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Cohomology Classification of Self Maps of Sphere Bundles over Spheres

Dedicated to Professor Yasutoshi Nomura on his 60th birthday

By

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§ 1. Introduction and Statement of Results

Let $\mathcal{X} \in \pi_{n-1}(SO(q+1))$. We denote the induced q-sphere bundle over the n-sphere by $E(\mathcal{X})$ or simply E. The purpose of this note is to study the image of the function

$$[E(\mathcal{X}),E(\mathcal{X})] \to \mathrm{Hom}(\tilde{H}^*(E(\mathcal{X}))/\mathrm{Tor}, \tilde{H}^*(E(\mathcal{X}))/\mathrm{Tor})$$

which assigns the induced homomorphism, where $\tilde{H}^m(X)$ is the reduced m-th cohomology group of a space X with values in $\mathbb{Z}$, the group of integers. Let p_1: $SO(q+1) \to S^q$ be the canonical projection. We denote $p_{1*}(\mathcal{X})$ by $\alpha \in \pi_{n-1}(S^q)$.

According to [13],

$$E(\mathcal{X}) = S^q \cup_\alpha e^n \cup_\rho e^{n+q},$$

where ρ is the attaching map of the top cell of $E(\mathcal{X})$. Let $Y = S^q \cup_\alpha e^n$. When $n=1$ or when $q=1$ and $\alpha=0$, the function is surjective by [5]. In this note, if we do not specify otherwise, we will always assume

$$q \geq 2,\ n \geq 2, \text{ and } \alpha = 0 \text{ provided } n = q+1.$$

In this case, note from [5, 6] that

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$$\tilde{H}^*(Y) = \mathbb{Z}\{x_q, y_n\},$$
$$\tilde{H}^*(E(\chi)) = \mathbb{Z}\{x_q, y_n, x_q y_n\},\ y_n^2 = 0,$$

where $\deg(x_q) = q$ and $\deg(y_n) = n$. Let k, l be integers. When $q \neq n$, a self map f of $E(\chi)$ or Y is called an $M(k,l)$-structure if $f^*(x_q) = kx_q$ and $f^*(y_n) = ly_n$. Let (a_{ij}) be a 2×2-matrix whose entries a_{ij} are integers. When $q = n$, a self map f of $E(\chi)$ or $Y = S^n \vee S^n$ is called an (a_{ij})-structure with respect to $\{x_n, y_n\}$ if $f^*(x_n) = a_{11}x_n + a_{12}y_n$ and $f^*(y_n) = a_{21}x_n + a_{22}y_n$. When no confusion will occur, we will omit the words "with respect to $\{x_n, y_n\}$". Notice that when $q \neq n$, an $M(k,l)$-structure is an M_θ-structure [4] for any $\theta : \{1,2,\ldots\} \to \mathbb{Z}$ with $\theta(q) = k$ and $\theta(n) = l$.

We will study conditions on the existence of an $M(k,l)$-structure and an (a_{ij})-structure on $E(\chi)$ in § 2 and § 3, respectively. Our results are partial when $E(\chi)$ does not have a section. To state our results, we need some notations. When $E(\chi)$ has a section, we denote by ξ an element of $\pi_{n-1}(SO(q))$ such that $i_*(\xi) = \chi$, where $i : SO(q) \to SO(q+1)$ is the inclusion. Let ι_m denote the identity map of S^m and $J : \pi_r(SO(m)) \to \pi_{r+m}(S^m)$ the J-homomorphism.

Theorem 1. *When $q \neq n$ and $E(\chi)$ has a section, $E(\chi)$ has an $M(k,l)$-structure if and only if*

$$klJ(\xi) - k\iota_q \circ J(\xi) = k[\iota_q, \beta] \text{ for some } \beta \in \pi_n(S^q).$$

In particular, when $q > n$, $E(\chi)$ has an $M(k,l)$-structure if and only if

$$k(l-1)J(\chi) = 0.$$

Theorem 2. *When $q = n$, there exists a basis $\mathfrak{B} = \{x_n, y_n\}$ such that $E(\chi)$ has an (a_{ij})-structure with respect to $\mathfrak{B}$ if and only if one of the following holds.*

(1) $n = 1, 3, 7$ *and* (a_{ij}) *is arbitrary.*

(2) $n \equiv 1 \pmod 2$ *with* $n \neq 1, 3, 7$, $\chi = 0$ *and* $a_{11}a_{21} \equiv a_{12}a_{22} \equiv 0 \pmod 2$.

(3) $n \equiv 0 \pmod 2$, $\chi = 0$ *and* $a_{11}a_{21} = a_{12}a_{22} = 0$.

(4) $n = 2$, $\chi \neq 0$ *and*

$$a_{11}^2 + a_{11}a_{21} = a_{11}a_{22} + a_{12}a_{21},$$
$$a_{12}(a_{12} + 2a_{22}) = a_{11}(a_{12} - a_{21}) = 0.$$

(5) $n = 4, 8$, $\chi = m\theta (m \neq 0)$, *where θ is a generator such that $-J(\theta)$ is the suspension of the Hopf map, and*

$$ma_{11}^2+2a_{11}a_{21} = m(ma_{11}a_{21}+a_{11}a_{22}+a_{12}a_{21}),$$

$$ma_{11}(a_{11}-1)+2a_{11}a_{21} \equiv ma_{12} \equiv 0 \pmod{2b},$$

$$a_{12}(ma_{12}+2a_{22}) \equiv ma_{11}(a_{12}-a_{21}) = 0,$$

where b is 12 or 24 according as n is 4 or 8.

(6) $n \equiv 0 \pmod 2$ *with* $n \neq 2, 4, 8$, $\mathcal{X} \neq 0$ *and*

$$a_{11}a_{21} = a_{12}a_{22} = a_{11}(a_{22}-1)J(\mathcal{X}) = a_{12}J(\mathcal{X}) = 0.$$

(7) $n \equiv 1 \pmod 8 \geq 9$, $\mathcal{X} \neq 0$ *and* $a_{12} \equiv a_{11}a_{21} \equiv a_{11}(a_{22}-1) \equiv 0 \pmod 2$.

Theorem 3. *If $q \neq n$ and $E(\mathcal{X})$ has an $M(k,l)$-structure, then*

(1) $$k\iota_q \circ \alpha = l\alpha$$

and there exists an element $y \in \pi_{n+q}(S^n)$ such that

(2) $$(\Sigma\alpha)\circ y = k\iota_{q+1} \circ J(\mathcal{X}) - klJ(\mathcal{X}).$$

Theorem 4. *Suppose that there exist integers a,b,c such that $a\iota_q \circ \alpha = 0$, $b\iota_q \circ \bar{a} \circ \rho = 0$, and $c\chi = 0$, where $\bar{a} : Y \to S^q$ is an extension of $a\iota_q$. If $k \equiv 0 \pmod{ab}$ and $l \equiv 0 \pmod c$, then there exists an $M(k,l)$-structure on $E(\mathcal{X})$.*

These theorems except Theorem 2 will be proved in § 2. Theorem 2 will be proved in § 3. As applications of these theorems, we will give partial results on the Stiefel manifolds of 2-frames: $V_{n+2,2} = O(n+2)/O(n)$, $W_{n+2,2} = U(n+2)/U(n)$ and $X_{n+2,2} = Sp(n+2)/Sp(n)$ in § 4, § 5 and § 6, respectively. For example, in § 5, we will prove

Theorem 5. (1) *If n is 0 or 2, then $W_{n+2,2}$ has an $M(k,l)$-structure for all k and l.*

(2) *When n is even with $n \geq 4$, $W_{n+2,2}$ has an $M(k,l)$-structure if and only if*
$k(l-1) \equiv 0 \pmod 8$ *or* $k(l-5) \equiv 0 \pmod 8$.

(3) *$W_{3,2}$ has an $M(k,l)$-structure if and only if $k \equiv l \pmod 2$.*

(4) *If n is odd with* $n \geq 3$ and *$W_{n+2,2}$ has an $M(k,l)$-structure, then*
$k \equiv 0,1 \pmod 4$ *and* $k \equiv l \pmod 2$.

(5) *When n is odd with $n \geq 3$, $W_{n+2,2}$ has an $M(k,l)$-structure in the following two cases*:

$k \equiv 0 \pmod 4$ *and* $l \equiv 0 \pmod 2$,
k is the square of an odd integer and l is odd.

We use the following notations. Let $j: S^q \to Y$ be the inclusion. Given an element β of a group, we denote by $\#\beta$ the order of β or zero according as β has a finite order or not. Given a subset B of a group, $\langle B \rangle$ denotes the subgroup generated by B. We denote by H the Hopf invariant $\pi_{2n-1}(S^n) \to \mathbb{Z}$ and the 0-th Hopf-Hilton homomorphism $\pi_m(S^n) \to \pi_m(S^{2n-1})$ (cf., [22]).

In § 7, we will give results on cohomology classification of self maps of the suspension of $E(\mathcal{X})$.

§ 2. Generalities

The following lemma is probably well-known.

Lemma 2.1. *Let $q \geq 2$ and $n \geq q+1$. Assume that K is a $(q-1)$-connected CW-complex. Let $\beta: S^{n-1} \to K$ and let K^* be the mapping cone of β. For $r \leq n+q-3$, there exists an exact sequence which makes the following diagram commutative;*

$$
\begin{array}{ccccccccc}
\pi_{r+1}(K^*) & \xrightarrow{\Delta} & \pi_r(S^{n-1}) & \xrightarrow{\beta_*} & \pi_r(K) & \xrightarrow{j_*} & \pi_r(K^*) & \longrightarrow & \cdots \\
 & & \cong \downarrow \Sigma & & & & & & \\
\| & & \pi_{r+1}(S^n) & & \| & & \| & & \\
 & & \uparrow p_* & & & & & & \\
\pi_{r+1}(K^*) & \longrightarrow & \pi_{r+1}(K^*,K) & \xrightarrow{\partial} & \pi_r(K) & \xrightarrow{j_*} & \pi_r(K^*) & \longrightarrow & \cdots,
\end{array}
$$

where the lower horizontal sequence is the homotopy exact sequence of the pair (K^*,K) and p: $(K^*,K) \to (S^n, *)$ is the pinching map. Moreover if $\pi_q(K) = \mathbb{Z}\{\theta\}$, then

$$\operatorname{Ker}\{j_*: \pi_{n+q-2}(K) \to \pi_{n+q-2}(K^*)\} = \operatorname{Image} \beta_* + \langle [\theta,\beta] \rangle,$$

where $[\theta, \beta]$ is the Whitehead product of θ and β.

Proof. By Blakers-Massey theorem, $p_*: \pi_{r+1}(K^*,K) \to \pi_{r+1}(S^n)$ is isomorphic for $r \leq n+q-3$ and is epimorphic for $r=n+q-2$. Let F be the homotopy fiber of $j: K \to K^*$. Then there exists a map $f: F \to \Omega S^n$ such that the following diagram of fiber sequences commutes:

$$
\begin{array}{ccccccc}
\Omega K^* & \longrightarrow & F & \xrightarrow{i} & K & \xrightarrow{j} & K^* \\
\downarrow \Omega p & & \downarrow f & & \downarrow & & \downarrow p \\
\Omega S^n & = & \Omega S^n & \longrightarrow & * & \longrightarrow & S^n.
\end{array}
$$

Moreover the following diagram commutes:

$$\begin{array}{ccccc} \pi_{r+1}(S^n) & \xleftarrow{p_*} & \pi_{r+1}(K^*,K) & \xrightarrow{\partial} & \pi_r(K) \\ \downarrow\cong & & \downarrow\cong & & \| \\ \pi_r(\Omega S^n) & \xleftarrow{f_*} & \pi_r(F) & \xrightarrow{i_*} & \pi_r(K) \end{array}$$

where the vertical isomorphisms are canonical ones (see (8.20) of [22]). Let $\gamma \in \pi_{n-1}(F) \cong \mathbb{Z}$ be a generator, which corresponds to $\hat{\beta} \in \pi_n(K^*,K)$. Then, the above diagrams imply that $f_*\gamma = \Sigma_*(\iota_{n-1})$ *and* $i_*(\gamma) = \beta$, where $\Sigma : S^{n-1} \to \Omega S^n$ is the suspension. This implies that for $r=n-1$ the middle rectangle of the diagram in Lemma 2.1 commutes.

We define Δ to make the first rectangle of the diagram in Lemma 2.1 commutative.

Let $r \leq n+q-2$ and $a \in \pi_r(F)$. Since $f_*: \pi_r(F) \to \pi_r(\Omega S^n)$ and the suspension $\Sigma_*: \pi_r(S^{n-1}) \to \pi_r(\Omega S^n)$ are surjective, there exists an element $a_1 \in \pi_r(S^{n-1})$ such that $f_*(a) = \Sigma_*(a_1) = f_*(\gamma \circ a_1)$, hence $a - \gamma\circ a_1 \in \mathrm{Ker}(f_*)$. When $r \leq n+q-3$, since f_* and Σ_* are isomorphic, we have $a = \gamma\circ a_1$ so that $i_*(a) = i\circ\gamma\circ a_1 = \beta\circ a_1$. Therefore we have proved the commutativity and exactness of Lemma 2.1 for $r \leq n+q-3$. Let $r=n+q-2$ and suppose $\pi_q(K) = \mathbb{Z}\{\theta\}$. It then follows from the James exact sequence [11] that the kernel of $p_*: \pi_{n+q-1}(K^*,K) \to \pi_{n+q-1}(S^n)$ is generated by the relative Whitehead product $[\theta,\hat{\beta}]$, where $\hat{\beta}$ is the characteristic map of the cell of K^*, which is attached by β. We then have

$$i_*(a-\gamma\circ a_1) \in i_*(\mathrm{Ker}(f_*)) = \partial(\mathrm{Ker}(p_*)) = \partial\langle[\theta,\hat{\beta}]\rangle = \langle[\theta,\beta]\rangle$$

and $i_*(a-\gamma\circ a_1) = i_*(a) - i\circ\gamma\circ a_1 = i_*(a) - \beta\circ a_1$. Hence

$$\mathrm{Image}\{i_*: \pi_{n+q-2}(K) \to \pi_{n+q-2}(K^*)\} \subset \mathrm{Image}(\beta_*) + \langle[\theta,\beta]\rangle \subset \mathrm{Image}(i_*).$$

Thus $\mathrm{Image}(i_*) = \mathrm{Image}(\beta_*) + \langle[\theta,\beta]\rangle$ and the result follows from the equalities $\mathrm{Ker}(j_*) = \mathrm{Image}(\partial) = \mathrm{Image}(i_*)$. This completes the proof of Lemma 2.1. □

Lemma 2.2. (1) *Let* $q,n \geq 1$. *Then there exists a self map* h *of* Y *such that* $h\circ j = j\circ k\iota_q$ *and* $p\circ h = l\iota_n\circ p$ *if and only if* $k\iota_q\circ\alpha = l\alpha$, *where* $p: Y \to S^n$ *is the quotient map.*

(2) *When* $q \neq n$, $E(X)$ *has an* $M(k,l)$*-structure if and only if there exists an* $M(k,l)$*-structure* h *on* Y *such that* $h\circ\rho = kl\rho$.

Proof. When $n \leq q+1$ and $\alpha = 0$, (1) is obvious. When $n=q+1 \geq 2$ and $\alpha \neq 0$, (1) holds, since $k\iota_q\circ\alpha = l\alpha$ if and only if $k=l$. Let $n \geq q+2$. When $q=1$, the bundle is trivial so that (1) is obvious. Let $q \geq 2$. Suppose given a self map h of Y satisfying the properties in (1). Then, by Lemma 2.1, there is an integer m satisfying $k\iota_q \circ \alpha = \alpha \circ m\iota_{n-1}$ so that there exists a self map h' of Y such that $h' \circ j = j \circ k\iota_q$ and $m\iota_n \circ p = p \circ h'$. Since $j^*(h) = j^*(h')$, there exists an element $b \in \pi_n(Y)$ such that $h' = h^b$. Here h^b is the composition of

$$Y \xrightarrow{c} Y \vee S^n \xrightarrow{h \vee b} Y$$

where c is the cooperator [8]. Then $m\iota_n \circ p = p \circ h' = p \circ h^b = (p \circ h)^{p \circ b} = (l\iota_n + p \circ b) \circ p$. Considering the induced homomorphisms of these maps on cohomology, we have $m\iota_n = l\iota_n + p \circ b$. Since $p_*: \pi_n(Y, S^q) \cong \pi_n(S^n)$, it follows from the homotopy exact sequence of the pair (Y, S^q) that $p \circ b = x\iota_n$ with $x \equiv 0 \pmod{\#\alpha}$. Hence $m = l + x \equiv l \pmod{\#\alpha}$ so that $k\iota_q \circ \alpha = m\alpha = l\alpha$. Conversely if $k\iota_q \circ \alpha = l\alpha$, then there is a desired map. This ends the proof of (1).

To prove (2), suppose that $E(\mathcal{X})$ has an $M(k,l)$-structure f. Then $h = f|_Y$ is an $M(\mathrm{k},l)$-structure on Y. By 2.1, there exists an integer m with $h \circ \rho = \rho \circ m\iota_{n+q-1}$ so that there is a self map f' of $E(\mathcal{X})$ with $f' \circ j = j \circ h$, where $j: Y \to E(\mathcal{X})$ is the inclusion. By the method used above, we can prove $m \equiv kl \pmod{\#\rho}$ so that $h \circ \rho = kl\rho$ as desired. The converse is apparently true. This completes the proof of (2). □

Recall that if $E(\mathcal{X})$ has a section, then $\alpha = 0$, $Y = S^q \vee S^n$ and there exists an element $\xi \in \pi_{n-1}(SO(q))$ such that $i_*(\xi) = \mathcal{X} \in \pi_{n-1}(SO(q+1))$, where $i: SO(q) \to SO(q+1)$ is the inclusion. By James-Whitehead [13], we have

$$\rho = [i_q, i_n] + i_{q*}J(\xi), \tag{2.3}$$

where i_q and i_n are the obvious inclusion maps.

Proof of Theorem 1. First suppose that $n \geq q+1$. Note that a map $h: S^q \vee S^n \to S^q \vee S^n$ gives an $M(k,l)$-structure if and only if $h \circ i_q = ki_q$ and $h \circ i_n = li_n + i_q \circ \beta$ for some $\beta \in \pi_n(S^q)$. Therefore,

$$\begin{aligned} h \circ [i_q, i_n] &= [h \circ i_q, h \circ i_n] = [ki_q, li_n + i_q \circ \beta] \\ &= kl[i_q, i_n] + k[i_q, i_q \circ \beta] = kl[i_q, i_n] + k(i_q \circ [\iota_q, \beta]). \end{aligned}$$

On the other hand, $h \circ i_q \circ J(\xi) = i_q \circ k\iota_q \circ J(\xi)$. By using (2.3), we get $kl\rho - h \circ \rho = i_{q*}(klJ(\xi) - k\iota_q \circ J(\xi) + k[\iota_q, \beta])$. Since i_{q*} is monomorphic, we have the desired result by Lemma 2.2.

Second suppose that $n < q$. Let $h = h(k,x,l) \in [S^q \vee S^n, S^q \vee S^n]$ be the map corresponding to $k\iota_q \oplus x \oplus l\iota_n \in \pi_q(S^q) \oplus \pi_q(S^n) \oplus \pi_n(S^n)$ under the canonical isomorphism, that is $h \circ i_q = ki_q + i_n \circ x$ and $h \circ i_n = li_n$. It follows that $h_*[i_q, i_n] = [h \circ i_q, h \circ i_n] = [ki_q + i_n \circ x, li_n] = kl[i_q, i_n] + li_{n*}[x, \iota_n]$ and

$$\begin{aligned} h_* i_{q*} J(\xi) &= (ki_q + i_n \circ x) \circ J(\xi) \\ &= ki_q \circ J(\xi) + i_n \circ x \circ J(\xi), \quad \text{by p.534 in [22]}, \\ &= k(i_q \circ J(\xi)) + i_n \circ x \circ J(\xi) \end{aligned}$$

so that $kl\rho - h_*\rho = i_{q*}(k(l-1)J(\xi)) - i_{n*}(l[x,\iota_n] + x \circ J(\xi))$. Hence $kl\rho = h_*\rho$ if and only if $k(l-1)J(\xi) = 0$ and $l[x,\iota_n] + x \circ J(\xi) = 0$. If there exists an $M(k,l)$-structure on E, then there is an $M(k,l)$-structure g on Y with $kl\rho = g_*\rho$ by Lemma 2.2 so that $k(l-1)J(\xi) = 0$ by the above discussion. Conversely if $k(l-1)J(\xi) = 0$, then $l[0,\iota_n] + 0 \circ J(\xi) = 0$ and $kl\rho = h(k,0,l)_*\rho$ by the above discussion so that there is an $M(k,l)$-structure on E. Since $\Sigma J(\xi) = -J(\mathcal{X})$ and $\Sigma: \pi_{n+q-1}(S^q) \cong \pi_{n+q}(S^{q+1})$, it follows that $\#J(\xi) = \#J(\mathcal{X})$. This ends the proof of Theorem 1. □

Proof of Theorem 3. Since $E(\mathcal{X})$ has an $M(k,l)$-structure, Y has also an $M(k,l)$ -structure. Then (1) follows from Lemma 2.2(1).

In order to prove (2), we consider the following commutative diagram:

$$\begin{array}{ccccc} S^{n-1} & \xrightarrow{\alpha} & S^q & \xrightarrow{j} & Y \\ \downarrow & & \downarrow & & \downarrow i \\ * & \longrightarrow & E & = & E \\ \downarrow & & \downarrow & & \downarrow \\ S^n & \xrightarrow{i_n} & E/S^q & \xrightarrow{p'} & S^{n-q} \\ \| & & \downarrow g & & \downarrow \Sigma\rho \\ S^n & \xrightarrow{\Sigma\alpha} & S^{q+1} & \xrightarrow{\Sigma j} & \Sigma Y, \end{array}$$

where all the straight lines are cofiber sequences. Since $p \circ \rho = 0$, where p is the bundle projection, we see that $E/S^q = S^n \vee S^{n+q}$. Thus we can write as $g = \Sigma\alpha \vee \omega$ for some $\omega: S^{n-q} \to S^{q+1}$, in other words, there exists a map $i_{n+q}: S^{n+q} \to E/S^q$ such that $p' \circ i_{n-q} = \iota_{n-q}$ and $g \circ i_{n+q} = \omega$. This implies that $(\Sigma j)_*(\omega) = \Sigma\rho$. On the other hand, by [12], we know that $\Sigma\rho = (\Sigma j)_*(J(\mathcal{X}))$. So we have $(\Sigma j)_*(\omega) = (\Sigma j)_*(J(\mathcal{X}))$. Applying Lemma 2.1 for the case $\beta = \Sigma\alpha$, we get

$$\omega = J(\mathcal{X}) + (\Sigma\alpha)_*(x) + m[\iota_{q+1}, \Sigma\alpha], \tag{2.4}$$

for some $x \in \pi_{n+q}(S^n)$ and $m \in \mathbb{Z}$. Now suppose that there exists an $M(k,l)$-structure on E. Then, there exists the following commutative diagram:

$$\begin{array}{ccccccc} S^q & \longrightarrow & E & \longrightarrow & E/S^q & \xrightarrow{\Sigma\alpha \vee \omega} & S^{q+1} \\ \downarrow k & & \downarrow f_{k,l} & & \downarrow f & & \downarrow k \\ S^q & \longrightarrow & E & \longrightarrow & E/S^q & \xrightarrow{\Sigma\alpha \vee \omega} & S^{q+1} \end{array}$$

where $f\circ i_n = li_n$ and $f\circ i_{n+q} = kli_{n+q}+i_n\circ y$ *for some* $y\in\pi_{n+q}(S^n)$.
Hence $\Sigma\alpha\circ y+kl\omega = k\iota_{q+1}\circ\omega$. From this and (2.4),

$$
\begin{aligned}
&klJ(\mathcal{X})-k\iota_{q+1}\circ J(\mathcal{X})\\
&\quad = kl(\omega-(\Sigma\alpha)\circ x-m[\iota_{q+1},\Sigma\alpha])-k\iota_{q+1}\circ(\omega-(\Sigma\alpha)\circ x-m[\iota_{q+1},\Sigma\alpha])\\
&\quad \equiv k\iota_{q+1}\circ(\Sigma\alpha)\circ x+m(k^2-kl)[\iota_{q+1},\Sigma\alpha] \pmod{(\Sigma\alpha)_*\pi_{n+q}(S^n)}\\
&\quad \equiv k\iota_{q+1}\circ(\Sigma\alpha)\circ x \pmod{(\Sigma\alpha)_*\pi_{n+q}(S^n)},\quad \text{since}\quad (k-l)\Sigma\alpha=0,\\
&\quad \equiv \Sigma\alpha\circ(k\iota_n\circ x) \pmod{(\Sigma\alpha)_*\pi_{n+q}(S^n)}\\
&\quad \equiv 0 \pmod{(\Sigma\alpha)_*\pi_{n+q}(S^n)}.
\end{aligned}
$$

This implies (2) and completes the proof of Theorem 3. □

From now on, we consider the sufficient conditions for the existence of $M(k,l)$-structure on $E(\mathcal{X})$.

Proposition 2.5. *Let $q\geq 2$ and $n\geq 2$. Assume that there exists a non-zero integer a such that $a\iota_q\circ\alpha = 0$. Then, there exists an extension of $a\iota_q$ to Y, say $\bar{a}: Y\to S^q$. Suppose that there exists a non-zero integer b such that $b\iota_q\circ\bar{a}\circ\rho = 0$. Then there exists a map $f_1: E(\mathcal{X})\to S^q\times S^n$ such that the following diagram commutes*:

$$
\begin{array}{ccccc}
S^{n+q-1} & \xrightarrow{\rho} & Y & \longrightarrow & E(\mathcal{X})\\
\downarrow{\scriptstyle ab\iota_{n+q-1}} & & \downarrow{\scriptstyle (b\iota_q\circ\bar{a})\vee p} & & \downarrow{\scriptstyle f_1}\\
S^{n+q-1} & \xrightarrow{[i_q,i_n]} & S^q\vee S^n & \xrightarrow{i} & S^q\times S^n,
\end{array}
$$

where p is the restriction of the bundle projection $p: E(\mathcal{X})\to S^n$.

Proof. It is clear that $p\circ\rho = 0$. Since dim $Y < n+q$, the map $(b\iota_q\circ\bar{a})\times p: Y\to S^q\times S^n$ goes through $S^q\vee S^n$. From the assumption, $i\circ((b\iota_q\circ\bar{a})\vee p)\circ\rho = 0$, where $i: S^q\vee S^n\to S^q\times S^n$ is the inclusion map. Recall that the Whitehead product $[i_q,i_n]$ is the attaching map of the top cell of $S^q\times S^n$. Thus from Lemma 2.1, there exists an integer m such that $[i_q,i_n]= ((b\iota_q\circ\bar{a})\vee p)\circ\rho$. By the method used in the proof of Lemma 2.2, we have $m=ab$. We omit the details. □

Proposition 2.6. *Let $q,n\geq 1$. Suppose that there exists a non-zero integer x such that $x\mathcal{X}=0$. Then there exists a coextension of $x\iota_n$, say $\tilde{x}: S^n\to Y$, and a map $f_2: S^q\times S^n\to E(\mathcal{X})$ such that the following diagram commutes*:

$$
\begin{array}{ccccc}
S^{n+q-1} & \xrightarrow{[i_q,i_n]} & S^q\vee S^n & \xrightarrow{i} & S^q\times S^n\\
\downarrow{\scriptstyle x\iota_{n+q-1}} & & \downarrow{\scriptstyle j\vee\tilde{x}} & & \downarrow{\scriptstyle f_2}\\
S^{n+q-1} & \xrightarrow{\rho} & Y & \xrightarrow{i} & E(\mathcal{X}),
\end{array}
$$

where $j: S^q \to Y = S^q \cup_\alpha e^n$ *is the bottom inclusion map.*

Proof. The assumption $x\mathcal{X}=0$ implies that the bundle induced from $E(\mathcal{X})$ by the map of degree x is trivial. So there exists a bundle map:

$$\begin{array}{ccc} S^q \times S^n & \xrightarrow{f_2} & E(\mathcal{X}) \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle p} \\ S^n & \xrightarrow{x\iota_n} & S^n, \end{array}$$

where pr_2 is the projection to the second factor. By restricting this bundle map f_2 to the $n+q-1$-skeleton, we get the map $S^q \vee S^n \to Y$. Then clearly, this map is described as $j \vee \tilde{x}$ by some coextension $\tilde{x}: S^n \to Y$. Since $i_*((j \vee \tilde{x})\circ[i_q, i_n]) = 0$, there exist an integer m and a map $f: S^q \times S^n \to E(\mathcal{X})$ such that $m\rho = (j \vee \tilde{x})\circ [i_q, i_n]$, $f\circ i = i\circ(j \vee \tilde{x})$ and $m\iota_{q+n}\circ p = p\circ f$. Using cohomology, we then have $m=x$. This ends the proof. □

Corollary 2.7. *Under the assumption of the above proposition, we have* $x\rho = [j, \tilde{x}]$. *Moreover there exists an element* $\xi' \in \pi_{n-1}(SO(q))$ *with* $i_*(\xi') = (\#\alpha)\mathcal{X}$ *and* $(\#\alpha)\rho = [j, \widetilde{\#\alpha}] + j_* J(\xi')$, *where* $i: SO(q) \to SO(q+1)$ *is the inclusion and* $\widetilde{\#\alpha}: S^n \to Y$ *is a coextension of* $(\#\alpha)\iota_n$.

Proof. First assertion is obvious from the above diagram in Proposition 2.6. We show the second assertion. Let E' be the induced bundle from $E(\mathcal{X})$ by the map of degree $\#\alpha$. Then E' has a section, that is, $E' = (S^q \vee S^n) \cup_{\rho'} e^{n+q}$. The existence of the bundle map $E' \to E$ implies that there exists a following commutative diagram:

$$\begin{array}{ccc} S^{n+q-1} & \xrightarrow{(\#\alpha)\iota_{n+q-1}} & S^{n+q-1} \\ \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} \\ S^q \vee S^n & \xrightarrow{j \vee \widetilde{\#\alpha}} & S^q \cup_\alpha e^n. \end{array}$$

Therefore, using (2.3), we have the desired result. □

Proof of Theorem 4. Consider the composite:

$$E(\mathcal{X}) \xrightarrow{f_1} S^q \times S^n \xrightarrow{k/(ab) \times l/x} S^q \times S^n \xrightarrow{f_2} E(\mathcal{X}),$$

where f_1 and f_2 are maps in Propositions 2.5 and 2.6. This gives the desired $M(k,l)$-structure. □

Remark 2.8. Suppose that there exists an integer m such that $m\mathcal{X} = \mathcal{X}$. Then there exists an $M(1,m)$-structure on $E(\mathcal{X})$.

Proposition 2.9. *Suppose $q \geq 2$, $n \geq q+1$ and $\alpha = 0$ provided $n = q+1$. Let $h : Y \to Y$ be an $M(k,l)$-structure on Y. Then we have*

$$h \circ \rho - kl\rho = j_* \beta \quad \textit{for some } \beta \in \pi_{n+q-1}(S^q).$$

Besides, if $x\mathcal{X} = 0$ for an integer x and $\pi_n(S^q)$ is generated by $\alpha \circ \eta_{n-1}$, then $xj_* \beta = 0$. Here $\eta_2 : S^3 \to S^2$ is the Hopf map and $\eta_m = \Sigma^{m-2}\eta_2$ for $m \geq 2$.

Proof. Let $\hat{\alpha} \in \pi_n(Y, S^q) \cong \mathbb{Z}$ be a characteristic map of the top cell of Y. Then we have $h_*(\hat{\alpha}) = l\hat{\alpha}$. Let $i : (Y, \emptyset) \to (Y, S^q)$ be the inclusion. Consider the commutative diagram:

$$\begin{array}{ccccc}
\pi_{n+q-1}(S^q) & \xrightarrow{j_*} & \pi_{n+q-1}(Y) & \xrightarrow{i_*} & \pi_{n+q-1}(Y, S^q) \\
\downarrow (k\iota_q)_* & & \downarrow h_* & & \downarrow h_* \\
\pi_{n-q-1}(S^q) & \xrightarrow{j_*} & \pi_{n+q-1}(Y) & \xrightarrow{i_*} & \pi_{n+q-1}(Y, S^q)
\end{array}$$

We have

$$\begin{aligned}
i_* h_*(\rho) &= h_* i_*(\rho) \\
&= h_*(-[\iota_q, \hat{\alpha}]), \quad \text{by [14]}, \\
&= -kl[\iota_q, \hat{\alpha}] \\
&= i_*(kl\rho),
\end{aligned}$$

and $h_*(\rho) - kl\rho \in \mathrm{Ker}(i_*) = \mathrm{Image}(j_*)$, so there exists $\beta \in \pi_{n+q-1}(S^q)$ such that $h_*(\rho) = kl\rho + j_*(\beta)$. Now assume that $x\mathcal{X} = 0$ and $\pi_n(S^q)$ is generated by $\alpha \circ \eta_{n-1}$. We have $p_* h_* \tilde{x} = xl\iota_n = p_*(l\tilde{x})$, where $p : Y = S^q \cup_\alpha e^n \to S^n$ is the pinching map of S^q. From the assumption, it follows that $p_* : \pi_n(Y) \to \pi_n(S^n)$ is injective, so that $h_* \tilde{x} = l\tilde{x}$. Then from Corollary 2.7, we have $xh_*(\rho) = h_*[j, \tilde{x}] = [kj, l\tilde{x}] = kl[j, \tilde{x}] = xkl\rho$ so that $xj_*\beta = 0$. □

Proposition 2.10. *Suppose that $q \geq 2$, $n = q+1$ and $\alpha \neq 0$. Then q is odd and there exists an M_k-structure [4] on E for $k \equiv 0,1 \pmod{\#\rho}$.*

Proof. Recall from [5] that $p^* : H^*(S^{n+q}) \cong H^*(E)/\mathrm{Tor}$. By [15], $\pi_q(SO(q+1))$ is finite for q even. Hence q is odd under the assumption. Since $\pi_{2q}(S^q \cup_\alpha e^{q+1})$ is finite by a Serre's theorem, the order of ρ, $\#\rho$, is finite. Hence, when g is 0 if $k \equiv 0 \pmod{\#\rho}$ and id if $k \equiv 1 \pmod{\#\rho}$, there exists a self map f of E which makes the following diagram commutative:

$$\begin{array}{ccccc} S^{2q} & \xrightarrow{\rho} & S^q \cup e^{q+1} & \longrightarrow & E \\ \downarrow{\scriptstyle k\iota_{2q}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ S^{2q} & \xrightarrow{\rho} & S^q \cup e^{q+1} & \longrightarrow & E \end{array}$$

It is obvious that f is an M_k-structure. □

§ 3. Proof of Theorem 2

In this section we assume $q=n$. Let $i_j: S^n \to S^n \vee S^n$ be the inclusion to the j-th component for $j=1,2$. Given a self map a of $S^n \vee S^n$, we define an integral 2×2-matrix (a_{ij}) by $a \circ i_j = a_{j1} i_1 + a_{j2} i_2$ for $j = 1,2$. This defines a bijection between $[S^n \vee S^n, S^n \vee S^n]$ and the set of 2×2 integral matrices.

Lemma 3.1. (1) *For any* $x \in \pi_{2n-1}(S^n)$ *and* $a,b \in \mathbb{Z}$, *we have*

$$(ai_1+bi_2)\circ x = i_{1*}(ax+\binom{a}{2}H(x)[\iota_n,\iota_n])+i_{2*}(bx+\binom{b}{2}H(x)[\iota_n,\iota_n])+abH(x)[i_1,i_2].$$

(2) *E has an (a_{ij})-structure with respect to a basis $\mathfrak{B} = \{x,y\}$ if and only if there exists an (a_{ij})-structure g on Y with respect to $\mathfrak{B}$ such that*

$$g \circ \rho = (a_{11}a_{21}a+a_{11}a_{22}+(-1)^n a_{12}a_{21}+a_{12}a_{22}b)\rho,$$

where a and b are defined by $x^2=axy$ and $y^2=bxy$.

(3) *There exist bases* $\mathfrak{B} = \{x_n, y_n\}$ *and* $\mathfrak{B}' = \{x_n', y_n\}$ *of* $H^n(E(\mathcal{X}))$ *such that*

$$a = H(J(\xi)), \quad i.e., \quad x_n^2 = H(J(\xi))x_n y_n,$$

$$a' = \begin{cases} 1 & if\ n = 2,4,8 \text{ and } H(J(\xi)) \equiv 1 \pmod 2 \\ 0 & otherwise \end{cases},$$

$$b = b' = 0, \quad i.e., \quad y_n^2 = 0.$$

Proof. Under the notations in (1), it follows from Theorem 8.5 on p.534 in [22] that we have $(ai_1+bi_2)\circ x = ai_1\circ x + bi_2 \circ x + [ai_1, bi_2]\circ H(x)\iota_{2n-1}$. We also have

$$ai_1\circ x = i_{1*}(a\iota_n \circ x) = i_{1*}(ax+\binom{a}{2}[\iota_n,\iota_n]\circ H(x)\iota_{2n-1})$$

$$= i_{1*}(ax+\binom{a}{2}H(x)[\iota_n,\iota_n])$$

and similarly

$$b i_2 \circ x = i_{2*}(bx + \binom{b}{2} H(x)[\iota_n, \iota_n]).$$

Hence (1) follows.

Let $\mathfrak{B} = \{x,y\}$ be a basis of $H^n(E)$. Note that if n is odd, then $x^2 = y^2 = 0$. If f is an (a_{ij})-structure on $E(\mathcal{X})$ with respect to $\mathfrak{B}$, then $g = f| Y$ is an (a_{ij})-structure on $Y = S^n \vee S^n$ with respect to $\mathfrak{B}$. By Lemma 2.1, there exists an integer m such that $g \circ \rho = m\rho$. Thus there is a map f': $E(\mathcal{X}) \to E(\mathcal{X})$ which makes the following diagram of cofibre sequences commutative:

$$\begin{array}{ccccccc} S^{2n-1} & \xrightarrow{\rho} & S^n \vee S^n & \longrightarrow & E(\mathcal{X}) & \longrightarrow & S^{2n} \\ \downarrow{\scriptstyle m\iota_{2n-1}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle m\iota_{2n}} \\ S^{2n-1} & \xrightarrow{\rho} & S^n \vee S^n & \longrightarrow & E(\mathcal{X}) & \longrightarrow & S^{2n} \end{array}$$

Since

$$\begin{aligned} f'^*(xy) &= f'^*(x) f'^*(y) \\ &= (a_{11}x + a_{12}y)(a_{21}x + a_{22}y) \\ &= a_{11}a_{21}x^2 + a_{11}a_{22}xy + a_{12}a_{21}yx + a_{12}a_{22}y^2 \\ &= \{a_{11}a_{21}a + a_{11}a_{22} + (-1)^n a_{12}a_{21} + a_{12}a_{22}b\} xy, \end{aligned}$$

we have $m = a_{11}a_{21}a + a_{11}a_{22} + (-1)^n a_{12}a_{21} + a_{12}a_{22}b$. This has proved a half of (2). The other half is obvious.

By (2.3), we have the following commutative diagram of the cofibre sequences:

$$\begin{array}{ccccccc} S^{2n-1} & \xrightarrow{\rho} & S^n \vee S^n & \longrightarrow & E & \longrightarrow & S^{2n} \\ \| & & \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle f'} & & \| \\ S^{2n-1} & \xrightarrow{J(\xi)} & S^n & \longrightarrow & C & \longrightarrow & S^{2n} \end{array}$$

Here pr_1 is the first projection. Choose generators $z_n \in H^n(C) \cong \mathbb{Z}$ and $z_{2n} \in H^{2n}(C) \cong \mathbb{Z}$ such that $z_n^2 = H(J(\xi))z_{2n}$. Set $x_n = f^*(z_n)$. Let y_n be the image of a generator of $H^n(S^n)$ under the bundle projection and satisfy $f^*(z_{2n}) = x_n y_n$. Then $x_n^2 = H(J(\xi))x_n y_n$. As is well-known, the image of $H \circ J : \pi_{n-1}(SO(n)) \to \mathbb{Z}$ is $\mathbb{Z}$ (if $n = 2,4,8$), $2\mathbb{Z}$ (if n is even and not 2,4,8), or 0 (if n is odd). It follows easily that the following element has the desired property.

$$x_n' = \begin{cases} x_n - \{(H(J(\xi))-1)/2\}y_n & \text{if} \quad H(J(\xi)) \equiv 1 \pmod 2 \\ x_n - \{(H(J(\xi))/2\}y_n & \text{if} \quad H(J(\xi)) \equiv 0 \pmod 2 \end{cases}$$

This completes the proof of Lemma 3.1. □

Remark 3.2. When we say "the basis $\mathfrak{B}$ in Lemma 3.1(3)", it is the one defined in the proof of Lemma 3.1(3). This satisfies the following: $x_n = \pm pr_1^*[S^n]$ and $y_n = \pm pr_2^*[S^n]$, *where* $pr_j: S^n \vee S^n \to S^n$ is the j-th projection and $[S^n]$ is a generator of $H^n(S^n)$.

Lemma 3.3. *Let* $\mathfrak{B} = \{x_n, y_n\}$ *be the basis in Lemma* 3.1(3). *Then* $E(\mathcal{X})$ *has an* (a_{ij})*-structure with respect to* $\mathfrak{B}$ *if and only if*

(i) $$a_{11}J(\xi) + \{\binom{a_{11}}{2}H(J(\xi)) + a_{11}a_{21}\}[\iota_n, \iota_n] = \{a_{11}a_{21}H(J(\xi)) + a_{11}a_{22} + (-1)^n a_{12}a_{21}\}J(\xi),$$

(ii) $$a_{12}J(\xi) + \{\binom{a_{12}}{2}H(J(\xi)) + a_{12}a_{22}\}[\iota_n, \iota_n] = 0,$$

(iii) $$a_{11}(a_{12} - a_{21})H(J(\xi)) = 0.$$

Proof. Let $\mathfrak{B} = \{x_n, y_n\}$ be the basis in Lemma 3.1(3). By Lemma 3.1, there exists an (a_{ij})-structure on E with respect to $\mathfrak{B}$ if and only if

(3.4) $$g_*(\rho) = \{a_{11}a_{21}H(J(\xi)) + a_{11}a_{22} + (-1)^n a_{12}a_{21}\}\rho,$$

where g is the (a_{ij})-structure on $S^n \vee S^n$ with respect to $\mathfrak{B}$. We have

$$\begin{aligned} g_*(\rho) &= g_* i_{1*}(J(\xi)) + g_*([i_1, i_2]) \\ &= (a_{11}i_1 + a_{12}i_2)\circ J(\xi) + [a_{11}i_1 + a_{12}i_2,\ a_{21}i_1 + a_{22}i_2] \\ &= i_{1*}\{a_{11}J(\xi) + (\binom{a_{11}}{2}H(J(\xi)) + a_{11}a_{21})[\iota_n, \iota_n]\} \\ &\quad + i_{2*}\{a_{12}J(\xi) + (\binom{a_{12}}{2}H(J(\xi)) + a_{12}a_{22})[\iota_n, \iota_n]\} \\ &\quad + \{a_{11}a_{12}H(J(\xi)) + a_{11}a_{22} + (-1)^n a_{12}a_{21}\}[i_1, i_2] \end{aligned}$$

and the right hand term of (3.4) is equal to $i_{1*}(mJ(\xi)) + m[i_1, i_2]$, where $m = a_{11}a_{21}H(J(\xi)) + a_{11}a_{22} + (-1)^n a_{12}a_{21}$. Since the homomorphism $\psi: \pi_{2n-1}(S^n) \oplus \pi_{2n-1}(S^n) \oplus \mathbb{Z} \to \pi_{2n-1}(S^n \vee S^n)$ which is defined by $\psi(u, v, w) = i_{1*}(u) + i_{2*}(v) + w[i_1, i_2]$ is an isomorphism, it follows that (3.4) holds if and only if the three equations in Lemma 3.3 hold. This completes the proof of Lemma 3.3. □

Proof of Theorem 2. Let $\mathfrak{B}$ be the basis in Lemma 3.1(3). As is well-known

$$\pi_{n-1}(SO(n+1)) = \begin{cases} \mathbb{Z} & \text{if} \quad n \equiv 0 \pmod 4 \\ \mathbb{Z}_2 & \text{if} \quad n \equiv 1,2 \pmod 8 \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Note that all possible cases of n and $\mathcal{X}$ are given in (1), ..., (7).

When $n=1,3,7$, the bundle $E(\mathcal{X})$ is trivial and S^n is an H-space. Hence every self map of $S^n \vee S^n$ can be extended to a self map of $S^n \times S^n$. Thus $E(\mathcal{X})$ has an (a_{ij})-structure for every (a_{ij}) for $n=1,3,7$.

Let $n \neq 1,3,7$ and $\mathcal{X}=0$. Taking $\xi=0$, it follows from 3.3 that $a_{11}a_{21}[\iota_n,\iota_n] = a_{12}a_{22}[\iota_n,\iota_n] = 0$. Hence $E(\mathcal{X})$ has an (a_{ij})-structure when (2) or (3) happens.

In the rest of the proof we always assume $\mathcal{X} \neq 0$.

Let $i : SO(n) \to SO(n+1)$ be the inclusion and $\Delta : \pi_n(S^n) \to \pi_{n-1}(SO(n))$ the connecting homomorphism for the bundle $SO(n+1) \to S^n$.

Let $n=2,4,8$ and $\tau : S^{2n-1} \to S^n$ the Hopf map such that $H([\iota_n,\iota_n]) = 2H(\tau) = 2$. Recall the following:

$$\pi_{2n-1}(S^n) = \mathbb{Z}\{\tau\} \oplus \mathbb{Z}_b, \quad b = \begin{cases} 1 & \text{if} \quad n=2 \\ 12 & \text{if} \quad n=4 \\ 120 & \text{if} \quad n=8. \end{cases}$$

Let $\theta'' \in \pi_{n-1}(SO(n+1))$ be a generator satisfying $J(\theta'') = -\Sigma\tau$. Let $\theta' \in \pi_{n-1}(SO(n))$ be an element satisfying $i_*(\theta') = \theta''$. Then $J(\theta') - \tau \in \operatorname{Ker} \Sigma$. Hence $J(\theta') - \tau = a[\iota_n,\iota_n]$ for some $a \in \mathbb{Z}$ by the EHP-sequence. Set $\theta = \theta' - a\Delta\iota_n$. Then

$$\pi_{n-1}(SO(n)) = \begin{cases} \mathbb{Z}\{\theta\} & \text{if } n=2 \\ \mathbb{Z}\{\Delta\iota_n\} \oplus \mathbb{Z}\{\theta\} & \text{if } n=4,8 \end{cases}, \quad J(\theta) = \tau, \text{ and } i_*(\theta) = \theta''.$$

Since $H([\iota_n,\iota_n] - 2\tau) = 0$, there exists $\omega \in \pi_{2n-2}(S^{n-1}) = \mathbb{Z}_b$ with $[\iota_n,\iota_n] - 2\tau = \Sigma\omega$. Hence $-2\Sigma\tau = \Sigma^2\omega$ so that $\#\Sigma^2\omega = (1/2)\#\Sigma\tau = b$. Therefore ω is a generator. Let $\mathcal{X} = m\theta''$ with $m=1$ for $n=2$ and $m \neq 0$ for $n=4,8$. Set $\xi = m\theta$. Then the three equations in 3.3 are equivalent to the following:

$$ma_{11}^2 + 2a_{11}a_{21} = m(ma_{11}a_{21} + a_{11}a_{22} + a_{12}a_{21}),$$
$$m\binom{a_{11}}{2} + a_{11}a_{21} \equiv m\binom{a_{12}}{2} + a_{12}a_{22} \equiv 0 \pmod b,$$
$$ma_{12}^2 + 2a_{12}a_{22} = ma_{11}(a_{12} - a_{21}) = 0.$$

Hence $E(\mathcal{X})$ has an (a_{ij})-structure when (4) or (5) happens.

Let $n \equiv 0 \pmod 2$ and $n \neq 2,4,8$. Since we have assumed $\mathcal{X} \neq 0$, it follows that $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 8$. Then

$$\pi_{2n-1}(S^n) = \mathbb{Z}\{[\iota_n,\iota_n]\} \oplus T, \quad \Sigma : T \cong \pi_{2n}(S^{n+1}),$$
$$\pi_{n-1}(SO(n)) = \mathbb{Z}\{[\Delta\iota_n]\} \oplus \langle\beta\rangle, \quad \pi_{n-1}(SO(n+1)) = \langle i_*\beta\rangle,$$
$$\langle\beta\rangle \cong \langle i_*\beta\rangle \cong \begin{cases} \mathbb{Z} & \text{if} \quad n \equiv 0 \pmod 4 \\ \mathbb{Z}_2 & \text{if} \quad n \equiv 2 \pmod 8 \end{cases}, \quad [\iota_n,\iota_n] = J\Delta\iota_n.$$

Choose ξ such that $J(\xi) \in T$. Then $\#J(\xi) = \#J(\mathcal{X})$ and the three equations in 3.3 are equivalent to the following:

$$a_{11}J(\xi) = (a_{11}a_{22}+a_{12}a_{21})J(\xi), \quad a_{11}a_{21} = a_{12}a_{22} = a_{12}J(\xi) = 0.$$

Thus $E(\mathcal{X})$ has an (a_{ij})-structure when (6) happens.

Let $n \equiv 1 \pmod 8 \geq 9$. Then

$$\pi_{n-1}(SO(n)) = \mathbb{Z}_2\{\Delta\iota_n\} \oplus \mathbb{Z}_2\{\beta\}, \quad \pi_{n-1}(SO(n+1)) = \mathbb{Z}_2\{i_*\beta\},$$

and $J(i_*\beta) \neq 0$ by Adams [1]. Let $\mathcal{X} = i_*\beta$ and $\xi = \beta$. Suppose the three equations in 3.3 hold. Applying Σ to them, we have $a_{12} \equiv a_{11}(a_{22}-1) \equiv a_{11}a_{21} \equiv 0 \pmod 2$ since $\Sigma J(\xi) = -J(\mathcal{X})$ whose order is 2. Conversely if these equations hold, then so do the three equations in 3.3. Thus $E(\mathcal{X})$ has an (a_{ij})-structure when (7) hold. This completes the proof of Theorem 2. □

§ 4. Real Stiefel Manifolds of 2-Frames

Lemma 4.1 ([7,9,10,16]).
(1) $[\iota_m,\eta_m]=0$ *if and only if* $m \equiv 3 \pmod 4$ *or* $m=2,6$.
(2) $[\iota_m,\eta_m^2]=0$ *if and only if* $m \equiv 2,3 \pmod 4$ *or* $m=5$.

Proposition 4.2. *Let* $n \geq 2$ *be even. Then* $V_{n+2,2}$ *has an* $M(k,l)$*-structure if and only if one of the following holds*:
(1) $n=2,6$ *and* k, l *are arbitrary*;
(2) $n \equiv 0 \pmod 4$ *and* $k \equiv 0 \pmod 4$;
(3) $n \equiv 0 \pmod 4$ *and* k *and* l *are odd*;
(4) $n \equiv 2 \pmod 4$ *with* $n \geq 10$ *and* $k \equiv 0 \pmod 4$;
(5) $n \equiv 2 \pmod 4$ *with* $n \geq 10$ *and* l *is odd.*

Proof. When $n=2,6$, $\pi_n(SO(n+1)) = 0$ so that $V_{n+2,2} = S^n \times S^{n+1}$ and $V_{n+2,2}$ has an $M(k,l)$-structure for every k,l.

In the rest of the proof, we assume $n \neq 2,6$. By Nomura [19], we have
(i) $HJ(\xi) = 0$ and $2J(\xi) = [\eta_n,\iota_n]$ for $n \equiv 2 \pmod 4$ with $n \geq 10$;

(ii) $HJ(\xi)=\eta_{2n-1}$ and $J(\xi)$ is of order 2 for $n\equiv 0 \pmod 4$.
Given integers k,l, we then have

$$\begin{aligned}\delta &:= klJ(\xi)-k\iota_n\circ J(\xi)\\ &= k(l-1)J(\xi)-\begin{cases}0 & \text{if } \ n\equiv 2 \pmod 4\\ \binom{k}{2}[\eta_n,\iota_n] & \text{if } \ n\equiv 0 \pmod 4).\end{cases}\end{aligned}$$

By Theorem 1, there exists an $M(k,l)$-structure on $V_{n+2,2}$ if and only if $\delta\in k\langle[\eta_n,\iota_n]\rangle$. It follows from Lemma 4.1(1) and the above Nomura's result (i) that, for $n\equiv 2 \pmod 4)\geq 10$, $V_{n+2,2}$ has an $M(k,l)$-structure if and only if $k(l-1)\equiv 0 \pmod 4)$ or $k(l-3)\equiv 0 \pmod 4)$ if and only if $k\equiv 0 \pmod 4)$ or $l\equiv 1 \pmod 2)$. When $n\equiv 0 \pmod 4)$, (ii) and the equation $\Sigma J(\xi)=[\iota_{n+1},\iota_{n+1}]$ imply that $J(\xi)$ and $[\eta_n,\iota_n]$ are linearly independent over $\mathbb{Z}_2$. The assertion then follows easily in this case. This completes the proof of Proposition 4.2. □

§ 5. Complex Stiefel Manifolds of 2-Frames

Let $\mathcal{X}\in\pi_{2n+2}(SO(2n+2))$ be the characteristic map of the bundle

$$S^{2n+1}\longrightarrow W_{n+2,2}\xrightarrow{\ p\ } S^{2n+3}.$$

Then, the following lemma is known.

Lemma 5.1. *Let $n\geq 4$ be an even integer. Then the bundle $W_{n+2,2}\to S^{2n+3}$ has a section and there is a generator $\xi\in\pi_{2n+2}(SO(2n+1))\cong\mathbb{Z}_8$, [15], such that*

(1) ([15]) *$i_*(\xi)=\mathcal{X}$, where i:$SO(2n+1)\to SO(2n+2)$ is the inclusion map;*

(2) ([15,20]) *$J(\xi)\in\pi_{4n+3}(S^{2n+1})$ can be desuspended;*

(3) ([7,15,21,23]) *$4J(\xi)=J(\triangle(\eta^2_{2n+1}))=[\iota_{2n+1},\eta^2_{2n+1}]\neq 0$, where $\triangle$ is the connecting homomorphism of the bundle $SO(2n+1)\to SO(2n+2)\to S^{2n+3}$.*

Proofs of Theorem 5 (1), (2). *Let n be even.* Then (1) follows, since $W_{n+2,2}=S^{2n+1}\times S^{2n+3}$ for $n=0,2$. Suppose $n\geq 4$. Applying Theorem 1 and Lemma 5.1, we see that $W_{n+2,2}$ has an $M(k,l)$-structure if and only if $klJ(\xi)-kJ(\xi)=4kxJ(\xi)$ for some $x\in\mathbb{Z}$. Since $J(\xi)$ is of order 8, the proof of (2) follows easily. □

Lemma 5.2. *Let n be odd. Then, since $p_{1*}(\mathcal{X})=\eta_{2n+1}$, the bundle $W_{n+2,2}\to S^{2n+3}$ can not have a section.*

(1) *$\mathcal{X}\in\pi_{2n+2}(SO(2n+2))$ is of order 2.*

(2) *Let $\bar{2}:S^{2n+1}\cup_{\eta_{2n+1}}e^{2n+3}\to S^{2n+1}$ be an extension of $2\iota_{2n+1}$. Then $\bar{2}\circ\rho\neq 0$ and $2\iota_{2n+1}\circ\bar{2}\circ\rho=0$.*

Proof. The assertion (1) follows from [15]. We will show (2). We have

$$\begin{aligned}\Sigma(\bar{2}\circ\rho) &= 2\iota_{2n+2}\circ J(\chi), \quad \text{since } \Sigma\rho = \Sigma j\circ J(\chi) \text{ by [12]},\\ &= 2J(\chi)+[\iota_{2n+2}, \iota_{2n+2}]\circ\eta_{4n+3}, \quad \text{since } H(J(\chi)) = \eta_{4n+3},\\ &= [\iota_{2n+2}, \iota_{2n+2}]\circ\eta_{4n+3}, \quad \text{since } 2J(\chi) = 0,\\ &= [\iota_{2n+2}, \eta_{2n+2}]\\ &\neq 0,\end{aligned}$$

so that $\bar{2}\circ\rho \neq 0$. In the exact sequence

$$\pi_{4n+5}(S^{4n+3}) \xrightarrow{P} \pi_{4n+3}(S^{2n+1}) \xrightarrow{\Sigma} \pi_{4n+4}(S^{2n+2}),$$

we have $P(\eta_{4n+3}^2) = [\iota_{2n+1}, \eta_{2n+1}^2] = 0$ so that Σ is injective, and

$$\Sigma(2\iota_{2n+1}\circ\bar{2}\circ\rho) = 2\Sigma(\bar{2}\circ\rho) = 2([\iota_{2n+2}, \eta_{2n+2}]) = 0.$$

Hence $2\iota_{2n+1}\circ\bar{2}\circ\rho = 0$. □

Proof of Theorem 5 (4). Suppose that $n\geq 3$ is odd and $W_{n+2,2}$ has an $M(k,l)$-structure. Applying Theorem 3, we see that $k \equiv l \pmod 2$ and there exists an element $y \in \pi_{4n+4}(S^{2n+3})$ such that $\eta_{2n+2}\circ y = k\iota_{2n+2}\circ J(\chi) - klJ(\chi)$. Now, from [22], we have $k\iota_{2n+2}\circ J(\chi) = kJ(\chi) + \binom{k}{2}[\iota_{2n+2}, \iota_{2n+2}]\circ H(J(\chi))$. On the other hand, since $H(J(\chi)) = \eta_{4n+3}$, $J(\chi)$ is of order 2 and $k \equiv l \pmod 2$, it follows that

$$\eta_{2n+2}\circ y = \binom{k}{2}[\iota_{2n+2}, \iota_{2n+2}]\circ\eta_{4n+3}.$$

However, by Nomura [18], this can occur only when $\binom{k}{2} \equiv 0 \pmod 2$ or $n = 1$. Since in our case $n\geq 3$, it follows that $k \equiv 0$ or $1 \pmod 4$. This proves (4). □

To prove Theorem 5 (3),(5), we need some preliminaries. Set $Y_m = S^m \cup_{\eta_m} e^{m+2}$ for $m \geq 2$. Let $j: S^m \to Y_m$ and $p: Y_m \to S^{m+2}$ the inclusion and the quotient maps, respectively.

Lemma 5.3. *Let* $m\geq 2$.

(1) *We have* $\pi_m(Y_m) = \mathbb{Z}\{j\}, [Y_m, S^{m+2}] = \mathbb{Z}\{p\}$ *and*

$$\pi_{m+2}(Y_m) = \begin{cases} \mathbb{Z}\{\tilde{2}\} & \text{if } m \geq 3 \\ 0 & \text{if } m = 2 \end{cases}, \quad [Y_m, S^m] = \begin{cases} \mathbb{Z}\{\bar{2}\} & \text{if } m \geq 3 \\ 0 & \text{if } m = 2 \end{cases}$$

where $p_*\tilde{2} = 2\iota_{m+2}$ *and* $j^*\bar{2} = 2\iota_m$.

(2) *The set* $[Y_m, Y_m]$ *has a structure of an abelian group such that the following is a short exact sequence of groups*:

$$0 \longrightarrow \pi_{m+2}(Y_m) \xrightarrow{p^*} [Y_m, Y_m] \xrightarrow{j^*} \pi_m(Y_m) \longrightarrow 0,$$

which is natural under the suspension.

(3) *When* $m \geq 3$, *every element of* $[Y_m, Y_m]$ *has a form*

$$h_{k,l} = k \cdot id + \{(l-k)/2\} \cdot p^*\tilde{2}, \quad k \equiv l \pmod 2),$$

Proof. The assertion (1) is well-known.

Recall that $Y_2 = P(\mathbf{C}^3)$, the complex projective plane, so that $\pi_4(P(\mathbf{C}^3)) = 0$, $\pi_2(P(\mathbf{C}^3)) = \mathbb{Z}\{j\}$, and $k\iota_2 \circ \eta_2 = k^2\eta_2$ for any integer k. These and the following commutative diagram imply the assertion when $m=2$.

$$\begin{array}{ccccc} [P(\mathbf{C}^3),P(\mathbf{C}^3)] & = & [P(\mathbf{C}^3),P(\mathbf{C}^\infty)] & = & H^2(P(\mathbf{C}^3)) \\ \downarrow j^* & & & & \cong \downarrow j^* \\ [S^2,P(\mathbf{C}^3)] & = & [S^2,P(\mathbf{C}^\infty)] & = & H^2(S^2) \end{array}$$

Since $Y_3 = \Sigma Y_2$ is cogroup-like and $SU(3)$ is group-like, $[Y_3, SU(3)]$ is an abelian group so that an isomorphism

$$(5.4) \qquad [Y_3, Y_3] \cong [Y_3, SU(3)]$$

induced by the inclusion $Y_3 \subset Y_3 \cup e^8 = SU(3)$ gives $[Y_3, Y_3]$ an abelian group structure. Since $\pi_4(SU(3))=0$, by applying $[-, SU(3)]$ to the cofibration $S^4 \xrightarrow{\eta} S^3 \to Y_3$, we have an exact sequence of groups

$$0 \to \pi_5(SU(3)) \to [Y_3, SU(3)] \to \pi_3(SU(3)) \to 0.$$

The assertion (2) then follows from (5.4) when $m=3$.

When $m \geq 4$, $[Y_m, Y_m]$ is stable so that the assertion (2) follows easily by applying $\{-, Y_m\} = \lim_k [\Sigma^k(-), \Sigma^k Y_m]$ to the cofibration $S^{m+1} \xrightarrow{\eta} S^m \to Y_m$. This proves (2).

From now on we suppose $m \geq 3$. By (1) and (2), we have

$$[Y_m, Y_m] = \mathbb{Z}\{p^*\tilde{2}\} \oplus \mathbb{Z}\{id\}.$$

Applying $H^*(-)$, for every integers x,y, we have a commutative diagram:

$$\begin{array}{ccccc} S^m & \xrightarrow{j} & S^m \cup_\eta e^{m+2} & \longrightarrow & S^{m+2} \\ \downarrow{\scriptstyle x\iota_m} & & \downarrow{\scriptstyle x\cdot id + y\cdot p^*\tilde{2}} & & \downarrow{\scriptstyle (x+2y)\iota_{m+2}} \\ S^m & \xrightarrow{j} & S^m \cup_\eta e^{m+2} & \longrightarrow & S^{m+2} \end{array}$$

Hence (3) follows. □

The following is obvious from Proposition 2.9 and Lemma 5.2(1).

Lemma 5.5. *Let $n \geq 3$ be odd and $k \equiv l$ (mod 2). Then Y has an $M(k,l)$-structure $h_{k,l}$ such that $h_{k,l}{}^*(\rho) = kl\rho + j^*(\beta_{k,l})$ for some $\beta_{k,l} \in \pi_{4n+3}(S^{2n+1})$ which satisfies $2j^*(\beta_{k,l}) = 0$.*

Proofs of Theorem 5 (3),(5). If $W_{3,2} = SU(3)$ has an $M(k,l)$-structure, then $k \equiv l$ (mod 2), by Theorem 3 (1). Conversely assume $k \equiv l$ (mod 2). Since $\pi_7(SU(3)) = 0$, it follows from the next diagram that $i \circ h_{k,l}$ can be extended to an $M(k,l)$-structure on SU (3).

$$\begin{array}{ccccc} S^7 & \xrightarrow{\rho} & Y_3 & \xrightarrow{i} & SU(3) \\ & & \downarrow{\scriptstyle h_{kl}} & & \\ & & Y_3 & \xrightarrow{i} & SU(3) \end{array}$$

This proves (3).

Let n be odd. When $k \equiv l$ (mod 2), we have

$$\begin{aligned} h_{k^2,l^2*}(\rho) &= h_{k,l*} h_{k,l*}(\rho) \\ &= kl h_{k,l*}(\rho) + h_{k,l*}(j^*(\beta_{k,l})) \\ &= k^2l^2\rho + klj^*(\beta_{k,l}) + j^*(k\iota_{2n+1} \circ \beta_{k,l}) \\ &= k^2l^2\rho + klj^*(\beta_{k,l}) + j^*(k\beta_{k,l}) \\ &= k^2l^2\rho + k(l-1)j^*(\beta_{k,l}) \\ &= k^2l^2\rho, \end{aligned}$$

where the 4-th equality follows from Lemma 4.1(2). Hence $W_{n+2,2}$ has an $M(k^2,l^2)$-structure when $k \equiv l$ (mod 2). In particular there is an $M(m^2,1)$-structure $f_{m^2,1}$ for m odd. Now from Remark 2.8, it follows that there exists an $M(1,l)$-structure $f_{1,l}$ for l odd. Hence, when m and l are odd, $f_{m^2,1} \circ f_{1,l}$ is a desired $M(k^2,l)$-structure. When $k \equiv 0$ (mod 4) and $l \equiv 0$ (mod 2), we have an $M(k,l)$-structure by Theorem 4 and Lemma 5.2. This completes the proof of Theorem 5.

Problem 5.6. *Does there exist an $M(4m+1,1)$-structure on $W_{n+2,2}$ for n odd?*

Proposition 5.7. *There is a central extension of groups*:

$$0 \to \pi_8(SU(3)) = \mathbb{Z}_{12} \xrightarrow{q^*} [SU(3), SU(3)] \to [Y_3, Y_3] = \mathbb{Z} \oplus \mathbb{Z} \to 0.$$

Proof. Applying $[-, SU(3)]$ to the cofibration $S^7 \xrightarrow{\rho} Y_3 \to SU(3)$, we have an exact sequence of groups:

$$[Y_4, SU(3)] \xrightarrow{(\Sigma\rho)^*} \pi_8(SU(3)) \to [SU(3), SU(3)] \to [Y_3, SU(3)] \to \pi_7(SU(3)).$$

Then we obtain the desired exact sequence, since $(\Sigma\rho)^*$ is factored as

$$[Y_4, SU(3)] \xrightarrow{j^*} \pi_4(SU(3)) = 0 \xrightarrow{J(X)^*} \pi_8(SU(3)),$$

and since $\pi_7(SU(3)) = 0$ and $i_*: [Y_3, Y_3] \cong [Y_3, SU(3)]$. The sequence is central by (3.10) on page 465 in [22]. □

Remark 5.8. We can determine the group $[SU(3), SU(3)]$ which is non-abelian. Details will appear elsewhere.

§ 6. Quaternionic Stiefel Manifolds of 2-Frames

Recall that $Y = S^{4n+3} \cup_\alpha e^{4n+7}$ and

$$\alpha = \begin{cases} (n+2)\nu_{4n+3} & \text{if } n \geq 1 \\ \omega & \text{if } n = 0 \end{cases}$$

where $\nu_4: S^7 \to S^4$ is the Hopf map, $\nu_m = \Sigma^{m-4}\nu_4$ for $m \geq 4$, and ω is a generator of $\pi_6(S^3) = \mathbb{Z}_{12}$ and $\Sigma^2\omega = 2\nu_5$. Recall that $\#[\nu_{2n}, \iota_{2n}]$ is 12 or 24 for $n \geq 2$, since $H[\nu_{2n}, \iota_{2n}] = H([\iota_{2n}, \iota_{2n}] \circ \nu_{4n-1}) = 2\nu_{4n-1}$. Let (m, m') denote the greatest commom divisor of integers m, m'. The purpose of this section is to prove the following two results.

Proposition 6.1. *We have*

$$(1)\ \#\mathcal{X} = \begin{cases} 4 \cdot 3 & \textit{if } n = 0 \\ 8 \cdot 3/(n+2, 3) & \textit{if } n \equiv 1 \pmod 2 \textit{ or } n = 2 \\ 16 \cdot 3/(n+2, 3) & \textit{otherwise.} \end{cases}$$

$$(2)\ \#J(\mathcal{X}) = \begin{cases} \#\mathcal{X}/2 & \textit{if } n \equiv 0 \pmod 2 \geq 4 \textit{ and } \#[\nu_{4n+4}, \iota_{4n+4}] = 12, \\ \#\mathcal{X} & \textit{otherwise.} \end{cases}$$

Proposition 6.2. (1) *$Sp(2)$ has an $M(k,l)$-structure if and only if $k \equiv l$ (mod 12).*
(2) *If there is an $M(k,l)$-structure on $X_{n+2,2}$, then*

$$k \equiv l \pmod{24/(n+2,24)},$$
$$k(l-1) \equiv 0 \begin{cases} \pmod{2(n+2,8)} & \text{if } n \equiv 0 \pmod 2 \geq 4 \text{ and } \#[\nu_{4n+4}, \iota_{4n+4}] = 24 \\ \pmod{(n+2,8)} & \text{if } n \equiv 0 \pmod 2 \geq 4 \text{ and } \#[\nu_{4n+4}, \iota_{4n+4}] = 12. \end{cases}$$

(3) *If $k \equiv 0 \pmod{\#\mathcal{X}}$ for n even and $k \equiv 0 \pmod{2\#\mathcal{X}}$ for n odd and $l \equiv 0 \pmod{\#\mathcal{X}}$, then there is an $M(k,l)$-structure on $X_{n+2,2}$.*

(4) *When $n+2 \equiv 0$ (mod 24), $X_{2+2,2}$ has an $M(k,l)$-structure if and only if $k(l-1)J(\mathcal{X})=0$.*

Proof of Proposition 6.1. *Let $\mathcal{X}_{Sp} \in \pi_{4n+6}(Sp(n+1))$ be the characteristic* element of the bundle $Sp(n+2)\to S^{4n+7}$. This is a generator and $i_*(\mathcal{X}_{Sp}) = \mathcal{X}$, where $i: Sp(n+1) \to SO(4n+4)$ is the inclusion.

The case $n=0$ follows from the following commutative diagram.

$$\begin{array}{ccccc} \pi_6(Sp(1)) & \xrightarrow{i^*} & \pi_6(SO(4)) & \xrightarrow{J} & \pi_{10}(S^4) \\ \downarrow \cong & & \downarrow p_* & & \cong \downarrow H \\ \pi_6(S^3) & \xrightarrow{=} & \pi_6(S^3) & \xrightarrow{-\Sigma^4} & \pi_{10}(S^7) \end{array}$$

In the rest of the proof we suppose $n \geq 1$. Set

$$\epsilon(n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 2 \geq 2 \\ 1 & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

Applying $\pi_*(-)$ to the diagram

$$\begin{array}{ccccc} Sp(n+1) & \longrightarrow & Sp(n+2) & \longrightarrow & S^{4n+7} \\ \downarrow i' & & \downarrow & & \| \\ SU(2n+3) & \longrightarrow & SU(2n+4) & \longrightarrow & S^{4n+7} \end{array}$$

we see that $i'_*: \pi_{4n+6}(Sp(n+1)) \to \pi_{4n+6}(SU(2n+3))$ is surjective. Since $\pi_{4n+6}(SO(4n+6)/SU(2n+3)) = 0$ by [3], the inclusion induces a surjection $\pi_{4n+6}(SU(2n+3)) \to \pi_{4n+6}(SO(4n+6))$. Hence the composite of the following is surjective: $\pi_{4n+6}(Sp(n+1)) \xrightarrow{i_*} \pi_{4n+6}(SO(4n+4)) \xrightarrow{i_{1*}} \pi_{4n+6}(SO(4n+5)) \xrightarrow{i_{2*}} \pi_{4n+6}(SO(4n+6))$, where i_1, i_2 are inclusions. Let $i_0: SO(4n+3) \to SO(4n+4)$ be also the inclusion. Consider the commutative diagram where $m=4n+3$:

$$
\begin{array}{ccccccc}
 & & \pi_{m+3}(SO(m)) & & & & \\
 & & \downarrow & & & & \\
\pi_{m+4}(S^{m+1}) & \xrightarrow{\triangle} & \pi_{m+3}(SO(m+1)) & \longrightarrow & \pi_{m+3}(SO(m+2)) & \longrightarrow & \pi_{m+3}(S^{m+1}) \\
\| & & \downarrow p_* & & \downarrow & & \| \\
\pi_{m+4}(S^{m+1}) & \xrightarrow{\triangle'} & \pi_{m+3}(S^m) & \longrightarrow & \pi_{m+3}(V_{m+2,2}) & \longrightarrow & \pi_{m+3}(S^{m+1})
\end{array}
$$

We have $\triangle'(\nu_{4n+4}) = \triangle'(\iota_{4n+4})\circ\nu_{4n+3} = \pm 2\nu_{4n+3}$, where the second equality follows from the fact that $\pi_{4n+3}(V_{4n+5,2}) = \mathbb{Z}_2$ so that $\triangle'(\iota_{4n+3}) = \pm 2\iota_{4n+3}$. It then follows from [2] and [15] that

$$
\begin{aligned}
\pi_{4n+6}(SO(4n+3)) &= \begin{cases} \mathbb{Z}_{8\epsilon(n)}\{a\} & \text{if } n \neq 2 \\ \mathbb{Z}_8\{a\} & \text{if } n \neq 2 \end{cases}; \\
\pi_{4n+6}(SO(4n+4)) &= \begin{cases} \mathbb{Z}_{8\epsilon(n)}\{i_{0*}a\} \oplus \mathbb{Z}_{24/\epsilon(n)}\{[\epsilon(n)\nu_{4n+3}]\} & \text{if } n \neq 2 \\ \mathbb{Z}_8\{i_{0*}a\} \oplus \mathbb{Z}_{12}\{[2\nu_{11}]\} & \text{if } n = 2 \end{cases}; \\
\pi_{4n+6}(SO(4n+5)) &= \mathbb{Z}_8\{(i_1\circ i_0)_* a\}; \\
\pi_{4n+6}(SO(4n+6)) &= \mathbb{Z}_4\{(i_2\circ i_1\circ i_0)_* a\},
\end{aligned}
$$

where $p_*[\epsilon(n)\nu_{4n+3}] = \epsilon(n)\nu_{4n+3}$ and $i_{1*}[\epsilon(n)\nu_{4n+3}] = 0$. Write $\mathcal{X} = x\cdot i_{0*}a + y[\epsilon(n)\nu_{4n+3}]$. Since $(i_2\circ i_1)_*\mathcal{X}$ is a generator and $p_*(\mathcal{X}) = (n+2)\nu_{4n+3}$, we have $x \equiv 1 \pmod 2$ and $y \equiv (n+2)/\epsilon(n) \pmod{24/\epsilon(n)}$. Hence (1) follows and

$$J(\mathcal{X}) = xJ(i_{0*}a) + \{(n+2)/\epsilon(n)\}J([\epsilon(n)\nu_{4n+3}]).$$

Since $HJ(\mathcal{X}) = -\Sigma^{4n+4}p_*(\mathcal{X}) = -(n+2)\nu_{8n+7}$, we have $3/(3,n+2) \mid \# J(\mathcal{X})$ so that the 3-component of $\# J(\mathcal{X})$ is $3/(3,n+2)$ by (1). Since $HJ[\epsilon(n)\nu_{4n+3}] = -\epsilon(n)\nu_{8n+7}$, we have $24/\epsilon(n) \mid \# J[\epsilon(n)\nu_{4n+3}]$. Hence $\# J[\epsilon(n)\nu_{4n+3}] = 24/\epsilon(n)$.

When $n \equiv 1 \pmod 2$, (2) follows easily from the above calculations.

Suppose $n \equiv 0 \pmod 2$. Write $\Delta(\nu_{4n+4}) = u\cdot i_{0*}a + v[2\nu_{4n+3}]$. Applying p_* to it, we have $v \equiv \pm 1 \pmod{12}$. Since $\#\Delta(\nu_{4n+4})$ is 12 if $n=2$ and 24 if $n>2$, it follows that u is even if $n=2$ and $2 \pmod 4$ if $n>2$. We then have $[\nu_{4n+4}, \iota_{4n+4}] = J\Delta(\nu_{4n+4}) = uJ(i_{0*}a) \pm J[2\nu_{4n+3}]$, hence

$$12[\nu_{4n+4}, \iota_{4n+4}] = 12uJ(i_{0*}a).$$

If $\# [\nu_{4n+4}, \iota_{4n+4}] = 24$, then $n > 2$ and $\# J(i_{0*}a) = 16$ so that $\# J(\mathcal{X}) = 16\cdot 3/(3, n+2)$. Suppose $\# [\nu_{4n+4}, \iota_{4n+4}] = 12$. Then $8J(i_{0*}a) = 0$, hence $\# J(\mathcal{X}) \mid 8\cdot 3/(3,n+2)$. We have $-4\Sigma J(i_{0*}a) = J(4i_{1*}i_{0*}a) = J\Delta(\eta^2_{4n+5}) = [\eta^2_{4n+5}, \iota_{4n+5}] \neq 0$. Hence $4J(i_{0*}a) \neq 0$ so that $\# J(\mathcal{X}) = 8\cdot 3/(3,n+2)$ as desired. This completes the proof of

Proposition 6.1. □

Set $c_n = \# \alpha = 24/(24, n+2)$. Let $\tilde{c}_n: S^{4n+7} \to Y$ (or $X_{n+2,2}$) be a coextension of $c_n \iota_{4n+7}$.

Lemma 6.3. (1) $[Y, Y] = \mathbb{Z}\{id\} \oplus \mathbb{Z}\{\tilde{c}_n \circ p\}$ *as an abelian group.*

(2) *Y has an $M(k,l)$-structure if and only if $k \equiv l \pmod{c_n}$. When $k \equiv l \pmod{c_n}$, the map*

$$h_{k,l} = k \cdot id + (l-k)/c_n \cdot \tilde{c}_n \circ p$$

is the unique $M(k,l)$-structure up to homotopy.

(3) *$Sp(2)$ has an $M(k,l)$-structure if and only if Y does.*

Proof. We will prove this only for $n=0$. Other is easier. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} \pi_4(Sp(2)) & \longrightarrow & \pi_7(Sp(2)) & \longrightarrow & [Y, Sp(2)] & \longrightarrow & \pi_3(Sp(2)) & \longrightarrow & \pi_6(Sp(2)) \\ & & \uparrow \cong & & \uparrow \cong & & \uparrow \cong & & \\ & & \pi_7(Y) & \xrightarrow{p^*} & [Y, Y] & \xrightarrow{j^*} & \pi_3(Y) & & \end{array}$$

Since $\pi_4(Sp(2)) = \mathbb{Z}_2$, $\pi_7(Sp(2)) = \mathbb{Z}\{\widetilde{12}\}$, $\pi_3(Y) = \mathbb{Z}\{j\}$ and $\pi_6(Sp(2)) = 0$, the lower sequence is short exact. Also it is central by p.465 in [22]. Hence $[Y, Y]$ is abelian and $[Y, Y] = \mathbb{Z}\{id\} \oplus \mathbb{Z}\{\widetilde{12} \circ p\}$. Consider the following commutative square for $m = 3,7$:

$$\begin{array}{ccc} [Y, Y] & \longrightarrow & \mathrm{Hom}(H^m(Y), H^m(Y)) \\ i_* \downarrow \cong & & \cong \downarrow \mathrm{Hom}(i^*, id) \\ [Y, Sp(2)] & \longrightarrow & \mathrm{Hom}(H^m(Sp(2)), H^m(Y)) \end{array}$$

This diagram shows that $(f+g)^* = f^* + g^*$ for any self maps f, g of Y. It then follows that, for $a, b \in \mathbb{Z}$, $a \cdot id + b \cdot \widetilde{12} \circ p$ is an $M(a, a+12b)$-structure on Y. This ends the proof for $n=0$. □

Proof of Proposition 6.2. (1) is Lemma 6.3(2),(3). To prove (2), suppose that there exists an $M(k,l)$-structure on $X_{n+2,2}$. Then Y has an $M(k,l)$-structure so that the first part follows from Lemma 6.3(2). By Theorem 3, there exists an element $y \in \pi_{8n+10}(S^{4n+7})$ such that $(n+2)\nu_{4n+4} \circ y = k\iota_{4n+4} \circ J(\mathcal{X}) - klJ(\mathcal{X})$. Since y is stable, it follows that $(n+2)\nu_{4n+4}\circ y = (n+2)(\nu_{4n+4}\circ y)$. Also

$$k\iota_{4n+4}\circ J(\mathcal{X}) = kJ(\mathcal{X})+\binom{k}{2}[\iota_{4n+4},\iota_{4n+4}]\circ HJ(\mathcal{X})$$
$$= kJ(\mathcal{X})-\binom{k}{2}(n+2)[\nu_{4n+4},\iota_{4n+4}].$$

Hence

$$k(l-1)J(\mathcal{X}) = (n+2)\{-\binom{k}{2}[\nu_{4n+4},\iota_{4n+4}]-\nu_{4n+4}\circ y\}.$$

Since $c_n(=24/(24,n+2))$-times of the right hand term is zero, it follows that

$$k(l-1)c_n \equiv 0 \pmod{\#J(\mathcal{X})}.$$

By Proposition 6.1, this gives us a non trivial information only when $n \equiv 0 \pmod 2) \geq 4$. The result (2) then follows easily.

Note that $n+2 \equiv 0 \pmod{24}$ if and only if $X_{n+2,2}$ has a section. Thus, since $J(\xi)$ can be desuspended in this case, $X_{n+2,2}$ has an $M(k,l)$-structure if and only if $k(l-1)J(\xi)=0$ by Theorem 1.

Since $\pi_{8n+11}(S^{8n+7})=0$, the suspension $\Sigma: \pi_{8n+9}(S^{4n+3}) \to \pi_{8n+10}(S^{4n+4})$ is injective. Hence $\#J(\xi)=\#J(\mathcal{X})$ provided $n+2 \equiv 0 \pmod{24}$ so that (4) follows. We also have

$$\Sigma(\bar{c}_n\circ\rho) = \Sigma\,\bar{c}_n\circ\Sigma\rho$$
$$= \Sigma\bar{c}_n\circ\Sigma j\circ J(\mathcal{X}), \quad \text{by [12]},$$
$$= c_n\iota_{4n+4}\circ J(\mathcal{X}) = c_nJ(\mathcal{X})+\binom{c_n}{2}[\iota_{4n+4},\iota_{4n+4}]\circ HJ(\mathcal{X})$$
$$= J(c_n\mathcal{X})+(n+2)\binom{c_n}{2}[\nu_{4n+4},\iota_{4n+4}].$$

Let b denote $\#\mathcal{X}/\#\alpha$ or $2\cdot\#\mathcal{X}/\#\alpha$ according as n is even or odd. Then $c_nb\mathcal{X}=0$ and $b(n+2)\binom{c_n}{2}$ is 0 (mod 48) or 0 (mod 24) according as n is even or odd. Hence $\Sigma(b\iota_{4n+3}\circ\bar{c}_n\circ\rho) = b\Sigma(\bar{c}_n\circ\rho)=0$ so that $b\iota_{4n+3}\circ\bar{c}_n\circ\rho=0$. Thus (3) follows from Theorem 4. This completes the proof of Proposition 6.2. □

§ 7. Self Maps of $\Sigma E(\mathcal{X})$

In this section we assume $n \geq q+2$ and $q \geq 2$.

A self map f of ΣE is called an $M(k,l,m)$-structure if $f^*: H^r(\Sigma E) \to H^r(\Sigma E)$ is the multiplication by k, l or m according as r is $q+1$, $n+1$ or $q+n+1$. We use the word $M(k,l)$-structure on ΣY in the obvious sense. Let $j: S^{q+1} \to \Sigma Y$ and $j': \Sigma Y \to \Sigma E$ be the inclusions and $p: \Sigma Y \to S^{n+1}$ and $p': \Sigma Y \to S^{q+n+1}$ the quotient maps.

Proposition 7.1. *Suppose $n \geq q+2$ and $q \geq 2$.*

(1) *There is an exact sequence of groups:*

$$0 \to \pi_{n+1}(S^{q+1})/\langle \Sigma\alpha\circ\eta_n, \eta_{q+1}\circ\Sigma^2\alpha\rangle \oplus \mathbb{Z}\{\beta\} \underset{\phi}{\to} [\Sigma Y, \Sigma Y] \underset{\phi}{\to} \mathbb{Z} \to 0,$$

where $\phi(x) = p^* j_*(x)$ for $x \in \pi_{n+1}(S^{q+1}), \phi(\beta) = \widetilde{\#\Sigma}\, \alpha\circ p$, and $\phi(f) = j_*^{-1} j^*(f)$.

(2) *There exists an $M(k,l)$-structure on ΣY if and only if $k \equiv l \pmod{\#\Sigma\alpha}$.*

(3) *There exists an $M(k,l,m)$-structure on ΣE if and only if*

(i) $k \equiv l \pmod{\#\Sigma\alpha}$ *and*

(ii) $mJ(\mathcal{X}) - k\iota_{q+1}\circ J(\mathcal{X}) \in \operatorname{Ker}\{j_*: \pi_{q+n}(S^{q+1}) \to \pi_{q+n}(\Sigma Y)\}$
$$= \Sigma\alpha\circ\pi_{q+n}(S^n) + \langle[\iota_{q+1}, \Sigma\alpha]\rangle.$$

Corollary 7.2. *When E is $W_{n+2,2}$ or $X_{n+2,2}$, there is an $M(k,l,m)$-structure on ΣE if and only if $k \equiv l \pmod{\#\Sigma\alpha}$ and $k \equiv m \pmod{\#\Sigma\rho}$.*

When n is odd, $\#\Sigma\rho$ in 7.2 was determined by Mukai [17]. When n is even, we don't know the value of $\#\Sigma\rho$ except the following cases.

Proposition 7.3. (1) *If $n \geq 4$ is even and $E = W_{n+2,2}$, then $\#\Sigma\rho = 4$ and $\#\Sigma^2\rho = 2$.*

(2) *If $n+2 \equiv 0 \pmod{24}$ and $E = X_{n+2,2}$, then $\#\Sigma\rho = 16$ and $\#\Sigma^2\rho = 8$ (provided $\#[\iota_{4n+4}, \nu_{4n+4}] = 24$).*

Proof of Proposition 7.1. We can prove (1) by using Puppe sequences. We omit its proof.

We have (2) by Lemma 2.2(1).

The equality in (3)(ii) follows from Lemma 2.1.

Suppose given an $M(k,l,m)$-structure f on ΣE. Write $h = f|_{\Sigma Y}$. This is an $M(k,l)$-structure on ΣY. Hence (i) follows from (2). By Lemma 2.1, there is an integer m' with $m'\Sigma\rho = h\circ\Sigma\rho$. Hence there is a self map f' of ΣE such that $j'\circ h = f'\circ j'$ and $p'\circ f' = m'\iota_{q+n+1}\circ p'$. By the method used in the proof of Lemma 2.2, we have $m \equiv m'$ (mod $\#\Sigma\rho$) so that $h\circ\Sigma\rho = m\Sigma\rho$. Since $\Sigma\rho = j_* J(\mathcal{X})$, we then have (ii).

Conversely suppose (i) and (ii). By (2), there is an $M(k,l)$-structure h on ΣY. Then $m\Sigma\rho = j_*(mJ(\mathcal{X})) = j_*(k\iota_{q+1}\circ J(\mathcal{X})) = h\circ j\circ J(\mathcal{X}) = h\circ\Sigma\rho$. Hence there is a self map f of ΣE such that $j'\circ h = f\circ j'$ and $p'\circ f = m\iota_{q+n+1}\circ p'$. Clearly f is an $M(k,l,m)$-structure on ΣE. This ends the proof of (3) and completes the proof of Lemma 7.1. □

Proof of Corollary 7.2. This follows from 7.1 and the following two equalities:

$$k\iota_{q+1}\circ J(\mathcal{X}) = kJ(\mathcal{X}) \pm \binom{k}{2}[\iota_{q+1}, \Sigma\alpha] \quad \text{and} \quad j_* J(\mathcal{X}) = \Sigma\rho. \qquad \square$$

Proof of Proposition 7.3. We prove only (2), because (1) can be proved similarly. Since $\Sigma\rho = j_* J(\mathcal{X})$ and $\Sigma^2\rho = (\Sigma j)_* \Sigma J(\mathcal{X})$ and since j_* and $(\Sigma j)_*$ are injective, it suffices to prove the assertions replacing $\#\Sigma\rho$ and $\#\Sigma^2\rho$ by $\# J(\mathcal{X})$ and $\#\Sigma J(\mathcal{X})$, respectively. By the proof of 6.1, we have $\# J(\mathcal{X}) = 16$. Since $\pi_{4n+6}(SO(4n+5)) = \mathbb{Z}_8$, we have $\Sigma(8J(\mathcal{X})) = -8J(i_*(\mathcal{X})) = 0$, where $i: SO(4n+4) \to SO(4n+5)$ is the inclusion. Hence $8J(\mathcal{X}) = 12[\iota_{4n+4}, \nu_{4n+4}]$, since $\mathrm{Ker}\,\Sigma = \langle[\iota_{4n+4}, \nu_{4n+4}]\rangle$. To induce a contradiction, assume $4\Sigma J(\mathcal{X}) = 0$. Then $4J(\mathcal{X}) = c[\iota_{4n+4}, \nu_{4n+4}]$ with $2c \equiv 12$ (mod 24). By applying H, we have $0 = 12\nu_{8n+7}$ which is a contradiction. Hence $\#\Sigma J(\mathcal{X}) = 8$. This ends the proof of (2). □

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