Publ. RIMS, Kyoto Univ.
32 (1996), 235–276

Irreducible Modules of Quantized Enveloping Algebras at Roots of 1

By

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Introduction

Let **A** be an associative algebra over a field. An interesting problem is to understand the structure of irreducible modules of **A** (of finite dimensions). More or less, this is equivalent to understand the structure of maximal left ideals of **A** (of finite codimensions). For the latter, it would be helpful if we know the generators of the maximal left ideals.

In Lie theory, there are some infinite dimensional algebras associated to a semisimple Lie algebra $\mathfrak{g}$ over **C**. We shall be only concerned with the following four of them.

(i) The universal enveloping algebra $\mathfrak{U}$ of $\mathfrak{g}$.

(ii) The hyperalgebra $\mathfrak{U}_{\mathfrak{k}} := \mathfrak{U}_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathfrak{k}$, where $\mathfrak{U}_{\mathbf{Z}}$ is the Kostant **Z**-form of $\mathfrak{U}$ and $\mathfrak{k}$ is an algebraically closed field of prime characteristic.

(iii) The quantized enveloping algebra U (over $\mathbf{Q}(v)$, v is an indeterminate) of $\mathfrak{g}$.

(iv) The quantized hyperalgebra $U_\xi := U_{\mathbf{Q}[v,v^{-1}]} \otimes_{\mathbf{Q}[v,v^{-1}]} \mathbf{Q}(\xi)$, where $\xi \in \mathbf{C}^*$ and $U_{\mathbf{Q}[v,v^{-1}]}$ is a $\mathbf{Q}[v, v^{-1}]$-form of U [L1, Section 4.1, p. 243], and $\mathbf{Q}(\xi)$ is regarded as a $\mathbf{Q}[v, v^{-1}]$-algebra through the **Q**-algebra homomorphism $\mathbf{Q}[v, v^{-1}] \to \mathbf{Q}(\xi)$, $v \to \xi$.

We are mainly interested in finite dimensional irreducible modules of these algebras, or equivalently, in maximal left ideals of the algebras of finite codimensions. The generators of maximal left ideals of $\mathfrak{U}$ with finite codimensions are known more than forty years ago [HC, Lemma 15, p. 42]. Thanks to the works [L1, Theorem 4.12, p. 247] and [APW, Corollary 7.7, p. 40], a similar result holds for maximal left ideals of U and of U_ξ with finite codimensions provided that ξ is not a root of 1 or $\xi^2 = 1$. We will review these results in Section 1.2.

The purpose of the paper is to find out the counterparts of the above

Communicated by M. Kashiwara, March 13, 1995. Revised August 11, 1995.
1991 Mathematics Subject Classification(s): 17B37, 17B10.
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results for the hyperalgebra $\mathfrak{U}_{\mathfrak{k}}$ and for the quantized hyperalgebra U_ξ when ξ is a root of 1 of order ≥ 3. The main results might lead to a way to compute the characters of finite dimensional irreducible modules of $\mathfrak{U}_{\mathfrak{k}}$ and of U_ξ.

The basic idea is simple. When ξ is a root of 1 of order ≥ 3, the algebra U_ξ has a Frobenius kernel $\mathbf{u}_\xi$ [L4, Theorem 8.3, p. 107]. The Frobenius kernel $\mathbf{u}_\xi$ is a $\mathbf{Q}(\xi)$-algebra of finite dimension. Moreover, the algebra $\mathbf{u}_\xi$ has a triangular decomposition $\mathbf{u}_\xi = \mathbf{u}_\xi^- \mathbf{u}_\xi^0 \mathbf{u}_\xi^+$. Each Verma module of $\mathbf{u}_\xi$ has a unique irreducible submodule, and each irreducible $\mathbf{u}_\xi$-module L is an irreducible submodule of certain Verma module Z of $\mathbf{u}_\xi$. As $\mathbf{u}_\xi^-$-modules, Z is isomorphic to $\mathbf{u}_\xi^-$. Therefore there exists an element x in $\mathbf{u}_\xi^-$ such that as $\mathbf{u}_\xi^-$-modules L is isomorphic to $\mathbf{u}_\xi^- x$. It turns out that the element x is a monomial of the generators of U_ξ^- (the negative part of U_ξ). So the generators of the maximal left ideal of $\mathbf{u}_\xi$ corresponding to L can be described explicitly (Theorem 5.3). But L is a restriction to $\mathbf{u}_\xi$ of certain irreducible U_ξ-module [L2, Prop. 7.1 (c), p. 70]. Using tensor product theorem [L2, Theorem 7.4, p. 73], we can give the generators of maximal left ideals of U_ξ of finite codimensions (Theorem 5.4). The same idea is valid to the hyperalgebra $\mathfrak{U}_{\mathfrak{k}}$.

The paper is organized as follows. In Section 1 we recall some basic definitions and review some results in [APW, HC, L1-L4]. In Section 2 we consider the Frobenius kernel $\mathbf{u}_\xi$. In Section 3 we consider the category of finite dimensional U_ξ-modules of type 1. In Section 4 we prove that certain monomials in U_ξ^- are actually in $\mathbf{u}_\xi^-$. For a technical reason we require that every simple component of $\mathfrak{g}$ is not of type G_2. In Section 5 we give the main theorems of the paper. In Section 6 we consider the hyperalgebra $\mathfrak{U}_{\mathfrak{k}}$. In Section 7 we give some questions.

Acknowledgement

I would like to express my hearty thanks to H.H. Andersen, F. Hirzebruch, G. Lusztig and B. Srinivasan for their kind encouragements. I wish to thank G. Lusztig for helpful comments.

I am grateful to Max-Planck-Institut für Mathematik for financial support and hospitality. Part of the work was done during my visit at IHES, Bures-sur-Yvette, 1993 Fall. I am grateful to IHES for financial support and hospitality.

I am very grateful to the referee for helpful comments.

§1. Quantized Hyperalgebra

1.1. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with rank n and let (a_{ij}) be the Cartan matrix associated to $\mathfrak{g}$. We can find integers d_i in $\{1, 2, 3\}$ such that $(d_i a_{ij})$ is a symmetric matrix. Assume that the sum of all d_i is as small as possible.

Let U be the quantized enveloping algebra of $\mathfrak{g}$ over $\mathbf{Q}(v)$ with parameter v (v an indeterminate). By definition, U is an associative $\mathbf{Q}(v)$-algebra and has generators E_i, F_i, K_i, K_i^{-1} ($i = 1, 2, \dots, n$), which satisfy certain relations (see for example, [L4, 1.1, p. 90]). The algebra U is in fact a Hopf algebra, the coproduct Δ, antipode S, counit ε are defined as follows:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i,$$
$$S(E_i) = -K_i^{-1}E_i, \quad S(F_i) = -F_iK_i, \quad S(K_i) = K_i^{-1},$$
$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.$$

We need some notations to introduce quantized hyperalgebras and for later uses. Given an integer a and positive integers b, d, set

$$[a]_d := \frac{v^{ad} - v^{-ad}}{v^d - v^{-d}}, \quad [b]_d^! := \prod_{h=1}^{b} \frac{v^{hd} - v^{-hd}}{v^d - v^{-d}}, \quad [0]_d^! := 1, \quad [-b]_d^! = (-1)^b [b]_d^!;$$
$$\begin{bmatrix} a \\ b \end{bmatrix}_d := \prod_{h=1}^{b} \frac{v^{(a-h+1)d} - v^{-(a-h+1)d}}{v^{hd} - v^{-hd}}, \quad \begin{bmatrix} a \\ 0 \end{bmatrix}_d := 1, \quad \begin{bmatrix} a \\ -b \end{bmatrix}_d := 0.$$

Note that $\begin{bmatrix} a \\ b \end{bmatrix}_d$ is in $\mathbf{Q}[v, v^{-1}]$, we shall denote $\begin{bmatrix} a \\ b \end{bmatrix}_{d,\xi}$ the evaluation of $\begin{bmatrix} a \\ b \end{bmatrix}_d$ at ξ for any ξ in $\mathbf{C}^* \cup \{v\}$. Of course, we have $\begin{bmatrix} a \\ b \end{bmatrix}_{d,v} = \begin{bmatrix} a \\ b \end{bmatrix}_d$.

The quantized hyperalgebra U_ξ ($\xi \in \mathbf{C}^*$) is defined as follows. Let $U_{\mathbf{Q}[v,v^{-1}]}$ be the $\mathbf{Q}[v, v^{-1}]$-subalgebra of U generated by the elements $E_i^{(a)} := E_i^a/[a]_{d_i}^!$, $F_i^{(a)} := F_i^a/[a]_{d_i}^!$, K_i, K_i^{-1} for $i = 1, 2, \dots, n$, $a \geq 0$. Regard $\mathbf{Q}(\xi)$ as a $\mathbf{Q}[v, v^{-1}]$-algebra through the $\mathbf{Q}$-algebra homomorphism $\mathbf{Q}[v, v^{-1}] \to \mathbf{Q}(\xi)$, $v \to \xi$. Define $U_\xi := U_{\mathbf{Q}[v,v^{-1}]} \otimes_{\mathbf{Q}[v,v^{-1}]} \mathbf{Q}(\xi)$ and call U_ξ a quantized hyperalgebra (associated to (a_{ij}) with parameter ξ). For convenience, set $U_v := U$. The algebra U_ξ inherits a Hopf algebra structure from that of $U_{\mathbf{Q}[v,v^{-1}]}$, denote again by Δ the coproduct, by S the antipode and by ε the counit. The tensor product of two U_ξ-modules then has a natural U_ξ-module structure by means of the coproduct, and the antipode can be used to define the dual module of a U_ξ-module.

For an integer c and a positive integer a we set

$$\begin{bmatrix} K_i, c \\ a \end{bmatrix} := \prod_{h=1}^{a} \frac{K_i v^{(c-h+1)d_i} - K_i^{-1} v^{-(c-h+1)d_i}}{v^{hd_i} - v^{-hd_i}} \quad \text{and} \quad \begin{bmatrix} K_i, c \\ 0 \end{bmatrix} := 1.$$

We have $\begin{bmatrix} K_i, c \\ a \end{bmatrix} \in U_{\mathbf{Q}[v,v^{-1}]}$ [L1, Lemma 4.4, p. 244]. For simplicity, the images in U_ξ of $E_i^{(a)}$, $F_i^{(a)}$, K_i, K_i^{-1}, $\begin{bmatrix} K_i, c \\ a \end{bmatrix}$, etc. will be denoted by the same notations respectively. For convenience, we set $E_i^{(a)} := 0$, $F_i^{(a)} := 0$ for all i and $a < 0$.

The algebra U_ξ has a triangular decomposition. Let U_ξ^+ (resp. U_ξ^- ; U_ξ^0) be the subalgerba of U_ξ generated by the elements $E_i^{(a)}$ $\left(\text{resp. } F_i^{(a)}; K_i, K_i^{-1}, \begin{bmatrix} K_i, c \\ a \end{bmatrix}, c \in \mathbf{Z}\right)$ for $i = 1, 2, \ldots, n$, $a \geq 0$. The multiplication in U_ξ defines a $\mathbf{Q}(\xi)$-space isomorphism between $U_\xi^- \otimes U_\xi^0 \otimes U_\xi^+$ and U_ξ.

1.2. Given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{Z}_+^n$, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \{\pm 1\}^n$, let $I_{\lambda,\sigma}^+$ be the left ideal of U_ξ generated by the elements $E_i^{(a)}$, $K_i - \sigma_i \xi^{\lambda_i d_i}$, $\begin{bmatrix} K_i, c \\ a \end{bmatrix} - \sigma_i^a \begin{bmatrix} \lambda_i + c \\ a \end{bmatrix}_{d_i,\xi}$ for $i = 1, 2, \ldots, n$, $a \geq 1$, $c \in \mathbf{Z}$, and let I_λ^- be the left ideal of U_ξ^- generated by the elements $F_i^{(a_i)}$ for all i and $a_i \geq \lambda_i + 1$. Then let $I_{\lambda,\sigma}$ be the left ideal of U_ξ generated by all elements in $I_{\lambda,\sigma}^+ \cup I_\lambda^-$. Then

(i) The U_ξ-module $V_\xi(\lambda, \sigma) := U_\xi / I_{\lambda,\sigma}$ is of finite dimension and has a unique irreducible quotient module, denoted by $L_\xi(\lambda, \sigma)$. The dimension of $V_\xi(\lambda, \sigma)$ is given by Weyl's character formula. [L1, Theorem 4.12, p. 247]. We shall denote $v_{\lambda,\sigma}$ the image in $V_\xi(\lambda, \sigma)$ of the neutral element $1 \in U_\xi$, and denote $\bar{v}_{\lambda,\sigma}$ the image in $L_\xi(\lambda, \sigma)$ of $v_{\lambda,\sigma}$.

Sometimes we call $V_\xi(\lambda, \sigma)$ a Weyl module of U_ξ.

(ii) The map $(\lambda, \sigma) \to L_\xi(\lambda, \sigma)$ defines a bijection between the set $\mathbf{Z}_+^n \times \{\pm 1\}^n$ and the set of isomorphism classes of irreducible U_ξ-modules of finite dimensions. [L1, Prop. 2.6 and Prop. 3.2, p. 241] and [L2, Prop. 6.4, p. 69].

(iii) One has

$$V_\xi(\lambda, \sigma) \simeq V_\xi(\lambda, \mathbb{1}) \otimes \mathbf{Q}(\xi)_\sigma, \quad L_\xi(\lambda, \sigma) \simeq L_\xi(\lambda, \mathbb{1}) \otimes \mathbf{Q}(\xi)_\sigma,$$

where $\mathbb{1} = (1, 1, \ldots, 1) \in \{\pm 1\}^n$ and $\mathbf{Q}(\xi)_\sigma$ is the one dimensional U_ξ-module on which all $E_i^{(a)}$, $F_i^{(a)}$ ($i = 1, 2, \ldots, n$, $a \geq 1$) act by scalar zero and K_i, $\begin{bmatrix} K_i, c \\ a \end{bmatrix}$ ($i = 1, 2, \ldots, n$, $c \in \mathbf{Z}$, $a \in \mathbf{N}$) act by scalar σ_i, $\sigma_i^a \begin{bmatrix} c \\ a \end{bmatrix}_{d_i,\xi}$ respectively. [APW, 1.6, pp. 6–7].

(iv) Provided that ξ is not a root of 1 or $\xi^2 = 1$, then $V_\xi(\lambda, \sigma)$ is irreducible, i.e. $V_\xi(\lambda, \sigma) \simeq L_\xi(\lambda, \sigma)$. And every finite dimensional U_ξ-module is completely reducible. [L4, 7.2, pp. 105–106; APW, Corollary 7.7, p. 40].

Therefore, the theory of finite dimensional U_ξ-modules is well understood when ξ is not a root of 1 or $\xi^2 = 1$. When ξ is a root of 1 of order ≥ 3 we do not know much about the irreducible module $L_\xi(\lambda, \sigma)$. In Section 5 we shall describe the generators of the maximal left ideal $J_{\lambda,\sigma}$ of U_ξ corresponding to $L_\xi(\lambda, \sigma)$. To have a look what the generators are we introduce some monomials of $F_i^{(a)}$ $(i = 1, 2, \ldots, n,\ a \geq 0)$. These monomials play a central role in the paper.

1.3. Set $\alpha_i = (a_{1i}, a_{2i}, \ldots, a_{ni}) \in \mathbf{Z}^n$. For every $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbf{Z}^n$, we also write $\langle \mu, \alpha_i^\vee \rangle$ for μ_i. Define $s_i \colon \mathbf{Z}^n \to \mathbf{Z}^n$ by $s_i\mu = \mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i$. The reflections $s_1, s_2, \ldots, s_n$ generate the Weyl group W of the Cartan matrix (a_{ij}).

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{Z}_+^n$. Assume that $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression of an element w in W. Set $\lambda_{\mathbf{i},1} = \lambda_{i_1}$, $\lambda_{\mathbf{i},2} = \langle s_{i_1}\lambda, \alpha_{i_2}^\vee \rangle, \ldots, \lambda_{\mathbf{i},k} = \langle s_{i_{k-1}} \cdots s_{i_1}\lambda, \alpha_{i_k}^\vee \rangle$, where $\mathbf{i} = (i_1, i_2, \ldots, i_k)$. Define

$$x_{\lambda,w,\mathbf{i}} := F_{i_1}^{(\lambda_{\mathbf{i},1})} F_{i_2}^{(\lambda_{\mathbf{i},2})} \cdots F_{i_k}^{(\lambda_{\mathbf{i},k})}, \quad \text{and} \quad x'_{\lambda,w^{-1},\mathbf{i}} := F_{i_k}^{(\lambda_{\mathbf{i},k})} F_{i_{k-1}}^{(\lambda_{\mathbf{i},k-1})} \cdots F_{i_1}^{(\lambda_{\mathbf{i},1})}.$$

Depending on the contexts, the monomials will be regarded as elements in U_ξ ($\xi \in \mathbf{C}^*$) or elements in U. Note that in the universal enveloping algebra $\mathfrak{U}$ of $\mathfrak{g}$ similar elements are defined by Verma [V, Theorem 4, p. 162].

Lemma 1.4. *The elements $x_{\lambda,w,\mathbf{i}}$ and $x'_{\lambda,w^{-1},\mathbf{i}}$ are independent of the choice of the reduced expression of w, only depend on λ and w. We shall denote them $x_{\lambda,w}$ and $x'_{\lambda,w^{-1}}$ respectively. When w is the longest element w_0 of W, we simply write x_λ and x'_λ for $x_{\lambda,w}$ and $x'_{\lambda,w}$ respectively.*

Proof. Use the quantum Verma identity [L7, Prop. 39.3.7, p. 313].

1.5. From now on ξ will be a root of 1 with order $l \geq 3$. Let l_i be the order of ξ^{2d_i} and set $\kappa := (l_1 - 1, l_2 - 1, \ldots, l_n - 1)$. We say that an element $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{Z}_+^n$ is $\mathbf{l}$-restricted if $\lambda_1 \leq l_1 - 1, \ldots, \lambda_n \leq l_n - 1$. For each $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbf{Z}^n$ we set $\mathbf{l}\mu := (l_1\mu_1, l_2\mu_2, \ldots, l_n\mu_n)$.

Let $\lambda, \mu \in \mathbf{Z}_+^n$, $\sigma \in \{\pm 1\}^n$ and assume that λ is $\mathbf{l}$-restricted. Let $J_{\mathbf{l}\mu+\lambda,\sigma}$ be the left ideal of U_ξ generated by all elements in $I_{\mathbf{l}\mu+\lambda,\sigma}$ and elements F in U_ξ^- such that $Fx_{\kappa-\lambda} = 0$, one main result of the paper says that $U_\xi/J_{\mathbf{l}\mu+\lambda,\sigma} \simeq L_\xi(\mathbf{l}\mu + \lambda, \sigma)$ (Theorem 5.4) provided that every indecomposable component of the Cartan matrix (a_{ij}) is not of type G_2. One key step to reach the result is the assertion that $x'_{\kappa+w_0\lambda} (= x_{\kappa-\lambda})$ belongs to the Frobenius kernel (Theorem 4.2).

1.6. *Remark.* Some results in [L1-L4] are stated and proved in full generality in [L7]. The other results in [L1-L4] can be stated and proved in full generality along the same ways in [L1-L4]. Therefore the author feels free to quote the results in [L1-L4] in full generality forms.

§2. Frobenius Kernel

2.1. Recall that ξ is a root of 1 with order $l \geq 3$ and l_i is the order of ξ^{2d_i}. Let R^+ be the set of positive roots of the root system $R := W\{\alpha_1, \alpha_2, \ldots,$

$\alpha_n\} \subset \mathbf{Z}^n$. Set $l_\alpha := l_i$, $d_\alpha := d_i$ if $\alpha = w(\alpha_i)$ for some w in W. For each positive root α in R^+, let E_α, F_α be the root vectors defined in [L4, Theorem 6.6 (iii), p. 104]. For a non-negative integer a, we also write $E_\alpha^{(a)}$, $F_\alpha^{(a)}$ for $E_\alpha^a/[a]_{d_\alpha}^!$, $F_\alpha^a/[a]_{d_\alpha}^!$ respectively.

Let $U_{\xi,l}$ be the subalgebra of U_ξ generated by the elements $E_i^{(al_i)}$, $F_i^{(al_i)}$, $K_i^{\pm l_i}$, $\begin{bmatrix} K_i, c \\ al_i \end{bmatrix}$ for $i = 1, 2, \ldots, n$, $c \in \mathbf{Z}$, $a \in \mathbf{N}$. The positive part $U_{\xi,l}^+$, the negative part $U_{\xi,l}^-$ and the zero part $U_{\xi,l}^0$ of $U_{\xi,l}$ are defined in an obvious way. Let $\mathbf{u}_\xi$ be the subalgebra of U_ξ generated by the elements E_α, F_α, K_i, K_i^{-1} for $\alpha \in R_l^+ := \{\alpha \in R^+ \mid l_\alpha \geq 2\}$ and $i = 1, 2, \ldots, n$. The algebra is called the Frobenius kernel of U_ξ. The Frobenius kernel $\mathbf{u}_\xi$ is a Hopf algebra and $\dim_{\mathbf{Q}(\xi)} \mathbf{u}_\xi = 2^n \prod_{i=1}^n l_i \prod_{\alpha \in R^+} l_\alpha^2$ [L4, 8.11, p. 111, and Theorem 8.3, p. 107]. We define the positive part $\mathbf{u}_\xi^+$, the negative part $\mathbf{u}_\xi^-$ and the zero part $\mathbf{u}_\xi^0$ of $\mathbf{u}_\xi$ in an obvious manner.

2.2. The following are some properties concerned with the algebras $U_{\xi,l}$ and $\mathbf{u}_\xi$, which are due to Lusztig.

(i) Assume that (a_{ij}) is indecomposable. Then there exists a unique $\mathbf{Q}(\xi)$-algebra homomorphism $U_{\xi,l} \to U_{\xi^*}^* \otimes_{\mathbf{Q}} \mathbf{Q}(\xi)$ such that $E_i^{(al_i)} \to E_i^{(a)}$, $F_i^{(al_i)} \to F_i^{(a)}$, $K_i^{\pm l_i} \to K_i^{\pm 1}$, $\begin{bmatrix} K_i, 0 \\ al_i \end{bmatrix} \to \begin{bmatrix} K_i, 0 \\ a \end{bmatrix}$, and $E_i^{(b_i)} \to 0$, $F_i^{(b_i)} \to 0$, $\begin{bmatrix} K_i, 0 \\ b_i \end{bmatrix} \to 0$, for $i = 1, 2, \ldots, n$, $a \in \mathbf{N}$, $b_i \in \mathbf{N} - l_i\mathbf{N}$, where $U_{\xi^*}^* = U_{\xi^*}$ when $l_1 = l_2 = \cdots = l_n$ and $U_{\xi^*}^*$ is the quantized hyperalgebra associated to the transpose matrix of (a_{ij}) with parameter ξ^* when $l_k = d_m l_m$ for some k, m with $1 = d_k < d_m \in \{2, 3\}$, and $\xi^* = \xi^{l_1^2}$ when $l_1 = l_2 = \cdots = l_n$ and $\xi^* = \xi^{d_m l_m^2}$ when $l_k = d_m l_m$ for some k, m with $1 = d_k < d_m \in \{2, 3\}$. (Note that $l_i = l_j$ if α_i, α_j are conjugate under W. So ξ^* does not depend on the choice of m and is well defined.) [L7, Theorems 35.1.9, p. 270; 35.5.2, p. 279; L4, Theorem 8.10, p. 110].

We always have $\xi^* = \pm 1$. Actually, if $l_1 = l_2 = \cdots = l_n$, choose i such that $d_i = 1$, then $\xi^* = \xi^{l_i^2} = (\pm 1)^{l_i} = \pm 1$, if $l_k = d_m l_m$ for some k, m with $1 = d_k < d_m \in \{2, 3\}$, then $\xi^* = \xi^{d_m l_m^2} = (\pm 1)^{l_m} = \pm 1$.

(ii) Let $\{x_a\}$ be a $\mathbf{Q}(\xi)$-basis of $\mathbf{u}_\xi^-$ and $\{y_b\}$ be a $\mathbf{Q}(\xi)$-basis of $U_{\xi,l}^-$, then $\{x_a y_b\}$ is a $\mathbf{Q}(\xi)$-basis of U_ξ^-, so is $\{y_b x_a\}$ [L4, Lemma 8.8, p. 109; L7, Theorem 35.4.2 (b), p. 276, 35.5.2, p. 279].

(iii) The elements $\prod_{\alpha \in R_l^+} F_\alpha^{(a_\alpha)} \prod_{i=1}^n K_i^{b_i} \prod_{\alpha \in R_l^+} E_\alpha^{(a'_\alpha)}$ $(0 \leq a_\alpha, a'_\alpha \leq l_\alpha - 1, 0 \leq b_i \leq 2l_i - 1)$ form a $\mathbf{Q}(\xi)$-basis of $\mathbf{u}_\xi$; the elements $\prod_{\alpha \in R_l^+} F_\alpha^{(a_\alpha)}$ $(0 \leq a_\alpha \leq l_\alpha - 1)$ form a $\mathbf{Q}(\xi)$-basis of $\mathbf{u}_\xi^-$; the elements $\prod_{i=1}^n K_i^{b_i}$ $(0 \leq b_i \leq 2l_i - 1)$ form a $\mathbf{Q}(\xi)$-basis of $\mathbf{u}_\xi^0$; the elements $\prod_{\alpha \in R_l^+} E_\alpha^{(a'_\alpha)}$ $(0 \leq a'_\alpha \leq l_\alpha - 1)$ form a $\mathbf{Q}(\xi)$-basis of $\mathbf{u}_\xi^+$. [L4, Theorem 8.3, p. 107].

(iv) Let $\lambda, \mu \in \mathbf{Z}_+^n$ and $\sigma \in \{\pm 1\}^n$. Assume that λ is $\mathbf{l}$-restricted. Then [L2, Theorem 7.4, p. 73]

$$L_\xi(\mathbf{l}\mu + \lambda, \sigma) \simeq L_\xi(\mathbf{l}\mu, \sigma) \otimes L_\xi(\lambda, \mathbf{1}) \simeq L_\xi(\mathbf{l}\mu, \mathbf{1}) \otimes L_\xi(\lambda, \sigma).$$

(v) Assume that (a_{ij}) is indecomposable. The restriction to $U_{\xi,l}$ of $L_\xi(\mathbf{l}\mu, \sigma)$ is an irreducible $U_{\xi,l}$-module, and $E_\alpha L_\xi(\mathbf{l}\mu, \sigma) = F_\alpha(L_\xi(\mathbf{l}\mu, \sigma) = 0$ for all $\alpha \in R_l^+$. By this and (i) we see there is a $U_{\xi^*}^* \otimes_{\mathbf{Q}} \mathbf{Q}(\xi)$-module structure on $L_\xi(\mathbf{l}\mu, \sigma)$ which is compatible with the homomorphism in (i). Moreover, $L_\xi(\mathbf{l}\mu, \sigma)$ is an irreducible $U_{\xi^*}^* \otimes_{\mathbf{Q}} \mathbf{Q}(\xi)$-module corresponding to (μ, σ') for a suitable $\sigma' \in \{\pm 1\}^n$. So dimension of $L_\xi(\mathbf{l}\mu, \sigma)$ can be computed through Weyl's character formula. [L7, Prop. 35.3.2, p. 273, Corollary 35.3.4, p. 275; L2, Prop. 7.5 (b), p. 74].

(vi) As a $\mathbf{u}_\xi$-module, $L_\xi(\lambda, \sigma)$ is irreducible if λ is $\mathbf{l}$-restricted. The map $(\lambda, \sigma) \to L_\xi(\lambda, \sigma)$ defines a bijections between the set $\mathbf{Z}^n_{+,\mathbf{l}} \times \{\pm 1\}^n$ and the set of isomorphism classes of irreducible $\mathbf{u}_\xi$-modules, where $\mathbf{Z}^n_{+,\mathbf{l}}$ is the set of all $\mathbf{l}$-restricted elements in $\mathbf{Z}^n_+$ [L3, Prop. 5.11, p. 291].

According to (i-vi), the algebra $\mathbf{u}_\xi$ is a key to understand U_ξ. For convenience, we consider the subalgebra $\tilde{\mathbf{u}}_\xi$ of U_ξ generated by all elements in $\mathbf{u}_\xi \cup U_\xi^0$. One has $\tilde{\mathbf{u}}_\xi = \mathbf{u}_\xi^- U_\xi^0 \mathbf{u}_\xi^+$. By (vi) we see

(vii) Assume that $(\lambda, \sigma) \in \mathbf{Z}^n_{+\mathbf{l}} \times \{\pm 1\}^n$, then the restriction to $\tilde{\mathbf{u}}_\xi$ of the irreducible U_ξ-module $L_\xi(\lambda, \sigma)$ is an irreducible $\tilde{\mathbf{u}}_\xi$-module, denoted by $\underset{\sim}{L}_\xi(\lambda, \sigma)$.

2.3. To go further we need some notions. Let $\gamma \in \mathbf{Z}R$. An element x in U_ξ is said to have degree γ if $K_i x K_i^{-1} = \xi^{-\langle \gamma, \alpha_i^\vee \rangle d_i} x$ and $\begin{bmatrix} K_i, c \\ a \end{bmatrix} x = x \begin{bmatrix} K_i, c - \langle \gamma, \alpha_i^\vee \rangle \\ a \end{bmatrix}$ for $i = 1, 2, \ldots, n$, $c \in \mathbf{Z}$, $a \in \mathbf{N}$. We also call x a homogenous element (of degree γ) and write $\deg(x) = \gamma$.

Let U_ξ' be a subalgebra of U_ξ containing U_ξ^0 and let M be a U_ξ'-module. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{Z}^n$, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \{\pm 1\}^n$. An element $m \in M$ is called to have weight (λ, σ) if

$$K_i m = \sigma_i \xi^{\lambda_i d_i} m, \quad \begin{bmatrix} K_i, c \\ a \end{bmatrix} m = \sigma_i^a \begin{bmatrix} \lambda_i + c \\ a \end{bmatrix}_{d_i, \xi} m$$

for $i = 1, 2, \ldots, n$, $c \in \mathbf{Z}$, $a \in \mathbf{N}$. Denote by $M_{\lambda,\sigma}$ the set of all elements in M of weight (λ, σ). We call (λ, σ) a weight of M if $M_{\lambda,\sigma}$ is not zero. If an element x in U_ξ' has degree γ, then obviously $x M_{\lambda,\sigma} \subseteq M_{\lambda - \gamma, \sigma}$.

As usual, for $(\lambda, \sigma), (\mu, \tau) \in \mathbf{Z}^n \times \{\pm 1\}^n$, we write $(\lambda, \sigma) \le (\mu, \tau)$ as well as $\lambda \le \mu$ if $\mu - \lambda \in \mathbf{N}R^+$ and $\sigma = \tau$. This defines a partial order in $\mathbf{Z}^n \times \{\pm 1\}^n$ as well as in $\mathbf{Z}^n$.

2.4. Now we return to the algebra $\tilde{\mathbf{u}}_\xi$. Assume that $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbf{Z}^n$ and $\tau = (\tau_1, \tau_2, \ldots, \tau_n) \in \{\pm 1\}^n$. Let $\tilde{I}^+_{\mu,\tau}$ be the left ideal of $\tilde{\mathbf{u}}_\xi$ generated by the elements E_α, $K_i - \tau_i \xi^{\mu_i d_i}$, $\begin{bmatrix} K_i, c \\ a \end{bmatrix} - \tau_i^a \begin{bmatrix} \mu_i + c \\ a \end{bmatrix}_{d_i, \xi}$ for $\alpha \in R_l^+$ and $i = 1, 2, \ldots, n$, $c \in \mathbf{Z}$, $a \in \mathbf{N}$. Denote by $\tilde{Z}_\xi(\mu, \tau)$ the Verma module $\tilde{\mathbf{u}}_\xi / \tilde{I}^+_{\mu,\tau}$ of $\tilde{\mathbf{u}}_\xi$ with highest weight (μ, τ). We shall write $\tilde{1}_{\mu,\tau}$ for the image in $\tilde{Z}_\xi(\mu, \tau)$ of the neutral element $1 \in \tilde{\mathbf{u}}_\xi$. By 2.2 (iii), $\tilde{Z}_\xi(\mu, \tau)$ has dimension $\prod_{\alpha \in R^+} l_\alpha$. We denote by $\mathbf{Q}(\xi)_{\mathbf{l}\mu,\tau}$ the one dimensional $\tilde{\mathbf{u}}_\xi$-module on which all E_α, F_α ($\alpha \in R_l^+$) act by scalar zero and K_i, $\begin{bmatrix} K_i, c \\ a \end{bmatrix}$ ($i = 1, 2, \ldots, n$, $c \in \mathbf{Z}$, $a \in \mathbf{N}$) act by scalar τ_i, $\tau_i^a \begin{bmatrix} l_i \mu_i + c \\ a \end{bmatrix}_{d_i, \xi}$, respectively.

Let (λ, σ), $(\mu, \tau) \in \mathbf{Z}^n \times \{\pm 1\}^n$. Then (cf. [X1, Prop. 2.4, Prop. 2.9])
(i) The Verma module $\tilde{Z}_\xi(\lambda, \sigma)$ has a unique irreducible quotient module, denoted by $\tilde{L}_\xi(\lambda, \sigma)$ (this notation would cause no confusion with that in 2.2 (vii) since both are isomorphic when $\lambda \in \mathbf{Z}^n_{+,\mathbf{l}}$). Moreover $\tilde{L}_\xi(\mathbf{l}\mu, \tau) \simeq \mathbf{Q}(\xi)_{\mathbf{l}\mu,\tau}$.
(ii) We have

$$\tilde{Z}_\xi(\lambda + \mathbf{l}\mu, \sigma\tau) \simeq \tilde{Z}_\xi(\lambda, \sigma) \otimes \tilde{L}_\xi(\mathbf{l}\mu, \tau) \simeq \tilde{Z}_\xi(\lambda, \tau) \otimes \tilde{L}_\xi(\mathbf{l}\mu, \sigma),$$

where the meaning of $\sigma\tau \in \{\pm 1\}^n$ is obvious.
(iii) We have

$$\tilde{L}_\xi(\lambda + \mathbf{l}\mu, \sigma\tau) \simeq \tilde{L}_\xi(\lambda, \sigma) \otimes \tilde{L}_\xi(\mathbf{l}\mu, \tau) \simeq \tilde{L}_\xi(\lambda, \tau) \otimes \tilde{L}_\xi(\mathbf{l}\mu, \sigma).$$

(iv) Let L be an irreducible $\tilde{\mathbf{u}}_\xi$-module such that L is the direct sum of its weight spaces, then L is isomorphic to certain $\tilde{L}_\xi(\lambda, \sigma)$. Two irreducible $\tilde{\mathbf{u}}_\xi$-modules $\tilde{L}_\xi(\lambda, \sigma)$ and $\tilde{L}_\xi(\mu, \tau)$ are isomorphic if and only if $(\lambda, \sigma) = (\mu, \tau)$.

(v) *Remark.* There is a natural bijection between the set of isomorphism classes of irreducible $\tilde{\mathbf{u}}_\xi$-modules and the set of isomorphism classes of irreducible U_ξ^0-modules (or equivalently, the set of maximal ideals of U_ξ^0 since U_ξ^0 is commutative). Note that the subalgebra $U'^0_{\xi,l}$ of U_ξ generated by $\begin{bmatrix} K_i, 0 \\ l_i \end{bmatrix}$ ($i = 1, 2, \ldots, n$) is isomorphic to a polynomial ring over $\mathbf{Q}(\xi)$ in n variables, and U_ξ^0 is generated by all elements in $U'^0_{\xi,l} \cup \mathbf{u}_\xi^0$.

Proof. Let P be an irreducible U_ξ^0-module. We regard P as a $\tilde{\mathbf{u}}_\xi^b = \mathbf{u}_\xi^+ U_\xi^0$-module by defining $E_\alpha P = 0$ for all $\alpha \in R_l^+$. Let $\tilde{Z}(P) = \tilde{\mathbf{u}}_\xi \otimes_{\tilde{\mathbf{u}}_\xi^b} P$. Then $\tilde{Z}(P)$ is a $\tilde{\mathbf{u}}_\xi$-module. Denote again by P the image in $\tilde{Z}(P)$ of P. Then $\tilde{Z}(P) = \mathbf{u}_\xi^- P$. Let M be a submodule of $\tilde{Z}(P)$. If $M \cap P \neq 0$, then $P \subseteq M$ since P is an irreducible U_ξ^0-module. Thus $\tilde{Z}(P) = \mathbf{u}_\xi^- P \subseteq M$. Therefore, if M is a proper submodule of $\tilde{Z}(P)$, then $M \cap P = 0$. Thus $\tilde{Z}(P)$ has a unique maximal submodule, which is the sum of all proper submodules of $\tilde{Z}(P)$. Denote by $M(P)$ the maximal submodule of $\tilde{Z}(P)$ and denote by $\tilde{L}(P)$ the irreducible $\tilde{\mathbf{u}}_\xi$-module $\tilde{Z}(P)/M(P)$.

Let P, Q be irreducible U_ξ^0-modules. It is easy to see that $\tilde{L}(P) \simeq \tilde{L}(Q)$ if and only if $P \simeq Q$.

Now let $\tilde{L}$ be an irreducible $\tilde{\mathbf{u}}_\xi$-module. Let $P = \{x \in \tilde{L} \mid E_\alpha x = 0$ for all $\alpha \in R_l^+\}$. Let y be a nonzero element in $\tilde{L}$. Then $\mathbf{u}_\xi^+ y$ is of finite dimension and we can find a nonzero element x in $\mathbf{u}_\xi^+ y$ such that $E_\alpha x = 0$ for all $\alpha \in R_l^+$. Therefore P is a nonzero space. Obviously P is stable under U_ξ^0. Let P' be a proper submodule of P. Then $\tilde{\mathbf{u}}_\xi P' \cap P = P'$. So $\tilde{\mathbf{u}}_\xi P'$ is a proper submodule of $\tilde{L}$. But $\tilde{L}$ is an irreducible $\tilde{\mathbf{u}}_\xi$-module, so $\tilde{\mathbf{u}}_\xi P' = 0$. In particular, $P' = 0$. Hence P is an irreducible U_ξ^0-module. We have a natural $\tilde{\mathbf{u}}_\xi$-homomorphism $\tilde{Z}(P) = \tilde{\mathbf{u}}_\xi \otimes_{\tilde{\mathbf{u}}_\xi^b} P \to \tilde{L}$, $u \otimes x \to ux$, which gives rise to an isomorphism between $\tilde{L}(P)$ and $\tilde{L}$.

The assertion is proved.

We need the following result to see that $\tilde{Z}_\xi(\lambda, \sigma)$ has a unique irreducible submodule.

Lemma 2.5. *Given a nonzero element y in $\mathbf{u}_\xi^-$ we can find an element x in $\mathbf{u}_\xi^-$ such that $xy = F_\kappa$, where $F_\kappa = \prod_{\alpha \in R_l^+} F_\alpha^{(l_\alpha - 1)}$, the product takes the order opposite to that in* [L4, 4.3, pp. 93–94].

Proof. Set $r := |R^+|$. Let β_{r-q+1} be the q-th root in the total order on R^+ arranged in [loc. cit]. Then $\beta_1, \beta_2, \ldots, \beta_r$ give rise to a total order on R^+ opposite to that in [loc. cit]. By 2.2 (iii),

$$y = \sum_{\substack{0 \le a_m \le l_{\beta_m} - 1 \\ 1 \le m \le r}} A(a_1, a_2, \ldots, a_r) F_{\beta_1}^{(a_1)} F_{\beta_2}^{(a_2)} \cdots F_{\beta_r}^{(a_r)}, \qquad A(a_1, a_2, \ldots, a_r) \in \mathbf{Q}(\xi).$$

Let $(b_1, b_2, \ldots, b_r)$ be the minimal element in $\{(a_1, a_2, \ldots, a_r) \in \mathbf{Z}_+^r \mid A(a_1, a_2, \ldots, a_r) \neq 0\}$. (Here we use the lexicographical order in $\mathbf{Z}_+^r$ such that $(0, 0, \ldots, 0, 1) < (0, 0, \ldots, 1, 0) < \cdots < (0, 1, \ldots, 0, 0) < (1, 0, \ldots, 0, 0)$.) Set $c_1 = l_{\beta_1} - 1 - b_1, \ldots, c_r = l_{\beta_r} - 1 - b_r$ and let $x' = F_{\beta_r}^{(c_r)} \cdots F_{\beta_2}^{(c_2)} F_{\beta_1}^{(c_1)}$. Using commutation relations in [L4, 5.3–4, pp. 95–97] and [L4, Theorem 6.6 (iii), p. 104], we see $x'y = A(b_1, b_2, \ldots, b_r) x' F_{\beta_1}^{(b_1)} F_{\beta_2}^{(b_2)} \cdots F_{\beta_r}^{(b_r)} = \theta F_\kappa$ for some nonzero number θ in $\mathbf{Q}(\xi)$. Then the element $x := \theta^{-1} x'$ satisfies our requirements.

Proposition 2.6. *Let $(\lambda, \sigma) \in \mathbf{Z}^n \times \{\pm 1\}^n$, then*

(i) *The Verma module $\tilde{Z}_\xi(\lambda, \sigma)$ has a unique irreducible submodule.*

(ii) *Assume that λ is* **l**-*restricted. Then the unique irreducible submodule of $\tilde{Z}_\xi(2\kappa + w_0 \lambda, \sigma)$ is isomorphic to $\tilde{L}_\xi(\lambda, \sigma)$, where w_0 is the longest element of W.*

Proof. (i) By Lemma 2.5, each submodule of $\tilde{Z}_\xi(\lambda, \sigma)$ contains the element $F_\kappa \tilde{1}_{\lambda,\sigma}$. So $\tilde{Z}_\xi(\lambda, \sigma)$ has a unique irreducible submodule which is generated by $F_\kappa \tilde{1}_{\lambda,\sigma}$.

(ii) Since F_κ has degree 2κ, so $F_\kappa \tilde{1}_{2\kappa + w_0\lambda, \sigma}$ has weight $(w_0 \lambda, \sigma)$.

According to the symmetries [L7, Prop. 5.2.7, p. 45], the lowest weight of $L_\xi(\lambda, \sigma)$ is $(w_0\lambda, \sigma)$. According to 2.2 (vii), 2.4 (iii-iv) and the proof of (i) we see that the unique irreducible submodule of $\tilde{Z}_\xi(2\kappa + w_0\lambda, \sigma)$ is isomorphic to $\tilde{L}_\xi(\lambda, \sigma)$.

Corollary 2.7. *Assume that λ is $\mathbb{l}$-restricted. Then*
(i) *There exists a nonzero element y_λ in $\mathbf{u}_\xi^-$ (unique up to a scalar) such that $y_\lambda \tilde{1}_{2\kappa + w_0\lambda,\sigma}$ has weight (λ, σ) and $E_\alpha y_\lambda \tilde{1}_{2\kappa + w_0\lambda,\sigma} = 0$ for all $\alpha \in R_l^+$. Necessarily $y_\lambda \tilde{1}_{2\kappa + w_0\lambda,\sigma}$ generates the unique irreducible submodule of $\tilde{Z}_\xi(2\kappa + w_0\lambda, \sigma)$.*

(ii) *There exists a nonzero element y'_λ in $\mathbf{u}_\xi^-$ (unique up to a scalar) such that $y'_\lambda \tilde{1}_{\kappa + \lambda,\sigma}$ has weight $(\kappa + w_0\lambda, \sigma)$ and $E_\alpha y'_\lambda \tilde{1}_{\kappa + \lambda,\sigma} = 0$ for all $\alpha \in R_l^+$. Necessarily $y'_\lambda \tilde{1}_{\kappa + \lambda,\sigma}$ generates the unique irreducible submodule of $\tilde{Z}_\xi(\kappa + \lambda, \sigma)$.*

We shall see that $y'_\lambda = \eta x'_\lambda$ for some non-zero number η in $\mathbf{Q}(\xi)$ provided that every indecomposable component of the Cartan matrix (a_{ij}) is not of type G_2 (Theorem 4.2 (ii), see 1.4 for the definition of x'_λ).

Proposition 2.8. *Let $\sigma \in \{\pm 1\}^n$. Then*
(i) *The Verma module $\tilde{Z}_\xi(\kappa, \sigma)$ is an irreducible $\tilde{\mathbf{u}}_\xi$-module, i.e. $\tilde{Z}_\xi(\kappa, \sigma) \simeq \tilde{L}_\xi(\kappa, \sigma)$.*
(ii) *As a $\tilde{\mathbf{u}}_\xi$-module, $V_\xi(\kappa, \sigma)$ is isomorphic to $\tilde{Z}_\xi(\kappa, \sigma)$. In particular, $V_\xi(\kappa, \sigma)$ is an irreducible U_ξ-module (cf.* [L7, Prop. 35.4.4, p. 277].)
(iii) *For every $\mu \in \mathbf{Z}_+^n$, the module $V_\xi(\mathbb{l}\mu + \kappa, \sigma)$ is an irreducible U_ξ-module.*

Proof. (i) Note that $w_0\kappa = -\kappa$. By Prop. 2.6 (ii), the unique irreducible submodule of $\tilde{Z}_\xi(\kappa, \sigma)$ is isomorphic to $\tilde{L}_\xi(\kappa, \sigma)$. But $\tilde{Z}_\xi(\kappa, \sigma)_{\kappa,\sigma}$ is of one dimension, so the irreducible submodule of $\tilde{Z}_\xi(\kappa, \sigma)$ is generated by $\tilde{1}_{\kappa,\sigma}$. Hence $\tilde{Z}_\xi(\kappa, \sigma)$ is irreducible and isomorphic to $\tilde{L}_\xi(\kappa, \sigma)$.

(ii) By the definitions of $\tilde{Z}_\xi(\kappa, \sigma)$ and of $V_\xi(\kappa, \sigma)$, we have a natural $\tilde{\mathbf{u}}_\xi$-module homomorphism $\tilde{Z}_\xi(\kappa, \sigma) \to V_\xi(\kappa, \sigma)$, $\tilde{1}_{\kappa,\sigma} \to v_{\kappa,\sigma}$. The homomorphism is surjective according to 2.2 (ii) and to the definition of $V_\xi(\kappa, \sigma)$. Weyl's character formula tells us that the dimension of $V_\xi(\kappa, \sigma)$ is $\prod_{\alpha \in R^+} l_\alpha$. So the homomorphism is a $\tilde{\mathbf{u}}_\xi$-module isomorphism. This proves (ii).

(iii) By 1.2 (i), $L_\xi(\mathbb{l}\mu + \kappa, \sigma)$ is the unique irreducible quotient module of $V_\xi(\mathbb{l}\mu + \kappa, \sigma)$. Using (ii) and 2.2 (iv) we see that $L_\xi(\mathbb{l}\mu + \kappa, \sigma)$ is isomorphic to $L_\xi(\mathbb{l}\mu, \sigma) \otimes V_\xi(\kappa, \mathbb{1})$. Combining 2.2 (v), 1.2 (i) and 1.2 (iv), we know that the dimensions of $V_\xi(\mathbb{l}\mu + \kappa, \sigma)$ and $L_\xi(\mathbb{l}\mu, \sigma) \otimes V_\xi(\kappa, \mathbb{1})$ can be calculated by means of Weyl's character formula, they are equal. Hence $V_\xi(\mathbb{l}\mu + \kappa, \sigma)$ is an irreducible U_ξ-module.

The proposition is proved.

The following result will not be used in the sequel of the paper.

Theorem 2.9. (i) *The algebra* $\mathbf{u}_\xi$ *is symmetric.*
(ii) *Let* $\mathbf{k}$ *be the two sided ideal of* $\mathbf{u}_\xi$ *generated by* $K_1^{l_1}-1, K_2^{l_2}-1,\dots,K_n^{l_n}-1$. *Then the algebra* $\mathbf{u}'_\xi := \mathbf{u}_\xi/\mathbf{k}$ *is symmetric provided that all* $K_i^{l_i}$ *are central in* $\mathbf{u}_\xi$.

Remark. The theorem was proved in [X1, Theorem 3.5] with some restrictions on l. Since [X1] is unpublished and Theorem 3.5 in [X1] was quoted in some papers, it might be good to represent here a version without restrictions on l. The proof is the same as in [X1].

Proof. (i) We need to construct a $\mathbf{Q}(\xi)$-bilinear form φ on $\mathbf{u}_\xi$ such that
(a) φ is associative, that is, $\varphi(xy, z) = \varphi(x, yz)$ for any $x, y, z \in \mathbf{u}_\xi$;
(b) φ is no-degenerate, i.e. if $\varphi(x, x') = 0$ (resp. $\varphi(x', x) = 0$) for all $x' \in \mathbf{u}_\xi$, then $x = 0$;
(c) φ is symmetric, that is, $\varphi(x, y) = \varphi(y, x)$.

Let $\beta_1, \beta_2, \dots, \beta_r$ be as in the proof of Lemma 2.5. Set

$$\mathbf{Z}^r_{+,\mathbf{l}} := \{(a_1, a_2, \dots, a_r) \in \mathbf{Z}^r \mid 0 \le a_1 \le l_{\beta_1} - 1, \dots, 0 \le a_r \le l_{\beta_r} - 1\},$$
$$\mathbf{Z}^n_{+,2\mathbf{l}} := \{(h_1, h_2, \dots, h_n) \in \mathbf{Z}^n \mid 0 \le h_1 \le 2l_1 - 1, \dots, 0 \le h_n \le 2l_n - 1\}.$$

For $A = (a_1, a_2, \dots, a_r) \in \mathbf{Z}^r_{+,\mathbf{l}}$ and $H = (h_1, h_2, \dots, h_r) \in \mathbf{Z}^n_{+,2\mathbf{l}}$, we shall write $F_A, F'_A; E_A, E'_A; K_H$ for $F_{\beta_1}^{(a_1)} F_{\beta_2}^{(a_2)} \cdots F_{\beta_r}^{(a_r)}, F_{\beta_r}^{(a_r)} \cdots F_{\beta_2}^{(a_2)} F_{\beta_1}^{(a_1)}; E_{\beta_r}^{(a_r)} E_{\beta_{r-1}}^{(a_{r-1})} \cdots E_{\beta_1}^{(a_1)}, E_{\beta_1}^{(a_1)} \cdots E_{\beta_{r-1}}^{(a_{r-1})} E_{\beta_r}^{(a_r)}; K_1^{h_1} K_2^{h_2} \cdots K_n^{h_n}$, respectively. Let φ_0 be the $\mathbf{Q}(\xi)$-linear function of $\mathbf{u}_\xi$ defined by

$$\varphi_0(F_A K_H E_{A'}) = \begin{cases} 1, & \text{if } F_A K_H E_{A'} = F_\iota E_\iota, \\ 0, & \text{otherwise,} \end{cases}$$

where $\iota = (l_{\beta_1} - 1, l_{\beta_2} - 1, \dots, l_{\beta_r} - 1) \in \mathbf{Z}^r_{+,\mathbf{l}}$. Set $\varphi(x, y) := \varphi_0(xy)$. Obviously φ is an associative $\mathbf{Q}(\xi)$-bilinear form on $\mathbf{u}_\xi$. We now show that φ is non-degenerate on $\mathbf{u}_\xi$.

Let

$$x = \sum_{\substack{A, A' \in \mathbf{Z}^r_{+,\mathbf{l}} \\ H \in \mathbf{Z}^n_{+,2\mathbf{l}}}} \theta(A, H, A') F_A K_H E_{A'} \ne 0, \qquad \theta(A, H, A') \in \mathbf{Q}(\xi).$$

Let $B = (b_1, b_2, \dots, b_r)$ be the minimal element in $\{A \in \mathbf{Z}^r_{+,\mathbf{l}} \mid \theta(A, H, A') \ne 0$ for some $H, A'\}$ (for the lexicographical order on $\mathbf{Z}^r_+$ defined in the proof of Lemma 2.5), and let $B' = (b'_1, b'_2, \dots, b'_r)$ be the minimal element in $\{A' \in \mathbf{Z}^r_{+,\mathbf{l}} \mid \theta(B, H, A') \ne 0$ for some $H\}$ (for the lexicographical order on $\mathbf{Z}^r_+$ opposite to that defined in the proof of Lemma 2.5). Set

$$y_1 = F_{\beta_r}^{(c_r)} \cdots F_{\beta_2}^{(c_2)} F_{\beta_1}^{(c_1)}, \quad y_2 = E_{\beta_1}^{(c'_1)} \cdots E_{\beta_{r-1}}^{(c'_{r-1})} E_{\beta_r}^{(c'_r)},$$

where $c_1 = l_{\beta_1} - 1 - b_1, \dots, c_r = l_{\beta_r} - 1 - b_r$, and $c'_1 = l_{\beta_1} - 1 - b'_1, \dots, c'_r = l_{\beta_r} - 1 - b'_r$. By the proof of Lemma 2.5 we have

$$y_2y_1x = \theta_1 y_2 \sum_{\substack{A'\in \mathbf{Z}^r_{+,l}\\ H\in \mathbf{Z}^n_{+,2l}}} \theta(B, H, A')F_l K_H E_{A'}$$

for some nonzero number $\theta_1 \in \mathbf{Q}(\xi)$.

Recall that in U we have

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{v^{d_i} - v^{-d_i}}.$$

This implies the following

(∗) Let $\alpha, \beta \in R^+$. Then in U we have

$$E_\alpha F_\beta = F_\beta E_\alpha + \sum \sigma(F, K, E)FKE, \quad \sigma(F, K, E)\in \mathbf{Q}[v, v^{-1}],$$

where in the sum K runs through a finite subset of $U^0 \cap U_{\mathbf{Q}[v,v^{-1}]}$, F (resp. E) runs through a finite subset of homogenous elemets of $U^- \cap U_{\mathbf{Q}[v,v^{-1}]}$ (resp. $U^+ \cap U_{\mathbf{Q}[v,v^{-1}]}$), and $\sigma(F, K, E) = 0$ if $\deg(F) \geq \beta$ or $\deg(E) \leq -\alpha$.

From (∗) and 2.2 (iii) we get

(★) Let $\alpha, \beta \in R_l^+$. Then in $\mathbf{u}_\xi$ we have

$$E_\alpha F_\beta = F_\beta E_\alpha + \sum_{\substack{A, A'\in \mathbf{Z}^r_{+,l}\\ H\in \mathbf{Z}^n_{+,2l}}} \sigma(A, H, A')F_A K_H E_{A'}, \qquad \sigma(A, H, A')\in \mathbf{Q}(\xi),$$

where $\sigma(A, H, A') = 0$ if $\deg(F_A) \geq \beta$ or $\deg(E_{A'}) \leq -\alpha$.

Repeatedly using (★) we get

(†) Let $B, C \in \mathbf{Z}^r_{+,l}$ and let $y = E_B$ or E'_B, $z = F_C$ or F'_C. Then in $\mathbf{u}_\xi$ we have

$$yz = zy + \sum_{\substack{A, A'\in \mathbf{Z}^r_{+,l}\\ H\in \mathbf{Z}^n_{+,2l}}} \sigma(A, H, A')F_A K_H E_{A'}, \qquad \sigma(A, H, A')\in \mathbf{Q}(\xi),$$

where $\sigma(A, H, A') = 0$ if $\deg(F_A) \geq \deg(z)$ or $\deg(E_{A'}) \leq -\deg(y)$.

By (†) we get

$$y_2F_l = F_l y_2 + \sum_{\substack{A, A'\in \mathbf{Z}^r_{+,l}\\ A\neq l\\ H\in \mathbf{Z}^n_{+,2l}}} \eta(A, H, A')F_A K_H E_{A'}, \qquad \eta(A, H, A')\in \mathbf{Q}(\xi).$$

As in the proof of Lemma 2.5 we see $\theta(B, H, A')y_2E_{A'} \neq 0$ implies that $A' = B'$ and $y_2E_{B'} = \theta_2 E_l$ for some nonzero number $\theta_2 \in \mathbf{Q}(\xi)$. Thus

$$y_2y_1x = \theta_1\theta_2 \sum_{H\in \mathbf{Z}^n_{+,2l}} \theta(B, H, B')F_l K_H E_l + \sum_{\substack{A, A'\in \mathbf{Z}^r_{+,l}\\ A\neq l\\ H\in \mathbf{Z}^n_{+,2l}}} \eta'(A, H, A')F_A K_H E_{A'},$$

where $\eta'(A, H, A')\in \mathbf{Q}(\xi)$. Let $I \in \mathbf{Z}^n_{+,2l}$ be such that $\theta(B, I, B') \neq 0$. By the definition of φ we see $\varphi(K_I^{-1}y_2y_1, x) \neq 0$. We also have $\varphi(x, K_I^{-1}y_2y_1) \neq 0$ since φ is symmetric by the following argument.

Note that the elements $E'_A K_H F'_{A'}$ $(A, A' \in \mathbf{Z}^r_{+,1}, H \in \mathbf{Z}^n_{+,2l})$ also form a $\mathbf{Q}(\xi)$-basis of $\mathbf{u}_\xi$. Let $A = (a_k)$, $B = (b_k)$, $P = (p_k)$, $Q = (q_k)$ be elements in $\mathbf{Z}^r_{+,l}$ and let $H = (h_i)$, $H' = (h'_i)$ be elements in $\mathbf{Z}^n_{+,2l}$. Using commutation relations in [L4, 5.3–5.4, pp. 95–97], Theorem 6.6 in [L4, pp. 103–104] and using (†), we see that $\varphi(F_A K_H E_P, E'_Q K_{H'} F'_B) = \varphi(E'_Q K_{H'} F'_B, F_A K_H E_P) = 0$ if one of the following three cases happens: (a) $K_H K_{H'} \neq 1$, (b) $\sum_{k=1}^r (a_k + b_k)\beta_k \neq 2\kappa$, (c) $\sum_{k=1}^r (p_k + q_k)\beta_k \neq 2\kappa$. Using [L4, Theorem 6.6, pp. 103–104] and commutation relations in [L4, 5.3–5.4, pp. 95–97] and induction on Q (resp. B) we know that (d) $E_P E'_Q = E'_Q E_P$ (resp. $F_A F'_B = F'_B F_A$) if $\sum_{k=1}^r (p_k + q_k)\beta_k = 2\kappa$ (resp. $\sum_{k=1}^r (a_k + b_k)\beta_k = 2\kappa$). By this and (†), and noting that the coefficients of $E_P E'_Q$, $F'_B F_A$ in $K_H E_P E'_Q K_H^{-1}$, $K_H^{-1} F'_B F_A K_H$ are the same when $\sum_{k=1}^r (a_k + b_k)\beta_k = \sum_{k=1}^r (p_k + q_k)\beta_k = 2\kappa$, we see that $\varphi(F_A K_H E_P, E'_Q K_{H'} F'_B) = \varphi(E'_Q K_{H'} F'_B, F_A K_H E_P)$ if $\sum_{k=1}^r (a_k + b_k)\beta_k = \sum_{k=1}^r (p_k + q_k)\beta_k = 2\kappa$ and $K_H K_{H'} = 1$. Therefore φ is symmetric. Part (i) is proved.

(ii) Since all $K_i^{l_i}$ are central in $\mathbf{u}_\xi$, the images in $\mathbf{u}'_\xi$ of the elements $F_A K_H E_{A'}$ $(A, A' \in \mathbf{Z}^r_{+,l}, H \in \mathbf{Z}^n_{+,l})$ form a $\mathbf{Q}(\xi)$-basis of $\mathbf{u}'_\xi$, the proof of (i) is also valid to $\mathbf{u}'_\xi$.

The theorem is proved.

§3. Category of Finite Dimensional U_ξ-modules of Type 1

3.1. Let M be a finite dimensional U_ξ-module. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n$, $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ in $\{\pm 1\}^n$, let $M_{(\lambda,\sigma)}$ be the set of all elements m in M satisfying

$$K_i m = \sigma_i \xi^{d_i \lambda_i} m, \quad \left(\begin{bmatrix} K_i, c \\ a \end{bmatrix} - \sigma_i^a \begin{bmatrix} \lambda_i + c \\ a \end{bmatrix}_{d_i, \xi} \right)^k m = 0,$$

for $i = 1, 2, \dots, n$, $c \in \mathbf{Z}$ and some $k \in \mathbf{N}$. Then we have

(i) $M = \bigoplus_{\substack{\lambda \in \mathbf{Z}^n \\ \sigma \in \{\pm 1\}^n}} M_{(\lambda,\sigma)}$, and $E_i^{(a)} M_{(\lambda,\sigma)} \subseteq M_{(\lambda + a\alpha_i, \sigma)}$, $F_i^{(a)} M_{(\lambda,\sigma)} \subseteq M_{(\lambda - a\alpha_i, \sigma)}$. Therefore, for a fixed $\sigma \in \{\pm 1\}^n$, the space $M^\sigma = \bigoplus_{\lambda \in \mathbf{Z}^n} M_{(\lambda,\sigma)}$ is a submodule of M and $M = \bigoplus_{\sigma \in \{\pm 1\}^n} M^\sigma$. [L2, Prop. 5.1 and its proof, pp. 65–67].

(ii) Obviously, $M_{\lambda,\sigma} \neq 0$ if and only if $M_{(\lambda,\sigma)} \neq 0$. So the set $P(M) := \{\lambda \in \mathbf{Z}^n \mid M_{(\lambda,\sigma)} \neq 0 \text{ for some } \sigma \in \{\pm 1\}^n\}$ is stable under the action of W [L7, Prop. 5.2.7, p. 45].

In U_ξ we have $K_i^{2l_i} = 1$. For each $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ in $\{\pm 1\}^n$, set $\tilde{M}^\sigma := \{m \in M \mid K_i^{l_i} m = \sigma_i m \text{ for } i = 1, 2, \dots, n\}$. When all $K_i^{l_i}$ are central in U_ξ, $\tilde{M}^\sigma$ is a submodule of M and $M = \bigoplus_{\sigma \in \{\pm 1\}^n} \tilde{M}^\sigma$. If l is odd (prime to 3 if there is type G_2 involved), all l_i are equal to l and all K_i^l are central in U_ξ, in this case we have $M^\sigma = \tilde{M}^\sigma$. When some $K_i^{l_i}$ are not central in U_ξ, in general $\tilde{M}^\sigma$ is not a U_ξ-submodule of M, since we can find j in $[1, n]$ such that $K_i^{l_i}E_j = \xi^{d_i a_{ij} l_i} E_j K_i^{l_i} = -E_j K_i^{l_i}$ and $K_i^{l_i} F_j = \xi^{-d_i a_{ij} l_i} F_j K_i^{l_i} = -F_j K_i^{l_i}$.

We say that M has type σ if $M = M^\sigma$. All finite dimensional U_ξ-modules of type σ with usual U_ξ-module homomorphisms form a category of U_ξ-modules, denoted by $\mathscr{C}_\sigma$. Clearly, the map $M \to M \otimes \mathbf{Q}(\xi)_\sigma$ gives rise to an isomorphism between the categories $\mathscr{C}_\mathbf{1}$ and $\mathscr{C}_\sigma$ [APW, 1.6, pp. 6–7]. What is more, the $\mathbf{Q}(\xi)$-algebra automorphism $U_\xi \to U_\xi$ defined by $E_i^{(a)} \to \sigma_i^a E_i^{(a)}$, $F_i^{(a)} \to F_i^{(a)}$, $K_i \to \sigma_i K_i$ $(i = 1, 2, \ldots, n,\ a \geq 0)$ interchanges the U_ξ-modules of type $\mathbf{1}$ to those of type σ [L2, 4.6, p. 65].

Therefore, it suffices to work with the category $\mathscr{C}_\mathbf{1}$ of U_ξ-modules. Note that $V_\xi(\lambda, \mathbf{1})$, $L_\xi(\lambda, \mathbf{1}) \in \operatorname{ob} \mathscr{C}_\mathbf{1}$ for each $\lambda \in \mathbf{Z}_+^n$. We shall drop the index $\mathbf{1}$ in all notations involving it. So $\mathscr{C}$, $V_\xi(\lambda)$, $L_\xi(\lambda)$, v_λ, etc. will stand for $\mathscr{C}_\mathbf{1}$, $V_\xi(\lambda, \mathbf{1})$, $L_\xi(\lambda, \mathbf{1})$, $v_{\lambda,\mathbf{1}}$, etc. respectively. One main result of the section is the following, which will be proved after establishing Lemma 3.4.

Theorem 3.2. *Let $\mu \in \mathbf{Z}_+^n$.*
(i) *The module $V_\xi(\mathbf{l}\mu + \kappa)$ is injective as well as projective in the category $\mathscr{C}$.*
(ii) *The category $\mathscr{C}$ has enough injective objects and enough projective objects as well.*
(iii) *In $\mathscr{C}$ each injective object is also a projective object and vice versa.*
(iv) *Every module M in ob $\mathscr{C}$ is integrable (i.e. $M = \bigoplus_{\lambda \in \mathbf{Z}^n} M_\lambda$ and $E_i^{(a)}$, $F_i^{(a)}$ are locally nilpotent on M for $i = 1, 2, \ldots, n$, $a \geq 1$).*
(v) *If M is a finite dimensional U_ξ-module, then $M = \bigoplus_{(\lambda,\sigma) \in \mathbf{Z}^n \times \{\pm 1\}^n} M_{\lambda,\sigma}$, i.e. M is integrable.*
(vi) *Let E be an injective object in $\mathscr{C}$, then E has a submodule filtration $0 = E_k \subset E_{k-1} \subset \cdots \subset E_2 \subset E_1 = E$ such that $E_a/E_{a+1} \simeq V_\xi(v_a)$ for some $v_a \in \mathbf{Z}_+^n$, $a = 1, \ldots, k-1$.*

Remark. When l is a power of a prime number, the theorem is proved in [APW, 9.8, p. 44; 9.12, p. 45].

3.3. Let M be a U_ξ-module of type $\mathbf{1}$. A nonzero element m in M is called primitive if $m \in M_\lambda$ for some $\lambda \in \mathbf{Z}^n$ and $E_i^{(a)} m = 0$ for $i = 1, 2, \ldots, n$, $a \geq 1$. We have

(i) Let M be an integrable or finite dimensional U_ξ-module of type $\mathbf{1}$. Assume that m is a primitive element of weight λ. Then $\lambda \in \mathbf{Z}_+^n$ and there is a unique U_ξ-module homomorphism $V_\xi(\lambda) \to M$ which carries v_λ to m. [L7, Prop. 3.5.8, p. 33].

Given a finite dimensional U_ξ-module E of type $\mathbf{1}$, we define the dual modules E^*, $E^\star$ as in [APW, 1.18, p. 9] by means of the antipode S of U_ξ and its inverse S^{-1} respectively. Then [APW, 1.18, p. 9–10]
(ii) We have $(E^*)^\star \simeq E \simeq (E^\star)^*$.
(iii) For any U_ξ-modules M, N, one has

$$\mathrm{Hom}_{U_\xi}(M, N \otimes E) \simeq \mathrm{Hom}_{U_\xi}(M \otimes E^\star, N),$$
$$\mathrm{Hom}_{U_\xi}(E^* \otimes M, N) \simeq \mathrm{Hom}_{U_\xi}(M, E \otimes N).$$

The following assertion is well known.

(iv) For $\lambda \in \mathbf{Z}_+^n$ we have

$$L_\xi(\lambda)^* \simeq L_\xi(\lambda)^\star \simeq L_\xi(-w_0\lambda).$$

For later uses we prove that some U_ξ-modules admit filtrations of Weyl modules.

For U-modules we define weight spaces, primitive elements, and their types as usual. Here we only consider U-modules of type $\mathbf{1}$. For a U-module M of type $\mathbf{1}$ we denote M_λ the λ-weight space of M.

Let $\mathscr{A} = \mathbf{Z}[v, v^{-1}]$ and let $U_\mathscr{A}$ be the $\mathscr{A}$-subalgebra of U generated by the elements $E_i^{(a)}$, $F_i^{(a)}$, K_i, K_i^{-1} for $i = 1, \ldots, n$, $a \in \mathbf{N}$. For $\lambda \in \mathbf{Z}_+^n$, let $V(\lambda)$ be an irreducible U-module of type $\mathbf{1}$ with highest weight λ, and let $v_\lambda \in V(\lambda)$ be a nonzero element of weight λ. Define $V(\lambda)_\mathscr{A} = U_\mathscr{A} v_\lambda$. Let $B(\lambda)$ be the canonical basis of $V(\lambda)$ containing v_λ, then $B(\lambda)$ is an $\mathscr{A}$-basis of $V(\lambda)_\mathscr{A}$ (see [K, Section 7.2 and Lemma 7.3.1; L5, Theorem 8.10; L7, 19.3.1, p. 170]).

We have

(v) Let (M, B) be a based module of U (see [L7, 27.1.2, p. 214] for definition) and let $M_\mathscr{A}$ be the $\mathscr{A}$-submodule of M generated by B. (By definition $M_\mathscr{A}$ is stable under $U_\mathscr{A}$.) Then $M_\mathscr{A}$ has a filtration of $U_\mathscr{A}$-submodules

$$0 = M_{\mathscr{A},0} \subset M_{\mathscr{A},1} \subset M_{\mathscr{A},2} \subset \cdots \subset M_{\mathscr{A},h} = M_\mathscr{A}$$

such that all $M_{\mathscr{A},1}, \ldots, M_{\mathscr{A},h}$ are free $\mathscr{A}$-modules and $M_{\mathscr{A},a}/M_{\mathscr{A},a-1} \simeq V(\delta_a)_\mathscr{A}$ for some $\delta_a \in \mathbf{Z}_+^n$, $a = 1, \ldots, h$.

Proof. Let $\lambda \in \mathbf{Z}_+^n$ be such that $M_\lambda \neq 0$ and such that λ is maximal with this property. Let $b \in B \cap M_\lambda$ and let $M_1 = Ub$ be the submodule of M generated by b. We have $E_i b = 0$ for all i in $[1, n]$ by the maximality of λ. Hence there is a unique U-homomorphism $\phi\colon V(\lambda) \to M$ which carries v_λ to b [L7, Prop. 3.5.8, p. 33].

By [L7, Prop. 27.1.7, p. 215], $B_1 = B \cap M_1$ is a basis of M_1 and ϕ defines an isomorphism $V(\lambda) \simeq M_1$ which carries $B(\lambda)$ to B_1. Since $B(\lambda)$ is an $\mathscr{A}$-basis of $V(\lambda)_\mathscr{A}$, we see B_1 is an $\mathscr{A}$-basis of $U_\mathscr{A} b$. Let $M_{\mathscr{A},1} = U_\mathscr{A} b$ and let $M' = M/M_1$. Denote by B' the image in M' of $B - B_1$. Then (M', B') is a based module [L7, 27.1.4, p. 215]. Since $M_\mathscr{A}/M_{\mathscr{A},1} \simeq M'_\mathscr{A}$, using induction on $\dim M$ we see the required filtration exists.

The assertion is proved.

For $a = 1, 2, \ldots, h$, let $M_{\xi,a} = M_{\mathscr{A},a} \otimes_\mathscr{A} \mathbf{Q}(\xi)$, and set $M_\xi = M_\mathscr{A} \otimes_\mathscr{A} \mathbf{Q}(\xi)$. Then $M_{\xi,a}$ is a U_ξ-module. Noting that $V_\xi(\lambda) \simeq V(\lambda)_\mathscr{A} \otimes_\mathscr{A} \mathbf{Q}(\xi)$, by (v) we see

(vi) The U_ξ-module M_ξ has a filtration of submodules

$$0 = M_{\xi,0} \subset M_{\xi,1} \subset M_{\xi,2} \subset \cdots \subset M_{\xi,h} = M_\xi$$

such that $M_{\xi,a}/M_{\xi,a-1} \simeq V_\xi(\delta_a)$ for some $\delta_a \in \mathbf{Z}^n_+$, $a = 1,\dots,h$.

Let λ, $\mu \in \mathbf{Z}^n_+$ and let $M = V(\lambda) \otimes V(\mu)$. According to [L7, 27.3.3, p. 221], there exists a basis B of M such that (M, B) is a based module and such that the $\mathscr{A}$-submodule of M generated by B is equal to $V(\lambda)_{\mathscr{A}} \otimes_{\mathscr{A}} V(\mu)_{\mathscr{A}}$. (Note that $B(\lambda)$ and $B(\mu)$ are $\mathscr{A}$-bases of $V(\lambda)_{\mathscr{A}}$ and $V(\mu)_{\mathscr{A}}$ respectively.) Since $V_\xi(\lambda) \otimes V_\xi(\mu) \simeq (V(\lambda)_{\mathscr{A}} \otimes_{\mathscr{A}} V(\mu)_{\mathscr{A}}) \otimes_{\mathscr{A}} \mathbf{Q}(\xi)$, by (vi) we get

(vii) Let λ, $\mu \in \mathbf{Z}^n_+$. Then the U_ξ-module $V := V_\xi(\lambda) \otimes V_\xi(\mu)$ has a filtration of submodules

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_h = V$$

such that $V_a/V_{a-1} \simeq V_\xi(\delta_a)$ for some $\delta_a \in \mathbf{Z}^n_+$, $a = 1,\dots,h$.

Lemma 3.4. *Let M be a finite dimensional U_ξ-module of type* $\mathbf{1}$ *and let* $\mu \in \mathbf{Z}^n_+$.

(i) *Assume that $V_\xi(\mathbf{l}\mu + \kappa)$ is a submodule of M. Then $V_\xi(\mathbf{l}\mu + \kappa)$ is a direct summand of M, i.e. there exists a submodule M' of M such that M is isomorphic to $V_\xi(\mathbf{l}\mu + \kappa) \oplus M'$.*

(ii) *Assume that $V_\xi(\mathbf{l}\mu + \kappa)$ is a quotient module of M. Then $V_\xi(\mathbf{l}\mu + \kappa)$ is a direct summand of M.*

(iii) *Assume that $V_\xi(\mathbf{l}\mu + \kappa)$ is a composition factor of M. Then $V_\xi(\mathbf{l}\mu + \kappa)$ is a direct summand of M.*

Proof. Let $\nu = -w_0(\mu)$. By 3.2 (iv) we have $V_\xi(\mathbf{l}\mu + \kappa)^* \simeq V_\xi(\mathbf{l}\mu + \kappa)^\star \simeq V_\xi(\mathbf{l}\nu + \kappa)$ since $V_\xi(\mathbf{l}\mu + \kappa)$ is irreducible (Prop. 2.8 (iii)) and $w_0(\mathbf{l}\mu + \kappa) = -\mathbf{l}\nu - \kappa$. Thus part (i) and part (ii) are equivalent by 3.3 (ii). We give a proof of part (i).

(i) By induction on $\dim_{\mathbf{Q}(\xi)} M$ we may assume that $M/V_\xi(\mathbf{l}\mu + \kappa)$ is irreducible. One of the following three cases must happen.

(a) There is a maximal weight λ in $P(M)$ such that $\lambda \neq \mathbf{l}\mu + \kappa$.

(b) $\mathbf{l}\mu + \kappa$ is the unique maximal weight in $P(M)$ and $\dim_{\mathbf{Q}(\xi)} M_{(\mathbf{l}\mu+\kappa)} = 1$.

(c) $\mathbf{l}\mu + \kappa$ is the unique maximal weight in $P(M)$ and $\dim_{\mathbf{Q}(\xi)} M_{(\mathbf{l}\mu+\kappa)} = 2$.

Case (a). By 3.1 (ii), $M_\lambda \neq 0$. Choose a nonzero element m in M_λ, then m is a primitive element. Let M' be the submodule of M generated by m. We claim that $M' \cap V_\xi(\mathbf{l}\mu + \kappa) = \{0\}$. Otherwise, $M' \cap V_\xi(\mathbf{l}\mu + \kappa) = V_\xi(\mathbf{l}\mu + \kappa)$. Then we can find an element y in U_ξ^- such that $v_{\mathbf{l}\mu+\kappa} = ym$. Note that $F_\kappa v_{\mathbf{l}\mu+\kappa} \neq 0$. Using 2.2 (ii-iii) we see that $F_\kappa v_{\mathbf{l}\mu+\kappa} = F_\kappa y' m$ for some element y' in $U_{\xi,l}^-$. Therefore $\lambda = \mathbf{l}\tau + \mathbf{l}\mu + \kappa$ for certain nonzero element τ in $\mathbf{Z}^n$. By Prop. 2.8 (iii) and 3.3 (i), M' is irreducible. A contradiction to the assumption $M' \cap V_\xi(\mathbf{l}\mu + \kappa) = V_\xi(\mathbf{l}\mu + \kappa)$. Hence $M' \cap V_\xi(\mathbf{l}\mu + \kappa) = \{0\}$. In addition we have $M \simeq V_\xi(\mathbf{l}\mu + \kappa) \oplus M'$ and M' is irreducible.

Case (b). By 3.1 (ii), all the four spaces $M_{\mathbf{l}\mu+\kappa}$, $M_{(\mathbf{l}\mu+\kappa)}$, $M_{-\mathbf{l}\nu-\kappa}$, $M_{(-\mathbf{l}\nu-\kappa)}$ are of one dimension. So $M^*_{\mathbf{l}\nu+\kappa}$ is equal to $M^*_{(\mathbf{l}\nu+\kappa)}$ and is of one dimension. By 3.1 (ii), $\mathbf{l}\nu + \kappa$ is the unique maximal weight in $P(M^*) = -P(M)$. Let M_1 be an irreducible submodule of M^* such that M^*/M_1 is isomorphic to $V_\xi(\mathbf{l}\mu + \kappa)^* (\simeq V_\xi(\mathbf{l}\nu + \kappa))$. By our assumptions on M we have $\mathbf{l}\nu + \kappa \notin P(M_1)$. Choose a nonzero element m in $M^*_{\mathbf{l}\nu+\kappa}$, then m is a primitve element in M^* and generates a submodule M_2 of M^*. By Prop. 2.8 (iii) and 3.3 (i), M_2 is isomorphic to $V_\xi(\mathbf{l}\nu + \kappa) (\simeq V_\xi(\mathbf{l}\mu + \kappa)^*)$. Hence M^* is isomorphic to $V_\xi(\mathbf{l}\nu + \kappa) \oplus M_1$. Note that $V_\xi(\mathbf{l}\nu + \kappa)^\star \simeq V_\xi(\mathbf{l}\mu + \kappa)$, by 3.3 (ii) we see that M is isomorphic to $V_\xi(\mathbf{l}\mu + \kappa) \oplus M_1^\star$.

Case (c). Set $\mu_i := \langle \mu, \alpha_i^\vee \rangle$ for $i = 1, 2, \ldots, n$. By 3.1 (ii) we have $F_i^{(l_i\mu_i + l_i)} M_{(\mathbf{l}\mu+\kappa)} = 0$ for all i. Using 3.1 (i) and our assumption on $\mathbf{l}\mu + \kappa$ we see

$$\begin{bmatrix} K_i, 0 \\ \mu_i l_i + l_i \end{bmatrix} M_{(\mathbf{l}\mu+k)} = E_i^{(l_i\mu_i + l_i)} F_i^{(l_i\mu_i + l_i)} M_{(\mathbf{l}\mu+\kappa)} = 0.$$

But in U_ξ we have [L2, 4.2 (f), p. 63; L7, Lemma 34.1.2 (b), p. 265]

$$\begin{bmatrix} K_i, 0 \\ \mu_i l_i + l_i \end{bmatrix} = \frac{\xi^{-\frac{1}{2} d_i l_i^2 \mu_i(\mu_i+1)}}{(\mu_i + 1)!} \prod_{j=0}^{\mu_i} \begin{bmatrix} K_i, -jl_i \\ l_i \end{bmatrix}.$$

By [L2, 4.2 (d), 4.2 (c), p. 63, L7, Lemma 34.1.2 (b), p. 265] we see

(*) On $M_{(\mathbf{l}\mu+\kappa)}$ the following equality holds

$$\begin{bmatrix} K_i, -jl_i \\ l_i \end{bmatrix} = \xi^{-jl_i^2 d_i} \left(\begin{bmatrix} K_i, 0 \\ l_i \end{bmatrix} - \xi^{d_i l_i^2(\mu_i+1) + d_i l_i(l_i-1)} j \right).$$

By definition, $\begin{bmatrix} K_i, 0 \\ l_i \end{bmatrix} - \xi^{d_i l_i^2(\mu_i+1) + d_i l_i(l_i-1)} \mu_i$ is nilpotent on $M_{(\mathbf{l}\mu+\kappa)}$, so $\begin{bmatrix} K_i, -jl_i \\ l_i \end{bmatrix}$ is invertible on $M_{(\mathbf{l}\mu+\kappa)}$ for $j \neq \mu_i$. Thus we get

$$\left(\begin{bmatrix} K_i, 0 \\ l_i \end{bmatrix} - \xi^{d_i l_i^2(\mu_i+1) + d_i l_i(l_i-1)} \mu_i \right) M_{(\mathbf{l}\mu+\kappa)} = 0.$$

So $M_{(\mathbf{l}\mu+\kappa)} = M_{\mathbf{l}\mu+\kappa}$ since all K_i, $\begin{bmatrix} K_i, 0 \\ l_i \end{bmatrix}$ generate the algebra U_ξ^0. Therefore M is isomorphic to $V_\xi(\mathbf{l}\mu + \kappa) \oplus V_\xi(\mathbf{l}\mu + \kappa)$.

(iii) Let M_1 be a submodule of M such that $V_\xi(\mathbf{l}\mu + \kappa)$ is a quotient module of M_1. By (ii), $V_\xi(\mathbf{l}\mu + \kappa)$ is isomorphic to a submodule of M_1. Since M_1 is a submodule of M, by (i), M is isomorphic to $V_\xi(\mathbf{l}\mu + \kappa) \oplus M'$ for some submodule M' of M.

The lemma is proved.

3.5. Now we prove Theorem 3.2. Part (i) is a trivial consequence of Lemma 3.4 (iii).

(ii) According to part (i) and 3.3 (iii), for any finite dimensional U_ξ-module M of type $\mathbb{1}$, the modules $V_\xi(\mathbb{l}\mu+\kappa)\otimes M$ and $M\otimes V_\xi(\mathbb{l}\mu+\kappa)$ are projevtive and injective as well in the category $\mathscr{C}$. For any λ in $\mathbf{Z}_+^n$, choose $\nu=(a,a,\ldots,a)$ in $\mathbf{Z}_+^n$ such that $\mathbb{l}\nu+\kappa-\lambda\in\mathbf{Z}_+^n$. By 3.3 (i) we have a nonzero U_ξ-homomorphism $V_\xi(\mathbb{l}\nu+\kappa)\to L_\xi(\lambda)\otimes V_\xi(\mathbb{l}\nu+\kappa-\lambda)$. By 3.3 (iii), this gives rise to a nonzero U_ξ-homomorphism $V_\xi(\mathbb{l}\nu+\kappa)\otimes V_\xi(\mathbb{l}\nu+\kappa-\lambda)^\star\to L_\xi(\lambda)$, which is necessarily surjective. Further, this surjective gives rise to a nonzero U_ξ-homomorphism $L_\xi(-w_0\lambda)\simeq L_\xi(\lambda)^*\to(V_\xi(\mathbb{l}\nu+\kappa)\otimes V_\xi(\mathbb{l}\nu+\kappa-\lambda)^\star)^*\simeq V_\xi(\mathbb{l}\nu+\kappa)\otimes V_\xi(\mathbb{l}\nu+\kappa-\lambda)$. (Note that $w_0(\nu)=-\nu$ for our choice.) Therefore the category $\mathscr{C}$ has enough injective objects and enough projective objects as well. Part (ii) is proved.

(iii) The U_ξ-modules $V_\xi(\mathbb{l}\mu+\kappa)^*$, $V_\xi(\mathbb{l}\mu+\kappa)^\star$ are isomorphic to $V_\xi(\mathbb{l}\nu+\kappa)$, where $\nu=-w_0(\mu)$. So for each $M\in\operatorname{ob}\mathscr{C}$, the modules $V_\xi(\mathbb{l}\mu+\kappa)\otimes M$ and $(V_\xi(\mathbb{l}\mu+\kappa)\otimes M)^*=V_\xi(\mathbb{l}\nu+\kappa)\otimes M^*$ are projective and injective as well in the category $\mathscr{C}$. By the proof of (ii) we see that (iii) is true.

(iv) We have seen that each indecomposable injective object is a direct summand of $V_\xi(\mathbb{l}\nu+\kappa)\otimes V_\xi(\delta)$ for some $\nu,\delta\in\mathbf{Z}_+^n$. So each injective object in $\operatorname{ob}\mathscr{C}$ is an integrable U_ξ-module. Let M be a finite dimensional U_ξ-module of type $\mathbb{1}$ and let M' be the maximal completely reducible submodule of M. By (ii), we can find an injective object E in $\operatorname{ob}\mathscr{C}$ and an injective U_ξ-homomorphism $M'\hookrightarrow E$. Since E is injective in the category, the above injection can be extended to an injective U_ξ-homomorphism $M\hookrightarrow E$. Therefore M is integrable since E is integrable.

According to the statements in 3.1 we see that (v) is an immediate consequence of (iv).

(vi) It is no harm to assume that E is indecomposable, then E is a direct summand of $V:=V_\xi(\mathbb{l}\nu+\kappa)\otimes V_\xi(\delta)$ for some $\nu,\delta\in\mathbf{Z}_+^n$. By 3.3 (vii), V has a submodule filtration $0=V_h\subset V_{h-1}\subset\cdots\subset V_2\subset V_1=V$ such that $V_a/V_{a+1}\simeq V_\xi(\delta_a)$ for some $\delta_a\in\mathbf{Z}_+^n$, $a=1,\ldots,h-1$. Since E is a direct summand of V, according to the following Lemma 3.6, the required filtration exists.

The theorem is proved.

Lemma 3.6. *Let M be a finite dimensional U_ξ-module of type $\mathbb{1}$. Assume that M has a submodule filtration $0=M_h\subset M_{h-1}\subset\cdots\subset M_2\subset M_1=M$ such that $M_a/M_{a+1}\simeq V_\xi(\delta_a)$ for some $\delta_a\in\mathbf{Z}_+^n$, $a=1,\ldots,h-1$. (We say that M has a filtration of Weyl modules.)*

(i) *Let λ be a maximal weight of M and m be a non-zero element in M_λ. Then the submodule $U_\xi m$ of M generated by m is isomorphic to $V_\xi(\lambda)$.*

(ii) *Let M', M'' be two U_ξ-modules. Assume that M is isomorphic to $M' \oplus M''$, then both M' and M'' have filtrations of Weyl modules.*

Proof. We copy the arguments in [J3, 3.5 and 3.6, pp. 279–280].

(i) Choose k such that m is in M_k but not in M_{k-1}. Then the image $\bar{m}$ in $M_k/M_{k-1} \simeq V_\xi(\delta_k)$ of m is not zero. Since λ is a maximal weight of M, we necessarily have $\delta_k = \lambda$. Thus we get a surjective U_ξ-homomorphism $U_\xi m \to V_\xi(\lambda)$, $m \to \bar{m}$. According to 3.3 (i), the homomorphism is an isomorphism.

(ii) Choose a maximal weight λ of M. It is no harm to assume that $M'_\lambda \neq 0$. Let m be a non-zero element in M'_λ. By (i), $U_\xi m$ is isomorphic to $V_\xi(\lambda)$. By the argument in (i), the module $M/U_\xi m$ has a filtration of Weyl modules. But $M/U_\xi m$ is isomorphic to $(M'/U_\xi m) \oplus M''$. Using induction on $\dim_{\mathbf{Q}(\xi)} M$ we see that both M' and M'' have filtrations of Weyl modules.

The lemma is proved.

Another main result of the section is the following.

Theorem 3.7. *Let $\lambda \in \mathbf{Z}^n_{+,\mathbf{l}}$, $\mu \in \mathbf{Z}^n_+$. Then*
(i) *The module $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$ contains a unique irreducible submodule.*
(ii) *The irreducible submodule of $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$ has highest weight $\mathbf{l}\mu + \kappa + w_0\lambda$ and is generated by $y'_\lambda v_{\mathbf{l}\mu+\kappa+\lambda}$. (See* Corollary 2.7 (ii) *for the definition of y'_λ.)*

Proof. (i) In the proof of Theorem 3.5 (ii) we have seen that $V_\xi(\mathbf{l}\mu + \kappa) \otimes V_\xi(\lambda)$ is an injective object in the category $\mathscr{C}$. According to 3.3 (vii) and Lemma 3.6 (i), the submodule of $V_\xi(\mathbf{l}\mu + \kappa) \otimes V_\xi(\lambda)$ generated by $v_{\mathbf{l}\mu+\kappa} \otimes v_\lambda$ is isomorphic to $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$. Let E be the indecomposable direct summand of $V_\xi(\mathbf{l}\mu + \kappa) \otimes V_\xi(\lambda)$ containing $v_{\mathbf{l}\mu+\kappa} \otimes v_\lambda$, then $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$ is isomorphic to a submodule of E. The module E contains a unique irreducible submodule since E is an indecomposable injective object in the category $\mathscr{C}$. This forces that $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$ contains a unique irreducible submodule.

(ii) We need to prove that
(a) The element $y'_\lambda v_{\mathbf{l}\mu+\kappa+\lambda}$ is a primitive element in $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$.
(b) The element $y'_\lambda v_{\mathbf{l}\mu+\kappa+\lambda}$ generates an irreducible submodule of $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$.

By Prop. 2.8 (ii), $F_\kappa v_\kappa \neq 0$, so $F_\kappa \notin I^-_\kappa$. But $I^-_{\kappa+\lambda} \subseteq I^-_\kappa$. Hence $F_\kappa \notin I^-_{\kappa+\lambda}$. This implies that in $V_\xi(\kappa + \lambda)$ we have $F_\kappa v_{\kappa+\lambda} \neq 0$. Since $F_\alpha F_\kappa \in \mathbf{u}^-_\xi$ if $\alpha \in R^+_l$ and F_κ has the maximal degree in $\mathbf{u}^-_\xi$ (recall the definition of degree in 2.3), we see $F_\alpha F_\kappa = 0$ in U_ξ for all $\alpha \in R^+_l$. In particular, $F_\alpha F_\kappa v_{\kappa+\lambda} = 0$ for all $\alpha \in R^+_l$. Noting that F_κ has degree 2κ and $\xi^{d_i l_i \langle \alpha^\vee_i, 2\kappa \rangle} = 1$ ($i \in [1, n]$), by [L7, Theorem 35.4.2 (a), p. 276], $F^{(l_i)}_i F_\kappa - F_\kappa F^{(l_i)}_i \in \mathbf{u}^-_\xi$. But $F^{(l_i)}_i F_\kappa - F_\kappa F^{(l_i)}_i$ has degree $2\kappa + l_i\alpha_i > 2\kappa$, so $F^{(l_i)}_i F_\kappa = F_\kappa F^{(l_i)}_i$. Since $F_\kappa = xF^{(l_i-1)}_i$ for some x in $\mathbf{u}^-_\xi$ (see Lemma 2.5), we get $F^{(l_i)}_i F_\kappa = F_\kappa F^{(l_i)}_i = xF^{(l_i-1)}_i F^{(l_i)}_i = xF^{(2l_i-1)}_i$ for $i = 1, 2, \ldots, n$ (cf. [L2, 3.2 (c), p. 62]). Thus $F^{(l_i)}_i F_\kappa v_{\kappa+\lambda} = xF^{(2l_i-1)}_i v_{\kappa+\lambda} = 0$ for $i = 1, 2, \ldots, n$. Therefore, $-\kappa + \lambda$ (the weight of $F_\kappa v_{\kappa+\lambda}$) is the lowest weight of the submodule M' of $V_\xi(\kappa + \lambda)$ generated by $F_\kappa v_{\kappa+\lambda}$.

By the proof of Prop. 2.6 (ii), the submodule M' of $V_\xi(\kappa + \lambda)$ is an irreducible module of highest weight $\kappa + w_0\lambda$. By Corollary 2.7 (ii), the irreducible module M' is generated by $y'_\lambda v_{\kappa+\lambda}$. Thus

(c) $y'_\lambda v_{\kappa+\lambda}$ is a primitive element in $V_\xi(\kappa + \lambda)$ (since it has weight $\kappa + w_0\lambda$) and generates the irreducible submodule of $V_\xi(\kappa + \lambda)$.

According to the proof of Theorem 3.2 (ii), $M := V_\xi(\kappa) \otimes V_\xi(\lambda) \otimes L_\xi(\mathbb{l}\mu)$ is injective in $\mathscr{C}$. By Theorem 3.2 (vi) and Lemma 3.6 (i), the submodule N of M generated by $v_\kappa \otimes v_\lambda \otimes \bar{v}_{\mathbb{l}\mu}$ is isomorphic to $V_\xi(\mathbb{l}\mu + \kappa + \lambda)$. Since the submodule of $V_\xi(\kappa) \otimes V_\xi(\lambda)$ generated by $v_\kappa \otimes v_\lambda$ is isomorphic to $V_\xi(\kappa + \lambda)$ (see 3.3 (vii) and Lemma 3.6 (i)), by (c) and 2.2 (v), we get

(d) The element $m' := y'_\lambda(v_\kappa \otimes v_\lambda \otimes \bar{v}_{\mathbb{l}\mu}) = (y'_\lambda(v_\kappa \otimes v_\lambda)) \otimes \bar{v}_{\mathbb{l}\mu}$ is primitive in N.

Since $\kappa + w_0\lambda$ is $\mathbb{l}$-restricted, by (c), (d) and 2.2 (iv), we see that m' generates an irreducible submodule of N. This completes the proof of (ii).

§4. The Elements x'_λ

4.1. Recall that in 1.4 we have defined the element $x'_\lambda \in U_\xi^-$ and in Corollary 2.7 (ii) defined the element $y'_\lambda \in \mathbf{u}_\xi^-$ for every λ in $\mathbf{Z}^n_+$. The main result of this section is Theorem 4.2. We prove it after establishing several lemmas. It is a pity that the author could not find a simple proof of Theorem 4.2 except for type A_n, B_2 and could not prove it for type G_2. For convenience we say that the quantized hyperalgebra U_ξ has no factors of type G_2 if any indecomposable component of the Cartan matrix (a_{ij}) is not of type G_2.

Theorem 4.2. *Assume that U_ξ has no factors of type G_2. Let $\lambda \in \mathbf{Z}^n_{+,\mathbb{l}}$, $\mu \in \mathbf{Z}^n_+$. Then*

(i) *The element $x'_\lambda v_{\mathbb{l}\mu+\kappa+\lambda}$ is a primitive element in $V_\xi(\mathbb{l}\mu + \kappa + \lambda)$.*

(ii) *We have $x'_\lambda = \eta y'_\lambda$ for some nonzero number $\eta \in \mathbf{Q}(\xi)$. In particular, x'_λ is in $\mathbf{u}_\xi^-$.*

Lemma 4.3. *Let M be an integrable U_ξ-module of type $\mathbb{1}$ and let $m \in M_\mu$ ($\mu \in \mathbf{Z}^n$). Let i, j be integers in $[1, n]$ and let a, b, c be non-negative integers.*

(i) *Assume that $E_i^{(h)}m = 0$ for $h \geq 1$. Then $F_i^{(a)}F_j^{(b)}F_i^{(c)}m = 0$ if $a + \langle\alpha_j, \alpha_i^\vee\rangle b + c > \langle\mu, \alpha_i^\vee\rangle$. In particular, $F_i^{(a)}F_j^{(b)}m = 0$ if $a + \langle\alpha_j, \alpha_i^\vee\rangle b > \langle\mu, \alpha_i^\vee\rangle$.*

(ii) *Assume that $E_i^{(h)}m = 0$, $E_j^{(h)}m = 0$ for $h \geq 1$. Then $F_i^{(a)}E_j^{(b)}F_j^{(c)}m = 0$ if $a + \langle\alpha_j, \alpha_i^\vee\rangle(c - b) > \langle\mu, \alpha_i^\vee\rangle$.*

Proof. (i) By the commutation relations in [L4, 5.3–5.4, pp. 95–97], the element $F_i^{(a)}F_j^{(b)}F_i^{(c)}$ is in the left ideal of U_ξ^- generated by $F_i^{(h)}$, $h \geq a + \langle\alpha_j, \alpha_i^\vee\rangle b + c > \langle\mu, \alpha_i^\vee\rangle$. Now applying 3.3 (i) to the subalgebra of U_ξ generated by the elements $E_i^{(h)}$, $F_i^{(h)}$, K_i, K_i^{-1} ($h \geq 0$), we see that (i) is true.

(ii) If $b > c$, then

$$F_i^{(a)}E_j^{(b)}F_j^{(c)}m = F_i^{(a)} \sum_{0 \le h \le c} F_j^{(c-h)} \begin{bmatrix} K_j, 2h-c-b \\ h \end{bmatrix} E_j^{(b-h)}m = 0.$$

If $b \le c$, using (i), we see

$$F_i^{(a)}E_j^{(b)}F_j^{(c)}m = F_i^{(a)}F_j^{(c-b)} \begin{bmatrix} K_j, b-c \\ b \end{bmatrix} m = \begin{bmatrix} \langle \mu, \alpha_i^\vee \rangle + b - c \\ b \end{bmatrix}_{d_j, \xi} F_i^{(a)}F_j^{(c-b)}m = 0.$$

The lemma is proved.

Lemma 4.4. *Let $\lambda \in \mathbf{Z}^n_{+,\mathbf{l}}$, $\mu \in \mathbf{Z}^n_+$, $w \in W$. Then*
(i) *In $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$ we have $x'_{\lambda,w}v_{\mathbf{l}\mu+\kappa+\lambda} \ne 0$.*
(ii) *If $l_i \ge 2$, then $E_i x'_{\lambda,w}v_{\mathbf{l}\mu+\kappa+\lambda} = 0$.*
(iii) *If $l_\alpha \ge 2$, then $E_\alpha x'_{\lambda,w}v_{\mathbf{l}\mu+\kappa+\lambda} = 0$.*
(iv) *Assume that $x'_{\lambda,w} = F_{i_1}^{(a_1)}F_{i_2}^{(a_2)}\cdots F_{i_k}^{(a_k)}$. Given non negative integers b_1, $b_2, \ldots, b_k$, if $a_h - b_h \in l_{i_h}\mathbf{Z}$ for $h = 1, \ldots, k$, then $E_i F_{i_1}^{(b_1)}F_{i_2}^{(b_2)}\cdots F_{i_k}^{(b_k)}v_{\mathbf{l}\mu+\kappa+\lambda} = 0$ when $l_i \ge 2$.*

Proof. Set $m := v_{\mathbf{l}\mu+\kappa+\lambda}$.

(i) According to [L7, Lemma 39.1.2, p. 304], in $V_\xi(\lambda)$ we have $x'_{\lambda,w}v_\lambda \ne 0$. By 1.2 (i), this implies that $x'_{\lambda,w}m \ne 0$.

(ii) According to [L7, Lemma 39.1.4, p. 305], there exists z in U_ξ^- such that

$$E_i x'_{\lambda,w} = x'_{\lambda,w}E_i + z \begin{bmatrix} K_i, 1 - \langle \lambda, \alpha_i^\vee \rangle \\ 1 \end{bmatrix}.$$

Since $l_i \ge 2$, $\begin{bmatrix} K_i, 1 - \langle \lambda, \alpha_i^\vee \rangle \\ 1 \end{bmatrix} m = \begin{bmatrix} l_i\langle \mu, \alpha_i^\vee \rangle + l_i \\ 1 \end{bmatrix}_{d_i, \xi} m = 0$. Therefore $E_i x'_{\lambda,w}m = 0$.

(iii) It is no harm to assume that the Cartan matrix (a_{ij}) is indecomposable. When $\mathbf{u}_\xi^-$ is generated by the elements 1, F_i $(i = 1, 2, \ldots, n)$, the assertion is a simple consequence of (ii). When $\mathbf{u}_\xi^-$ is not generated by the elements 1, F_i $(i = 1, 2, \ldots, n)$, one of the following cases must happen,
(a) The Cartan matrix (a_{ij}) is of type B_n, C_n, F_4 and $l = 4$.
(b) The Cartan matrix (a_{ij}) is of type G_2 and $l = 3, 4, 6$.

The generators of $\mathbf{u}_\xi^-$ are described in [L4, 8.3, pp. 107–108] explicitly. Using induction on the height of α and using Theorem 6.6 in [L4, pp. 103–104], one can prove that $E_\alpha F_i^{(a)} - F_i^{(a)}E_\alpha \in U_\xi^- \mathbf{u}_\xi^0 \mathbf{u}_\xi^+$ for all $i \in [1, n]$, $a \in \mathbf{N}$. Using (ii), we then can prove (iii) by induction on $l(w)$ and on height of α.

Part (iv) is a simple consequence of (ii).

Lemma 4.5. *Let $\lambda \in \mathbf{Z}^n_{+,\mathbf{l}}$, $w \in W$. Assume that the Cartan matrix (a_{ij}) is* *symmetric. If* $s_j w \geq w$, *then* $E_j^{(a)} x'_{\lambda,w} v_{\kappa+\lambda} = 0$ *for all* $a \geq 1$. (*We also use* "$\geq$" *for the Bruhat order on* W.)

Proof. Set $m := v_{\kappa+\lambda}$. Noting that all $l_1, l_2, \ldots, l_n$ are equal, we simply write l' for any one of them. Since U_ξ^+ is generated by the elements E_i, $E_i^{(l')}$ for $i = 1, 2, \ldots, n$, [L2, Prop. 3.2 (b), p.62], by Lemma 4.4 (ii), it suffices to prove that $E_j^{(l')} x'_{\lambda,w} m = 0$. We use induction on the length $l(w)$ of w. Let $s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression of w. We shall write a_h instead of $\langle s_{i_{h+1}} \cdots s_{i_k} \lambda, \alpha_{i_h}^\vee \rangle$ for $h = 1, \ldots, k$. When $k = 0, 1$, nothing need to be proved. Now assume that $k \geq 2$. Set $i := i_1$ and let u be the shortest element of the coset $\langle s_i, s_j \rangle w$ (here $\langle s_i, s_j \rangle$ denotes the subgroup of W generated by s_i, s_j). Since the Cartan matrix is symmetric, $k - 1 \geq l(u) \geq k - 2$.

If $l(u) = k - 1$, then $u = s_{i_2} \cdots s_{i_k}$ and $s_j u \geq u$. Note that $i \neq j$, using induction hypothesis, we see $E_j^{(l')} x'_{\lambda,w} m = F_i^{(a_1)} E_j^{(l')} x'_{\lambda,u} m = 0$.

If $l(u) = k - 2$, we may assume that $i_2 = j$ and $u = s_{i_3} \cdots s_{i_k}$. Then $s_i u \geq u$, $s_j u \geq u$ and $E_i^{(a)} x'_{\lambda,u} m = 0$, $E_j^{(a)} x'_{\lambda,u} m = 0$ for all $a \geq 1$. So $E_j^{(l')} x'_{\lambda,w} m = F_i^{(a_1)} E_j^{(l')} F_j^{(a_2)} x'_{\lambda,u} m$. Noting that $a_1 = \langle s_j u\lambda, \alpha_i^\vee \rangle = \langle u\lambda, \alpha_i^\vee + \alpha_j^\vee \rangle = \langle u\lambda, \alpha_i^\vee \rangle + a_2$ and $x'_{\lambda,u} m$ has weight $\kappa + u\lambda$, by Lemma 4.3 (ii) we see $E_j^{(l')} x'_{\lambda,w} m = 0$. The lemma is proved.

Lemma 4.6. *Let* $\lambda \in \mathbf{Z}^n_{+,1}$. *Assume that* U_ξ *has no factors of type* G_2. *Then in* $V_\xi(\kappa + \lambda)$ *the element* $x'_\lambda v_{\kappa+\lambda}$ *is primitive.*

Proof. Set $m := v_{\kappa+\lambda}$. Since U_ξ^+ is generated by the elements E_i, $E_i^{(l_i)}$ for $i = 1, 2, \ldots, n$, by Lemma 4.4 (ii), it suffices to prove that $E_i^{(l_i)} x'_\lambda m = 0$ for all i. Set $\lambda_i = \langle \lambda, \alpha_i^\vee \rangle$ for $i = 1, 2, \ldots, n$.

(a) Assume that (a_{ij}) is symmetric. Choose a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_r}$ of the longest element w_0 of W such that $i_1 = i$. Note that $a := \langle s_{i_2} \cdots s_{i_r} \lambda, \alpha_i^\vee \rangle < l_i$, so

$$E_i^{(l_i)} x'_\lambda m = \sum_{0 \leq h \leq a} F_i^{(a-h)} \begin{bmatrix} K_i, 2h - a - l_i \\ h \end{bmatrix} E_i^{(l_i - h)} x'_{\lambda,u} m,$$

where $u = s_{i_2} \cdots s_{i_r}$. By Lemma 4.5, $E_i^{(l_i - h)} x'_{\lambda,u} m = 0$ for $h = 0, 1, \ldots, a$. Therefore $E_i^{(l_i)} x'_\lambda m = 0$ for $i = 1, 2, \ldots, n$.

(b) Assume that (a_{ij}) is of type B_n. We number the simple roots in R^+ so that $\langle \alpha_2, \alpha_1^\vee \rangle = -2$, $\langle \alpha_1, \alpha_2^\vee \rangle = \langle \alpha_2, \alpha_3^\vee \rangle = \cdots = \langle \alpha_{n-1}, \alpha_n^\vee \rangle = -1$. We have $d_1 = 1$, $d_2 = \cdots = d_n = 2$, $l_2 = \cdots = l_n$, and $2l_j \geq l_1 \geq l_j$ for $j = 2, \ldots, n$. We use induction on n.

When $n = 2$, write $a := \langle \lambda, \alpha_1^\vee \rangle$, $b := \langle \lambda, \alpha_2^\vee \rangle$. Then

$$x'_\lambda = F_1^{(a)} F_2^{(a+b)} F_1^{(a+2b)} F_2^{(b)} = F_2^{(b)} F_1^{(a+2b)} F_2^{(a+b)} F_1^{(a)}.$$

Since $l_1 > a$, using Lemma 4.4 (ii) we see

$$E_1^{(l_1)} x'_\lambda m = F_1^{(a)} F_2^{(a+b)} E_1^{(l_1)} F_1^{(a+2b)} F_2^{(b)} m.$$

Note that $F_2^{(b)}m$ is a primitive element of weight $\kappa + \lambda - b\alpha_2$. Now

$$a + b - \langle \alpha_1, \alpha_2^\vee \rangle (l_1 - a - 2b) = l_1 - b > l_2 - 1 - b = \langle \kappa + \lambda - b\alpha_2, \alpha_2^\vee \rangle.$$

By Lemma 4.3 (ii) we have $E_1^{(l_1)} x'_\lambda m = 0$. Similarly we have

$$E_2^{(l_2)} x'_\lambda m = F_2^{(b)} F_1^{(a+2b)} E_2^{(l_2)} F_2^{(a+b)} F_1^{(a)} m = 0.$$

Now suppose that the lemma is true for type B_{n-1}. Let w be the longest element in $\langle s_1, s_2, \ldots, s_{n-1} \rangle$ (the subgroup of W generated by $s_1, s_2, \ldots, s_{n-1}$). Then

$$w_0 = s_n s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1} s_n w = w s_n s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1} s_n.$$

Set

$$a_h := \langle s_{h-1} \cdots s_2 s_1 s_2 \cdots s_{n-1} s_n w\lambda, \alpha_h^\vee \rangle = \lambda_h + \lambda_{h+1} + \cdots + \lambda_n, \qquad h = 2, \ldots, n,$$

and

$$b_1 := \langle s_2 \cdots s_{n-1} s_n w\lambda, \alpha_1^\vee \rangle = \lambda_1 + 2\lambda_2 + \cdots + 2\lambda_n,$$

$$b_2 := \langle s_3 \cdots s_{n-1} s_n w\lambda, \alpha_2^\vee \rangle = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$

$$b_h := \langle s_{h+1} \cdots s_{n-1} s_n w\lambda, \alpha_h^\vee \rangle = \lambda_1 + \lambda_2 + \cdots + \lambda_n + \lambda_2 + \cdots + \lambda_{h-1}, \quad h = 3, \ldots, n.$$

Then we have

(b1) $x'_\lambda = F_n^{(a_n)} \cdots F_2^{(a_2)} F_1^{(b_1)} F_2^{(b_2)} \cdots F_n^{(b_n)} x'_{\lambda,w}$.

By induction hypothesis, we have

(b2) The element $x'_{\lambda,w} m$ is primitive in $V_\xi(\kappa + \lambda)$ of weight $\kappa + w\lambda = (l_1 - 1 - \lambda_1, l_2 - 1 - \lambda_2, \ldots, l_{n-1} - 1 - \lambda_{n-1}, l_n - 1 + b_n)$.

Using Lemma 4.3 (ii) and Lemma 4.4 (ii) repeatedly we see

(b3) $E_i^{(l_i)} F_h^{(b_h)} F_{h+1}^{(b_{h+1})} \cdots F_n^{(b_n)} x'_{\lambda,w} m = 0$ for $h = 1, 2, \ldots, n$ and $i \neq h$.

(b4) $E_i^{(l_i)} F_h^{(a_h)} \cdots F_2^{(a_2)} F_1^{(b_1)} F_2^{(b_2)} \cdots F_n^{(b_n)} x'_{\lambda,w} m = 0$ for $i, h = 2, \ldots, n$ and $i \neq h$.

Since $a_n = \lambda_n < l_n$, by (b4) and Lemma 4.4 (ii) we know that $E_i^{(l_i)} x'_\lambda m = 0$ for $i = 2, \ldots, n$.

We need to do a little more to see that $E_1^{(l_1)} x'_\lambda m = 0$. Let u be the longest element of $\langle s_2, \ldots, s_n \rangle$. Then

$$w_0 = s_1 s_2 s_1 s_3 s_2 s_1 \cdots s_n \cdots s_2 s_1 u.$$

For $n \geq i \geq j \geq 1$, set

$$c_{ij} := \langle s_{j-1} \cdots s_1 s_{i+1} \cdots s_1 s_{i+2} \cdots s_1 \cdots s_n \cdots s_2 s_1 u\lambda, \alpha_j^\vee \rangle$$
$$= \lambda_1 + \cdots + \lambda_i + \lambda_{j+1} + \cdots + \lambda_i.$$

(Convention: $\lambda_{i+1} + \cdots + \lambda_i = 0$). Then we have

(b5) $x'_\lambda = F_1^{(c_{11})} F_2^{(c_{22})} F_1^{(c_{21})} \cdots F_n^{(c_{nn})} \cdots F_1^{(c_{n1})} x'_{\lambda,u}$.

By (a), we have

(b6) The element $x'_{\lambda,u}m$ is primitive in $V_\xi(\kappa+\lambda)$ of weight $\kappa+u\lambda=(l_1-1+c_{n1}, l_2-1-\lambda_n,\dots,l_n-1-\lambda_2)$.

Note that $c_{11}=\lambda_1<l_1$. Now we can see $E_1^{(l_1)}x'_\lambda m=0$ by repeatedly using Lemma 4.3 (ii) and Lemma 4.4 (ii). Thus we complete the proof type B_n.

(c) Similarly, we prove the lemma for type C_n.

(d) Similarly we prove the lemma for type F_4 but need a little patience. We number the simple roots so that $\langle\alpha_i,\alpha_{i+1}^\vee\rangle=-1$ for $i=1,2,3$. Then $d_1=d_2=1$, $d_3=d_4=2$. Moreover, $\alpha_1,\alpha_2,\alpha_3$ generate a root system of type C_3 and $\alpha_2,\alpha_3,\alpha_4$ generate a root system of type B_3. Let w be the longest element of the group $\langle s_1,s_2,s_3\rangle$. Then the longest element of W is $s_4s_3s_2s_3s_4s_1s_2s_3s_4s_2s_3s_1s_2s_3s_4w$. We shall also write $\lambda_{a,b,c,d}$ for $a\lambda_1+b\lambda_2+c\lambda_3+d\lambda_4$. For non-negative integers $\theta_1,\dots,\theta_4$, define

$$Y_1(\theta_1,\dots,\theta_4) := F_4^{(\lambda_4)}F_3^{(\lambda_3+\lambda_4-\theta_1l_3)}F_2^{(\lambda_2+2\lambda_3+2\lambda_4-\theta_2l_2)}F_3^{(\lambda_2+\lambda_3+\lambda_4-\theta_3l_3)}F_4^{(\lambda_2+2\lambda_3+\lambda_4-\theta_4l_4)},$$

$$Y_2(\theta_1,\dots,\theta_4):=F_1^{(\lambda_{1,1,2,2}-\theta_1l_1)}F_2^{(\lambda_{1,2,2,2}-\theta_2l_2)}F_3^{(\lambda_{1,2,3,2}-\theta_3l_3)}F_4^{(\lambda_{1,1,1,1}-\theta_4l_4)},$$

$$Y_3(\theta_2,\theta_3):=F_2^{(\lambda_{1,2,4,2}-\theta_2l_2)}F_3^{(\lambda_{1,1,2,1}-\theta_3l_3)},$$

$$Y_4(\theta_1,\dots,\theta_4):=F_1^{(\lambda_{1,3,4,2}-\theta_1l_1)}F_2^{(\lambda_{2,3,4,2}-\theta_2l_2)}F_3^{(\lambda_{1,2,2,1}-\theta_3l_3)}F_4^{(\lambda_{1,2,3,1}-\theta_4l_4)}.$$

We simply write Y_i for $Y_i(0,\dots,0)$, $i=1,2,3,4$.

Then we have

(d1) $x'_\lambda=Y_1Y_2Y_3Y_4x'_{\lambda,w}$.

According to (c), we get

(d2) The element $m':=x'_{\lambda,w}m$ is primitive in $V_\xi(\kappa+\lambda)$ and has weight $(l_1-1-\lambda_1, l_2-1-\lambda_2, l_3-1-\lambda_3, l_4-1+\lambda_{1,2,3,1})$.

Using (d2), Lemma 4.3 (ii) and Lemma 4.4 (ii) repeatedly, step by step, we obtain

(d3) $E_i^{(l_i)}Y_4x'_{\lambda,w}m$ for $i=2,3,4$.

(d4) $E_i^{(l_i)}Y_3Y_4x'_{\lambda,w}m=0$ for $i=1,3,4$.

(d5) $E_2^{(l_2)}Y_2Y_3Y_4x'_{\lambda,w}m=0$.

(d*) $E_1^{(l_1)}Y_1Y_2Y_3Y_4x'_{\lambda,w}m=0$.

To avoid more troubles we use the following consequence of Theorem 4.2 for type B_3 (cf. Corollary 4.10 (ii)).

(d6) $Y_1(0,1,0,0)F_2^{(\lambda_2)}F_3^{(\lambda_2+\lambda_3)}F_2^{(\lambda_2+2\lambda_3)}F_3^{(\lambda_3)}=0$.

Obviously, (d6) implies the following

(d7) $Y_1(0,1,0,0)Y_2Y_3Y_4x'_{\lambda,w}=0$.

Combining (d5) and (d7), also using Lemma 4.4 (ii) we see

(d★) $E_2^{(l_2)}Y_1Y_2Y_3Y_4x'_{\lambda,w}m=0$.

Let u be the longest element of $\langle s_2,s_3,s_4\rangle$. We may write down a presentation for x'_λ according to the reduced expression $s_1 s_2 s_3 s_2 s_1 s_4 s_3 s_2 s_1 s_3 s_2 s_4 s_3 s_2 s_1 u$. Using an argument similar to that for (d$*$) and (d$\bigstar$) we obtain (d$\dagger$) $E_i^{(l_i)} Y_1 Y_2 Y_3 Y_4 x'_{\lambda,w} m = 0$ for $i = 3, 4$.

The lemma is proved for type F_4 and the proof is completed.

Lemma 4.7. *Assume that (a_{ij}) is symmetric and $\lambda \in \mathbf{Z}_+^n$ is* $\mathbf{l}$*-restricted. Let $s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression of the longest element w_0. Set $a_h := \langle s_{i_{h+1}} s_{i_{h+2}} \cdots s_{i_r} \lambda, \alpha_{i_h}^\vee \rangle$ for $h = 1, 2, \ldots, r$. Given non-negative integers $\theta_1, \theta_2, \ldots, \theta_r$, we define*

$$Y_{\mathbf{i}}(\theta_1, \ldots, \theta_r) := F_{i_1}^{(a_1 - \theta_1 l')} F_{i_2}^{(a_2 - \theta_2 l')} \cdots F_{i_r}^{(a_r - \theta_r l')}.$$

Then in $V_\xi(\kappa + \lambda)$ we have $Y_{\mathbf{i}}(\theta_1, \ldots, \theta_r) v_{\kappa+\lambda} = 0$ if $\theta_1 + \cdots + \theta_r \geq 1$. Here $\mathbf{i} = (i_1, i_2, \ldots, i_r)$ and $l' := l_1 = \cdots = l_r$. (Recall that $F_i^{(a)} = 0$ for all i and $a < 0$, see 1.1.)

Proof. We use induction on h to prove that

(a) $Y_{\mathbf{i}}(\theta_1, \ldots, \theta_h, 0, \ldots, 0) v_{\kappa+\lambda} = 0$ if $\theta_1 + \cdots + \theta_h \geq 1$.

Since $a_1 < l'$, the assertion (a) is obvious when $h = 1$. Now assume that (a) is true for $h - 1$. That is, we assume the following is true.

(b) $Y_{\mathbf{i}}(\theta_1, \ldots, \theta_{h-1}, 0, \ldots, 0) v_{\kappa+\lambda} = 0$ if $\theta_1 + \cdots + \theta_{h-1} \geq 1$.

Set $m_{h+1} := F_{i_{h+1}}^{(a_{h+1})} \cdots F_{i_r}^{(a_r)} v_{\kappa+\lambda}$, then we have

(c) $\begin{bmatrix} K_{i_h}, \ \theta l' - (a_h - \theta' l') \\ \theta l' \end{bmatrix} m_{h+1} = \begin{pmatrix} \theta' + \theta \\ \theta \end{pmatrix} m_{h+1}$ for $\theta, \theta' \geq 0$. (Where $\begin{pmatrix} \theta' + \theta \\ \theta \end{pmatrix}$ is the ordinary binomial coefficient.)

Using (c), Lemma 4.4 (iv), Lemma 4.5 and Lemma 4.6, we get

(d) $E_{i_h}^{(l')} Y_{\mathbf{i}}(\theta_1, \ldots, \theta_{h-1}, 0, \ldots, 0) v_{\kappa+\lambda} = Y_{\mathbf{i}}(\theta_1, \ldots, \theta_{h-1}, 1, 0, \ldots, 0) v_{\kappa+\lambda} = 0$.

Assume that $\theta_h \geq 1$ and we have

(e) $Y_{\mathbf{i}}(\theta_1, \ldots, \theta_{h-1}, \theta_h - 1, 0, \ldots, 0) v_{\kappa+\lambda} = 0$ if $\theta_1 + \cdots + \theta_{h-1} + \theta_h - 1 \geq 1$.

Using (c), (e), Lemma 4.4 (iv), Lemma 4.5 and Lemma 4.6, we get

(f) $E_{i_h}^{(l')} Y_{\mathbf{i}}(\theta_1, \ldots, \theta_{h-1}, \ \theta_h - 1, \ 0, \ldots, 0) v_{\kappa+\lambda} = \begin{pmatrix} 2\theta_h - 1 \\ \theta_h \end{pmatrix} Y_{\mathbf{i}}(\theta_1, \ldots, \theta_{h-1}, \ \theta_h, 0, \ldots, 0) v_{\kappa+\lambda} = 0$.

Thus we have proved the assertion (a) by using induction on h as well as on θ_h. Take $h = r$, we obtain the lemma.

Lemma 4.8. *Let $\mu \in \mathbf{Z}_+^n$ and $\lambda \in \mathbf{Z}_{+,\mathbf{l}}^n$. Then the submodule of $L_\xi(\mathbf{l}\mu) \otimes V_\xi(\kappa + \lambda)$ generated by $\bar{v}_{\mathbf{l}\mu} \otimes v_{\kappa+\lambda}$ is isomorphic to $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$.*

Proof. By 3.3 (i), we have a U_ξ-homomorphism

$$V_1 := V_\xi(\mathbf{l}\mu + \kappa + \lambda) \to V := L_\xi(\mathbf{l}\mu) \otimes V_\xi(\kappa + \lambda),$$

which carries $m_1 := v_{\mathbf{l}\mu+\kappa+\lambda}$ to $m := \bar{v}_{\mathbf{l}\mu} \otimes v_{\kappa+\lambda}$. By 2.2 (v), $y'_\lambda m = \bar{v}_{\mathbf{l}\mu} \otimes y'_\lambda v_{\kappa+\lambda} \neq 0$. But $y'_\lambda m_1$ generates the unique irreducible submodule of V_1 (Theorem

3.7 (ii)). Therefore, the submodule of V generated by m is isomorphic to V_1. The lemma is proved.

4.9. Proof of Theroem 4.2. (A). We first prove that part (i) implies part (ii). Such an implication will be needed to simplify the induction proof of (i) for type B_n, C_n, F_4.

Assume that (i) is true. Since $x'_\lambda v_{\mathbf{l}\mu+\kappa+\lambda} \neq 0$ and has the same weight with $y'_\lambda v_{\mathbf{l}\mu+\kappa+\lambda}$, by (i) and Theorem 3.7, we can find a non-zero number $\eta \in \mathbf{Q}(\xi)$ such that $x'_\lambda - \eta y'_\lambda \in I^-_{\mathbf{l}\mu+\kappa+\lambda}$. Because of $\alpha < 6(\alpha_1 + \alpha_2 + \cdots + \alpha_n)$ and $l_\alpha \leq l$ for every $\alpha \in R^+$, we see

(a) $\lambda - w_0\lambda \leq \kappa - w_0\kappa = 2\kappa = \sum_{\alpha\in R^+} (l_\alpha - 1)\alpha < 6lr(\alpha_1 + \alpha_2 + \cdots + \alpha_n)$, where $r = |R^+|$.

Choose $\mu \in \mathbf{Z}^n_+$ such that $\langle \mu, \alpha_i^\vee \rangle > 6lr$ for all i. According to the definition of $I^-_{\mathbf{l}\mu+\kappa+\lambda}$ (see 1.2) we see that $x'_\lambda - \eta y'_\lambda \in I^-_{\mathbf{l}\mu+\kappa+\lambda}$ is equivalent to $x'_\lambda - \eta y'_\lambda = 0$. So part (i) implies part (ii).

Now we prove (i) case by case. According to Lemma 4.6 and Lemma 4.8, it suffices to prove the following assertion.

(*) In $L_\xi(\mathbf{l}\mu) \otimes V_\xi(\kappa + \lambda)$ we have $x'_\lambda(\bar{v}_{\mathbf{l}\mu} \otimes v_{\kappa+\lambda}) = \bar{v}_{\mathbf{l}\mu} \otimes x'_\lambda v_{\kappa+\lambda}$.
We need the following formula [L4, 1.3 (b) p. 91].

(†) In U_ξ we have $\Delta(F_i^{(a)}) = \sum_{h=0}^{a} \xi^{-d_i h(a-h)} F_i^{(h)} \otimes K_i^{-h} F_i^{(a-h)}$ for $i = 1, 2, \ldots, n$, $a \geq 0$.

Recall that we have (see 2.2 (v))

(‡) In $L_\xi(\mathbf{l}\mu)$, $F_\alpha \bar{v}_{\mathbf{l}\mu} = 0$ if $l_\alpha \geq 2$.

(B) Using (†), (‡), and Lemma 4.7 we see that (*) is true when the Cartan matrix (a_{ij}) is symmetric.

(C) Assume that the Cartan matrix (a_{ij}) is of type B_n. Keep the notations in (b) of the proof of Lemma 4.6. Given non-negative integers $\theta_n, \ldots, \theta_1, \theta'_2, \ldots, \theta'_n$, set

$$Y(\theta_n, \ldots, \theta_1, \theta'_2, \ldots, \theta'_n) := F_n^{(a_n - \theta_n l_n)} \cdots F_2^{(a_2 - \theta_2 l_2)} F_1^{(b_1 - \theta_1 l_1)} F_2^{(b_2 - \theta'_2 l_2)} \cdots F_n^{(b_n - \theta'_n l_n)} x'_{\lambda,w}.$$

Using (b3) and (b4) of the proof of Lemma 4.6, completely as the argument for Lemma 4.7 we get

(C1) $Y(\theta_n, \ldots, \theta_1, \theta'_2, \ldots, \theta'_n) v_{\kappa+\lambda} = 0$ if $\theta_n + \cdots + \theta_1 + \theta'_2 + \cdots + \theta'_n \geq 1$.

Regard $\{\alpha_1, -\alpha_1\}$ as a root system of type B_1, then obviously (i) is true for type B_1. Assume that (i) is true for type B_{n-1}. Then according to (A) we have $x'_{\lambda,w} \in \mathbf{u}^-_\xi$. Now using (†), (‡), and (C1) we see that (*) is true for type B_n.

(D) Similarly, we prove (*) for type C_n.

(E) Assume that the Cartan matrix (a_{ij}) is of type F_4. Keep the notations in (d) of the proof of Lemma 4.6. Given non-negatives integers $\theta_1, \ldots, \theta_{14}$, set

$$Y(\theta_1, \ldots, \theta_{14}) := Y_1(\theta_1, \ldots, \theta_4) Y_2(\theta_5, \ldots, \theta_8) Y_3(\theta_9, \theta_{10}) Y_4(\theta_{11}, \ldots, \theta_{14}) x'_{\lambda, w}.$$

For simplicity, we use the following consequence of Theorem 4.2 for type B_3 (cf. Corollary 4.10 (ii)).

(E1) $Y_1(\theta_1, \ldots, \theta_4) F_2^{(\lambda_2)} F_3^{(\lambda_2 + \lambda_3)} F_2^{(\lambda_2 + 2\lambda_3)} F_3^{(\lambda_3)} = 0$ if $\theta_1 + \cdots + \theta_4 \geq 1$.

Using (d3-d5) of the proof of Lemma 4.6 as well as (E1), completely as the argument for Lemma 4.7 we get

(E2) $Y(\theta_1, \ldots, \theta_{14}) v_{\kappa + \lambda} = 0$ if $\theta_1 + \cdots + \theta_{14} \geq 1$.

By (D) and (A) we know that $x'_{\lambda, w} \in \mathbf{u}_\xi^-$. Now using (†), (‡), and (E2) we see that (*) is true for type F_4.

The theorem is proved.

Corollary 4.10. *Keep the notations in* 4.7 *and* 4.9.
(i) *Assume that* (a_{ij}) *is symmetric. Then* $Y_{\mathbf{i}}(\theta_1, \ldots, \theta_r) = 0$ *if* $\theta_1 + \cdots + \theta_r \geq 1$.
(ii) *Assume that* (a_{ij}) *is of type* B_n. *Then* $Y(\theta_n, \ldots, \theta_1, \theta'_2, \ldots, \theta'_n) = 0$ *if* $\theta_n + \cdots + \theta_1 + \theta'_2 + \cdots + \theta'_n \geq 1$. *A similar result holds for type* C_n.
(iii) *Assume that* (a_{ij}) *is of type* F_4. *Then* $Y(\theta_1, \ldots, \theta_{14}) = 0$ *if* $\theta_1 + \cdots + \theta_{14} \geq 1$.
Where $Y_{\mathbf{i}}(\theta_1, \ldots, \theta_r)$, $Y(\theta_n, \ldots, \theta_1, \theta'_2, \ldots, \theta'_n)$, $Y(\theta_1, \ldots, \theta_{14})$ *are elements in* U_ξ^-. *Recall that* $F_i^{(a)} = 0$ *for all* i *and* $a < 0$, *see* 1.1.

Proof. We give a proof of (i). The proofs of (ii) and (iii) are similar. Using Lemma 4.7, (†) and (‡) in 4.9 we see that

$$Y_{\mathbf{i}}(\theta_1, \ldots, \theta_r)(\bar{v}_{\mathbf{1}\mu} \otimes v_{\kappa + \lambda}) = 0, \qquad \text{if } \theta_1 + \cdots + \theta_r \geq 1.$$

Using Lemma 4.8 and an argument as in 4.9 (A) we know that (i) is true.

4.11. By Lemma 4.4 (iii), Theorem 4.2 is actually equivalent to the assertion $x'_\lambda \in \mathbf{u}_\xi^-$ when λ is $\mathbf{l}$-restricted. For type B_2, using the commutation relation in [L4, 5.3 (i), p. 96] we see easily that if λ is $\mathbf{l}$-restricted then $x'_\lambda \in \mathbf{u}_\xi^-$. For type A_n there is a naive argument for the fact, which is based on the following Lemma 4.12. We need a notation. Given $i \in [1, n]$, let $\mathscr{H}_i$ be the $\mathbf{Q}(\xi)$-subspace of U_ξ^- spanned by the elements $F_{\beta_1}^{(a_1)} F_{\beta_2}^{(a_2)} \cdots F_{\beta_r}^{(a_r)}$ for $a_1, \ldots, a_r \in \mathbf{N}$ satisfying $a_h \leq l_{\beta_h} - 1$ whenever $\beta_h - \alpha_i \in \mathbf{N} R^+ (h = 1, \ldots, r)$. Obviously, $\bigcap_{i=1}^n \mathscr{H}_i = \mathbf{u}_\xi^-$.

Lemma 4.12. *Let* x *be an element in* U_ξ. *Assume that* x *is expressed as a* $\mathbf{Q}(\xi)$*-linear combination of some monomials* $z_1, \ldots, z_h$ *of* $F_\alpha^{(a)}$ $(\alpha \in R^+, a \in \mathbf{N})$.

Given $i \in [1, n]$. *If* $a \leq l_\alpha - 1$ *whenever* $F_\alpha^{(a)}$ *appears in some monomial* z_k *and* $\alpha - \alpha_i \in \mathbf{N}R^+$, *then* $x \in \mathscr{H}_i$.

Proof. Using commutation relations in [L4, 5.3–5.4, pp. 95–97] and [L4, Theorem 6.6, pp. 103–104].

4.13. Now we give a simple proof of Theorem 4.2 for type A_n by using Lemma 4.12. By Lemma 4.4 (ii), it suffices to prove that $x'_\lambda \in \mathfrak{u}_\xi^-$ when λ is l-restricted. We use induction on n. Set $\lambda_i := \langle \lambda, \alpha_i^\vee \rangle$ for $i = 1, 2, \ldots, n$. When $1 \leq i \leq j \leq n$ we also write $\lambda_{i,j}$ for $\lambda_i + \lambda_{i+1} + \cdots + \lambda_j$. Then

$$x'_\lambda = F_1^{(\lambda_n)} F_2^{(\lambda_{n-1,n})} \cdots F_n^{(\lambda_{1,n})} F_1^{(\lambda_{n-1})} F_2^{(\lambda_{n-2,n-1})} \cdots F_{n-1}^{(\lambda_{1,n-1})} \cdots F_1^{(\lambda_2)} F_2^{(\lambda_{1,2})} F_1^{(\lambda_1)}.$$

Note that $l_1 = \cdots = l_n$, by Lemma 4.12 we get

(a) $x'_\lambda \in \mathscr{H}_1$. Symmetrically, we have $x'_\lambda \in \mathscr{H}_n$.

Let $w = s_1 s_2 s_1 s_3 s_2 s_1 \cdots s_{n-1} \cdots s_2 s_1$ (the longest element of the group generated by $s_1, \ldots, s_{n-1}$). Set

$$y := F_1^{(\lambda_{n-1})} F_2^{(\lambda_{n-2,n-1})} F_1^{(\lambda_{n-2})} F_3^{(\lambda_{n-3,n-1})} F_2^{(\lambda_{n-3,n-2})} F_1^{(\lambda_{n-3})} \cdots F_{n-2}^{(\lambda_{2,n-1})} \cdots F_1^{(\lambda_2)},$$

$$y' := F_{n-1}^{(\lambda_{1,n-1})} \cdots F_2^{(\lambda_{1,2})} F_1^{(\lambda_1)}.$$

Then $x'_{\lambda,w} = yy'$. By induction hypothesis, $y, x'_{\lambda,w} \in \mathfrak{u}_\xi^-$. By 2.2 (ii), then $x'_{\lambda,w} = yz$ for some $z \in \mathfrak{u}_\xi^-$. Note that

$$x'_\lambda = F_1^{(\lambda_n)} F_2^{(\lambda_{n-1,n})} \cdots F_{n-1}^{(\lambda_{2,n})} y F_n^{(\lambda_{1,n})} z \quad \text{and} \quad F_1^{(\lambda_n)} F_2^{(\lambda_{n-1,n})} \cdots F_{n-1}^{(\lambda_{2,n})} y = x'_{v,w},$$

where $v := (\lambda_2, \ldots, \lambda_n, \lambda_1)$. According to induction hypothesis, $x'_{v,w} \in \mathfrak{u}_\xi^-$. Now by Lemma 4.12, $x'_\lambda = x'_{v,w} F_n^{(\lambda_{1,n})} z \in \bigcap_{i=1}^{n-1} \mathscr{H}_i$. Combine this and (a) we see $x'_\lambda \in \bigcap_{i=1}^n \mathscr{H}_i = \mathfrak{u}_\xi^-$.

§5. Main Results

5.1. In this section we give the main results of the paper. Essentially, they re-express some results in previous sections. Recall that in 1.4 we have defined the element $x_\lambda \in U_\xi^-$ for every λ in $\mathbf{Z}_+^n$.

Theorem 5.2. *Assume that* U_ξ *has no factors of type* G_2. *If* λ *is* l*-restricted, then* x_λ *and* x'_λ *are elements in* $\mathfrak{u}_\xi^-$.

Proof. By Theorem 4.2 (ii), x'_λ is an element in $\mathfrak{u}_\xi^-$. We have $x_\lambda = x'_{-w_0\lambda}$. Note that $-w_0\lambda$ is also l-restricted, by Theorem 4.2 (ii), $x_\lambda \in \mathfrak{u}_\xi^-$.

Theorem 5.3. *Assume that* U_ξ *has no factors of type* G_2. *Let* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{Z}_+^n$ *be* l*-restricted and let* $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \{\pm 1\}^n$. *Denote by* $\mathfrak{u}_\xi(\lambda, \sigma)$ *the left ideal of* $\mathfrak{u}_\xi$ *generated by the elements* $E_\alpha, K_i - \sigma_i \xi^{\lambda_i d_i}$ $(\alpha \in R_l^+, i = 1, 2, \ldots, n)$ *and elements* $F \in \mathfrak{u}_\xi^-$ *such that* $F x_{\kappa - \lambda} = 0$. *Then*

(i) $\mathbf{u}_\xi/\mathbf{u}_\xi(\lambda, \sigma)$ *is an irreducible* $\mathbf{u}_\xi$*-module. Moreover, as* $\mathbf{u}_\xi$*-modules,* $L_\xi(\lambda, \sigma)$ *is isomorphic to* $\mathbf{u}_\xi/\mathbf{u}_\xi(\lambda, \sigma)$.

(ii) *For any* $\gamma \in \mathbf{N}R^+$, *denote* $\mathbf{u}_{\xi,\gamma}^-$ *the set of all elements in* $\mathbf{u}_\xi^-$ *of degree* γ, *and set* $\mathbf{n}_\xi(\lambda, \gamma) := \{F \in \mathbf{u}_{\xi,\gamma}^- \mid F x_{\kappa-\lambda} = 0\}$. *Then*

$$\dim_{\mathbf{Q}(\xi)} L_\xi(\lambda, \sigma)_{\lambda-\gamma,\sigma} = \dim_{\mathbf{Q}(\xi)} \mathbf{u}_{\xi,\gamma}^- - \dim_{\mathbf{Q}(\xi)} \mathbf{n}_\xi(\lambda, \gamma).$$

In particular, we have

$$\dim_{\mathbf{Q}(\xi)} L_\xi(\lambda, \sigma) = \prod_{\alpha \in R^+} l_\alpha - \dim_{\mathbf{Q}(\xi)} \{F \in \mathbf{u}_\xi^- \mid F x_{\kappa-\lambda} = 0\}.$$

Proof. (i) Let $\tilde{J}_{\lambda,\sigma}$ be the left ideal of $\tilde{\mathbf{u}}_\xi$ generated by the elements E_α, $K_i - \sigma_i \xi^{\lambda_i d_i}$, $\begin{bmatrix} K_i, c \\ a \end{bmatrix} - \sigma_i^a \begin{bmatrix} \lambda_i + c \\ a \end{bmatrix}_{d_i,\xi}$ $(\alpha \in R_l^+,\ i = 1, 2, \ldots, n,\ c \in \mathbf{Z},\ a \in \mathbf{N})$ and elements $F \in \mathbf{u}_\xi^-$ such that $F x_{\kappa-\lambda} = 0$. Since $x_{\kappa-\lambda} = x'_{\kappa+w_0\lambda}$, by Theorem 4.2 (ii), Corollary 2.7 (ii) and Prop. 2.6 (ii) we see that $\tilde{\mathbf{u}}_\xi/\tilde{J}_{\lambda,\sigma} \simeq \tilde{L}_\xi(\lambda, \sigma)$. But λ is $\mathbf{l}$-restricted, so the restriction to $\mathbf{u}_\xi$ of $\tilde{L}_\xi(\lambda, \sigma)$ is an irreducible $\mathbf{u}_\xi$-module. Obviously, the restriction is isomorphic to $\mathbf{u}_\xi/\mathbf{u}_\xi(\lambda, \sigma)$. Since $\tilde{L}_\xi(\lambda, \sigma)$ is the restriction to $\tilde{\mathbf{u}}_\xi$ of the irreducible U_ξ-module $L_\xi(\lambda, \sigma)$, so as $\mathbf{u}_\xi$-modules, $L_\xi(\lambda, \sigma)$ is isomorphic to $\mathbf{u}_\xi/\mathbf{u}_\xi(\lambda, \sigma)$.

Part (ii) is an immediate consequence of part (i).

The theorem is proved.

Theorem 5.4. *Assume that* U_ξ *has no factors of type* G_2. *Let* $\lambda, \mu \in \mathbf{Z}_+^n$ *and let* $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \{\pm 1\}^n$. *Assume that* λ *is* $\mathbf{l}$*-restricted. Denote by* $J_{\mathbf{l}\mu+\lambda,\sigma}$ *the left ideal of* U_ξ *generated by all elements in* $I_{\mathbf{l}\mu+\lambda,\sigma}$ *(see* 1.2 *for the definition) and elements* $F \in \mathbf{u}_\xi^-$ *such that* $F x_{\kappa-\lambda} = 0$. *Then* $U_\xi/J_{\mathbf{l}\mu+\lambda,\sigma} \simeq L_\xi(\mathbf{l}\mu + \lambda, \sigma)$.

Proof. Since $L_\xi(\mathbf{l}\mu + \lambda, \sigma)$ is a quotient module of $V_\xi(\mathbf{l}\mu + \lambda, \sigma)$, we have $I_{\mathbf{l}\mu+\lambda,\sigma}\bar{v}_{\mathbf{l}\mu+\lambda,\sigma} = 0$. Noting that $L_\xi(\mathbf{l}\mu + \lambda, \sigma) \simeq L_\xi(\mathbf{l}\mu, \mathbf{1}) \otimes L_\xi(\lambda, \sigma))$ (see 2.2 (iv)), by 2.2 (v) and Theorem 5.3 (i) we know that $F\bar{v}_{\mathbf{l}\mu+\lambda,\sigma} = 0$ if $F \in \mathbf{u}_\xi^-$ and $F x_{\kappa-\lambda} = 0$. Therefore we have $J_{\mathbf{l}\mu+\lambda,\sigma}\bar{v}_{\mathbf{l}\mu+\lambda,\sigma} = 0$. Note that

(a) $L_\xi(\mathbf{l}\mu, \mathbf{1}) \otimes L_\xi(\kappa, \sigma) \simeq L_\xi(\mathbf{l}\mu, \mathbf{1}) \otimes V_\xi(\kappa, \sigma) \simeq V_\xi(\mathbf{l}\mu + \kappa, \sigma)$.

Let $z_1, z_2, \ldots, z_k, \ldots,$ be a $\mathbf{Q}(\xi)$-basis of $U_{\xi,l}^-$ such that

(b) The elements $z_b\bar{v}_{\mathbf{l}\mu}$ $(b = 1, 2, \ldots, k)$ form a $\mathbf{Q}(\xi)$-basis of the irreducible module $L_\xi(\mathbf{l}\mu)$, and $z_b\bar{v}_{\mathbf{l}\mu} = 0$ for $b = k + 1, k + 2 \ldots$

Let $\bar{I}$ be the $\mathbf{Q}(\xi)$-space spanned by the elements $z_h F$ $(1 \le h \le k,\ F \in \mathbf{u}_\xi^-)$. According to (a) and (b) we have $\bar{I} + I_{\mathbf{l}\mu+\kappa,\sigma}^- = U_\xi^-$. Since $I_{\mathbf{l}\mu+\kappa,\sigma}^- \subseteq I_{\mathbf{l}\mu+\lambda,\sigma}$, as $\mathbf{Q}(\xi)$-spaces we have

$$U_\xi/J_{\mathbf{l}\mu+\lambda,\sigma} \simeq U_\xi^-/U_\xi^- \cap J_{\mathbf{l}\mu+\lambda,\sigma} \simeq \bar{I}/\bar{I} \cap J_{\mathbf{l}\mu+\lambda,\sigma}.$$

By Theorem 5.3, $\dim_{\mathbf{Q}(\xi)} \bar{I} \cap J_{\mathbf{l}\mu+\lambda,\sigma} \ge k(\dim_{\mathbf{Q}(\xi)} \mathbf{u}_\xi^- - \dim_{\mathbf{Q}(\xi)} L_\xi(\lambda, \sigma))$. Since

$\dim_{\mathbf{Q}(\xi)} \bar{I} = k \dim_{\mathbf{Q}(\xi)} \mathbf{u}_\xi^-$, we have

$$\dim_{\mathbf{Q}(\xi)} U_\xi / J_{\mathbf{l}\mu+\lambda,\sigma} \leq k \dim_{\mathbf{Q}(\xi)} L_\xi(\lambda, \sigma) = \dim_{\mathbf{Q}(\xi)} L_\xi(\mathbf{l}\mu + \lambda, \sigma).$$

This forces that $U_\xi / J_{\mathbf{l}\mu+\lambda,\sigma}$ and $L_\xi(\mathbf{l}\mu + \lambda, \sigma)$ have the same dimension and as U_ξ-modules, they are isomorphic.

The theorem is proved

From the above proof we get the following result.

Corollary 5.5. *Keep the setup in Theorem* 5.4. *Then the left ideal* $J_{\mathbf{l}\mu+\lambda,\sigma} \cap U_\xi^-$ *of* U_ξ^- *is generated by the elements* $F_i^{(l_i\mu_i+l_i)}$ $(i = 1, 2, \dots, n)$ *and elements* $F \in \mathbf{u}_\xi^-$ *such that* $F x_{\kappa-\lambda} = 0$.

§6. Hyperalgebra

6.1. In this section we consider the hyperalgebra $\mathfrak{U}_\mathfrak{k}$ along the same line in the previous sections, the discussion will be brief. We often omit those proofs which are essentially the same as in the previous sections.

Recall that $\mathfrak{g}$ is a semisimple Lie algebra over $\mathbf{C}$ and $\mathfrak{U}$ is the universal enveloping algebra of $\mathfrak{g}$. Let e_α, f_α, h_i $(\alpha \in R^+, i = 1, 2, \dots, n)$ be a Chevalley basis of $\mathfrak{g}$. We also write e_i, f_i for $e_{\alpha_i}, f_{\alpha_i}$ $(i = 1, 2, \dots, n)$. The Kostant $\mathbb{Z}$-form $\mathfrak{U}_\mathbf{Z}$ of $\mathfrak{U}$ is the $\mathbf{Z}$-subalgebra of $\mathfrak{U}$ generated by the elements $e_\alpha^{(k)} := e_\alpha^k / k!$, $f_\alpha^{(k)} := f_\alpha^k / k!$ for $\alpha \in R^+$ and $k \in \mathbf{N}$. Set

$$\binom{h_i + c}{k} := \frac{(h_i + c)(h_i + c - 1) \cdots (h_i + c - k + 1)}{k!},$$

then $\binom{h_i+c}{k} \in \mathfrak{U}_\mathbf{Z}$, for $i = 1, 2, \dots, n$, $c \in \mathbf{Z}$, $k \in \mathbf{N}$. Let $\mathfrak{k}$ be an algebraically closed field of prime characteristic p. Define $\mathfrak{U}_\mathfrak{k} := \mathfrak{U}_\mathbf{Z} \otimes \mathfrak{k}$ and call $\mathfrak{U}_\mathfrak{k}$ the hyperalgebra associated to $\mathfrak{g}$ and $\mathfrak{k}$. Let $\mathfrak{U}_\mathfrak{k}^+, \mathfrak{U}_\mathfrak{k}^-, \mathfrak{U}_\mathfrak{k}^0$ be the positive part, negative part, zero part of $\mathfrak{U}_\mathfrak{k}$ respectively. To simplify notation, the images in $\mathfrak{U}_\mathfrak{k}$ of $e_\alpha^{(k)}, f_\alpha^{(k)}, \binom{h_i+c}{k}$, etc. will be denoted by the same notations respectively. The algebra $\mathfrak{U}_\mathfrak{k}$ is a Hopf algebra, the coproduct, denoted also by Δ, is defined as follows:

$$\Delta(e_\alpha^{(k)}) = \sum_{q=0}^{k} e_\alpha^{(q)} \otimes e_\alpha^{(k-q)}, \quad \Delta(f_\alpha^{(k)}) = \sum_{q=0}^{k} f_\alpha^{(q)} \otimes f_\alpha^{(k-q)}.$$

The tensor product of two $\mathfrak{U}_\mathfrak{k}$-modules then has a natural $\mathfrak{U}_\mathfrak{k}$-module structure by means of the coproduct, and the antipode can be used to define the dual module of a $\mathfrak{U}_\mathfrak{k}$-module.

Given a positive integer a, let $\mathfrak{u}_a$ be the a-th Frobenius kernel of $\mathfrak{U}_\mathfrak{k}$. By definition, $\mathfrak{u}_a$ is the subalgebra of $\mathfrak{U}_\mathfrak{k}$ generated by the elements $e_\alpha^{(k)}, f_\alpha^{(k)}, \binom{h_i}{k}$ for $\alpha \in R^+$, $i = 1, 2, \dots, n$, $0 \leq k < p^a$. Denote by $\mathfrak{u}_a^+, \mathfrak{u}_a^-, \mathfrak{u}_a^0$ the positive part, negative part, zero part of $\mathfrak{u}_a$ respectively. Let $\tilde{\mathfrak{u}}_a$ be the subalgebra of $\mathfrak{U}_{\mathfrak{k}}$ generated by all elements in $\mathfrak{u}_a \cup \mathfrak{U}_{\mathfrak{k}}^0$, then $\tilde{\mathfrak{u}}_a = \mathfrak{u}_a^- \mathfrak{U}_{\mathfrak{k}}^0 \mathfrak{u}_a^+$. Let $\mathfrak{U}_{\mathfrak{k},a}$ be the subalgebra of $\mathfrak{U}_{\mathfrak{k}}$ generated by the elements 1, $e_\alpha^{(p^b)}$, $f_\alpha^{(p^b)}$, $\binom{h_i}{p^b}$ for $\alpha \in R^+$, $i = 1, 2, \ldots, n$, $b \geq a$. Let $\mathfrak{U}_{\mathfrak{k},a}^+$, $\mathfrak{U}_{\mathfrak{k},a}^-$, $\mathfrak{U}_{\mathfrak{k},a}^0$ be the positive part, negative part, zero part of $\mathfrak{U}_{\mathfrak{k},a}$ respectively. The following results are easy to check.

(i) Let $g \in \mathfrak{u}_a$. We have $e_i^{(p^a)} g - g e_i^{(p^a)} \in \mathfrak{u}_a$ and $f_i^{(p^a)} g - g f_i^{(p^a)} \in \mathfrak{u}_a$.
(ii) Let $\{g_k\}$ be a basis of $\mathfrak{u}_a^-$ and $\{G_q\}$ be a basis of $\mathfrak{U}_{\mathfrak{k},a}^-$, then $\{g_k G_q\}$ is a basis of $\mathfrak{U}_{\mathfrak{k}}^-$, so is $\{G_q g_k\}$.
(iii) There exists a unique surjective $\mathfrak{k}$-algebra homomorphism $\mathfrak{U}_{\mathfrak{k}} \to \mathfrak{U}_{\mathfrak{k}}$ such that $e_\alpha^{(kp^a)} \to e_\alpha^{(k)}$, $f_\alpha^{(kp^a)} \to f_\alpha^{(k)}$, $\binom{h_i}{kp^a} \to \binom{h_i}{k}$ for $\alpha \in R^+$, $i = 1, 2, \ldots, n$, $k \in \mathbf{N}$, and such that $e_\alpha^{(k)} \to 0$, $f_\alpha^{(k)} \to 0$, $\binom{h_i}{k} \to 0$ if k is not divisible by p^a. In particular $\mathfrak{U}_{\mathfrak{k},a}$ is isomorphic to $\mathfrak{U}_{\mathfrak{k}}$.

Proof. The $\mathfrak{k}$-algebra homomorphism is obtained from the a-th Frobenius map of the simply connected, semisimple algebraic group (associated to $\mathfrak{g}$) over $\mathfrak{k}$. One also can see (iii) by using the commutation relations among the generators of $\mathfrak{U}_{\mathfrak{k}}$.

We order R^+ so that $R^+ = \{\beta_1, \beta_2, \ldots, \beta_r\}$ where $\beta_i \leq \beta_j$ implies that $i \geq j$. For $a \in \mathbf{N}$, set $f_{(p^a-1)\rho} = f_{\beta_1}^{(p^a-1)} f_{\beta_2}^{(p^a-1)} \cdots f_{\beta_r}^{(p^a-1)}$. For $b \geq a$ we set $f'_{p^a(p^{b-a}-1)\rho} = f_{\beta_r}^{(p^b-p^a)} f_{\beta_{r-1}}^{(p^b-p^a)} \cdots f_{\beta_1}^{(p^b-p^a)}$. Since in $\mathfrak{k}$ we have $\binom{p^b-1}{p^a-1} = 1$ and $f'_{p^a(p^{b-a}-1)\rho} f_{(p^a-1)\rho} \in \mathfrak{u}_b^-$, we get
(iv) Let $a, b \in \mathbf{N}$ with $b \geq a$. Then

$$f_{(p^b-1)\rho} = f'_{p^a(p^{b-a}-1)\rho} f_{(p^a-1)\rho}.$$

Using commutation relations among $f_\alpha^{(k)}$ ($\alpha \in R^+$, $k \in \mathbf{N}$) and using induction on $j \in [1, r]$ we get
(v) Let $a, k \in \mathbf{N}$ with $k \geq p^a$ and let $j \in [1, r]$. Then for each i in $[1, n]$ we have

$$f_i^{(k)} f_{\beta_1}^{(p^a-1)} \cdots f_{\beta_j}^{(p^a-1)} = f_{\beta_1}^{(p^a-1)} \cdots f_{\beta_j}^{(p^a-1)} f_i^{(k)}.$$

In particular,

$$f_i^{(k)} f_{(p^a-1)\rho} = f_{(p^a-1)\rho} f_i^{(k)}.$$

6.2. Let $\mathfrak{U}'_{\mathfrak{k}}$ be a subalgebra of $\mathfrak{U}_{\mathfrak{k}}$ containing $\mathfrak{U}_{\mathfrak{k}}^0$ and let M be a $\mathfrak{U}'_{\mathfrak{k}}$-module. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{Z}^n$. An element $m \in M$ is called to have weight λ if $\binom{h_i}{k} m = \binom{\lambda_i}{k} m$ for $i = 1, 2, \ldots, n$, $k \in \mathbf{N}$. Denote by M_λ the set of all elements in M of weight λ. We call λ a weight of M if M_λ is not zero. An element $g \in \mathfrak{U}_{\mathfrak{k}}$ is said to have degree $\gamma \in \mathbf{Z}R$ if $\binom{h_i}{k} g = g \binom{h_i - \langle \gamma, \alpha_i^\vee \rangle}{k}$ for $i = 1, 2, \ldots, n$, $k \geq 0$. If an element g in $\mathfrak{U}'_{\mathfrak{k}}$ has degree γ, then obviously $g M_\lambda \subseteq M_{\lambda-\gamma}$. We list some well known properties and supply proofs for a few of them. The letters a, b will stand for positive integers.

(i) If M is a finite dimensional $\mathfrak{U}_\mathfrak{k}$-module, then $\dim M_\lambda = \dim M_{w\lambda}$ for all $\lambda \in \mathbf{Z}^n$, $w \in W$.

Given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{Z}_+^n$, let $\mathfrak{J}_\lambda^+$ be the left ideal of $\mathfrak{U}_\mathfrak{k}$ generated by the elements $e_i^{(k)}$, $\binom{h_i}{k} - \binom{\lambda_i}{k}$ for $i = 1, 2, \ldots, n$, $k \geq 1$, and let $\mathfrak{J}_\lambda^-$ be the left ideal of $\mathfrak{U}_\mathfrak{k}^-$ generated by the elements $f_i^{(k_i)}$ for $i = 1, 2, \ldots, n$, $k_i \geq \lambda_i + 1$. Then let $\mathfrak{J}_\lambda$ be the left ideal of $\mathfrak{U}_\mathfrak{k}$ generated by all elements in $\mathfrak{J}_\lambda^+ \cup \mathfrak{J}_\lambda^-$. Then

(ii) The $\mathfrak{U}_\mathfrak{k}$-module $V_\mathfrak{k}(\lambda) := \mathfrak{U}_\mathfrak{k}/\mathfrak{J}_\lambda$ is of finite dimension and has a unique irreducible quotient module, denoted by $L_\mathfrak{k}(\lambda)$. The dimension of $V_\mathfrak{k}(\lambda)$ is given by Weyl's character formula. By abuse of notations, we also denote v_λ the image in $V_\mathfrak{k}(\lambda)$ of the neutral element $1 \in \mathfrak{U}_\mathfrak{k}$, and denote $\bar{v}_\lambda$ the image in $L_\mathfrak{k}(\lambda)$ of v_λ. The map $\lambda \to L_\mathfrak{k}(\lambda)$ defines a bijection between the set $\mathbf{Z}_+^n$ and the set of isomorphism classes of irreducible $\mathfrak{U}_\mathfrak{k}$-modules of finite dimensions.

(iii) When λ is p^a-restricted (i.e. $0 \leq \langle \lambda, \alpha_i^\vee \rangle < p^a$ for $i = 1, 2, \ldots, n$), the restriction to $\mathfrak{u}_a$ (resp. $\tilde{\mathfrak{u}}_a$) of $L_\mathfrak{k}(\lambda)$ is an irreducible $\mathfrak{u}_a$-module (resp. $\tilde{\mathfrak{u}}_a$-module), denote the restriction by $L_{\mathfrak{k},a}(\lambda)$ (resp. $\tilde{L}_{\mathfrak{k},a}(\lambda)$).

(iv) Assume $\lambda \in p^a \mathbf{Z}_+^n$, then $e_\alpha^{(k)} m = f_\alpha^{(k)} m = 0$ for all $m \in L_\mathfrak{k}(\lambda)$, $\alpha \in R^+$, $1 \leq k < p^a$.

Proof. Use 6.1 (iii) and 6.2 (ii).

(v) Assume that $\delta_0, \delta_1, \ldots, \delta_b \in \mathbf{Z}_+^n$ are p-restricted and set $\lambda := p^b\delta_b + \cdots + p\delta_1 + \delta_0$. Then (Steinberg's tensor product theorem)

$$L_\mathfrak{k}(\lambda) \simeq L_\mathfrak{k}(p^b\delta_b) \otimes \cdots \otimes L_\mathfrak{k}(p\delta_1) \otimes L_\mathfrak{k}(\delta_0).$$

Proof. Use (iv) and the trick in the proof of [L2, Theorem 7.4, p. 73].

Let M be a $\mathfrak{U}_\mathfrak{k}$-module (resp. $\tilde{\mathfrak{u}}_a$-module). A nonzero element m in M is called primitive if $m \in M_\lambda$ for some $\lambda \in \mathbf{Z}^n$ and $e_i^{(k)} m = 0$ for $i = 1, 2, \ldots, n$, $k \geq 1$ (resp. $e_\alpha^{(k)} m = 0$ for all $\alpha \in R^+$, $1 \leq k \leq p^a - 1$).

(vi) Let M be a finite dimensional $\mathfrak{U}_\mathfrak{k}$-module. Assume that $m \in M$ is a primitive element of weight λ. Then $\lambda \in \mathbf{Z}_+^n$ and there is a unique $\mathfrak{U}_\mathfrak{k}$-module homomorphism $V_\mathfrak{k}(\lambda) \to M$ which carries v_λ to m.

Proof. By (i) we see $s_i\lambda \leq \lambda$ for $i = 1, 2, \ldots, n$, that is $\lambda \in \mathbf{Z}_+^n$. Assume that $f_i^{(k)} m$ is not zero, again by (i) we see $s_i(\lambda - k\alpha_i) \leq \lambda$, i.e., $k \leq \langle \lambda, \alpha_i^\vee \rangle$. According to the definition of $V_\mathfrak{k}(\lambda)$ we know that the required $\mathfrak{U}_\mathfrak{k}$-homomorphism exists.

Given $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbf{Z}^n$, let $\tilde{\mathfrak{J}}_{\mu,a}^+$ be the left ideal of $\tilde{\mathfrak{u}}_a$ generated by the elements $e_\alpha^{(k)}$, $\binom{h_i}{k'} - \binom{\lambda_i}{k'}$ for $\alpha \in R^+$, $i = 1, 2, \ldots, n$, $1 \leq k < p^a$, $k' \in \mathbf{N}$. Denote by $\tilde{Z}_{\mathfrak{k},a}(\mu)$ the Verma module $\tilde{\mathfrak{u}}_a/\tilde{\mathfrak{J}}_{\mu,a}^+$ of $\tilde{\mathfrak{u}}_a$ with highest weight μ. We shall denote $\tilde{1}_{\mu,a}$ the image in $\tilde{Z}_{\mathfrak{k},a}(\mu)$ of the neutral element $1 \in \tilde{\mathfrak{u}}_a$. We have

(vii) Each Verma module of $\tilde{\mathfrak{u}}_a$ has a unique irreducible submodule. Assume that λ is p^a-restricted. Then the irreducible $\tilde{\mathfrak{u}}_a$-submodule of $\tilde{Z}_{\mathfrak{k},a}((p^a - 1)\rho + \lambda)$ (resp. $\tilde{Z}_{\mathfrak{k},a}(2(p^a - 1)\rho + w_0\lambda)$) is isomorphic to $\tilde{L}_{\mathfrak{k},a}((p^a - 1)\rho + w_0\lambda)$ (resp. $\tilde{L}_{\mathfrak{k},a}(\lambda)$) and is generated by $f_{(p^a-1)\rho}\tilde{1}_{(p^a-1)\rho+\lambda,a}$ (resp. $f_{(p^a-1)\rho}\tilde{1}_{2(p^a-1)\rho+w_0\lambda,a}$), where $\rho = (1, \ldots, 1) \in \mathbf{Z}_+^n$. In particular, $\tilde{Z}_{\mathfrak{k},a}((p^a - 1)\rho)$ is isomorphic to

$\tilde{L}_{\mathfrak{t},a}((p^q-1)\rho)$. [J4, 6.2 (1) p. 190].

One also can see (vii) as the same way of proving Prop. 2.6.

(viii) The module $V_{\mathfrak{t}}((p^a-1)\rho)$ is irreducible. And as a $\tilde{\mathfrak{u}}_a$-module, $V_{\mathfrak{t}}((p^a-1)\rho)$ is isomorphic to $\tilde{Z}_{\mathfrak{t},a}((p^a-1)\rho)$.

Proof. Use (iii) and (vii).

By (vii) we get

(ix) Assume that $\lambda \in \mathbf{Z}_+^n$ is p^a-restricted. Then there exists a nozero element $\mathfrak{y}'_\lambda$ in $\mathfrak{u}_a^-$ (unique up to a scalar) of degree $\lambda - w_0\lambda$ such that $\mathfrak{y}'_\lambda \tilde{1}_{\mu,a}$ is primitive in $\tilde{Z}_{\mathfrak{t},a}((p^a-1)\rho+\lambda)$, where $\mu=(p^a-1)\rho+\lambda$. Necessarily, $\mathfrak{y}'_\lambda \tilde{1}_{\mu,a}$ generates the unique irreducible submodule of $\tilde{Z}_{\mathfrak{t},a}((p^a-1)\rho+\lambda)$.

Using 6.1 (v), in the same way as the proof of Theorem 3.7 (ii) we get

(x) Assume that λ is p^a-restricted. Then $\mathfrak{y}'_\lambda v_\mu$ is primitive in $V_{\mathfrak{t}}(\mu)$, where $\mu=(p^a-1)\rho+\lambda$ and $\mathfrak{y}'_\lambda v_\mu$ generates an irreducible submodule of $V_{\mathfrak{t}}(\mu)$, which is isomorphic to $L_{\mathfrak{t}}((p^a-1)\rho+w_0\lambda)$. (Cf. [J4, Section 6.3, p. 191].)

Assume that $\lambda \in \mathbf{Z}_+^n$ is p^a-restricted and $b \geq a$. Let $M := L_{\mathfrak{t}}(p^a(p^{b-a}-1)\rho) \otimes V_{\mathfrak{t}}((p^a-1)\rho+\lambda)$ and $m := \bar{v}_{p^a(p^{b-a}-1)\rho} \otimes v_{(p^a-1)\rho+\lambda}$.

(xi) Keep the notation above. Regarding M as a $\tilde{\mathfrak{u}}_b$-module, we have a unique $\tilde{\mathfrak{u}}_b$-homomorphism $\tilde{Z}_{\mathfrak{t},b}((p^b-1)\rho+\lambda) \to M$ which carries $\tilde{1}_{(p^b-1)\rho+\lambda,b}$ to m. We claim the homomorphism is injective.

Proof. By (viii) and 6.1 (iii), the elements $f_{\beta_r}^{(k_r p^a)} f_{\beta_{r-1}}^{(k_{r-1} p^a)} \cdots f_{\beta_1}^{(k_1 p^a)} \bar{v}_{p^a(p^{b-a}-1)\rho}$ $(0 \leq k_r, \ldots, k_1 \leq p^{b-a}-1)$ form a basis of $L_{\mathfrak{t}}(p^a(p^{b-a}-1)\rho)$. Since $\mathfrak{I}^-_{(p^a-1)\rho+\lambda} \subseteq \mathfrak{I}^-_{(p^a-1)\rho}$, by (vii) and (viii), the elements $f_{\beta_1}^{(k_1)} f_{\beta_2}^{(k_2)} \cdots f_{\beta_r}^{(k_r)} v_{(p^a-1)\rho+\lambda}$ $(0 \leq k_1, \ldots, k_r \leq p^a-1)$ are linearly independent in $V_{\mathfrak{t}}((p^a-1)\rho+\lambda)$. Combining these and using (iv) and 6.1 (iv), we see

$$f_{(p^b-1)\rho}\, m = f'_{p^a(p^{b-a}-1)\rho}(\bar{v}_{p^a(p^{b-a}-1)\rho} \otimes f_{(p^a-1)\rho}\, v_{(p^a-1)\rho+\lambda}) \neq 0.$$

By (vii) we see the homomorphism is injective.

Since $\mathfrak{y}'_\lambda m = \bar{v}_{p^a(p^{b-a}-1)\rho} \otimes \mathfrak{y}'_\lambda v_{(p^a-1)\rho+\lambda}$ is primitive in M (see (x)) and $\mathfrak{y}'_\lambda \in \mathfrak{u}_a^- \subseteq \mathfrak{u}_b^-$, using (xi) we see $\mathfrak{y}'_\lambda \tilde{1}_{(p^b-1)\rho+\lambda,b}$ is primitive in $\tilde{Z}_{\mathfrak{t},b}((p^b-1)\rho+\lambda)$. Applying (ix) we get

(xii) Assume that $\lambda \in \mathbf{Z}_+^n$ is p^a-restricted and $b \geq a$. Let $\mathfrak{y}'_\lambda$ be as in (ix). Then $\mathfrak{y}'_\lambda \tilde{1}_{(p^b-1)\rho+\lambda,b}$ is primitive in $\tilde{Z}_{\mathfrak{t},b}((p^b-1)\rho+\lambda)$ and generates the unique irreducible submodule of $\tilde{Z}_{\mathfrak{t},b}((p^b-1)\rho+\lambda)$ which is isomorphic to $\tilde{L}_{\mathfrak{t},b}((p^b-1)\rho+w_0\lambda)$.

Applying (x) we get

(xiii) Assume that $\lambda \in \mathbf{Z}_+^n$ is p^a-restricted and $b \geq a$. Let $\mathfrak{y}'_\lambda$ be as in (ix). Then $\mathfrak{y}'_\lambda v_{(p^b-1)\rho+\lambda}$ is primitive in $V_{\mathfrak{t}}((p^b-1)\rho+\lambda)$ and generates an irreducible submodule of $V_{\mathfrak{t}}((p^b-1)\rho+\lambda)$ with highest weight $(p^b-1)\rho+w_0\lambda$.

Remark. Let λ be p^a-restricted. According to 3.3 (viii), $M' := V_{\mathfrak{t}}((p^a-1)\rho) \otimes V_{\mathfrak{t}}(\lambda)$ has a filtration of Weyl modules. So $V_{\mathfrak{t}}((p^a-1)\rho+\lambda)$ is isomorphic to a submodule of M' (cf. Lemma 3.6 (i)). It is well known that as a $\tilde{\mathfrak{u}}_a$-module, M' is projective and injective. Therefore $V_{\mathfrak{k}}((p^a-1)\rho+\lambda)$ has a unique irreducible $\mathfrak{U}_{\mathfrak{k}}$-module if as a $\tilde{\mathfrak{u}}_a$-module $V_{\mathfrak{k}}((p^a-1)\rho+\lambda)$ is indecomposable.

6.3. Given $\lambda\in\mathbf{Z}_+^n$, $w\in W$, define the monomials $\mathfrak{r}_{\lambda,w}$, $\mathfrak{r}'_{\lambda,w}$, $\mathfrak{r}_\lambda$, $\mathfrak{r}'_\lambda$ of $f_i^{(k)}$ ($i=1,2,\dots,n$, $k\geq 0$) in the same way as 1.4. Depending on the contexts, the monomials will be regarded as elements in $\mathfrak{U}_{\mathfrak{k}}$ or elements in $\mathfrak{U}$. We state the analogues of a few results in Section 4 and Section 5. The letters a, b will stand for positive integers.

Lemma 6.4. *Assume that $\lambda\in\mathbf{Z}_+^n$ is p^a-restricted and $b>6a|R^+|$. Let $w\in W$ and $\mu=(p^b-1)\rho+\lambda$. Then*
(i) *In $V_{\mathfrak{k}}(\mu)$ we have $\mathfrak{r}'_{\lambda,w}v_\mu\neq 0$.*
(ii) *If $k\geq 1$, then in $V_{\mathfrak{k}}(\mu)$ we have $e_i^{(k)}\mathfrak{r}'_{\lambda,w}u_\mu=0$ for $i=1,2,\dots,n$. That is, $\mathfrak{r}'_{\lambda,w}v_\mu$ is primitive in $V_{\mathfrak{k}}(\mu)$. In particular we have*
(iii) *The element $\mathfrak{r}'_\lambda v_\mu$ is primitive in $V_{\mathfrak{k}}(\mu)$.*

Proof. Part (i) is obvious (cf. Lemma 4.4 (i) and its proof). Now we prove (ii). Set $\lambda_i:=\langle\lambda,\alpha_i^\vee\rangle$ for $i=1,2,\dots,n$. Use induction on $l(w)$ we see that
(a) There exist $g_1,g_2,\dots,g_k$ in $\mathfrak{U}_{\mathfrak{k}}$ such that

$$e_i^{(k)}\mathfrak{r}'_{\lambda,w}=\mathfrak{r}'_{\lambda,w}e_i^{(k)}+g_1\begin{pmatrix}h_i+1-\lambda_i\\ 1\end{pmatrix}+g_2\begin{pmatrix}h_i+2-\lambda_i\\ 2\end{pmatrix}+\cdots+g_k\begin{pmatrix}h_i+k-\lambda_i\\ k\end{pmatrix}.$$

When $b>6a|R^+|$, we must have $\langle u\lambda,\alpha_j^\vee\rangle<p^b$ for all $u\in W$ and $j=1,2,\dots,n$. According to the definition of $\mathfrak{r}'_{\lambda,w}$, we may require that $g_{k'}=0$ when $k'\geq p^b$. Note that $g_{k'}\binom{h_i+k'-\lambda_i}{k'}v_\mu=0$ if $1\leq k'<p^b$. Now according to (a) we get (ii).

The lemma is proved.

Theorem 6.5. (i) *Assume that $\lambda\in\mathbf{Z}_+^n$ is p^a-restricted. Then $\mathfrak{r}_\lambda$ and $\mathfrak{r}'_\lambda$ are elements in $\mathfrak{u}_a^-$.*
(ii) *$\mathfrak{r}'_\lambda\tilde{1}_{(p^a-1)\rho+\lambda}$ is primitive in $\tilde{Z}_{\mathfrak{k},a}((p^a-1)\rho+\lambda)$ and generates the unique irreducible submodule of $\tilde{Z}_{\mathfrak{k},a}((p^a-1)\rho+\lambda)$, which is isomorphic to $\tilde{L}_{\mathfrak{k},a}((p^a-1)\rho+w_0\lambda)$.*

Proof. Let $b>6a|R^+|$. Since $\mathfrak{I}^-_{(p^b-1)\rho+\lambda}\subseteq\mathfrak{I}^-_{(p^b-1)\rho}$, we see the $\tilde{\mathfrak{u}}_b$-homomorphism $\tilde{Z}_{\mathfrak{k},b}((p^b-1)\rho+\lambda)\to V_{\mathfrak{k}}((p^b-1)\rho+\lambda)$, $\tilde{1}_{(p^b-1)\rho+\lambda,b}\to v_{(p^b-1)\rho+\lambda}$, is injective. By our choice of b we see $\mathfrak{r}'_\lambda\in\tilde{\mathfrak{u}}_b^-$. By Lemma 6.4 (iii) we see $\mathfrak{r}'_\lambda\tilde{1}_{(p^b-1)\rho+\lambda,b}$ is primitive in $\tilde{Z}_{\mathfrak{k},b}((p^b-1)\rho+\lambda)$. Since $\mathfrak{r}'_\lambda$ has degree $\lambda-w_0\lambda$, by 6.2 (xii), 6.2 (vii) and 6.2 (ix) (replacing a by b), we see $\mathfrak{r}'_\lambda=\theta\mathfrak{y}'_\lambda\in\tilde{\mathfrak{u}}_a^-$ for some nonzero $\theta\in\mathfrak{k}$. We have $\mathfrak{r}_\lambda=\mathfrak{r}'_{-w_0\lambda}$, since $-w_0\lambda$ is also p^a-restricted, so $\mathfrak{r}_\lambda\in\mathfrak{u}_a^-$. (i) is proved.

(ii) follows from 6.2 (xii) and the proof of (i).

6.6. *Remark.* We also can prove Theorem 6.5 (i) by using Theorem 5.2 provided that every simple component of $\mathfrak{g}$ is not of type G_2.

If p is odd, choose a p^a-th primitive root ξ of 1. If $p=2$, choose a 2^{a+1}-th primitive root ξ of 1. Let U'_ξ be the $\mathbf{Z}[\xi]$-subalgebra of U_ξ generated by the elements $E_i^{(k)}$, $F_i^{(k)}$, K_i, K_i^{-1} for $i=1,2,\dots,n$, $k\geq 0$. Consider the $\mathfrak{k}$-algebra $\mathscr{U}'_{\mathfrak{k}}:=U'_\xi\otimes_{\mathbf{Z}[\xi]}\mathfrak{k}$, where $\mathfrak{k}$ is regarded as a $\mathbf{Z}[\xi]$-algebra through the ring homomorphism $\mathbf{Z}[\xi]\to\mathfrak{k}$, $\xi\to 1$. For simplicity, the images in $\mathscr{U}'_{\mathfrak{k}}$ of $E_i^{(k)}$, $F_i^{(k)}$, K_i, K_i^{-1}, etc. will be denoted by the same notations respectively.

Let $\mathscr{K}'$ be the two-sided ideal of $\mathscr{U}'_{\mathfrak{k}}$ generated by $K_1-1,\dots,K_n-1$. Set $\mathscr{U}_{\mathfrak{k}}:=\mathscr{U}'_{\mathfrak{k}}/\mathscr{K}'$. Again for simplicity, the images in $\mathscr{U}_{\mathfrak{k}}$ of $E_i^{(k)}$, $F_i^{(k)}$, K_i, K_i^{-1}, etc. will be denoted by the same notations respectively. The following result is due to Lusztig [L3, 6.7 (d), p. 295] (cf. 1.6).

(i) There is a unique $\mathfrak{k}$-algebra isomorphism $\mathscr{U}_{\mathfrak{k}}\to\mathfrak{U}_{\mathfrak{k}}$ such that $E_i^{(k)}$ maps to $e_i^{(k)}$, $F_i^{(k)}$ maps to $f_i^{(k)}$, $\left[{K_i,0 \atop k}\right]$ maps to $\binom{h_i}{k}$, for $i=1,2,\dots,n$, $k\in\mathbf{N}$.

When $\mathfrak{g}$ is of type A_n, D_n, E_n; or B_n, C_n, F_4 and p is odd, Theorem 6.5 (i) is a simple consequence of (i) and Theorem 5.2. When $\mathfrak{g}$ is of type B_n, C_n, F_4 and $p=2$, one may prove Theorem 6.5 (i) by direct calculations.

Theorem 6.7. *Assume that $\lambda\in\mathbf{Z}_+^n$ is p^a-restricted.*

(i) *Let $\mathfrak{J}'_\lambda$ be the left ideal of $\mathfrak{U}_{\mathfrak{k}}$ generated by the elements $e_i^{(k)}$, $\binom{h_i}{k}-\binom{\langle\lambda,\alpha_i^\vee\rangle}{k}$, $f_i^{(k_i)}$ $(i=1,2,\dots,n,\ k\geq 1,\ k_i\geq p^a)$ and elements $f\in\mathfrak{u}_a^-$ such that $f\mathfrak{r}_{(p^a-1)\rho-\lambda}=0$, then $\mathfrak{U}_{\mathfrak{k}}/\mathfrak{J}'_\lambda\simeq L_{\mathfrak{k}}(\lambda)$.*

(ii) *Let $\mathfrak{u}_a(\lambda)$ be the left ideal of $\mathfrak{u}_a$ generated by the elements $e_\alpha^{(k)}$, $\binom{h_i}{k}-\binom{\langle\lambda,\alpha_i^\vee\rangle}{k}$ $(\alpha\in R^+,\ i=1,2,\dots,n,\ 1\leq k\leq p^a-1)$ and elements $f\in\mathfrak{u}_a^-$ such that $f\mathfrak{r}_{(p^a-1)\rho-\lambda}=0$, then $\mathfrak{u}_a/\mathfrak{u}_a(\lambda)\simeq L_{\mathfrak{k},a}(\lambda)$.*

(iii) *For any $\gamma\in\mathbf{N}R^+$, denote $\mathfrak{u}_{a,\gamma}^-$ the set of all elements in $\mathfrak{u}_a^-$ of degree γ and denote $\mathfrak{n}_a(\lambda,\gamma)$ the set $\{f\in\mathfrak{u}_{a,\gamma}^-\mid f\mathfrak{r}_{(p^a-1)\rho-\lambda}=0\}$, then*

$$\dim L_{\mathfrak{k}}(\lambda)_{\lambda-\gamma}=\dim\mathfrak{u}_{a,\gamma}^--\dim\mathfrak{n}_a(\lambda,\gamma).$$

In particular, we have

$$\dim L_{\mathfrak{k}}(\lambda)=p^{a|R^+|}-\dim\{f\in\mathfrak{u}_a^-\mid f\mathfrak{r}_{(p^a-1)\rho-\lambda}=0\}.$$

Proof. Since $\mathfrak{r}_{(p^a-1)\rho-\lambda}=\mathfrak{r}'_{(p^a-1)\rho+w_0\lambda}$, (ii) follows from Theorem 6.5 (ii) and 6.2 (iii). (i) and (iii) follow from (ii).

§7. Questions

7.1. Recall that ξ is a root of 1 of order $l\geq 3$. For $i\in[1,n]$, $k\in\mathbf{N}$, denote $\Theta_{i,k}$ the $\mathbf{Q}(\xi)$-linear homomorphism $U_\xi\to U_\xi$, $x\to xF_i^{(k)}$. The kernel and the image of $\Theta_{i,k}$ are easily described by means of PBW Theorem. Assume that $\lambda\in\mathbf{Z}_+^n$ is $\mathfrak{l}$-restricted. Let $s_{i_1}s_{i_2}\cdots s_{i_r}$ be a reduced expression of the longest element of W. Set $k_h:=\langle s_{i_{h-1}}\cdots s_{i_1}(\kappa-\lambda),\alpha_{i_h}^\vee\rangle$, $\delta_h:=k_1\alpha_{i_1}+\cdots+k_h\alpha_{i_h}$,

$h = 1, \dots, r$. Recall that for any $\gamma \in \mathbf{N}R^+$ we denote $\mathbf{u}^-_{\xi,\gamma}$ the set of all elements in $\mathbf{u}^-_\xi$ of degree γ. Given $\beta \in \mathbf{N}R^+$, set

$$D_{0,\beta} = \dim_{\mathbf{Q}(\xi)} \mathbf{u}^-_{\xi,\beta},$$
$$D_{1,\beta} = \dim_{\mathbf{Q}(\xi)} \Theta_{i_1,k_1}(\mathbf{u}^-_{\xi,\beta}),$$
$$D_{2,\beta} = \min\{D_{1,\beta}, \dim_{\mathbf{Q}(\xi)} \Theta_{i_2,k_2}(\mathbf{u}^-_{\xi,\beta+\delta_1})\},$$
$$\dots\dots$$
$$D_{h,\beta} = \min\{D_{h-1,\beta}, \dim_{\mathbf{Q}(\xi)} \Theta_{i_h,k_h}(\mathbf{u}^-_{\xi,\beta+\delta_{h-1}})\},$$
$$\dots\dots$$
$$D_{r,\beta} = \min\{D_{r-1,\beta}, \dim_{\mathbf{Q}(\xi)} \Theta_{i_r,k_r}(\mathbf{u}^-_{\xi,\beta+\delta_{r-1}})\}.$$

Conjecture A. *The number $D_{r,\beta}$ is independent of the choice of the reduced expression of w_0 and* $\dim_{\mathbf{Q}(\xi)} L_\xi(\lambda)_{\lambda-\beta} = D_{r,\beta}$.

7.2. For $i \in [1, n]$, $k \in \mathbf{N}$, denote $\theta_{i,k}$ the $\mathfrak{k}$-linear homomorphism $\mathfrak{U}_\mathfrak{k} \to \mathfrak{U}_\mathfrak{k}$, $x \to x f_i^{(k)}$. The kernel and the image of $\theta_{i,k}$ are easily described by means of PBW Theorem. Assume that $\lambda \in \mathbf{Z}^n_+$ is p^a-restricted. Let $s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression of the longest element of W. Set $k_h := \langle s_{i_{h-1}} \cdots s_{i_1}((p^a - 1)\rho - \lambda), \alpha^\vee_{i_h}\rangle$, $\delta_h := k_1\alpha_{i_1} + \cdots + k_h\alpha_{i_h}$, $h = 1, \dots, r$. Recall that for any $\gamma \in \mathbf{N}R^+$ we denote $\mathrm{u}^-_{a,\gamma}$ the set of all elements in u^-_a of degree γ. Given $\beta \in \mathbf{N}R^+$, set

$$\mathfrak{d}_{0,\beta} = \dim \mathrm{u}^-_{\xi,\beta},$$
$$\mathfrak{d}_{1,\beta} = \dim \theta_{i_1,k_1}(\mathrm{u}^-_{\xi,\beta}),$$
$$\mathfrak{d}_{2,\beta} = \min\{\mathfrak{d}_{1,\beta}, \dim \theta_{i_2,k_2}(\mathrm{u}^-_{\xi,\beta+\delta_1})\},$$
$$\dots\dots$$
$$\mathfrak{d}_{h,\beta} = \min\{\mathfrak{d}_{h-1,\beta}, \dim \theta_{i_h,k_h}(\mathrm{u}^-_{\xi,\beta+\delta_{h-1}})\},$$
$$\dots\dots$$
$$\mathfrak{d}_{r,\beta} = \min\{\mathfrak{d}_{r-1,\beta}, \dim \theta_{i_r,k_r}(\mathrm{u}^-_{\xi,\beta+\delta_{r-1}})\}.$$

Conjecture B. *The number $\mathfrak{d}_{r,\beta}$ is independent of the choice of the reduced expression of w_0 and* $\dim L_{\mathfrak{k},a}(\lambda)_{\lambda-\beta} = \mathfrak{d}_{r,\beta}$ *provided that $p \geq$ the Coxeter number of the root system R associated to* $\mathfrak{g}$.

7.3. Let ϕ_l be the l-th cyclomatic polynomial (i.e. the minimal polynomial of ξ). Denote by $\mathscr{A}$ the localization of $\mathbf{Q}[v, v^{-1}]$ at its prime ideal generated by ϕ_l. Let $U_\mathscr{A}$ be the $\mathscr{A}$-subalgebra of U generated by the elements $E_i^{(a)}, F_i^{(a)}, K_i, K_i^{-1}$ for $i = 1, 2, \dots, n$, $a \geq 0$ and let $U^b_\mathscr{A}$ be the $\mathscr{A}$-subalgebra of $U_\mathscr{A}$ generated by the elements $F_i^{(a)}, K_i, K_i^{-1}, \begin{bmatrix} K_i, c \\ a \end{bmatrix}$ for $i = 1, 2, \dots, n$, $c \in \mathbf{Z}$,

$a \in \mathbf{N}$. Define the category $\mathscr{C}$ (resp. $\mathscr{C}^b$) of $U_{\mathscr{A}}$-modules (resp. $U^b_{\mathscr{A}}$-modules) as in [APW, 2.2, p. 17]. Then define the induction functor $H^0: \mathscr{C}^b \to \mathscr{C}$ as in [APW, 2.8, p. 19]. For each $k \in \mathbf{N}$ we then have a derived functor $H^k: \mathscr{C}^b \to \mathscr{C}$.

Given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{Z}^n$, let $F_i^{(a)}$ acts on $\mathscr{A}$ by scalar zero and let K_i, $\begin{bmatrix} K_i, c \\ a \end{bmatrix}$ act on $\mathscr{A}$ by scalar $v^{d_i \lambda_i}$, $\begin{bmatrix} \lambda_i + c \\ a \end{bmatrix}_{d_i}$ respectively, $i = 1, 2, \ldots, n$, $c \in \mathbf{Z}$, $a \geq 1$. This defines a $U^b_{\mathscr{A}}$-module structure on $\mathscr{A}$. We denote the $U^b_{\mathscr{A}}$-module by $\mathscr{A}_\lambda$. We shall simply write $H^k(\lambda)$ for $H^k(\mathscr{A}_\lambda)$.

Recall that $U_v = U$. We drop the index v and the index $\mathbf{l}$ in all notations involving them. So $V(\lambda)$ will stand for $V_v(\lambda)$. Let $\lambda, \mu \in \mathbf{Z}^n_+$. Assume that $\lambda \in \mathbf{Z}^n_+$ is $\mathbf{l}$-restricted. Given $w \in W$, set

$$H_w(\mathbf{l}\mu + \lambda) := \{ y v_{\mathbf{l}\mu + \lambda} \mid y \in U \text{ and } y x_{\kappa - \lambda, w} \in U_{\mathscr{A}} \}.$$

Then $H_w(\mathbf{l}\mu + \lambda)$ is a free $\mathscr{A}$-submodule of $V(\mathbf{l}\mu + \lambda)$.

Conjecture C. *The $U_{\mathscr{A}}$-module $H_w(\mathbf{l}\mu + \lambda)$ is the free part of the cohomology group $H^{l(w^{-1}w_0)}(w^{-1}w_0(\mathbf{l}\mu + \lambda + \rho) - \rho)$.*

7.4. Keep the notations in Section 6. Denote by A the localization of $\mathbf{Z}$ at its prime ideal generated by p. Let $\mathfrak{U}_A$ be the A-subalgebra of $\mathfrak{U}$ generated by the elements $e_i^{(k)}, f_i^{(k)}$ for $i = 1, 2, \ldots, n$, $k \geq 0$ and let $\mathfrak{U}^b_A$ be the A-subalgebra of $\mathfrak{U}_A$ generated by the elements $f_i^{(k)}$, $\binom{h_i + c}{k}$ for $i = 1, 2, \ldots, n$, $c \in \mathbf{Z}$, $k \in \mathbf{N}$. Define the category $\mathfrak{C}$ (resp. $\mathfrak{C}^b$) of $\mathfrak{U}_A$-modules (resp. $\mathfrak{U}^b_A$-modules) in a similar way of [APW, 2.2, p. 17]. Then define the induction functor $\mathscr{H}^0: \mathfrak{C}^b \to \mathfrak{C}$ as in [APW, 2.8, p. 19]. For each $k \in \mathbf{N}$ we then have a derived functor $\mathscr{H}^k: \mathfrak{C}^b \to \mathfrak{C}$.

Given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{Z}^n$, let $f_i^{(k)}$ acts on A by scalar zero and let $\binom{h_i + c}{k}$ acts on A by scalar $\binom{\lambda_i + c}{k}$ for $i = 1, 2, \ldots, n$, $c \in \mathbf{Z}$, $k \geq 1$. This defines a $\mathfrak{U}^b_A$-module structure on A. We denote the $\mathfrak{U}^b_A$-module by A_λ. We shall simply write $\mathscr{H}^k(\lambda)$ for $\mathscr{H}^k(A_\lambda)$.

For every $\lambda \in \mathbf{Z}^n_+$, denote $M(\lambda)$ an irreducible $\mathfrak{U}$-module of highest weight λ. Let m_λ be a nonzero element in $M(\lambda)$ of weight λ. Assume that $\lambda \in \mathbf{Z}^n_+$ is p^a-restricted. Given $w \in W$, set

$$\mathscr{H}_w(\lambda) := \{ y m_\lambda \mid y \in \mathfrak{U} \text{ and } y \mathfrak{x}_{(p^a - 1)\rho - \lambda, w} \in \mathfrak{U}_A \}.$$

Then $\mathscr{H}_w(\lambda)$ is a free A-submodule of $M(\lambda)$.

Conjecture D. (i) *The $\mathfrak{U}_A$-module $\mathscr{H}_w(\lambda)$ is well defined and is the free part of the cohomology group $\mathscr{H}^{l(w^{-1}w_0)}(w^{-1}w_0(\lambda + \rho) - \rho)$.*
(ii) *The module $\mathscr{H}^{l(w)}(w(\lambda + \rho) - \rho)$ is isomorphic to the cohomology group $H^{l(w)}(G_A/B_A, w(\lambda + \rho) - \rho)$ defined in* [*A, Section* 2, *p.* 501] (*which has a natural*

$\mathfrak{U}_A$-module structure), where G_A is the 'simply connected' Chevalley group over A and associated to $\mathfrak{g}$ and B_A is a suitable 'Borel subgroup' of G_A.

7.5. Keep the notations in 7.3. Let $s_{i_k} \cdots s_{i_2} s_{i_1}$ be a reduced expression of w. For $h = 1, \ldots, k$, set $a_h := \langle s_{i_{h-1}} s_{i_{h-2}} \cdots s_{i_1} \lambda, \alpha_{i_h}^\vee \rangle$, $v_h := \langle \mathbf{l}\mu + \kappa, \alpha_{i_h}^\vee \rangle + a_h$, $d_h' = d_{i_h}$. Then define

$$a_{\lambda, w} := \begin{bmatrix} v_1 \\ a_1 \end{bmatrix}_{d_1'} \begin{bmatrix} v_2 \\ a_2 \end{bmatrix}_{d_2'} \cdots \begin{bmatrix} v_k \\ a_k \end{bmatrix}_{d_k'}.$$

Conjecture E. *As $U_{\mathscr{A}}$-modules, $U_{\mathscr{A}} x'_{\lambda, w} v_{\mathbf{l}\mu + \kappa + \lambda} / a_{\lambda, w}$ is isomorphic to $H_w(\mathbf{l}\mu + \kappa + \lambda)$.*

7.6. Keep the notations in 7.4. Let $s_{i_k} \cdots s_{i_2} s_{i_1}$ be a reduced expression of w. For $h = 1, \ldots, k$, set $a_h := \langle s_{i_{h-1}} s_{i_{h-2}} \cdots s_{i_1} \lambda, \alpha_{i_h}^\vee \rangle$, $v_h := p^a - 1 + a_h$. And define

$$\tilde{a}_{\lambda, w} := \binom{v_1}{a_1} \binom{v_2}{a_2} \cdots \binom{v_k}{a_k}.$$

Conjecture F. *As $\mathfrak{U}_A$-modules, $\mathfrak{U}_A \mathfrak{r}'_{\lambda, w} v_\mu / \tilde{a}_{\lambda, w}$ is isomorphic to $\mathscr{H}_w(\mu)$, where $\mu = (p^a - 1)\rho + \lambda$.*

7.7. Keep the notations in 7.3. Let $\Omega\colon U_{\mathscr{A}} \to U_{\mathscr{A}}^{opp}$ be the $\mathbf{Q}$-algebra homomorphism defined by (cf. [L4, 1.1 (d1), p. 91])

$$\Omega E_i^{(a)} = F_i^{(a)},\ \Omega F_i^{(a)} = E_i^{(a)},\ \Omega K_i = K_i^{-1},\ \Omega v = v^{-1}.$$

Given $\lambda \in \mathbf{Z}_+^n$, we define $V_{\mathscr{A}}(\lambda) := U_{\mathscr{A}} v_\lambda$. There exists a unique $\mathbf{Q}$-bilinear form $(\ ,\ )\colon V_{\mathscr{A}}(\lambda) \times V_{\mathscr{A}}(\lambda) \to \mathscr{A}$ such that (cf. [CK, 1.9, p. 482])

(a) $(\varphi u, u') = \varphi(u, u')$, $(u, \varphi u') = \bar{\varphi}(u, u')$,

(b) $(u, u') = \overline{(u', u)}$,

(c) $(v_\lambda, v_\lambda) = 1$, $(xu, u') = (u, \Omega(x) u')$,

where $\varphi = \varphi(v) \in \mathscr{A}$ and $\bar{\varphi} = \varphi(v^{-1})$ (that is, $\bar{\ }$ denotes the $\mathbf{Q}$-algebra homomorphism $\mathscr{A} \to \mathscr{A}$, $v \to v^{-1}$); u, u' are elements in $V_{\mathscr{A}}(\lambda)$ and $x \in U_{\mathscr{A}}$.

Assume that λ is $\mathbf{l}$-restricted and $\mu \in \mathbf{Z}_+^n$. For each integer $k \in \mathbf{N}$, set $V_{\mathscr{A}}(\mathbf{l}\mu + \lambda)_k := \{u \in V_{\mathscr{A}}(\mathbf{l}\mu + \lambda) \mid (u, u') \in \phi_l^k \mathscr{A}$ for all elements u' in $V_{\mathscr{A}}(\mathbf{l}\mu + \lambda)\}$,

$$V_{\mathscr{A}}(\mathbf{l}\mu + \lambda)_k' := \{y v_{\mathbf{l}\mu + \lambda} \mid y \in U_{\mathscr{A}} \text{ and } y x_{\kappa - \lambda} \in \phi_l^k U_{\mathscr{A}}\}.$$

Assertion. *We have $V_{\mathscr{A}}(\mathbf{l}\mu + \lambda)_{q+1}' \subseteq \phi_l V_{\mathscr{A}}(\mathbf{l}\mu + \lambda)$, where $q = \#\{\alpha \in R^+ \mid \langle \kappa - \lambda, \alpha^\vee \rangle$ is not divisible by $l_\alpha\}$. In particular we have $V_{\mathscr{A}}(\mathbf{l}\mu + \lambda)_{r+1}' \subseteq \phi_l V_{\mathscr{A}}(\mathbf{l}\mu + \lambda)$ (recall that $r = |R^+|$).*

Proof. For each $x \in U_{\mathscr{A}}$, $x F_i^{(k)}$ is not in $\phi_l U_{\mathscr{A}}$ if $x \notin \phi_l U_{\mathscr{A}}$ and k is divisible by l_i. Now the assertion follows from Prop. 41.1.7 in [L7, p. 326] and the definition of $x_{\kappa-\lambda}$.

Conjecture G. $V_{\mathscr{A}}(\mathbf{l}\mu+\lambda)_k = V_{\mathscr{A}}(\mathbf{l}\mu+\lambda)'_k$ *for* $k = 0, 1, 2, \ldots, r, r+1$.

7.8. Keep the notations in 7.4. Let $\omega\colon \mathfrak{U}_A \to \mathfrak{U}_A^{opp}$ be the A-algebra homomorphism defined by

$$\omega e_\alpha^{(k)} = f_\alpha^{(k)}, \quad \omega f_\alpha^{(k)} = e_\alpha^{(k)}.$$

For each $\lambda \in \mathbf{Z}_+^n$, we set $M_A(\lambda) := \mathfrak{U}_A m_\lambda$. There exists a unique A-bilinear form $(\ ,\ )\colon M_A(\lambda) \times M_A(\lambda) \to A$ such that (cf. [J2, Section 2, p. 56])
(a) $(\theta m, m') = (m, \theta m') = \theta(m, m')$,
(b) $(m, m') = (m', m)$,
(c) $(m_\lambda, m_\lambda) = 1$, $(gm, m') = (m, \omega(g)m')$,
where $\theta \in A$, m, m' are elements in $M_A(\lambda)$ and $g \in \mathfrak{U}_A$.

Assume that λ is p^a-restricted. Following Jantzen [J2, Lemma 3, p. 56], for each integer $k \in \mathbf{N}$, set

$$M_A(\lambda)_k := \{m \in M_A(\lambda) \mid (m, m') \in p^k A \text{ for all elements } m' \text{ in } M_A(\lambda)\},$$
$$M_A(\lambda)'_k := \{gm_\lambda \mid g \in \mathfrak{U}_A \text{ and } g\mathfrak{x}_{(p^a-1)\rho-\lambda} \in p^k \mathfrak{U}_A\}.$$

Conjecture H. $M_A(\lambda)_k = M_A(\lambda)'_k$ *for every* $k \in \mathbf{N}$.

7.9. Recall that in U^- a monomial of $F_i^{(k)}$ $(i = 1, 2, \ldots, n, k \geq 0)$ is called to be tight (resp. semi-tight) [L8, Section 1, p. 108] if the monomial is an element of the canonical basis of U^- (resp. a $\mathbf{Z}$-linear combination of elements in the canonical basis of U^-).

It was hoped that for each $\lambda \in \mathbf{Z}_+^n$ and $w \in W$, the monomials $x_{\lambda,w}, x'_{\lambda,w} \in U^-$ are tight. This is true for type A_1, A_2, A_3, B_2 and A_4 (see [L5, 3.4; L8, Prop. 13; L6, 12.8, p. 64; X2]). But in general this is not true. For example, for type G_2, let $\lambda = (1, 0)$, then x_λ is semi-tight but not tight (see [X2]). I donot know whether all $x_{\lambda,w}$, $x'_{\lambda,w}$ are semi-tight, or equivalently all x'_λ are semi-tight.

We may express the elements x_λ as $\mathbf{Q}(v)$-linear combinations of various Poincaré-Birkhoff-Witt Bases. It is rather difficult to see relations between the coefficients and Kazhdan-Lusztig polynomials for affine Weyl groups, even for type A_2.

7.10. Recall that in 2.1 we have defined the integer l_α for each $\alpha \in R^+$. Assume that $\mathfrak{g}$ is simple. In $\mathbf{R}^n$, consider the hyperplanes

$$H_{\alpha,k} := \{e \in \mathbf{R}^n \mid \langle e + \rho, \alpha^\vee \rangle = kl_\alpha\}, \qquad \alpha \in R^+, k \in \mathbf{Z}.$$

Denote by $s_{\alpha,k}$ the corresponding reflections of $\mathbf{R}^n$, that is

$$s_{\alpha,k}(e) = e - (\langle e + \rho, \alpha^\vee \rangle - kl_\alpha)\alpha, \qquad e \in \mathbf{R}^n.$$

These reflections generate an affine Weyl group $W_{\mathfrak{l}}$, which is the affine Weyl group associated to the Cartan matrix (a_{ij}) when $l_1 = \cdots = l_n$, the affine Weyl group associated to the transpose matrix of the Cartan matrix (a_{ij}) when $l_i \neq l_j$ for some i, j.

Conjecture I. *The Conjecture* 8.2 *in* [L2, p. 75] *is true in terms of* $W_{\mathfrak{l}}$ *and* U_ξ.

7.11. It would be interesting to describe clearly the injective hull (or projective cover) in $\mathscr{C}$ of $L_\xi(\lambda)$ $(\lambda \in \mathbf{Z}_+^n)$.

It is known that the category $\tilde{\mathfrak{C}}_a$ of finite dimensional $\tilde{\mathfrak{u}}_a$-modules has enough injective and projective objects. Question: describe clearly the injective hulls (or projective covers) in $\tilde{\mathfrak{C}}_a$ of irreducible $\tilde{\mathfrak{u}}_a$-modules of finite dimension.

7.12. We give some indication of evidence and motivations for the conjectures above. All conjectures are true for type A_1. Conjectures C and G are true for $\lambda = \kappa$, Conjectures D and H are true for $\lambda = (p^a - 1)\rho$, Conjectures E and F are true for $\lambda = 0$.

For an irreducible U-module L of finite dimension, one may compute the character ch(L) of L through Weyl's character formula. In [L5, Theorem 8.13; L6, 12.5, p. 63], an effective algorithm for computing ch(L) has been established (except for type G_2). It would be interesting to find an effective algorithm for computing the character ch(L_ξ) (resp. ch$(L_{\mathfrak{k}})$) of an irreducible U_ξ-module L_ξ (resp. $\mathfrak{U}_{\mathfrak{k}}$-module) of finite dimension. For types A_2, B_2, the author also checked some cases for Conjectures A and B. In Conjecture B there is a restriction on p, which is based on the following example due to Andersen and Jantzen.

Assume that l is a prime number ≥ 3 and $0 < a < l - 1$. Let $\lambda = (a, l-1, l-1, \ldots, l-1, l-a-2)$ and $\lambda' = (l-a-2, a, l-1, \ldots, l-1, l-a-2, a)$ be elements in $\mathbf{Z}_+^{l+2}$. If char $\mathfrak{k} = l$, then for type A_{l+2} one has ch $L_\xi(\lambda) = $ ch $V_\xi(\lambda)$ and ch $L_{\mathfrak{k}}(\lambda) = $ ch $V_{\mathfrak{k}}(\lambda) - $ ch $V_{\mathfrak{k}}(\lambda')$.

The U_ξ-module $V_\xi(\mathfrak{l}\mu + 2\kappa + w_0\lambda)$ has a unique irreducible submodule which is isomorphic to $L_\xi(\mathfrak{l}\mu + \lambda)$ and is generated by $x_{\kappa-\lambda}v_{\mathfrak{l}\mu+2\kappa+w_0\lambda}$ (see Theorems 3.7 and 4.2). From this one should be able to show $H_{w_0}(\mathfrak{l}\mu + \lambda) \otimes_{\mathscr{A}} \mathbf{Q}(\xi)$ has a unique irreducible submodule which is isomorphic to $L_\xi(\mathfrak{l}\mu + \lambda)$. Thus $H_{w_0}(\mathfrak{l}\mu + \lambda)$ is isomorphic to $H^0(\mathfrak{l}\mu + \lambda)$. Of course we should have $H_e(\mathfrak{l}\mu + \lambda) \simeq H^{l(w_0)}(w_0(\mathfrak{l}\mu + \lambda + \rho) - \rho)$ (this is true when l is a prime number > 3, see [APW, Theorem 7.3, p. 39]). Another evidence is the comparison between the natural homomorphisms

$$H_w(\mathfrak{l}\mu + \lambda) \to H_{ww'}(\mathfrak{l}\mu + \lambda),$$

$$H^{l(w^{-1}w_0)}(w^{-1}w_0(\mathbf{l}\mu+\lambda+\rho)-\rho)\to H^{l(w'^{-1}w^{-1}w_0)}(w'^{-1}w^{-1}w_0(\mathbf{l}\mu+\lambda+\rho)-\rho),$$

where w, $w'\in W$ and $l(ww')=l(w)+l(w')$. For Conjecture D, the motivation is similar.

Conjectures E and F are true if $l(w)\leq 1$. I hope that it is not difficult to prove them for $w=w_0$.

It should not be difficult to prove that if $yv_{\mathbf{l}\mu+\lambda}\neq 0$ in $V(\mathbf{l}\mu+\lambda)$, then $yx_{\kappa-\lambda}v_{\mathbf{l}\mu+2\kappa+w_0\lambda}\neq 0$ in $V(\mathbf{l}\mu+2\kappa+w_0\lambda)$. Then one may prove Conjecture G for $k=1$ by using Theorems 4.2 and 3.7. The consideration for Conjecture H is similar.

The Conjecture I is a natural extension of Conjecture 8.2 in [L2, p. 75], which is proved (see [KL, Theorem 38.1, p. 438; KT, Theorem 4.1.2]). For type B_2, G_2, maybe Conjecture I could be proved in a similar way as [APW, Section 11, pp. 52–54]. The linkage principal is known (see [L9, 8.3, p. 244). One may try to compute the determinant of the contrivariant form of $V_{\mathscr{A}}(\lambda)$ in a similar way as [J1, Teil II, Satz 1, p. 48] (cf. [KC, Prop. 1.9, p. 483]), then get a sum formula. It would be more interesting to eliminate the restriction on l in [APW].

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