Publ. RIMS, Kyoto Univ.
32 (1996), 371–382

Reduced Free Products of Completely Positive Maps and Entropy for Free Product of Automorphisms

By

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Abstract

The reduced free product of unital completely positive maps is defined. An invariant state on a C^*-algebra for an automorphism (based on the free shift) is the composition of a state of a subalgebra with the reduced free product of expectations. Entropies for the reduced free product and the tensor product of an automorphism γ with the free shift coincide with the entropy of γ.

§1. Introduction

The reduced free product $(A, \phi) = (*A_i, *\phi_i)_{i\in I}$ of unital C^*-algebras $\{A_i\}_{i\in I}$ with respect to their states $\{\phi_i\}_{i\in I}$ is defined by Avitzour in [A] and Voiculescu in [V] independently.

For a unital C^*-algebra B_i, $(i \in I)$ with a state ψ_i, let T_i be a unital completely positive linear map from A_i to B_i with $\psi_i \cdot T_i = \phi_i$. Put $(B, \psi) = (*B_i, *\psi_i)_{i\in I}$. In §2, we define the reduced free product $T = *_{i\in I} T_i$, which is a unital completely positive map from A to B with $\psi \cdot T = \phi$. If B_i is a C^*-subalgebra of A_i and T_i is a conditional expectation E_i from A_i onto B_i, then $*_{i\in I} E_i$ is still a conditional expectation from A onto B. If $B_i = A_i$ and T_i is an automorphism θ_i, then $*_{i\in I}\theta_i$ is an automorphism of A.

If the index set I is the integers $\mathbb{Z}$ and $A_i = A_0$ for all $i \in \mathbb{Z}$, then we have the automorphism α of A which comes from the shift: $n \in \mathbb{Z} \to n+1$. The α is called the free shift [S]. In §3, using the reduced free product of some conditional expectations, we show an extended version of Avitzour's uniquely ergodic theorem [A: 4.1 Proposition] for the free shift.

Sauvageot and Thouvenot [ST] give a definition of entropy $H_\rho(\gamma)$ for a ρ-invariant automorphism γ of a unital C^*-algebra C with a state ρ. Their

Communicated by H. Araki, October 19, 1995.
1991 Mathematical Subject Classification: 46L55, 46L35, 46L40
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entropy coincides with that of Connes, Narnhofer and Thirring [CNT] if C is nuclear. In §4, as an application of the result in §3, we show that the free shift α does not change Sauvageot-Thouvenot entropy for the reduced free product and the tensor product:

$$H_{\phi*\rho}(\alpha*\gamma) = H_\rho(\gamma) = H_{\phi\otimes\rho}(\alpha\otimes\gamma)$$

for every ρ invariant automorphism γ of C. Applying this to the identity automorphism of C, we have $H_\phi(\alpha) = 0$ ([S: Remark 2]).

§2. Reduced Free Products of Completely Positive Maps

In this section, we define the reduced free product of unital completely positive linear maps. First, to fix notations, we recall definition of reduced free product of unital C^*-algebras. Let I be an index set. For each $i \in I$ let A_i be a unital C^*-algebra and $\Phi_i: A_i \to B(H_i)$ be a $*$-representation on the Hilbert space H_i with the distinguished unit vector ξ_i. The free product Hilbert space $(*H_i, *\xi_i)_{i\in I}$ is (H, ξ) with

$$H = \mathbb{C}\xi \oplus \bigoplus_{n\geq 1} \Big(\bigoplus_{i_1 \neq \cdots \neq i_n} H_{i_1}^\circ \otimes \cdots \otimes H_{i_n}^\circ \Big).$$

Here $H_i^\circ = H_i \ominus \mathbb{C}\xi_i$. Put

$$H(i) = \mathbb{C}\xi \oplus \bigoplus_{n\geq 1} \Big(\bigoplus_{i \neq i_1 \neq i_2 \neq \cdots i_n} H_{i_1}^\circ \otimes \cdots \otimes H_{i_n}^\circ \Big).$$

The unitary operator $V_i: H_i \otimes H(i) \to H$ is defined by

$\xi_i \otimes \xi \to \xi$

$H_i^\circ \otimes \xi \to H_i^\circ$ by $\eta \otimes \xi \to \eta$

$\xi_i \otimes (H_{i_1}^\circ \otimes \cdots \otimes H_{i_n}^\circ) \to H_{i_1}^\circ \otimes \cdots \otimes H_{i_n}^\circ$ by $\xi_i \otimes \eta \to \eta$, $(i_1 \neq i)$

$H_i^\circ \otimes (H_{i_1}^\circ \otimes \cdots \otimes H_{i_n}^\circ) \to H_i^\circ \otimes H_{i_1}^\circ \otimes \cdots \otimes H_{i_n}^\circ$ by $\psi \otimes \eta \to \psi \otimes \eta$, $(i_1 \neq i)$.

The representation $\lambda_i: A_i \to B(H)$ is defined by

$$\lambda_i(a) = V_i(\Phi_i(a) \otimes 1_{H(i)}) V_i^*, \qquad (a \in A_i).$$

The *free product representation* $*_{i\in I}\Phi_i$ is defined as the $*$-homomorphism $*_{i\in I}\lambda_i$ from the enveloping C^*-algebra $*_{i\in I}A_i$ of the $*$-algebra free product of the $\{A_i\}_{i\in I}$ to $B(H)$ using the universal property of the free product ([VDN]).

Now, let A_i be a unital C^*-algebra and ϕ_i a state on A_i. We denote the GNS representation of A_i, $(i \in I)$ with respect to ϕ_i by (π_i, H_i, ξ_i). Then we have the above representation $\lambda_i: A_i \to B(H)$ coming from π_i. The *reduced free product* $(A, \phi) = (*A_i, *\phi_i)_{i\in I}$ is the C^*-algebra A on H generated by

$\bigcup_{i\in I}\lambda_i(A_i)$ and the vector state ϕ: $\phi(a)=\langle a\xi, \xi\rangle$, $a\in A$ ([V]).

Usually, for $a\in A_i$, we denote $\pi_i(a)$ by a, and also $\lambda_i(\pi_i(a))$ by a, for simplicity of notation. For each $i\in I$, let

$$\mathring{A}_i=\{a\in A_i\colon \phi_i(a)=0\}$$

and

$$red(A)=\{a_{i_1}\cdots a_{i_n}\colon a_{i_k}\in \mathring{A}_{i_k},\ i_1\neq\cdots\neq i_n\}.$$

We call an $a\in red(A)$ a reduced word in A. Then $\mathbb{C}1+$ linear span $red(A)$ is dense in A and $\phi(a)=0$ for a reduced word a in A.

Proposition 2.1. *Let A_i and B_i be unital C^*-algebras with states ϕ_i and ψ_i, $(i\in I)$ respectively and let T_i be a unital completely positive map of A_i to B_i with $\psi_i\cdot T_i=\phi_i$. Let $(A,\phi)=(*A_i, *\phi_i)_{i\in I}$ and $(B,\psi)=(*B_i, *\psi_i)_{i\in I}$. Then there exists a unital completely positive map T of A to B which satisfies*

$$\psi\cdot T=\phi \quad \textit{and} \quad T(a)=T_i(a), \quad (a\in A_i)$$

and $T(red(A))\subset red(B)$, more precisely

$$T(a_{i_1}a_{i_2}\cdots a_{i_n})=T_{i_1}(a_{i_1})T_{i_2}(a_{i_2})\cdots T_{i_n}(a_{i_n}), \quad (i_k\neq i_{k+1},\ a_{i_j}\in\mathring{A}_{i_j}).$$

Proof. Let (H,ξ) be as above and (β_i, K_i, η_i) be the GNS representation of B_i by ψ_i. Then by Stinespring's dilation theorem, there exist a Hilbert space L_i, a $*$-representation Φ_i: $A_i\to B(L_i)$ and an isometry W_i: $K_i\to L_i$ such that

$$T_i(a)=W_i^*\Phi_i(a)W_i, \quad (a\in A_i).$$

Put

$$(K,\eta)=(*K_i, *\eta_i)_{i\in I},\ \zeta_i=W_i\eta_i,\ (i\in I) \quad \text{and} \quad (L,\zeta)=(*L_i, *\zeta_i)_{i\in I}.$$

Then we have an isometry W: $K\to L$ defined by

$$\begin{cases} W\eta=\zeta \\ W(v_{i_1}\otimes\cdots\otimes v_{i_n})=W_{i_1}v_{i_1}\otimes\cdots\otimes W_{i_n}v_{i_n},\ (i_i\neq i_{j+1},\ v_{i_j}\in K_{i_j}^\circ). \end{cases}$$

Let Φ be the free product representation $*_{i\in I}\Phi_i$. We define T by

$$T(a)=W^*\Phi(a)W, \quad (a\in A).$$

It is obvious that T: $A\to B(K)$ is a unital completely positive map.

For $a\in\mathring{A}_i$,

$$\langle\Phi_i(a)\zeta_i, \zeta_i\rangle=\langle T_i(a)\eta_i, \eta_i\rangle=\psi_i(T_i(a))=\phi_i(a)=0.$$

Hence $T_i(a)\eta_i\in K_i^\circ$, $\Phi_i(a)\zeta_i\in L_i^\circ$ for $a\in\mathring{A}_i$ and $T_i(\mathring{A}_i)\subset\mathring{B}_i$. For a while we denote Voiculescu's unitary V_i: $K_i\otimes K(i)\to K$ by V_i^B and the representation

λ_i of B_i into B by λ_i^B. The notations of unitary representation V_i and the representation λ_i are used for the free product representation $*_{i\in I}\Phi_i$. First we remark

$$T_{i_1}(a_{i_1})\eta_{i_1}\otimes\cdots\otimes T_{i_n}(a_{i_n})\eta_{i_n} = \lambda_{i_1}^B(T_{i_1}(a_{i_1}))\cdots\lambda_{i_n}^B(T_{i_n}(a_{i_n}))\eta, \quad (i_k\neq i_{k+1},\ a_{i_j}\in \mathring{A}_{i_j}). \tag{2.1.1}$$

Hence

$$T(a_{i_1}a_{i_2}\cdots a_{i_n})\eta = \lambda_{i_1}^B(T_{i_1}(a_{i_1}))\cdots\lambda_{i_n}^B(T_{i_n}(a_{i_n}))\eta.$$

Next we consider a vector $v = v_{j_1}\otimes\cdots\otimes v_{j_m}$, $(j_k\neq j_{k+1},\ v_{j_k}\in K_{j_k}^\circ)$.

If $j_1\neq i_n$. We have

$$\lambda_{i_1}^B(T_{i_1}(a_{i_1}))\cdots\lambda_{i_n}^B(T_{i_n}(a_{i_n}))v = T_{i_1}(a_{i_1})\eta_{i_1}\otimes\cdots\otimes T_{i_n}(a_{i_n})\eta_{i_n}\otimes v. \tag{2.1.2}$$

Hence we have

$$T(a_{i_1}a_{i_2}\cdots a_{i_n})v = \lambda_{i_1}^B(T_{i_1}(a_{i_1}))\cdots\lambda_{i_n}^B(T_{i_n}(a_{i_n}))v.$$

Assume $j_1 = i_n$ and $m = 1$ (hence $v = v_{i_n}$). We decompose $T_{i_n}(a_{i_n})v_{i_n}$ into the direct sum:

$$T_{i_n}(a_{i_n})v_{i_n} = c_{i_n}\eta_{i_n} + [T_{i_n}(a_{i_n})v_{i_n} - c_{i_n}\eta_{i_n}]\in\mathbb{C}\eta_{i_n}\oplus K_{i_n}^\circ,$$

then $\Phi_{i_n}(a_{i_n})W_{i_n}v_{i_n} = c_{i_n}\zeta_{i_n} + [\Phi_{i_n}(a_{i_n})W_{i_n}v_{i_n} - c_{i_n}\zeta_{i_n}]\in\mathbb{C}\zeta_{i_n}\oplus L_{i_n}^\circ$, where

$$c_i = \langle\Phi_i(a)W_iv_i,\ \zeta_i\rangle = \langle T_{i_n}(a_{i_n})v_{i_n},\ \eta_{i_n}\rangle.$$

Put

$$v'_{i_n} = \Phi_{i_n}(a_{i_n})W_{i_n}v_{i_n} - c_{i_n}\zeta_{i_n}.$$

Then $T_{i_n}(a_{i_n})v_{i_n} = c_{i_n}\eta_{i_n} + W_{i_n}^*v'_{i_n}$. Since by the above discussion,

$$\lambda_{i_1}^B(T_{i_1}(a_{i_1}))\cdots\lambda_{i_n}^B(T_{i_n}(a_{i_n}))v_{i_n} = c_{i_n}T(a_{i_1}\cdots a_{i_{n-1}})\eta + T(a_{i_1}\cdots a_{i_{n-1}})\eta\otimes W_{i_n}^*v'_{i_n},$$

we have

$$\lambda_{i_1}^B(T_{i_1}(a_{i_1}))\cdots\lambda_{i_n}^B(T_{i_n}(a_{i_n}))v = c_{i_n}T(a_{i_1}\cdots a_{i_{n-1}})\eta + T(a_{i_1}\cdots a_{i_{n-1}})\eta\otimes W_{i_n}^*v'_{i_n} \tag{2.1.3}$$

which implies

$$T(a_{i_1}a_{i_2}\cdots a_{i_n})v = \lambda_{i_1}^B(T_{i_1}(a_{i_1}))\cdots\lambda_{i_n}^B(T_{i_n}(a_{i_n}))v.$$

If $j_1 = i_n$ and $m\geq 2$. We use again the above direct decomposition of $\Phi_{i_n}(a_{i_n})v_{i_n}$ and $T_{i_n}(a_{i_n})v_{i_n}$. Put

$$v(2) = v_{j_2}\otimes\cdots\otimes v_{j_m},$$

then we have

$$(2.1.4)\qquad T(a_{i_1}a_{i_2}\cdots a_{i_n})v = c_{i_n}W^*(\Phi(a_{i_1})\cdots\Phi(a_{i_{n-1}})Wv(2)) + W^*(\Phi(a_{i_1})\cdots\Phi(a_{i_{n-1}})(v'_{i_n}\otimes Wv(2))).$$

We put

$$(*) = c_{i_n}W^*(\Phi(a_{i_1})\cdots\Phi(a_{i_{n-1}})Wv(2))$$

and

$$(**) = W^*(\Phi(a_{i_1})\cdots\Phi(a_{i_{n-1}})(v'_{i_n}\otimes Wv(2))).$$

Then $(**)$ is reduced to the first case.

Assume that $n = 1$. Since $W^*v'_{i_1} = [T_{i_1}(a_{i_1})v_{i_1} - c_{i_1}\eta_{i_1}]$, we have

$$\lambda^B_{i_1}(T_{i_1}(a_{i_1}))v = c_{i_1}v(2) + W^*v'_{i_1}\otimes v(2),$$

so that, by (2.1.4)

$$(2.1.5)\qquad T(a_{i_1})v = \lambda^B_{i_1}(T_{i_1}(a_{i_1}))v.$$

Assume that $n \geq 2$. We compute $(*)$ in the two cases $i_{n-1} = j_2$ and $i_{n-1} \neq j_2$, and iterate the above discussions. Then

$$T(a_{i_1}a_{i_2}\cdots a_{i_n})v = \lambda^B_{i_1}(T_{i_1}(a_{i_1}))\cdots\lambda^B_{i_n}(T_{i_n}(a_{i_n}))v.$$

As we mentioned before, we denote $\lambda^B_i(T_i(a_i))$ simply by $T_i(a_i)$ and we have

$$T(a_{i_1}a_{i_2}\cdots a_{i_n}) = T_{i_1}(a_{i_1})T_{i_2}(a_{i_2})\cdots T_{i_n}(a_{i_n}),\quad (i_k \neq i_{k+1},\ a_{i_j}\in \mathring{A}_{i_j}).$$

This relation implies that $T(red(A)) \subset red(B)$. If $a \in A_i$, then $T(a) = T(\phi_i(a)1 + [a - \phi(a)1]) = \phi_i(a)1 + T_i(a - \phi_i(a)1) = T_i(a)$. Since $\mathbb{C}1 +$ linear span $red(A)$ and $\mathbb{C}1 +$ linear span $red(B)$ are dense in A and B respectively, we have $T(A) \subset B$.

Let $a \in red(A)$, then $T(a) \in red(B)$ so that

$$\psi \cdot T(a) = 0 = \phi(a),\quad (a \in red(A)).$$

It implies $\psi \cdot T(a) = \phi(a)$ for all $a \in A$. □

Definition. *We call the T in Proposition 2.1 the reduced free product of completely positive maps $\{T_i\}_{i\in I}$ and denote it by $*_{i\in I}T_i$ or as $T_1 * \cdots * T_n$ for the reduced free product for a finite index set I.*

It is pointed out by Enomoto and Takehana that in [B: theorem 3.1] Boca shows the completely positivety of the free product (not reduced free product) of unital completely positive linear maps.

It is well known that conditional expectations and *-automorphisms are typical examples of unital completely positive maps. Applying Proposition 2.1 to these completely positive maps, we have the following two Corollaries:

Corollary 2.2. *Let* (A, ϕ) *be as in Proposition* 2.1. *If* E_i *is a conditional expectation of* A_i *onto a unital* C^**-subalgebra* B_i *of* A_i *with* $\phi_i \cdot E_i = \phi_i$ *then the reduced free product* $*_{i\in I} E_i$ *is a conditional expectation* E *of* A *onto* $B = (*B_i, *\phi_i)_{i\in I}$ *with* $\phi = \phi \cdot E$,

$$E(a_{i_1} a_{i_2} \cdots a_{i_n}) = E_{i_1}(a_{i_1}) E_{i_2}(a_{i_2}) \cdots E_{i_n}(a_{i_n}), \quad (i_k \neq i_{k+1},\ a_{i_j} \in \mathring{A}_{i_j})$$

and

$$E(a) = E_i(a), \quad (a \in A_i).$$

Proof. By Proposition 2.1, we only need to prove the conditional expectation property

$$E(ab) = E(a)b, \quad (a \in A,\ b \in B).$$

This follows from the argument on [CE: page 166], because E is completely positive and satisfies that $E(b) = b$, $(b \in B)$. □

Corollary 2.3. *Let* α_i *be a* $*$*-automorphism of* A_i *with* $\phi_i \cdot \alpha_i = \phi_i$. *Then the reduced free product* $*_{i\in I} \alpha_i$ *is a* $*$*-automorphism* α *of the reduced free product* (A, ϕ) *with* $\phi \cdot \alpha = \phi$ *and* $\alpha(a_i) = \alpha_i(a_i)$, $(a_i \in A_i)$.

Proof. Since $*_{i\in I} \alpha_i$ maps a reduced word in A to a reduced word in A, it is clear that $*_{i\in I} \alpha_i$ is an automorphism if every α_i is an automorphism. □

§3. Invariant States on Reduced Free Products

Let A_0 be a unital C^*-algebra with a state ϕ_0. Put $A_i = A_0$, $\phi_i = \phi_0$ $(i \in \mathbb{Z})$ and $(A, \phi) = (*A_i, *\phi_i)_{i\in\mathbb{Z}}$. The $*$-automorphism α of A, which arises from the shift: $n \to n+1$ on $\mathbb{Z}$, is called the free shift on $(A, \phi) = (*A_i, *\phi_i)_{i\in\mathbb{Z}}$.

We denote by E_ϕ the conditional expectation of A onto $\mathbb{C}1$ conditioned by ϕ, that is, $E_\phi(a) = \phi(a)1$. Then the reduced free product $E_\phi * id_C$ is a conditional expectation of $A * C$ onto C by Corolary 2.2.

The following theorem is an extended version of [A: 4.1 Proposition] and we prove it by an analogous method as the proof of Theorem 3 in early version of [S].

Theorem 3.1. *Let* B *and* C *be unital* C^**-algebras with states* μ *and* ρ *respectively. Let* α *be the free shift on* A, γ *a* ρ*-invariant automorphism on* C *and* β *a* μ*-invariant automorphism of* B. *For the reduced free product* $(A * C, \phi * \rho)$ *if* ψ *is a state of* $(A * C) \otimes B$ *with* $\psi \cdot ((\alpha * \gamma) \otimes \beta) = \psi$ *and* ψ_1 *is the restriction of* ψ *to* $C \otimes B$, *then*

$$\psi = \psi_1 \cdot F,$$

where F *is the conditional expectation of* $(A * C) \otimes B$ *onto* $C \otimes B$ *defined by*

$$F = (E_\phi * id_C) \otimes id_B.$$

Proof. Put $D_1 = A$ *and* $D_2 = C$. Let S be the set of reduced words in $A*C$ which are not contained in C and each component in $\mathring{A}$ is a reduced word in A too. That is, $y \in S$ if and only if y satisfies the following two conditions:

$$y = y_{i_1,1} y_{i_2,2} \cdots y_{i_n,n} \notin C, \quad (i_j \in \{1, 2\},\ i_j \neq i_{j+1},\ y_{i_j,j} \in \mathring{D}_{i_j})$$

and if $y_{i_j,j} \in \mathring{A}$ then $a_j = y_{i_j,j}$ has the form

$$a_j = a_{j_1} a_{j_2} \cdots a_{j_{n(j)}} \quad \text{for} \quad a_{j_k} \in \mathring{A}_{i(j_k)}, \quad (1 \le k \le n(j),\ i(j_k) \neq i(j_{k+1})). \tag{3.1.1}$$

Since $\mathbb{C}1$ + linear span $red(A)$ is dense in A and $\mathbb{C}1$ + linear span $red(A*C)$ is dense in $A*C$, it is enough to show that $\psi(x) = \psi \cdot F(x)$ for $x = \sum_{k=1}^n x_k \otimes b_k + \sum_{i=1}^m c_i \otimes b_i'$, $(x_k \in S,\ b_k,\ b_i' \in B,\ c_i \in \mathring{C})$. Since F is a conditional expectation onto $C \otimes B$ and transforms a reduced word to a reduced word,

$$F(x_k \otimes b_k) = E_\phi * id_C(x_k) \otimes b_k = 0, \quad (x_k \in S).$$

Hence we need only to show $\psi(x) = 0$ for a self adjoint x which has the form

$$x = \sum_{i=1}^n x_i \otimes b_i, \quad (x_i \in S,\ b_i \in B). \tag{3.1.2}$$

Let (H_i, ξ_i) and (L, ζ) be the GNS representation spaces and the vectors of A_i and C by ϕ_i and ρ, respectively. Let $(H, \xi) = (*H_i * \xi_i)_{i \in \mathbb{Z}}$ and $(K, \eta) = (H * L, \xi * \zeta)$.

We need the following lemma:

Lemma 3.2. *If x has the form* (3.1.2), *then there exists a subspace K_x of K such that*

$$x_i(K_x^\perp) \subset K_x, \quad (1 \le i \le n).$$

Proof. For an $y \in S$, let $J(y)$ be the set of indices $i(j_k)$ such that some $a_{j_k} \in \mathring{A}_{i(j_k)}$ appears in the decomposition (3.1.1) of a_j and the $a_j \in \mathring{A}$ is a component of a reduced word y. Put

$$I(x) = \bigcup_{k=1}^n J(x_k).$$

Let

$$H_x = \bigoplus_{m \ge 1} \Big(\bigoplus_{\substack{i_1 \neq \cdots \neq i_m \\ i_1 \in I(x)}} \mathring{H}_{i_1} \otimes \cdots \otimes \mathring{H}_{i_m} \Big)$$

and

$$H'_x = \bigoplus_{m \geq 1} \Big(\bigoplus_{\substack{i_1 \neq \cdots \neq i_m \\ i_1 \notin I(x)}} \mathring{H}_{i_1} \otimes \cdots \otimes \mathring{H}_{i_m} \Big).$$

Put $K_1 = \mathring{H}$ and $K_2 = \mathring{L}$. Let

$$K_x = \mathring{L} \otimes H_x \oplus \bigoplus_{r \geq 1} \Big(\bigoplus_{\substack{i_1 \neq \cdots \neq i_r \\ K_{i_1} = \mathring{L}}} \mathring{L} \otimes H_x \otimes K_{i_1} \otimes \cdots \otimes K_{i_r} \Big)$$

$$\oplus H_x \oplus \bigoplus_{r \geq 1} \Big(\bigoplus_{\substack{i_1 \neq \cdots \neq i_r \\ K_{i_1} = \mathring{L}}} H_x \otimes K_{i_1} \otimes \cdots \otimes K_{i_r} \Big), \quad (i_j \in \{1, 2\}).$$

Then

$$K_x^{\perp} = \mathbb{C}\eta \oplus \mathring{L} \oplus \mathring{L} \otimes H'_x \oplus \bigoplus_{r \geq 1} \Big(\bigoplus_{\substack{i_1 \neq \cdots \neq i_r \\ K_{i_1} = \mathring{L}}} \mathring{L} \otimes H'_x \otimes K_{i_1} \otimes \cdots \otimes K_{i_r} \Big)$$

$$\oplus H'_x \oplus \bigoplus_{r \geq 1} \Big(\bigoplus_{\substack{i_1 \neq \cdots \neq i_r \\ K_{i_1} = \mathring{L}}} H'_x \otimes K_{i_1} \otimes \cdots \otimes K_{i_r} \Big), \quad (i_j \in \{1, 2\}).$$

To show $x_i(K_x^{\perp}) \subset K_x$, we divide the form of x_{i_0} (put $= y$ for simplicity of notation) into the following two cases, where $c_j \in C$ and a_j has the given form (3.1.1).

Case 1: $y = a_1 c_1 \cdots a_n$, $(n \geq 1)$ or $y = c_1 a_1 \cdots a_n$, $(n \geq 1)$. Since a_j has the form $a_j = a_{j_1} a_{j_2} \cdots a_{j_{n(j)}}$ and $a_{j_l} \in \mathring{A}_{i(j_l)}$, $(1 \leq l \leq n(j)$, $i(j_k) \neq i(j_{k+1}))$, $a_j \xi = a_{j_1} \xi_{j_1} \otimes \cdots \otimes a_{j_{n(j)}} \xi_{j_{n(j)}} \in H_x$. Hence $y\eta = a_1 \xi \otimes c_1 \xi \otimes \cdots a_n \xi \in K_x$. Similarly for the second form. Let $v = v_{i_1} \otimes \cdots \otimes v_{i_r} \in K_{i_1} \otimes \cdots \otimes K_{i_r}$, $(i_j \neq i_{j+1}$, $v_{i_j} \in K_{i_j})$, which satisfies the conditions for the components in K_x. Then $yv = y\eta \otimes v \in K_x$.

Case 2: $y = a_1 \cdots a_{n-1} c_n$, $(n \geq 1)$ or $y = c_1 a_1 \cdots a_{n-1} c_n$, $(n \geq 2)$. For the vector η and $v = v_{i_1} \otimes \cdots \otimes v_{i_r} \in K_{i_1} \otimes \cdots \otimes K_{i_r}$, $(i_1 \neq \cdots \neq i_r$, $v_{i_j} \in K_{i_j}$, $K_{i_1} = H'_x)$, similar discussion to the Case 1 implies $y\eta \in K_x$ and $yv \in K_x$. Assume $v_{i_1} \in \mathring{L}$. Put $c_n v = s\zeta + [c_n v - s\zeta]$, where $s = \langle c_n v, \zeta \rangle$. Put $y = zc_n$. Then $z\eta$ has the form $a_1 \xi \otimes \cdots \otimes a_{n-1} \xi$ or $c_1 \zeta \otimes \cdots \otimes a_{n-1} \xi$. Hence $yv = sz\eta + z\eta \otimes [c_n v - s\zeta] \in K_x$. $\square$

Continuation of the Proof of Theorem 3.1. Let Y be the GNS representation space of B by μ. Let

$$Z_x = K_x \otimes Y.$$

Then $Z_x^{\perp} = K_x^{\perp} \otimes Y$ and by Lemma 3.2 $x(Z_x^{\perp}) \subset Z_x$. Since α is the free shift there exists integers $0 = n_1 < n_2 < \cdots < n_{20}$ such that if α_0 denotes the shift on $\mathbb{Z}$ then the sets $\alpha_0^{n_i}(I(x))$, $(1 \leq i \leq 20)$ are all disjoint. Put

$$x(i) = ((\alpha*\gamma)\otimes\beta)^{n_i}(x) \quad \text{and} \quad \hat{x} = (1/20)\sum_{k=i}^{20} x(i).$$

Since α maps a reduced word in A to a reduced word in A, by a property of the reduced free product $\alpha*\gamma(x(i))$ has a similar form (3.1.2) as x. Let e_i be the orthogonal projection of $K\otimes Y$ onto $K_{x(i)}\otimes Y$. Then $\{e_i;\ 1\le i\le 20\}$ are mutually orthogonal and

$$x(i)(e_i^{\perp}(K\otimes Y)) = x(i)(K_{x(i)}^{\perp}\otimes Y)\subset K_{x(i)}\otimes Y = e_i(K\otimes Y).$$

Let $v\in K\otimes Y$ with $\|v\|=1$, then there exists $p(1\le p\le 20)$ with $\|e_p(v)\|^2\le 1/20$. Since $x(p)$ is selfadjoint, by [P: Lemma 4] we have

$$|\langle x(p)v, v\rangle| \le \|x\|\{\|e_p v\|^2 + 2\|e_p v\|\,\|e_p^{\perp}v\|\}.$$

Hence

$$|\langle \hat{x}v, v\rangle| \le \left(\frac{19}{20}+\frac{1}{20}\left(\frac{1}{20}+2\left(\frac{1}{\sqrt{20}}\right)\right)\right)\|x\| \le \frac{39}{40}\|x\|.$$

It implies that $\|\hat{x}\|\le(39/40)\|x\|$ because $\hat{x}$ is self adjoint. Again, $\hat{x}$ has the form (3.1.2). We iterate this method and for a given $\varepsilon>0$ we have

$$\hat{\hat{x}}\in\operatorname{conv}\{((\alpha*\gamma)\otimes\beta)^n(x):\ n\in\mathbb{N}\}$$

which satisfies that $\|\hat{\hat{x}}\|\le\varepsilon\|x\|$. Since ψ is $((\alpha*\gamma)\otimes\beta)$-invariant, $\psi(\hat{\hat{x}})=\psi(x)$. Hence $|\psi(x)|=|\psi(\hat{\hat{x}})|\le\varepsilon\|x\|$. Since ε is arbitrary, we have $\psi(x)=0$. □

Corollary 3.3. *Under the same conditions as in Theorem* 3.1, ψ *is an* $(\alpha*\gamma)\otimes\beta$ *invariant state of* $(A*C)\otimes B$ *if and only if there exists a* $\gamma\otimes\beta$ *invariant state* ω *of* $C\otimes B$ *such that*

$$\psi = \omega\cdot F, \quad \textit{for} \quad F=(E_\phi * id_C)\otimes id_B.$$

Proof. Let ψ be an $(\alpha*\gamma)\otimes\beta$ invariant state of $(A*C)\otimes B$, then the restriction ω of ψ to $C\otimes B$ is a $\gamma\otimes\beta$ invariant state of $C\otimes B$ and by Theorem 3.1 $\psi=\omega\cdot F$. Conversely, let ω be a $\gamma\otimes\beta$ invariant state of $C\otimes B$ and put $\psi=\omega\cdot F$. Then $\psi((\alpha*\gamma)\otimes\beta(c\otimes b)) = \omega(F(\gamma(c)\otimes\beta(b))) = \omega(\gamma\otimes\beta(c\otimes b)) = \psi(c\otimes b)$ for $c\in C$ and $b\in B$. Let $x=\sum_{i=1}^n x_i\otimes b_i$ for $x_i\in S$ and $b_i\in B$. Then $\psi(x)=\sum_i\omega(F(x_i\otimes b_i))=0$. Since $(\alpha*\gamma)\otimes\beta(x)=\sum_{i=1}^n\alpha*\gamma(x_i)\otimes\beta(b_i)$ and $\alpha^*\gamma(x_i)\in S$, we have $\psi((\alpha*\gamma)\otimes(x))=0$. Hence $\psi=\psi\cdot(\alpha*\gamma)\otimes\beta$. □

§4. Entropy for Reduced Free Products and Tensor Products

In this section we show relations between Sauvageot and Thouvenot entropy for the reduced free products and tensor products of automorphisms with the free shifts. To fix notations, we first recall the definition of entropy in [ST]. Let A be a unital C^*-algebra with a state ϕ. A coupling of (A, ϕ) is a pair (ψ, B), where B is an abelian C^*-algebra and ψ is a state on the C^*-algebra $A \otimes B$ whose restriction to A (identified with $A \otimes 1$) is ϕ. We denote by μ the probability measure on B obtained from the restriction of ψ to B.

Let α be a ϕ invariant automorphism of A and (ψ, B) be a coupling of (A, ϕ). Let β be an automorphism of B such that ψ is $\alpha \otimes \beta$-invariant. For a finite partition $\mathscr{P}$ of B, which consists of orthogonal projections $\{p_t \in B: 1 \le i \le n, \sum_i p_i = 1\}$, put

$$H_\mu(\mathscr{P}) = \sum_{i=1}^{n} -\mu(p_i) \log \mu(p_i)$$

and

$$h'(\psi, \mathscr{P}) = \underline{\lim}_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} \beta^{-i}(\mathscr{P})\right) - H_\mu(\mathscr{P}) - S(\phi \otimes \mu|_{\mathscr{P}} \mid \psi|_{A\otimes\mathscr{P}}).$$

Here $S(\cdot \mid \cdot)$ is the relative entropy of states ([PW], [K]). Then Sauvageot and Thouvenot entropy is defined by

$$H_\phi(\alpha) = \sup h'(\psi, \mathscr{P}),$$

where the sup is taken over all couplings (ψ, B), partitions $\mathscr{P}$ and automorphisms β as above.

From now, let A_0 be a unital C^*-algebra with a state ϕ_0 and put $A_i = A_0$, $\phi_i = \phi_0$ for $i \in \mathbb{Z}$. Let $(A, \phi) = (*A_i, *\phi_i)_{i\in\mathbb{Z}}$.

Theorem 4.1. *If α is the free shift on A, then*

$$H_{\phi*\rho}(\alpha*\gamma) = H_\rho(\gamma),$$

where ρ is a state on a unital C^*-algebra C and γ is a ρ-invariant automorphism on C.

Proof. Let (ψ, B) be a coupling of $(A*C, \phi*\rho)$ such that ψ is $(\alpha*\gamma)\otimes\beta$ invariant for an automorphism β of B. We denote by ψ_1 the restriction of ψ to $C \otimes B$ and by μ the restriction of ψ to $1 \otimes B$. Let F be the conditional expectation $(E_\phi * id_C) \otimes id_B$ as in Theorem 3.1. Then $(\phi*\rho)\otimes\mu = (\rho\otimes\mu)\cdot F$ and $\psi = \psi_1 \cdot F$ by Theorem 3.1. Hence

$$S((\phi*\rho)\otimes\mu \mid \psi) = S((\rho\otimes\mu)\cdot F \mid \psi_1 \cdot F) = S(\rho\otimes\mu \mid \psi_1)$$

by the invariance property of the relative entropy (see, [CNT: I, (7)]). Taking $\sup h'(\cdot, \cdot)$, we have $H_{\phi*\rho}(\alpha*\gamma) \le H_\rho(\gamma)$. Conversely let B be a unital abelian C^*-algebra with a state μ and β a μ-invariant automorphism of B. If a

$\gamma\otimes\beta$-invariant state ψ' on $C\otimes B$ satisfies $\psi'(c\otimes 1)=\rho(c)$, $(c\in C)$ and $\psi'(1\otimes b)=\mu(b)$, $(b\in B)$, then ψ' is extended to an $(\alpha*\gamma)\otimes\beta$ invariant state ψ on $(A*C)\otimes B$ by Corollary 3.3. The restriction of ψ to $A*C$ (resp. B) is $\phi*\rho$ (resp. μ). Hence we have $H_{\phi*\rho}(\alpha*\gamma)\geq H_\rho(\gamma)$ by the invariance property of the relative entropy. Therefore $H_{\phi*\rho}(\alpha*\gamma)=H_\rho(\gamma)$. □

Remark 4.2. Let $C_r^*(G)$ be the reduced C^*-algebra of a discrete countable group G and ω_g be the state on $C_r^*(G)$ defined by $\omega_g(\cdot)=\langle\cdot\,\delta_g,\delta_g\rangle$, where δ_g is the characteristic function of $g\in G$. We denote the identity of G by e. Let $A_0=C_r^*(\mathbb{Z})$ and $\phi_0=\omega_e$. Put $A_i=A_0$, $\phi_i=\phi_0$ and $(A,\phi)=(*A_i,*\phi_i)_{i\in\mathbb{Z}}$. Then $(C_r^*(F_\infty),\omega_e)\cong(C_r^*(F_\infty)*C_r^*(\mathbb{Z}),\ \phi*\phi_0)\cong(A*A_0,\phi*\phi_0)$, where F_∞ is the free group on infinite generators. On a nuclear C^* algebra, Sauvegeot-Thouvenot entropy coincides with Connes-Narnhofer-Thirring entropy, which coincides with Kolmogorov-Sinai entropy in the case of abelian algebras. Hence we have many values of Sauvageot-Thouvenot entropy for automorphisms on $C_r^*(F_\infty)$ by Theorem 4.1.

Theorem 4.3. *If α is the free shift on A, then*

$$H_{\phi\otimes\rho}(\alpha\otimes\gamma)=H_\rho(\gamma),$$

where ρ is a state of a unital C^*-algebra C and γ is a ρ-invariant automorphism on C.

Proof. Let (ψ,B) be a coupling of $(A\otimes C,\phi\otimes\rho)$ such that ψ is $\alpha\otimes\gamma\otimes\beta$ invariant for a automorphism β of B. We denote by ψ_1 the state on $C\otimes B$ such that $\psi_1(y)=\psi(1\otimes y)$ for an $y\in C\otimes B$ and by F the conditional expectation $A\otimes C\otimes B$ onto $1\otimes C\otimes B$ with $F(a\otimes y)=\phi(a)1\otimes y$, $(a\in A,$ $y\in C\otimes B)$. Then $\phi\otimes\rho\otimes\mu=(\phi\otimes\rho\otimes\mu)|_{1\otimes C\otimes B}\cdot F$ and $\psi=\psi|_{1\otimes C\otimes B}\cdot F$ by Theorem 3.1. Hence

$$\begin{aligned}S(\phi\otimes\rho\otimes\mu\,|\,\psi)&=S((\phi\otimes\rho\otimes\mu)|_{1\otimes C\otimes B}\cdot F\,|\,\psi|_{1\otimes C\otimes B}\cdot F)\\&=S((\phi\otimes\rho\otimes\mu)|_{1\otimes C\otimes B}\,|\,\psi|_{1\otimes C\otimes B})\\&=S(\rho\otimes\mu\,|\,\psi_1)\end{aligned}$$

by the invariance property. This implies $H_{\phi\otimes\rho}(\alpha\otimes\gamma)\leq H_\rho(\gamma)$. The converse inequality is trivial also by the invariance property of the relative entropy, because if a $\gamma\otimes\mu$-invariant state ψ' on $C\otimes B$ satisfies $\psi'(c\otimes 1)=\rho(c)$, $(c\in C)$ and $\psi'(1\otimes b)=\mu(b)$, $(b\in B)$, then ψ' is extended to an $\alpha\otimes\gamma\otimes\beta$ invariant state ψ on $A\otimes C\otimes B$ by $\psi'\cdot E$ for the slice map (defined by ϕ) E of $A\otimes C\otimes B$ onto $C\otimes B$, and the restriction ψ to $A\otimes C$ (resp. B) is $\phi\otimes\rho$ (resp. μ). Hence we have the equality. □

Remark. If we put $C=\mathbb{C}1$ in Theorem 4.3, then we have [S: Remark 2].

Combining Theorem 4.1 and 4.3, we have:

Corollary 4.4. *If α is the free shift on A, then*

$$H_{\phi\otimes\rho}(\alpha\otimes\gamma)=H_\rho(\gamma)=H_{\phi*\rho}(\alpha*\gamma),$$

where ρ is a state of a unital C^*-algebra C and γ is a ρ-invariant automorphism on C.

Acknowledgments

This work was motivated by the talks of E. Størmer and D. Voiculescu in the workshop on Operator Algebra and Topology at Åarhus University in June of 1995. The author thanks U. Haagerup, J. Andersen and the members of GAP for inviting her to the workshop and their warm hospitalities. She also thanks E. Størmer for sending her his preprint [S] and the referee for pointing out the short proof of Cor. 2.2 & kind linguistic corrections.

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Addendum. (March 16, 1996) F. Boca has kindly informed the author about his related work to Proposition 2.1: Completely positive maps on amalgamated product C^*-algebras, Math. Scand., **72** (1993), 212–222.