

Cancellation and Non Cancellation Phenomena for Infinite Complexes

By

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Abstract

Let $L(I, J)$ be defined by the pull-back of $CP_1^\infty \longrightarrow K(Q, 2) \longleftarrow \Omega S^3$ where $\{I, J\}$ is a partition of all primes. We classify spaces $\{\Omega^k \Sigma^k C(f)\}$ of loop-suspension of mapping cone of phantom map $f: L(I, J) \longrightarrow S^3$ for $k=0, 1, \dots, \infty$ which have the same n -type for all n . In the category of finite CW -complexes, cancellation and non cancellation phenomena are well studied. In the category of infinite CW -complexes, the phenomena are less known. We study cancellation and non cancellation phenomena for infinite dimensional complexes by using above spaces.

Introduction

P. J. Hilton and J. Roitberg [3] constructed an H -space $E(5w)$ which does not have the homotopy type of a Lie group. It is given by a fiber bundle over S^7 with a fiber S^3 induced by $5i: S^7 \longrightarrow S^7$ and a fiber bundle $p: Sp(2) \longrightarrow S^7$. $Sp(2)$ and $E(5w)$ are not homotopy equivalent and have the same genus, that is, $Sp(2)_{(p)} \sim E(5w)_{(p)}$ for all prime p . These spaces satisfy also the following non cancellation properties:

$$\begin{aligned} Sp(2) \times Sp(2) &\sim E(5w) \times E(5w) \\ Sp(2) \times S^k &\sim E(5w) \times S^k \quad (k=3, 7) \end{aligned}$$

Many topologists, [1], [2], [6], [7], [12], [13], [14], studied cancellation and non cancellation phenomena for finite CW -complexes. It seems that examples of cancellation and non cancellation phenomena are less studied for infinite dimensional CW -complexes. Hence we study various examples of the phenomena and its related topics in this paper.

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Let CP^∞ be an infinite dimensional complex projective space and S^k a k -dimension sphere. The based homotopy set $[\Sigma^k CP^\infty, S^{k+3}]$ contains only phantom maps and is equal to a rational vector space Z^\wedge/Z as an additive group. Now we state our main results.

Theorem 0.1. *Let $\{f_i: i=1, \dots, n\}, \{g_j: j=1, \dots, n\}$ be two sets of maps from $\Sigma^k CP^\infty$ to S^{k+3} for $k \geq 0$. Assume that there is a homotopy equivalence between the two wedge sums of mapping cones:*

$$(1) \quad C(f_1) \vee \dots \vee C(f_n) \longrightarrow C(g_1) \vee \dots \vee C(g_n)$$

If exactly k members of the set $\{f_1, \dots, f_n\}$ are essential then the same is true of the set $\{g_1, \dots, g_n\}$. Relabelling the essential maps first, if necessary, it follows that

$$(2) \quad C(f_1) \vee \dots \vee C(f_k) \sim C(g_1) \vee \dots \vee C(g_k)$$

In other words, the numbers of trivial mapping cones occurring on each side of (1) are equal and they can be cancelled. Cancellation phenomenon for non trivial mapping cones is open. We prove the following Theorem 0.2. From the results, we see that $nC(f) \vee sC(g) \vee nC(f) \vee sC(g')$ implies $g \sim \pm g'$.

Theorem 0.2. *Let $f_i (i=1, \dots, n), g$ and g' be phantom maps from $\Sigma^k CP^\infty$ to S^{k+3} for $k \geq 0$. Assume that $f_i (i=1, \dots, n)$ are non trivial and there is a homotopy equivalence between $C(f_1) \vee \dots \vee C(f_n) \vee sC(g)$ and $C(f_1) \vee \dots \vee C(f_n) \vee sC(g')$. Then $g \sim \pm g'$.*

In [9], [10], we classified spaces $\{\Omega^k \Sigma^k C(f)\}$ of loop-suspension of mapping cone of phantom map $f: CP^\infty \longrightarrow S^3$ for $k=0, 1, \dots, \infty$ which have the same n -type for all n . In Section 3, we generalize the result.

Theorem 0.3. *Let $C(f)$ and $C(g)$ be mapping cones of phantom maps $f, g: L(I, J) \longrightarrow S^3$ respectively. Then, $\Omega^k \Sigma^k C(f)$ and $\Omega^k \Sigma^k C(g)$ are homotopy equivalent if and only if f and $\pm g$ are homotopic for $k=0, 1, \dots, \infty$.*

By using the theorem, we prove the following theorems and give examples of cancellation and non cancellation phenomena.

Theorem 0.4. *Let f and g be phantom maps from $\Sigma^k L(I, J)$ to S^{k+3} of order m and n respectively. Then, if m and n are relatively prime, $C(f) \vee C(g)$ is homotopy equivalent to $C(f+g) \vee C(0)$ for $k \geq 1$.*

Theorem 0.5. *Let f and g be phantom maps from $\Sigma^k L(I, J)$ to S^{k+3} for $k \geq 1$. Then, if $sC(f) \vee tC(0)$ and $sC(g) \vee tC(0)$ are homotopy equivalent, $\text{ord}(f)$ and $\text{ord}(g)$ are equal.*

The theorem above is best. We can not get cancellation phenomenon in this case. In fact, let f be a phantom map of order 5. We show that $C(f) \vee C(0)$ and $C(2f) \vee C(0)$ are homotopy equivalent but $C(f)$ and $C(2f)$ are not homotopy equivalent by Theorem 0.3.

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§1. Preliminary

We work in the category of CW-complexes with base point and base point preserving continuous maps. Let $X \vee Y$ be a wedge sum of X and Y . A map $f: X \vee Y \rightarrow Z$ with $f|_X = a, f|_Y = b$ is denoted by (a, b) . When X is a coH-space, a map $g = (x \vee y) \Delta: X \rightarrow X \vee X \rightarrow Z \vee W$ is denoted by ${}^t(x, y)$. When X, Y are coH-spaces, we define the following map:

$$\begin{array}{ccc}
 F : X \vee Y & \xrightarrow{\hspace{10em}} & Z \vee W \\
 \downarrow \Delta \vee \Delta & & \uparrow (a, b) \vee (c, d) \\
 X \vee X \vee Y \vee Y & \xrightarrow{\hspace{10em}} & X \vee Y \vee X \vee Y \\
 & \text{IVTVI} &
 \end{array}$$

which is denoted by the matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We can prove easily the following composition law.

Lemma 1.1. *Let X and Y be double suspended spaces, and Z and W suspended spaces. When $F: X \vee Y \rightarrow Z \vee W$ and $G: Z \vee W \rightarrow U \vee V$ are given by matrices where e, f, g and h suspension maps, the composition GF is given as follows:*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

For general case, the composition law is given by the same way as the multiplication of matrices.

We must remark that even though the assumptions of the lemma don't hold, the composition law is true in our applications by Theorem D of [15].

kX means a wedge sum of k copies of X and $fI: kX \rightarrow kX$ is a map representing a diagonal matrix $Diag(f, \dots, f)$ for $f: X \rightarrow X$:

$$\begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & f \end{pmatrix}$$

When a complex X can be decomposed as $X \sim Y \vee Z$ where Y and Z are not contractible, X is said "decomposable". If it is not so, it is said "prime".

Theorem 1.2. *Let $f: \Sigma^k CP^\infty \rightarrow S^{k+3}$ be a nontrivial phantom map for $k \geq 0$. Then $C(f)$ is prime, that is, non decomposable.*

Proof. If $C(f)$ is decomposed as $X \vee Y$ where X and Y are non trivial, there exist maps $\alpha: C(f) \rightarrow X \vee Y, \beta: X \vee Y \rightarrow C(f)$ such that $\alpha \beta \sim Id, \beta \alpha \sim Id$. It is sufficient to prove for the case of odd k . Since $K(\Sigma^{2k} CP^n \vee S^{2k+2}) = Z + \tilde{K}(\Sigma^{2k} CP^n) + \tilde{K}(S^{2k+2}) = Z\{\sigma\mu^j, \nu | j=0, 1, \dots, n\}$, we set $\beta'(\sigma\mu^j) = u_j, \beta'(\nu) = v$. Here we set $\mu = \eta - 1$ where η is a canonical complex line bundle over CP^∞ and σ is k -time suspension isomorphism. We get easily Adams operation:

$$(*) \quad \phi^h(\sigma\mu^j) = h^k \sigma\{(\mu+1)^h - 1\}^j, \quad \phi^h(\nu) = h^{k+1}\nu$$

By using the relations, we can get Adams operations of u_j and v . If $K(X)$ contains u_1 and $v, K(Y)$ contains all u_j by using $\phi^2(u_j)$. Hence $K(Y) = Z$ and Y is trivial. This is a contradiction. We may assume that $\tilde{K}(X)$ and $\tilde{K}(Y)$ contains $\xi = au_1 + bv$ and $\zeta = cu_1 + dv$ respectively where a, b, c, d are integers and $ad - bc = \pm 1$. We have $\phi^2(\xi) = 2^{k+1} \xi + 2^k au_2, \phi^2(\zeta) = 2^{k+1} \zeta + 2^k cv_2$. If $\tilde{K}(X)$ contains u_2 , we get $c=0$ and $a = \pm 1$ and $d = \pm 1$. Hence $\tilde{K}(X) = Z\{\xi, u_j | j=2, 3, \dots\}, \tilde{K}(Y) = Z\{v\}$. This implies that skeletons of X and Y have $\Sigma^{2k} CP^n$ and S^{2k+2} as retractions respectively. We get that $C(f)$ has retraction $r: C(f) \rightarrow S^{2k+2}$ and $Id = ri: S^{k+3} \rightarrow C(f) \rightarrow S^{k+3}$. By composing $f = rif: \Sigma^k CP^\infty \rightarrow S^{k+3} \rightarrow C(f) \rightarrow S^{k+3}$, we have $f \sim 0$. This is a contradiction. It is

the same for the case $K(Y) \ni u_2$. ■

Let $j: S^3 \rightarrow \Sigma CP^\infty$ be a canonical inclusion. By Theorem D of [15], a homotopy set $[\Sigma^k CP^\infty, S^{k+3}]$ is equal to Z^{\wedge}/Z which is a vector space over \mathbb{Q} with uncountable basis. We proved the following theorems in [9], [10].

Theorem 1.3. *The map $j: S^3 \rightarrow \Sigma CP^\infty$ induces a monomorphism for $k \geq 0$: $\Sigma^k j: [\Sigma^k CP^\infty, S^{k+3}] \rightarrow [\Sigma^k CP^\infty, \Sigma^{k+1} CP^\infty]$*

By using this theorem, we classified the homotopy type of some spaces of the same n -type for all n .

Theorem 1.4. *Let f and g be maps from CP^∞ to S^3 , and $C(f)$ and $C(g)$ mapping cones of f and g respectively. Then, $\Omega^k \Sigma^k C(f)$ and $\Omega^k \Sigma^k C(g)$ are homotopy equivalent, if and only if f and $\pm g$ are homotopic for $k = 0, 1, 2, \dots, \infty$.*

In Section 3, we generalize the above theorem and get examples of cancellation and non cancellation phenomena.

§2. Cancellation Phenomena

Let $f_i, g_j: \Sigma^k CP^\infty \rightarrow S^{k+3}$ be maps for $k \geq 0, i, j = 1, 2, \dots, n$. We propose the following problem:

Problem. Does a homotopy equivalence $C(f_1) \vee \dots \vee C(f_n) \sim C(g_1) \vee \dots \vee C(g_n)$ imply $f_i \sim g_{s(i)}$ where $s(i)$ means a permutation?

For the case $n = 1$, this is true by Theorem 1.4. For the case $n = 2$, we can easily give the following counter example.

Example. Let maps $A: \Sigma^k CP^\infty \vee \Sigma^k CP^\infty \rightarrow \Sigma^k CP^\infty \vee \Sigma^k CP^\infty$ and $B: S^{k+3} \vee S^{k+3} \rightarrow S^{k+3} \vee S^{k+3}$ be given by the matrices ($k > 0$). Here m and n are relatively prime and $ms + nt = 1$.

$$A = \begin{pmatrix} 1 & -nt \\ & ms \\ 1 & \end{pmatrix} \quad B = \begin{pmatrix} m & -t \\ n & s \end{pmatrix}$$

These maps are homotopy equivalences. Let f be a non trivial phantom map from $\Sigma^k CP^\infty$ to S^{k+3} . Then, $C(mf) \vee C(nf)$ and $C(f) \vee C(mnf)$ are homo-

topy equivalent. Since we have the commutativity :

$$\begin{pmatrix} mf & 0 \\ 0 & nf \end{pmatrix} \begin{pmatrix} 1 & -nt \\ 1 & ms \end{pmatrix} = \begin{pmatrix} m & -t \\ n & s \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & mnf \end{pmatrix}$$

we can easily get the commutativity between cofiber sequences and the homotopy equivalence between $C(mf) \vee C(nf)$ and $C(f) \vee C(mnf)$. By Theorem 1.4, $C(f)$, $C(mf)$, $C(nf)$ and $C(mnf)$ are not homotopy equivalent each others.

On the other side, we have the next partial answer for the problem.

Theorem 2.1. *Let $\{f_i, g_j; i, j = 1, 2, \dots, n\}$ be phantom map from $\Sigma^k CP^\infty$ to S^{k+3} such that any two maps of $\{f_i; i = 1, 2, \dots, n\}$ are linearly independent over \mathbb{Q} . Then, $C(f_1) \vee \dots \vee C(f_n)$ and $C(g_1) \vee \dots \vee C(g_n)$ are homotopy equivalent, if and only if f_i and $\pm g_{s(i)}$ are homotopic for $k \geq 0$ where $s(i)$ means a permutation.*

Proof. By the assumption, $f_i (i = 1, 2, \dots, n)$ are not 0-homotopic. Let $\psi: C(f_1) \vee \dots \vee C(f_n) \rightarrow C(g_1) \vee \dots \vee C(g_n)$ be a homotopy equivalence. Since the composition $n\Sigma^k CP^\infty \rightarrow nS^{k+3} \rightarrow C(f_1) \vee \dots \vee C(f_n) \rightarrow C(g_1) \vee \dots \vee C(g_n) \rightarrow n\Sigma^{k+1} CP^\infty$ is 0-homotopic, ψ induces a homotopy equivalence $A: nS^{k+3} \rightarrow nS^{k+3}$ by Theorem 1.3. ψ and A induce homotopy equivalences $B: n\Sigma^{k+1} CP^\infty \rightarrow n\Sigma^{k+1} CP^\infty$. Hence we get $\sum A \cdot \text{Diag}(f_1, \dots, f_n) = \text{Diag}(g_1, \dots, g_n)B$. Here we may assume that A and B are $n \times n$ -matrices with components a_{ij} and b_{ij} of integers respectively by Theorem D of [15]. If a_{ij} and a_{ik} are not 0, f_j and f_k are linearly dependent by $g_i b_{ij} = a_{ij} f_j$ and $g_i b_{ik} = a_{ik} f_k$. This is a contradiction. Each row of A has only one non-zero component which is ± 1 and also for B . Hence we get the result. ■

When $\{f_i; i = 1, 2, \dots, n\}$ are not linearly independent over \mathbb{Q} , we investigate cancellation phenomena in the following. The following theorem is proved by the same method as Theorem 2.1. Hence we omit the proof.

Theorem 2.2. *Let $\{f_{ij}; i = 1, \dots, m, j = 1, \dots, p_i\}$, $\{g_{ik}; i = 1, \dots, m, k = 1, \dots, q_i\}$ be the two sets of non trivial phantom maps from $\Sigma^k CP^\infty$ to S^{k+3} . Assume that $f_{ij} = d_{ij} f_i$, $g_{ik} = d'_{ik} f_i$ for non zero rational numbers d_{ij}, d'_{ik} for $i = 1, \dots, m, j = 1, \dots, p_i, k = 1, \dots, q_i$ and $\{f_i; i = 1, \dots, m\}$ are linearly independent over \mathbb{Q} . Then, a homotopy equivalence of the two wedge sums $\bigvee_{i,j} C(f_{ij}) \sim \bigvee_{i,k} C(g_{ik})$ implies $p_i = q_i$ for $i = 1, \dots, m$.*

Theorem 2.3. *Let $\{f_i, g_j; i = 1, \dots, m, j = 1, \dots, n\}$ be non trivial phantom*

maps from $\sum^k CP^\infty$ to S^{k+3} for $k \geq 0$. Then, a homotopy equivalence $C(f_1) \vee \dots \vee C(f_m) \vee sC(0) \sim C(g_1) \vee \dots \vee C(g_n) \vee tC(0)$ implies $m = n$, $s = t$ and $C(f_1) \vee \dots \vee C(f_n) \sim C(g_1) \vee \dots \vee C(g_n)$.

Proof. By the consideration of $k + 3$ dimension homology, we get $m + s = n + t$. We suppose $t > s$. When F is a diagonal matrix, we set $Diag(f_1, \dots, f_m)$ and $C(F) = C(f_1) \vee \dots \vee C(f_m)$ etc. Set an inclusion $i: C(F) \longrightarrow C(F) \vee sC(0)$ and a homotopy equivalence $\phi: C(F) \vee sC(0) \longrightarrow C(G) \vee tC(0)$ and a retraction $r: C(G) \vee tC(0) \longrightarrow C(G)$. We represent $H_{k+3}(\phi)$ by the following matrix where A_{ij} 's are small matrices of $H_{k+3}(\phi)$.

$$H_{k+3}(\phi) = (A_{ij})$$

where A_{ij} ($i, j = 1, 2, 3$ and 4) are defined as follows:

$$\begin{aligned} A_{i1}: H_{k+3}(mS^{k+3}) &\longrightarrow H_{k+3}(nS^{k+3}) \\ &\quad (H_{k+3}(tS^{k+3}), H_{k+3}(n \sum^{k+1} CP^\infty), H_{k+3}(t \sum^{k+1} CP^\infty) \text{ resp.}) \\ A_{i2}: H_{k+3}(sS^{k+3}) &\longrightarrow H_{k+3}(nS^{k+3}) \\ &\quad (H_{k+3}(tS^{k+3}), H_{k+3}(n \sum^{k+1} CP^\infty), H_{k+3}(t \sum^{k+1} CP^\infty) \text{ resp.}) \\ A_{i3}: H_{k+3}(m \sum^{k+1} CP^\infty) &\longrightarrow H_{k+3}(nS^{k+3}) \\ &\quad (H_{k+3}(tS^{k+3}), H_{k+3}(n \sum^{k+1} CP^\infty), H_{k+3}(t \sum^{k+1} CP^\infty) \text{ resp.}) \\ A_{i4}: H_{k+3}(s \sum^{k+1} CP^\infty) &\longrightarrow H_{k+3}(nS^{k+3}) \\ &\quad (H_{k+3}(tS^{k+3}), H_{k+3}(n \sum^{k+1} CP^\infty), H_{k+3}(t \sum^{k+1} CP^\infty) \text{ resp.}) \end{aligned}$$

By (co)homological reason, we get $A_{13} = 0, A_{14} = 0, A_{23} = 0, A_{24} = 0$. Consider the composition $r' \phi i: C(F) \longrightarrow C(F) \vee sC(0) \longrightarrow C(G) \vee tC(0) \longrightarrow (n + t) \sum^{k+1} CP^\infty$ which is equal to $C(F) \longrightarrow mS^{k+3} \longrightarrow (n + t) \sum^{k+1} CP^\infty$. Since $r' \phi iF$ is 0-homotopic, we get $A_{31} = 0, A_{41} = 0$ by Theorem 1.3. Similarly we get $A_{21} = 0$. If s is smaller than t , we see $\text{rank}(A_{11} \ A_{12}) \leq n$ and $\text{rank}(0 \ A_{22}) \leq s$. Hence the rank of the following matrix is smaller than $m + s$:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

This contradicts the assumption. Hence $m = n, s = t$ and A is non singular. Similarly, the following matrix B is non singular:

$$B = \begin{pmatrix} A_{33} & A_{34} \\ A_{43} & A_{44} \end{pmatrix}$$

Since A_{11} and A_{22} are non singular, we can easily construct a homotopy equivalence $\phi : C(F) \vee sC(0) \rightarrow C(F) \vee sC(0)$ such that $H_{k+3}(\phi)$ is equal to the following matrix:

$$\begin{pmatrix} I_m & -(A_{11})^{-1}A_{12} & 0 & 0 \\ 0 & I_s & 0 & 0 \\ 0 & \begin{pmatrix} A_{32} \\ A_{42} \end{pmatrix} & I_m & 0 \\ 0 & -B^{-1} \begin{pmatrix} A_{32} \\ A_{42} \end{pmatrix} & 0 & I_s \end{pmatrix}$$

$H_{k+3}(\phi\psi)$ is equal to the matrix:

$$\begin{pmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & A_{43} & A_{44} \end{pmatrix}$$

Consider the following diagram of cofiber sequences:

$$\begin{array}{ccccccc} (m+s) \Sigma^k CP^\infty & \rightarrow & (m+s) S^{k+3} & \rightarrow & C(F) \vee sC(0) & \rightarrow & (m+s) \Sigma^{k+1} CP^\infty \rightarrow (m+s) S^{k+4} \\ \downarrow A & & \downarrow \phi\psi & & \downarrow B & & \downarrow \Sigma A \\ (m+s) \Sigma^k CP^\infty & \rightarrow & (m+s) S^{k+3} & \rightarrow & C(G) \vee sC(0) & \rightarrow & (m+s) \Sigma^{k+1} CP^\infty \rightarrow (m+s) S^{k+4} \end{array}$$

By the commutativity of the above diagram, we get $A_{11}F = GA_{33}$ and $A_{34} = 0$. Since $\det(A_{33}) = \pm 1$, $\det(A_{44}) = \pm 1$, we see $C(F) \sim C(G)$. ▀

As corollaries of Theorem 2.3, we get the following results.

Corollary 2.4. *Let f and g be maps from $\Sigma^k CP^\infty$ to S^{k+3} . $mC(f) \vee sC(0)$ and $mC(g) \vee sC(0)$ are homotopy equivalent, if and only if f and $\pm g$ are homotopic*

for $k \geq 0$. In particular, $mC(f)$ and $mC(g)$ are homotopy equivalent, if and only if f and g are homotopic for $k \geq 0$.

Proof. If $f \sim 0$ and $g \sim 0$, it is true. We may assume that f or g are not 0-homotopic. By theorem 2.2, we get $mC(f) \sim mC(g)$. By using the same notation, we get $A_{11} f = g A_{33}$. We have $f I_m = g (A_{11})^{-1} A_{33}$. By comparing the both sides, we get $f \sim \pm g$. ■

Now we prove a theorem which gives another type of cancellation phenomenon. We prepare an elementary lemma of linear algebra.

Lemma 2.5. *Let X and Y be $n \times n$ matrices with components x_{ij}, y_{ij} of integers or rational numbers respectively and H a diagonal matrix $Diag(h_1, \dots, h_n)$ with component h_i in Z/Z . Then, $XH = HY$ implies $detX = detY$.*

Proof. $XH = HY$ implies $x_{ij} h_j = h_i y_{ij}$ for all i, j . For any permutation $\sigma = (j_1, \dots, j_n)$, we see $x_{1j_1} x_{2j_2} \dots x_{nj_n} = y_{1j_1} y_{2j_2} \dots y_{nj_n}$ and hence $detX = detY$.

Theorem 2.6. *Let $f_i (i = 1, \dots, m)$, g and g' be phantom map from $\Sigma^k CP^\infty$ to S^{k+3} and $f_i (1, \dots, m)$ non trivial phantom map. $C(f_1) \vee \dots \vee C(f_m) \vee sC(g)$ and $C(f_1) \vee \dots \vee C(f_m) \vee sC(g')$ are homotopy equivalent, if and only if g and $\pm g'$ are homotopic for $k \geq 0$.*

Proof. Let $\phi: C(f_1) \vee \dots \vee C(f_m) \vee sC(g) \longrightarrow C(f_1) \vee \dots \vee C(f_m) \vee sC(g')$ be a homotopy equivalence. If g is homotopic to a constant map, the theorem is true by using the proof of Theorem 2.3. We assume that g and g' are not homotopic to a constant map. A homotopy equivalence ϕ induces homotopy equivalences $\phi': (m+s)S^{k+3} \longrightarrow (m+s)S^{k+3}$ and $\phi'': (m+s)\Sigma^{k+1}CP^\infty \longrightarrow (m+s)\Sigma^{k+1}CP^\infty$ by the same way of Theorem 2.3. We represent $H_{k+3}(\phi)$ by a matrix as in Theorem 2.3, where A_{ij} is restriction of $2(m+s) \times (m+s)$ matrix $H_{k+3}(\phi) = (A_{ij})$. By the same method as Theorem 2.3, $\sum \phi' Diag(f_1, \dots, f_m, g, \dots, g) = Diag(f_1, \dots, f_m, g', \dots, g') \phi''$. By representing homotopy equivalence $\sum \phi$ and ϕ'' by matrices, we get the following equation where $F = Diag(f_1, \dots, f_m)$:

$$(*) \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & gI \end{pmatrix} = \begin{pmatrix} F & 0 \\ 0 & g'I \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

Here determinants of the following matrices are ± 1 and A and D are matrices of degree m and s respectively:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

From the equation (*), we have

$$(**) \quad AF = FA', Bg = FB', CF = g' C' \text{ and } Dg = g' D'.$$

If D is not zero matrix, we get $D' = dD$ and $g = dg'$ for a non-zero rational number d from $Dg = g' D'$. If D is a zero matrix, B, C, B' and C' are not zero matrices. We get $g = b_i f_i$ for some non zero b_i from $Bg = FB'$, and $g' = c_j f_j$ for some non zero c_j from $CF = g' C'$. If g and g' are linearly independent, it contradicts Theorem 2.2. Hence $g = dg'$ for a non-zero rational number d . From (*), we have the following equation:

$$(*) \quad \begin{pmatrix} A & dB \\ C & dD \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & g'I \end{pmatrix} = \begin{pmatrix} F & 0 \\ 0 & g'I \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

Hence we get the following equation from Lemma 2.5:

$$\det \begin{pmatrix} A & dB \\ C & dD \end{pmatrix} = \pm d^s = \pm 1 = \det \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

We see $d = \pm 1$ and hence $g = \pm g'$. ■

By using Theorem 2.2 and 2.6, we get Theorem 0.1 stated in Introduction and the following corollary.

Corollary 2.7. *Let f, g and g' be phantom map from $\Sigma^k CP^\infty$ to S^{k+3} . $mC(f) \vee sC(g)$ and $mC(f) \vee sC(g')$ are homotopy equivalent, if and only if g and $\pm g'$ are homotopic for $k \geq 0$.*

We can also prove the following cancellation theorem by the same way as Theorem 2.3.

Theorem 2.8. *Let f_i, g_i ($i = 1, \dots, m$) be non trivial phantom map from $\Sigma^k CP^\infty$ to S^{k+3} . $C(f_1) \vee \dots \vee C(f_m) \vee (S^{h_1} \vee \dots \vee S^{h_s}) \vee t \Sigma^{k+1} CP^\infty$ and $C(g_1) \vee \dots \vee C(g_m) \vee (S^{h_1} \vee \dots \vee S^{h_s}) \vee t \Sigma^{k+1} CP^\infty$ are homotopy equivalent, if and only if*

$C(f_1) \vee \dots \vee C(f_m)$ and $C(g_1) \vee \dots \vee C(g_m)$ are homotopy equivalent for $k + 3 \leq h_i \leq 2k + 3 (i = 1, \dots, s)$.

§3. Spaces of the Same n -Type for All n

Let $\{I, J\}$ be a partition of all primes and $L(I, J)$ defined by the pull-back of $CP^\infty \longrightarrow K(Q, 2) \longleftarrow \Omega S^3$. Clearly we have $L(I, J)_I = CP^\infty$ and $L(I, J)_J = \Omega S^3$. The space $L(I, J)$ is $\Omega M^1(J, I)$ in [11]. The spaces are also studied in [4], [8]. When J is empty, $L(I, J)$ is CP^∞ . The integral homology of $L(I, J)$ is a free abelian group of finite type. Hereafter, we assume that $\{I, J\}$ is a non-trivial partition. Clearly spaces $\{C(f) \mid \text{phantom map } f: \Sigma^k L(I, J) \longrightarrow S^{k+3}\}$ are of finite type and have the same n -type for all n . We shall classify the homotopy type of mapping cones $\{C(f) \mid \text{phantom map } f: \Sigma^k L(I, J) \longrightarrow S^{k+3}\}$. When J is empty, we classified the homotopy type in [9]. The homotopy set $[\Sigma^k L(I, J), S^{k+3}]$ is given by Milnor exact sequence, that is, it contains a homotopy set $\text{Ph}(\Sigma^k L(I, J), S^{k+3})$ of phantom maps and $\varprojlim [\Sigma^k L(I, J)^n, S^{k+3}]$. We can determine the former set. The following result is also proved by using Mayer-Vietoris sequence (cf. [8], [11]).

Lemma 3.1. *The homotopy set $\text{Ph}(\Sigma^k L(I, J), S^{k+3})$ of phantom maps is $Z_I^\wedge / Z = Z_I^\wedge / Z_I \oplus (\oplus Z/q^\infty, q \in J)$. The suspension homomorphism $S_* : \text{Ph}(\Sigma^k L(I, J), S^{k+3}) \longrightarrow \text{Ph}(\Sigma^{k+1} L(I, J), S^{k+4})$ is an isomorphism.*

Let $j: S^{k+3} \longrightarrow \Sigma^{k+1} L(I, J)$ be the canonical inclusion and $L(I, J)^n$ the $2n$ -skeleton of $L(I, J)$. The map j induces the map of inverse systems $(j_n)_* : \{[\Sigma^{k+1} L(I, J)^n, S^{k+3}]\} \longrightarrow \{[\Sigma^{k+1} L(I, J)^n, \Sigma^{k+1} L(I, J)]\}$ and $j_* : \text{Ph}(\Sigma^k L(I, J), S^{k+3}) \longrightarrow \text{Ph}(\Sigma^k L(I, J), \Sigma^{k+1} L(I, J))$. The following theorem is proved by the same way as Proposition 1.2 of [9]. By Proposition 3.2 and Lemma 3.3, we can also prove Lemma 3.1.

Proposition 3.2. *The canonical inclusion $j: S^{k+3} \longrightarrow \Sigma^{k+1} L(I, J)$ induces a monomorphism $j_* : \text{Ph}(\Sigma^k L(I, J), S^{k+3}) \longrightarrow \text{Ph}(\Sigma^k L(I, J), \Sigma^{k+1} L(I, J))$ for $k \geq 0$.*

Proof. It is sufficient to prove that the map $\{(j_n)_*\} : \{[\Sigma^{k+1} L(I, J)^n, S^{k+3}]\} \longrightarrow \{[\Sigma^{k+1} L(I, J)^n, \Sigma^{k+1} L(I, J)]\}$ induces an into-isomorphism of \varprojlim^1 -groups for $k \geq 0$. Since $[\Sigma^{k+1} L(I, J)^n, \Sigma^{k+1} L(I, J)] \otimes Q$ is equal to $[\bigvee_{j=1}^n (S^{k+2j+1})_Q, \bigvee_{j=1}^\infty (S^{k+2j+1})_Q]$, there exists a free generator $g: \Sigma^{k+1} L(I,$

$J)^n \longrightarrow \sum^{k+1} L(I, J)$ where g_q represents (Inclusion) \cdot (projection) : $\bigvee_{j=1}^n (S^{k+2j+1})_q \longrightarrow (S^{k+3})_q \longrightarrow \bigvee_{j=1}^\infty (S^{k+2j+1})_q$ up to a finite degree. The free part of $[\sum^{k+1} L(I, J)^n, S^{k+3}]$ is Z and is mapped into the free part G_n of $[\sum^{k+1} L(I, J)^n, \sum^{k+1} L(I, J)]$ which is generated by the map g . It is sufficient to prove that the homological of the image of $\kappa_n : G_n \longrightarrow [S^{k+3}, \sum^{k+1} L(I, J)]$ increasing as n is increasing for $k \geq 0$. Since this value is evaluated by the ordinary cohomology $H^{k+3}(\kappa_n)$ or the K -theory $K(\kappa_n)$, we obtain the result for $k > 1$ from the following Lemma 3.3. For $k = 1$, we project it to its abelianized group.

Lemma 3.3. *If a map $h : \sum^{k+1} L(I, J)^n \longrightarrow \sum^{k+1} L(I, J)$ satisfies $H^m(h) = 0$ for $m > k + 3$, then the degree $d(n, k)$ of the map $H^{k+3}(h) : H^{k+3}(\sum^{k+1} L(I, J)) \longrightarrow H^{k+3}(\sum^{k+1} L(I, J)^n)$ satisfies*

$$\nu_p(d(n, k)) \geq \text{Max}\{\nu_p(j) : j = 1, 2, \dots, n\}$$

for prime number p in I , where $\nu_p(j)$ is the exponent of p in the decomposition of j to prime factors. Moreover there exist maps h such that $\nu_q(d(n, k)) = 0$ for prime q in J and n, k .

Proof. The proof of the lemma is similar to Lemma 1.3 of [9]. We may prove this lemma for odd k by considering the suspension. We shall prove only in the case of $k = 1$, because in the other case the proof is similar. Since Chern character map is monomorphic in this case, we use a localized K -theory $K(-; Z_I) = K(-) \otimes Z_I$. We set in $K(-; Z_I)$:

$$h^1(B\mu) = \sum_{j=1}^n a_j B\mu^j \quad (a_j \in Z_I)$$

Clearly a_1 is equal to the degree of $H^4(h)$. Since the Chern character map is monomorphic, it holds $h^1(B\mu^j) = 0$ for $j > 1$ by the assumption. We calculate the next formulas. $\phi^2 h^1(B\mu) = \phi^2(\sum_{j=1}^n a_j B\mu^j) = 2B\phi^2(\sum_{j=1}^n a_j \mu^j) = 2\sum_{j=1}^n a_j B(2\mu + \mu^2)^j$ and $h^1 \phi^2(B\mu) = 2h^1 B\phi^2(\mu) = 2h^1(B\mu^2 + 2B\mu) = 4(\sum_{j=1}^n a_j B\mu^j)$. By $\phi^2 h^1 = h^1 \phi^2$, we obtain $T(\mu^2 + 2\mu) = 2T(\mu)$ where $T(\mu) = \sum_{j=1}^n a_j \mu^j$. Hence $T(\mu)$ must be $a_1 \text{Log}(1 + \mu) = a_1 \sum_{j=1}^\infty \{(-1)^{j+1}/j\} \mu^j \text{ mod } \mu^{n+1}$ and $a_j = a_1 (-1)^{j+1}/j$. Since $a_j = a_1 (-1)^{j+1}/j$ is in Z_I for $j = 1, 2, \dots, n$. We obtain the former result. For any map $h : \sum^{k+1} L(I, J)^n \longrightarrow \sum^{k+1} L(I, J)$, we can construct h' by $(h')_I = h/\prod \nu_q(\text{degree}(h))$ ($q \in J$) and $(h')_J$ is a map of degree $\prod \nu_p(\text{degree}(h))$ ($p \in I$). This is possible by James's splitting of the suspension of ΩS^j . Hence we get the result. ▀

For the case of empty set J , we classified spaces $\{C(f) | f\}$ by Theorem 2.1 of [9]. Since a homotopy set $\text{Ph}(\sum^k L(I, J), S^{k+3})$ of phantom maps is equal to Z_I^\wedge / Z , it contains torsion elements. When a map f satisfies $rf \sim 0$ for the smallest natural number r , we call the number r the order of f , denoted by $\text{ord}(f) = r$. For a generalization to the case of non empty set J , we must prepare the following proposition.

Proposition 3.4. *Let $f: \sum^k L(I, J) \longrightarrow S^{k+3}$ be a phantom of order r . Then there exists a map $r \sim : C(f) \longrightarrow S^{k+3}$ such that $H_{k+3}(r \sim) = 0$ on $H_{k+3}(\sum^k L(I, J))$ and degree mr on $H_{k+3}(S^{k+3})$ for $k \geq 0$ and m .*

Proof. By $mrf \sim 0$, there is a map $r \sim : C(f) \longrightarrow S^{k+3}$ such that the degree on $H_{k+3}(S^{k+3})$ is mr . The localized map f_I is a constant map, we have $C(f)_I = C(f_I) = (S^{k+3})_I \vee \sum^{k+1} CP_I^\infty$. Since a map from $\sum^{k+1} CP_I^\infty$ to $(S^{k+3})_I$ is a constant map, we have $(r \sim)_I \sim 0$. From this, we get the result. ■

By using a codiagonal map Δ for $k > 0$, we define $(Id, ir \sim) \Delta : C(f) \longrightarrow C(f) \vee C(f) \longrightarrow C(f)$ where i is an inclusion of S^{k+3} into $\text{Cone}(\sum^k L(I, J))$ of $C(f)$. We get the following result.

Proposition 3.5. *Let $f: \sum^k L(I, J) \longrightarrow S^{k+3}$ be a phantom map of order r for $k \geq 1$. Then there exists a homotopy equivalence $r' : C(f) \longrightarrow C(f)$ such that $H_{k+3}(r')$ is given by the following 2×2 matrix for m :*

$$\begin{pmatrix} Id & 0 \\ mr & Id \end{pmatrix} : H_{k+3}(S^{k+3}) \oplus H_{k+3}(\sum^k L(I, J)) \longrightarrow H_{k+3}(S^{k+3}) \oplus H_{k+3}(\sum^k L(I, J))$$

Now we classify the homotopy type of mapping cones $\{C(f) | \text{phantom map } f: \sum^k L(I, J) \longrightarrow S^{k+3}\}$.

Theorem 3.6. *For $k \geq 0$, let $f, g: \sum^k L(I, J) \longrightarrow S^{k+3}$ be phantom maps. Then $C(f)$ and $C(g)$ are homotopy equivalent if and only if f and $\pm g$ are homotopic.*

Proof. If f and $\pm g$ are homotopic, $C(f)$ and $C(g)$ are clearly homotopy equivalent. If f or g are infinite order, the proof of the theorem is the same as Theorem 2.1 of [9]. Hence we assume that f and g are finite order. If there is

a homotopy equivalence $\beta: C(f) \rightarrow C(g)$, we set $\beta^*(V) = aV + b \sum^{k+1} U$ and $\beta^*(\sum^{k+1} U) = cV + d \sum^{k+1} U$, $ad - bc = \pm 1$ where V is the generator of $H^{k+3}(S^{k+3}; Z)$ and $\sum^{k+1} U$ is the generator of $H^{k+3}(\sum^{k+1} L(I, J); Z)$. By a property of phantom maps, we have $b = 0$ and hence $a = \pm 1$, $d = \pm 1$ by $ad - bc = \pm 1$. The map β does not induce a map $\alpha: S^{k+3} \rightarrow S^{k+3}$ of degree ± 1 . But by modifying β , it induces a map $\alpha: S^{k+3} \rightarrow S^{k+3}$ of degree ± 1 and $c = 0$ as follows.

When $f \sim g \sim 0$, the statement is clear. We may assume that f is not homotopic to a constant map. $q\beta jf: \sum^k L(I, J) \rightarrow \sum^k L(I, J)$ is homotopic to the constant map by $jf \sim 0$. Since $q\beta jf$ is homotopic to $sf: \sum^k L(I, J) \rightarrow S^{k+3} \rightarrow \sum^{k+1} L(I, J)$ where $s: S^{k+3} \rightarrow \sum^{k+1} L(I, J)$ is the map of degree s , the map s is a map of multiple degree mr of $r = \text{ord}(f)$ by Proposition 3.3. By making use of $\beta: C(f) \rightarrow C(f)$ and $h: C(f) \rightarrow C(f)$ in Proposition 3.5, we may assume that βh induces the following commutative diagram where $\alpha, \beta h, \gamma$ and δ are maps of homotopy equivalences.

$$\begin{array}{ccccccc}
 & & f & j & p & & -\sum f \\
 \sum^k L(I, J) & \longrightarrow & S^{k+3} & \longrightarrow & C(f) & \longrightarrow & \sum^{k+1} L(I, J) & \longrightarrow & S^{k+4} & \longrightarrow & \\
 & & \downarrow \alpha & & \downarrow \beta h & & \downarrow \gamma & & \downarrow \delta & & \\
 \sum^k L(I, J) & \longrightarrow & S^{k+3} & \longrightarrow & C(f) & \longrightarrow & \sum^{k+1} L(I, J) & \longrightarrow & S^{k+4} & \longrightarrow & \\
 & & g & & k & & q & & -\sum g & &
 \end{array}$$

We have that $\sum f$ and $\sum g$ are equivalent under the action homotopy equivalences on $\text{Ph}(\sum^{k+1} L(I, J), S^{k+4}) = \text{Ext}(H_{k+3}(\sum^{k+1} L(I, J); Q), \pi_{k+4}(S^{k+4})) / \text{Im}(Z_f \frown Z) = \text{Ext}(Q, Z) / \text{Im}(Z_f \frown Z) = Z_f \frown Z$. From this, we have $\sum f \sim \pm \sum g$ and hence $f \sim \pm g$ by the suspension isomorphism. The case for $k = 0$ is proved by considering the suspension isomorphism between sets of phantom maps. ■

The next theorem is analogously proved by the method of Theorem 3.2 of [9] and Theorem 2.1 of [10]. Hence we omit the proof.

Theorem 3.7. *Let $f, g: L(I, J) \rightarrow S^3$ be phantom maps. Then $\Omega^m \sum^m C(f)$ and $\Omega^m \sum^m C(g)$ are homotopy equivalent if and only if f and $\pm g$ are homotopic for $0 \leq m \leq \infty$.*

By the same way as Theorem 1.2, we can get the primarity of mapping cone of a non trivial phantom map $f: \sum^k L(I, J) \rightarrow S^{k+3}$.

Theorem 3.8. *Let $f: \Sigma^k L(I, J) \longrightarrow S^{k+3}$ be a non trivial phantom map. Then $C(f)$ is prime, that is, $C(f)$ can not be decomposed as a wedge sum of non-trivial spaces.*

§4. Non Cancellation Phenomena

We got some results of cancellation phenomena in Section 2 for mapping cones of $f: \Sigma^k CP^\infty \longrightarrow S^{k+3}$. If the orders of f and g are infinite, the statements of Section 2 hold by the same method. Actually the following theorem corresponds to Theorem 2.6.

Theorem 4.1. *Let f_i ($i = 1, \dots, n$), g and g' be phantom maps from $\Sigma^k L(I, J)$ to S^{k+3} of infinite order for $k \geq 0$. Then $C(f_1) \vee \dots \vee C(f_n) \vee sC(g)$ and $C(f_1) \vee \dots \vee C(f_n) \vee sC(g')$ are homotopy equivalent, if and only if g and $\pm g'$ are homotopic.*

In this Section, we give some examples of non cancellation phenomena.

Theorem 4.2. *Let $f, g: \Sigma^k L(I, J) \longrightarrow S^{k+3}$ be phantom maps of order m and n for $k > 0$. If m and n are relatively prime and $ms + nt = 1$, the following formula hold:*

- (1) $C(f) \vee C(g) \sim C(f + g) \vee C(0)$
- (2) $C(f) \vee C(g) \sim C(f + sg) \vee C(0)$
- (3) $C(f) \vee C(g) \sim C(t^2 f + s^2 g) \vee C(0)$

Proof. Since m and n are relatively prime, there exist integers s, t such that $ms + nt = 1$. By the following calculations, we get the results respectively.

$$\begin{pmatrix} 1 & s \\ -m & nt \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} f+g & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} nt + mnst & ms^2 \\ -m & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & s \\ -m & nt \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} f+sg & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} nt & ms \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} t & s \\ -m & n \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} t^2 f + s^2 g & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} n & m \\ s & -t \end{pmatrix}$$



By using Theorem 4.2, we can get various examples.

Example. Let $f, g: \Sigma^k L(I, J) \longrightarrow S^{k+3}$ be phantom maps of order 3 and 5 for 3 and 5 in I . By $3 \times 2 + 5 \times (-1) = 1$ and Theorem 4.2, we have $C(f) \vee C(g) \sim C(f+g) \vee C(0) \sim C(f+2g) \vee C(0)$. By $C(f) \sim C(\pm f)$ and $C(jg) \sim C(\pm jg)$ ($j = 1, 2, 3, 4$). Spaces $\{C(if+jg) \vee C(0) \mid i = 1, 2, j = 1, 2, 3, 4\}$ are homotopy equivalent to $C(f) \vee C(g)$. The example shows that uniqueness of splitting and cancellation phenomena fail.

Theorem 4.3. Let $f, g: \Sigma^k L(I, J) \longrightarrow S^{k+3}$ be phantom maps for $k \geq 0$, and orders of f and g finite. If $sC(f) \vee tC(0)$ and $sC(g) \vee tC(0)$ are homotopy equivalent, $\text{ord}(f)$ and $\text{ord}(g)$ are equal.

Proof. If $sC(f) \vee tC(0)$ and $sC(g) \vee tC(0)$ are homotopy equivalent, we set a homotopy equivalence ϕ between them. We represent $H_{k+3}(\phi)$ as in Theorem 2.2. By (co)homological reason, we have $A_{13} = 0, A_{14} = 0, A_{23} = 0$ and $A_{24} = 0$. By Proposition 3.2, we have $A_{31} = mA'_{31}$ and $A_{41} = mA'_{41}$ where components of A'_{31} and A'_{41} are integers. By Proposition 3.5 and the method of Theorem 3.6, there is a homotopy equivalence $\psi: sC(f) \vee tC(0) \longrightarrow sC(f) \vee tC(0)$ such that the composed map $\phi\psi$ gives $A_{ij} = 0$ for $(i, j) = (3, 1), (3, 2), (4, 1), (4, 2), (1, 3), (1, 4), (2, 3)$ and $(2, 4)$ in $H_{k+3}(\phi\psi)$. Hence we get the following equation:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} fI & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} gI & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

By comparing the both side, we have $A_{11}f = gB_{11}, A_{21}f = 0$ and $gB_{12} = 0$. Set $m = \text{ord}(f) = ac$ and $n = \text{ord}(g) = bc$ where c is the greatest common divisor of $\text{ord}(f)$ and $\text{ord}(g)$. From the above relations, we get $A_{11} = aA'_{11}, B_{11} = bB'_{11}, A_{21} = mA'_{21}, B_{12} = nB'_{12}$. If $m \neq n$, that is, a or b is not 1, we see that $\det A$ or $\det B$ is not ± 1 . This is a contradiction. Hence we get the result. \blacksquare

Example. When the homotopy set $Ph(\Sigma^k L(I, J), S^{k+3})$ of phantom maps contains a subgroup $Z/5$ of order 5 generated by $f: \Sigma^k L(I, J) \longrightarrow S^{k+3}, C(f)$ and $C(2f)$ are not homotopy equivalent by Theorem 3.6. Let A and B be maps given by the following matrices respectively:

$$\begin{pmatrix} 1 & 5 \\ 1 & 4 \end{pmatrix} : 2 \Sigma^k L(I, J) \longrightarrow 2 \Sigma^k L(I, J),$$

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} : 2S^{k+3} \longrightarrow 2S^{k+3}$$

We get a homotopy equivalence between $C(f) \vee C(0)$ and $C(2f) \vee C(0)$ by $Diag(2f, 0) A = B Diag(f, 0)$. Hence cancellation phenomena does not hold. Let G and H be maps given by the following matrices respectively:

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} : 2\Sigma^k L(I, J) \longrightarrow 2\Sigma^k L(I, J),$$

$$\begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix} : 2S^{k+3} \longrightarrow 2S^{k+3}$$

By $Gf = 2fH$, we get a homotopy equivalence between $2C(f)$ and $2C(2f)$.

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