x^{\frac{1}{4}}\in L^2(X,\mu)$ and an easy computation shows that actually $0\in\rho(v)$. We have $v^{-n}(x)=x^{\frac{n}{4}}\in L^2(X,\mu)$, $\forall n\in\mathbb{N}$, but $v^2(x)=x^{-\frac{1}{2}}\notin L^2(X,\mu)$.

We will now prove that if the absolute value of a complex number z is bigger than both $\|X\|_R$ and $\|X\|_L$, for a certain fixed $X\in\mathscr{A}_R\cap\mathscr{A}_L$, then z belongs to a the domain of regularity of X.

In what follows we will set

$$\|A\|_{\bar\alpha}\equiv\|A^*\|_\alpha \quad \forall A\in\mathscr{A}$$

Lemma 6.9. *Let $X\in\mathscr{A}_R$, $C\in D(\overline{L_X})$, and let $z\in\mathbb{C}$ such that $|z|>\|X\|_R$. Therefore*

$$\|(X-z)\bullet C\|_\alpha\geq\|C\|_\alpha(|z|-\|X\|_R).$$

Analogously, if $X\in\mathscr{A}_L$, $B\in D(\overline{R_X})$, and let $\mu\in\mathbb{C}$ such that $|\mu|>\|X\|_L$ then

$$\|B\bullet(X-\mu)\|_{\bar\alpha}\geq\|B\|_{\bar\alpha}(|\mu|-\|X\|_L).$$

Proof. We start by proving the first statement for an element $C\in R\mathscr{A}$.

In our hypothesis, taking an $\Omega \in \mathscr{S}(\mathscr{A})$ we can easily prove the following inequality:

$$\Omega((X-z)C, (X-z)C) \geq \Omega(C, C)(|z| - \| X \|_R)^2$$

Taking the supremum over the family $\mathscr{S}(\mathscr{A})$ we get the statement for elements in $R\mathscr{A}$. The general result is obtained with a simple limit argument, using the fact that $\| \|$-convergence implies $\| \|_\alpha$-convergence.

The left counterpart of the lemma is proved in the very same way, starting with the state Ω^*, for a given $\Omega \in \mathscr{S}(\mathscr{A})$. $\square$

Lemma 6.10. *Let $(\mathscr{A}, *, R\mathscr{A}, \flat)$ be a *-semisimple CQ^*-algebra and $X \in \mathscr{A}$.*
i) *If $\| X \|_R < \infty$ and if $|z| > \| X \|_R$ then the set*

$$\text{Ran}\, L_{X-z} \equiv \{(X-z)B \ : B \in R\mathscr{A}\}$$

is $\| \|_\alpha$-dense in $\mathscr{A}$.
ii) *If $\| X \|_L < \infty$ and if $|z| > \| X \|_L$ then the set*

$$\text{Ran}\, R_{X-z} \equiv \{B(X-z) \ : B \in L\mathscr{A}\}$$

is $\| \|_\alpha$-dense in $\mathscr{A}$.

Proof. Were it not so, then there would exist a non zero $\| \|_\alpha$-continuous functional F on $\mathscr{A}$ such that $F((X-z)B) = 0 \, \forall B \in R\mathscr{A}$. Therefore we should have $F(XB) = zF(B) \, \forall B \in R\mathscr{A}$.
From the $\| \|_\alpha$-continuity of F we get $|F(XB)| \leq \| F \|^\alpha \| XB \|_\alpha$. Using the definition of $\| \|_R$ we deduce that $\| XB \|_\alpha \leq \| X \|_R \| B \|_\alpha$, so that

$$|F(XB)| \leq \| F \|^\alpha \| X \|_R \| B \|_\alpha.$$

Defining $F_X(B) \equiv F(XB)$, $\forall B \in R\mathscr{A}$ and computing $\| F \|_0^\alpha$, that is the norm of the functional restricted to $R\mathscr{A}$, we find the following contradictory inequality: $|z| \leq \| X \|_R$. In finding this result one also has to use that $F_X = zF$.

This proof can also be adapted with minor modifications to prove the left counterpart of the statement. $\square$

Proposition 6.11. *Let $(\mathscr{A}, *, R\mathscr{A}, \flat)$ be a *-semisimple and regular CQ^*-algebra. Let $X \in \mathscr{A}_R \cap \mathscr{A}_L$ and $z \in \mathbb{C}$ such that $|z| > \max\{\| X \|_R, \| X \|_L\}$. Then the inverse $(X-z)^{-1}$ exists. Moreover, $(X-z)^{-1} \in \mathscr{A}_\lambda \cap \mathscr{A}_\rho$ and*

$$\{z \in \mathbb{C} \ : |z| > \max\{\| X \|_R, \| X \|_L\}\} \subseteq \rho(X).$$

Proof. Using the previous Lemmas we can prove that the following sets

$$\text{Ran}\, \bar{L}_{X-z} \equiv \{(X-z) \bullet B \ : B \in D(\overline{L_X})\}$$

and

$$\mathrm{Ran}\,\bar{R}_{X-z} \equiv \{B \bullet (X-z)\ : B \in D(\overline{R_X})\}$$

both coincide with the whole space $\mathscr{A}$. The reason is that $\mathrm{Ran}\bar{L}_{X-z} \supset \mathrm{Ran}L_{X-z}$ and $\mathrm{Ran}\bar{R}_{X-z} \supset \mathrm{Ran}R_{X-z}$, so they are both dense in $\mathscr{A}$ due to Proposition 6.10. Using Lemma 6.9 one can prove that both the sets are $\|\ \|$-closed, so that $\mathrm{Ran}\,\bar{L}_{X-z} \equiv \mathrm{Ran}\,\bar{R}_{X-z} \equiv \mathscr{A}$. In this last step the regularity of the algebra plays a crucial role.

Since our CQ*-algebra contains the unity $\mathbb{I}$ we deduce that there exist $B_1 \in D(\overline{L_X})$ and $B_2 \in D(\overline{R_X})$ such that $(X-z) \bullet B_1 = \mathbb{I}$ and $B_2 \bullet (X-z) = \mathbb{I}$. In this way we have defined a left and a right inverse. Due to Proposition 2.6, which ensures the associativity of the product in this situation, we have

$$B_2 = B_2 \bullet \mathbb{I} = B_2 \bullet ((X-z) \bullet B_1) = (B_2 \bullet (X-z)) \bullet B_1 = \mathbb{I} \bullet B_1 = B_1.$$

Therefore the inverse of $(X-z)$ exists.
The maps

$$\Lambda_{(X-z)^{-1}}: A \in \mathscr{A} \mapsto (X-z)^{-1} \bullet A \in \mathscr{A}$$

and

$$P_{(X-z)^{-1}}: A \in \mathscr{A} \mapsto A \bullet (X-z)^{-1} \in \mathscr{A}$$

are, therefore, everywhere defined and as is easily seen, closed in $\mathscr{A}$. Hence $(X-z)^{-1} \in \mathscr{A}_\lambda \cap \mathscr{A}_\rho$.
Let now $z_0 \in \mathbb{C}$ satisfy $|z_0| > \max\{\|X\|_R, \|X\|_L\}$ and $z \in \mathbb{C}$ such that $|z-z_0| \leq \|(X-z_0)^{-1}\|_{\sharp}^{-1}$ then, by Proposition 5.9, the power (in strong sense) series

$$(X-z_0)^{-1}\left\{\mathbb{I} + \sum_{n=1}^{\infty}(z-z_0)^n (X-z_0)^{-n}\right\}$$

converges with respect to $\|\ \|_{\sharp}$ to an element Y of $\mathscr{A}_\lambda$. It is easily checked that $Y = (X-z)^{-1}$. Hence the function $f(z) = (X-z)^{-1}$ admits a Taylor expansion at z_0 and is, therefore, analytic. □

There are yet several aspects of the functional calculus on a CQ*-algebra that should be investigated in details: first of all the spectral characterization of positive elements. We hope to discuss them in a further paper.

Acknowledgment

We thank Prof. J.-P. Antoine and Prof. G. Epifanio for stimulating discussions. We also thank Prof. A. Inoue and the Referee for their suggestions on Examples 2.4 and 2.8.

References

[1] Lassner, G., Topological algebras and their applications in quantum statistics, *Wiss. Z. KMU, Leipzig, Math.-Nat. R.*, **6** (1981), 572–595.
[2] ———, Algebras of Unbounded Operators and Quantum Dynamics, *Physica*, **124A**

(1984), 471–480.
[3] Haag, R. and Kastler, D., An Algebraic Approach to Quantum Field Theory, *J. Math. Phys.,* **5** (1964), 848–861.
[4] Bagarello, F. and Trapani, C., States and representations of CQ^*-algebras, *Ann. Inst. Henri Poincaré*, **61** (1994), 103–133.
[5] ———, L^p-spaces as Quasi *-algebras, *J. Math. Anal. Appl.* **197** (1996), 810–824.
[6] Antoine, J.-P. and Karwowski, W., Partial *-Algebras of Closed Linear Operators in Hilbert Space, *Publ. RIMS, Kyoto Univ.,* **21** (1985), 205–236; Add./Err. ibid. **22** (1986), 507–511.
[7] Antoine, J.-P. and Mathot, F., Partial *-Algebras of Closed Operators and their Commutants, I. General Structure, *Ann. Inst. H. Poincaré,* **46** (1987), 299–324.
[8] Antoine, J.-P., Mathot, F. and Trapani, C., Partial *-Algebras of Closed Operators and their Commutants, II. Commutants and Bicommutants, *Ann. Inst. H. Poincaré,* **46** (1987), 325–351.
[9] Antoine, J.-P., Inoue, A. and Trapani, C., Partial *-Algebras of Closable Operators, I: The Basic Theory and the Abelian Case, *Publ. RIMS, Kyoto Univ.,* **26** (1990), 359–395.
[10] ———, Partial *-Algebras of Closable Operators, II: States and Representations of Partial *-Algebras, *Publ. RIMS, Kyoto Univ.,* **27** (1991), 399–430.
[11] Lassner, G. and Lassner, G. A., Qu*-algebras and Twisted Product, *Publ. RIMS, Kyoto Univ.,* **25** (1989), 279–299.
[12] Sakai, S., *C *-algebras and W *-algebras*, Springer-Verlag, New York, 1971.
[13] Zaanen, A. C., *Integration*, North Holland Publ. Co., Amsterdam, 1967.
[14] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics, vol.1*, Academic Press, New York, 1972.
[15] Kato, T., *Perturbation theory for linear operators*, Springer-Verlag, New York, 1966.
[16] Stratila, S. and Zsido, L., *Lectures on von Neumann Algebras*, Abacus Press, Tonbridge Wells, 1979.
[17] Inoue, A., Extension of unbounded left Hilbert algebras to partial *-algebras, *J. Math. Phys.,* **32** (1991), 323–331.
[18] Segal, I. E., A non-commutative extension of abstract integration, *Ann. Math.,* **57** (1953), 401–457.
[19] Inoue, A., On a class of unbounded operators II, *Pacific. J. Math.,* **66** (1976), 411–431.
[20] Köthe, G., *Topological Vector Spaces II*, Springer-Verlag, New York, 1979.
[21] Antoine, J.-P., Inoue, A. and Trapani, C., On the Regularity of the Partial O^*-Algebra Generated by a Closed Symmetric Operator, *Publ. RIMS, Kyoto Univ.,* **28** (1992), 757–774.