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Energy Decay and Asymptotic Behavior of Solutions to the Wave Equations with Linear Dissipation

By

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§1. Introduction

Let $\Omega \subset \mathbf{R}^N$ be an unbounded domain with smooth boundary $\partial\Omega$. We consider the mixed initial-boundary value problem

$$(1.1)\qquad \begin{cases} w_{tt} - \Delta w + b(x, t) w_t = 0, & (x, t) \in \Omega \times (0, \infty) \\ w(x, 0) = w_1(x),\ w_t(x, 0) = w_2(x), & x \in \Omega \\ w(x, t) = 0, & (x, t) \in \partial\Omega \times (0, \infty), \end{cases}$$

where $w_t = \partial w/\partial t$, $w_{tt} = \partial^2 w/\partial t^2$, Δ is the N-dimensional Laplacian and $b(x, t)$ is a nonnegative C^1-function.

Let $H^k(\Omega)$, $k = 0,1,2,\cdots$, be the usual Sobolev space with norm

$$\|f\|_{H^k} = \Big\{ \sum_{|\alpha| \le k} \int_\Omega |\nabla^\alpha f(x)|^2 dx \Big\}^{1/2} < \infty,$$

where α are multiindices. We write $H^0(\Omega) = L^2(\Omega)$ and $\|f\|_{L^2} = \|f\|$. $H_0^1(\Omega)$ is the completion in $H^1(\Omega)$ of the set of all smooth functions with compact support in Ω. Let E be the space of all pairs $f = \{f_1, f_2\}$ of functions such that

$$\|f\|_E^2 = \|\{f_1, f_2\}\|_E^2 = \frac{1}{2}(\|f_2\|^2 + \|\nabla f_1\|^2) < \infty.$$

For solution $w(t)$ of (1.1), we simply write

$$\|w(t)\|_E^2 = \|\{w(t), w_t(t)\}\|_E^2$$

and call it the energy of $w(t)$ at time t.

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Now, assume

$$\{w_1, w_2\} \in [H^2(\Omega)\cap H_0^1(\Omega)]\times H_0^1(\Omega). \tag{1.2}$$

Then as is well known, the initial-boundary value problem (1.1) has a global solution in the class

(1.3)
$$w(\cdot, t) \in C^0([0, \infty); H^2(\Omega))\cap C^1([0, \infty); H_0^1(\Omega))\cap C^2([0, \infty); L^2(\Omega)).$$

Moreover, we have the energy equation

$$\|w(t)\|_E^2+\int_0^t\int_\Omega b(x, \tau)w_t(x, \tau)^2dxd\tau=\|w(0)\|_E^2 \tag{1.4}$$

for any $t>0$.

Since $b(x, t)\geq 0$, $b(x, t)w_t$ represents a friction of viscous type, and we see from (1.4) that the energy $\|w(t)\|_E^2$ of solution $w(t)$ is decreasing in $t>0$. Thus, a question naturally rises whether it decays or not as t goes to infinity.

The decay and nondecay problems have been studied in works of Matsumura [1] and Mochizuki [2], [3] in case where $\Omega=\mathbb{R}^N$. It is proved in [1] that the energy decays if $b_0(1+r+t)^{-1}\leq b(x, t)\leq b_1$ ($r=|x|$) and $b_t(x, t)\leq 0$. (Note that Matsumura's result is restricted to the compactly supported initial data. Its noncompact version is given in [3].) On the other hand, it is proved in [2], [3] that if $0\leq b(x, t)\leq b_2(1+r)^{-\gamma}$, $\gamma>1$, then the energy does not in general decay and every solution with finite energy is asymptotically free as $t\to\infty$.

From these results we see that if $b(x, t)=O(r^{-\gamma})$ as $r=|x|\to\infty$, then $\gamma=1$ is the critical exponent of energy decay. Our purpose of the present paper is to improve this result. We consider the case $b(x, t)=o(r^{-1})$ and obtain the critical exponent of logarithmic order.

In order to state the assumption on $b(x, t)$, we define the positive number e_n and the function $\log^{[n]}$ ($n=0,1,2,\cdots$) by

$$e_0=1,\ e_1=e, \cdots, e_n=e^{e_{n-1}},$$

$$\log^{[0]}a=a,\ \log^{[1]}a=\log a, \cdots, \log^{[n]}a=\log\log^{[n-1]}a.$$

In the following we require one of the following (A1) and (A2).

(A1) There exist b_0, $b_1>0$ and a nonnegative integer n such that

$$b_0\{(e_n+r+t)\log(e_n+r+t)\cdots\log^{[n]}(e_n+r+t)\}^{-1}\leq b(x, t)\leq b_1.$$

Moreover,

$$b_t(x, t) \leq 0, \ (x, t) \in \Omega \times (0, \infty).$$

(A2) $N \geq 3$ and a $\mathbf{R}^N \setminus \Omega$ is starshaped with respect to the origin $x=0$. There exist $b_2>0$, $\gamma>1$ and a nonnegative integer n such that

$$0 \leq b(x, t) \leq b_2\{(e_n+r)\cdots\log^{[n-1]}(e_n+r)\,[\log^{[n]}(e_n+r)]^{\gamma}\}^{-1}.$$

Our results on the energy decay are summarized in the following

Theorem 1. *Assume* (A1). *Let* $\{w_1, w_2\}$ *satisfy* (1.2) *and*

$$\int_\Omega \log^{[n]}(e_n+r)\,\{w_2^2+|\nabla w_1|^2\}\,dx < \infty. \tag{1.5}$$

Then the energy of the solution to (1.1) *decays as t goes to infinity. More precisely, there exists a constant* $K=K(w_0, w_1, n)>0$ *such that*

$$\|w(t)\|_E^2 \leq K\{\log^{[n]}(e_n+t)\}^{-\mu}, \tag{1.6}$$

where $\mu=\min\{1, b_0/2\}$.

To state another theorem, we need a local decay estimate for the free wave equation in Ω:

$$\begin{cases} w_{0tt}-\Delta w_0=0, & (x, t)\in\Omega\times(0,\infty) \\ w_0(x, 0)=f_1(x),\ w_{0t}(x, 0)=f_2(x), & x\in\Omega \\ w(x, t)=0, & (x, t)\in\partial\Omega\times(0,\infty), \end{cases} \tag{1.7}$$

As we shall show in Lemma 3.3, if N and Ω satisfies the conditions in (A2), then we have

$$\int_0^\infty\int_\Omega \{(e_n+r)\log(e_n+r)\cdots[\log^{[n]}(e_n+r)]^{\gamma}\}^{-1} w_{0t}^2\,dxdt \leq C\|f\|_E^2 \tag{1.8}$$

for some $C>0$ independent of $f=\{f_1, f_2\}\in E$.

With this inequality, our results on energy nondecay and asymptotics are summarized in the following

Theorem 2. *Assume* (A2). (a) *Let* $f=\{f_1, f_2\}\in E$ *and* $w_0(t)$ *be the solution to* (1.7). *We choose* $\sigma>0$ *to satisfy*

$$\int_\sigma^\infty\int_\Omega \{(e_n+r)\log(e_n+r)\cdots[\log^{[n]}(e_n+r)]^{\gamma}\}^{-1} w_{0t}^2\,dxdt \leq 4b_2^{-1}\|f\|_E^2. \tag{1.9}$$

Let $w_\sigma(t)$ *be the solution to* (1.1) *with the initial data*

$$\{w_\sigma(0),\ w_{\sigma t}(0)\}=\{w_0(\sigma),\ w_{0t}(\sigma)\}. \tag{1.10}$$

Then the energy of this solution remains positive as t goes to infinity.

(b) *For any solution* $w(t)$ *of* (1.1) *with* $\{w_1,\ w_2\}\in E$, *there exists a pair* $f^+=\{f_1^+,f_2^+\}\in E$ *such that*

$$\|w(t)-w_0^+(t)\|_E\to 0 \quad as \quad t\to\infty, \tag{1.11}$$

where $w_0^+(t)$ *is the solution to* (1.7) *with f replaced by* f^+.

Our argument on the decay property is based on a weighted energy inequality. So, the same results as Theorem 1 can be obtained also for the problem with Neumann or Robin boundary condition. On the other hand, to show Theorem 2 we combine the usual energy estimate and inequality (1.8). A similar treatment is found e.g., in [3].

In the case where Ω is bounded, there are many works on the energy decay. However, in the case of unbounded domain there are not so many works other than [1], [3]. We refer here Nakao [5] and Zuazua [7], where are treated the Klein-Gordon equations with dissipative term. As for the energy nondecay, another approach is developed in Rauch-Taylor [6] for $b(x,\ t)$ with compact support in x.

Theorems 1 and 2 are proved in § 2 and § 3, respectively. In § 4 we remark that our proof of the energy decay can be applied to some quasilinear wave equations.

§2. Proof of Theorem 1

Let $\varphi(s)$, $s\geq 0$, be a smooth function satisfying

$$\varphi(s)\geq 1 \text{ and } \lim_{s\to\infty}\varphi(s)=\infty; \tag{2.1}$$

$$\varphi'(s)>0,\ \varphi''(s)\leq 0,\ \varphi'''(s)\geq 0 \text{ and they all are bounded in } s\geq 0; \tag{2.2}$$

$$2\varphi'(s)\varphi'''(s)-\varphi''(s)^2\geq 0. \tag{2.3}$$

With this $\varphi(s)$ we define a weighted energy of solutions at time t as follows:

$$\|w(t)\|_{E_\varphi}^2=\frac{1}{2}\int_\Omega \varphi(r+t)\,(w_t^2+|\nabla w|^2)\,dx, \tag{2.4}$$

where $r=|x|$. In order to show a energy decay property, the initial data are required other than (1.2) to satisfy

$$\|w(0)\|_{E_\varphi} < \infty \tag{2.5}$$

(cf., (1.5)).

Multiply by $\{\varphi(r+t)w\}_t$ on both sides of (1.1). It then follows that

$$X_t + \nabla\cdot Y + Z = 0, \tag{2.6}$$

where

$$\begin{aligned}
X &= \frac{1}{2}\varphi\{w_t^2 + |\nabla w|^2\} + \varphi' w_t w + \frac{1}{2}(\varphi' b - \varphi'')w^2,\\
Y &= -(\varphi w_t + \varphi' w)\nabla w,\\
Z &= (\varphi b - 2\varphi')w_t^2 + \frac{1}{2}\varphi'\left|\frac{x}{r}w_t + \nabla w + \frac{x}{r}\varphi'^{-1}\varphi'' w\right|^2\\
&\quad + \frac{1}{2}\{\varphi''' - \varphi'^{-1}\varphi''^2 - (\varphi' b)_t\}w^2 - \varphi'' w_t w.
\end{aligned}$$

Making use of the identity

$$-\varphi'' w_t w = -\frac{1}{2}\partial_t[\varphi'' w^2] + \frac{1}{2}\varphi''' w^2$$

and noting (2.3), we easily have

$$Z \geq (\varphi b - 2\varphi')w_t^2 - \frac{1}{2}(\varphi' b)_t w^2 - \frac{1}{2}\partial_t[\varphi'' w^2]. \tag{2.7}$$

Lemma 2.1. *For any $t>0$ and $0<\epsilon<1$, the solution $w(t)$ of* (1.1) *admits the inequality*

$$\begin{aligned}
&(1-\epsilon)\|w(t)\|_{E_\varphi}^2 + \frac{1}{2}\int_\Omega(-2\varphi'' + \varphi' b - \epsilon^{-1}\varphi^{-1}\varphi'^2)w^2 dx\\
&\quad + \int_0^t\int_\Omega\left\{(\varphi b - 2\varphi')w_t^2 - \frac{1}{2}(\varphi' b)_t w^2\right\}dx d\tau\\
&\leq (1+\epsilon)\|w(0)\|_{E_\varphi}^2 + \frac{1}{2}\int_\Omega(-2\varphi'' + \varphi' b + \epsilon^{-1}\varphi^{-1}\varphi'^2)w_1^2 dx.
\end{aligned} \tag{2.8}$$

Proof. Let $\Omega(R)=\{x\in\Omega\,;\,|x|<R\}$ and $S_\Omega(R)=\{x\in\Omega\,;\,|x|=R\}$. We integrate (2.6) over $\Omega(R)\times(0,t)$. Then integration by parts and (2.7) give

$$\int_{\Omega(R)}\left\{X(x,\tau)-\frac{1}{2}\varphi''(r+\tau)w(x,\tau)^2\right\}dx\Big|_{\tau=0}^{\tau=t}+\int_0^t\int_{S_\Omega(R)}\frac{x}{r}\cdot Y(x,\tau)\,dSd\tau$$
$$+\int_0^t\int_{\Omega(R)}\left\{(\varphi b-2\varphi')w_t^2-\frac{1}{2}(\varphi' b)_t w^2\right\}dxd\tau\leq 0. \tag{2.9}$$

By the Schwarz inequality

$$X(x,t)-\frac{1}{2}\varphi''(r+t)w(x,t)^2$$
$$\geq\frac{1-\epsilon}{2}\varphi\{w_t^2+|\nabla w|^2\}+\frac{1}{2}(-2\varphi''+\varphi' b-\epsilon^{-1}\varphi^{-1}\varphi'^2)w^2, \tag{2.10}$$

$$X(x,0)-\frac{1}{2}\varphi''(r)w(x,0)^2$$
$$\leq\frac{1+\epsilon}{2}\varphi\{w_2^2+|\nabla w_1|^2\}+\frac{1}{2}(-2\varphi''+\varphi' b+\epsilon^{-1}\varphi^{-1}\varphi'^2)w_1^2. \tag{2.11}$$

Similarly, we have

$$\left|\frac{x}{r}\cdot Y(x,\tau)\right|\leq\varphi(w_t^2+w_r^2)+\frac{1}{2}\varphi^{-1}\varphi'^2w^2. \tag{2.12}$$

Note here (1.3), (2.2) and that $\varphi(s)=O(s)$ as $s\to\infty$. Then (2.12) implies

$$\liminf_{R\to\infty}\int_0^t\int_{S_\Omega(R)}\left|\frac{x}{r}\cdot Y(x,\tau)\right|dSd\tau=0.$$

Thus, applying (2.10), (2.11) and letting $R\to\infty$ in (2.9), we conclude the assertion of the lemma. □

Lemma 2.2. *Let $w(t)$ be as in the above lemma. Suppose that*

$$\varphi(r+t)b(x,t)\geq 2\varphi'(r+t), \tag{2.13}$$

$$\{\varphi'(r+t)b(x,t)\}_t\leq 0 \tag{2.14}$$

for any $(x,t)\in\Omega\times(0,\infty)$. Then we have

$$\|w(t)\|_{E_\varphi}^2\leq 3\|w(0)\|_{E_\varphi}^2+2\int_\Omega\{-\varphi''(r)+\varphi'(r)\ b\ (x,0)\}w_1^2(x)\,dx<\infty. \tag{2.15}$$

Thus, the energy of $w(t)$ decays like

$$\|w(t)\|_E^2 = O(\varphi(t)^{-1}) \quad as \quad t \to \infty. \tag{2.16}$$

Proof. We put $\epsilon=1/2$ in (2.8). Then it follows from (2.2) and (2.13) that

$$-2\varphi''+\varphi' b-\epsilon^{-1}\varphi^{-1}\varphi'^2\geq 0, \quad -2\varphi''+\varphi' b+\epsilon^{-1}\varphi^{-1}\varphi'^2\leq 2(-\varphi''+\varphi' b).$$

Applying these inequalities and (2.13), (2.14) in (2.8), we obtain (2.15) and hence (2.16). ▢

Proof of Theorem 1. We choose

$$\varphi(s) = [\log^{[n]}(e_n+s)]^{\mu}. \tag{2.17}$$

Note that $\mu\leq 1$. Then (2.5) follows from condition (1.5). So, Theorem 1 is proved if we can verify that the above φ satisfies conditions (2.1) ~ (2.3) and (2.13), (2.14) of Lemma 2.2.

(2.1) is obvious from (2.17). Differentiating (2.17), we have

$$\varphi'=\mu[n]^{\mu-1}[n-1]^{-1}\cdots[2]^{-1}[1]^{-1}[0]^{-1}, \tag{2.18}$$

$$\begin{aligned}\varphi''=&-\mu[n]^{\mu-1}[n-1]^{-1}\cdots[2]^{-1}[1]^{-1}[0]^{-2}\\ &-\mu[n]^{\mu-1}[n-1]^{-1}\cdots[2]^{-1}[1]^{-2}[0]^{-2}\\ &\cdots\\ &-\mu[n]^{\mu-1}[n-1]^{-2}\cdots[2]^{-2}[1]^{-2}[0]^{-2}\\ &-\mu(1-\mu)[n]^{\mu-2}[n-1]^{-2}\cdots[2]^{-2}[1]^{-2}[0]^{-2},\end{aligned}$$

$$\begin{aligned}\varphi'''=&\left\{-2\sum_{i=0}^{n-1}[i]^{-1}\cdots[0]^{-1}-(2-\mu)[n]^{-1}\cdots[0]^{-1}\right\}\varphi''\\ &-\mu\sum_{k=1}^{n}[k]^{-1}\cdots[0]^{-1}\sum_{i=1}^{k}[n]^{\mu-1}\cdots[i]^{-1}[i-1]^{-2}\cdots[0]^{-2},\end{aligned}$$

where $[k]=\log^{[k]}(e_n+s)$ $(k=0,1\cdots, n)$. These show (2.2) except the inequality $\varphi'''(s)\geq 0$, which also holds true since we have

$$\begin{aligned}\varphi'''\geq&\left\{-2\sum_{i=0}^{n-1}[i]^{-1}\cdots[0]^{-1}-(2-\mu)[n]^{-1}\cdots[0]^{-1}\right\}\varphi''\\ &+\sum_{k=1}^{n}[k]^{-1}\cdots[0]^{-1}\varphi''\end{aligned}$$

$$=\left\{-2[0]^{-1}-\sum_{k=1}^{n-1}[k]^{-1}\cdots[0]^{-1}-(1-\mu)[n]^{-1}\cdots[0]^{-1}\right\}\varphi''>0.$$

Next, note

$$\frac{\varphi''}{\varphi'}=-\sum_{k=0}^{n-1}[k]^{-1}\cdots[0]^{-1}-(1-\mu)[n]^{-1}\cdots[0]^{-1},$$

$$\frac{\varphi'''}{\varphi''}\leq -2[0]^{-1}-\sum_{k=1}^{n-1}[k]^{-1}\cdots[0]^{-1}-(1-\mu)[n]^{-1}\cdots[0]^{-1}.$$

Then it follows that

$$\frac{2\varphi'''}{\varphi''}-\frac{\varphi''}{\varphi'}\leq -3[0]^{-1}-\sum_{k=1}^{n-1}[k]^{-1}\cdots[0]^{-1}-(1-\mu)[n]^{-1}\cdots[0]^{-1}\leq 0.$$

This proves (2.3).

(2.13) easily follows from (A1), (2.17) and (2.18) since $\mu \leq b_0/2$. (2.14) is obvious from (A1) and (2.2).

Thus, the assertion of Theorem 1 results from Lemma 2.2 if we choose

$$K=3\|w(0)\|_{E_\varphi}^2+2\int_\Omega\{-\varphi''(r)+\varphi'(r)b(x,0)\}w_1^2(x)\,dx$$

in (2.15). □

§3. Proof of Theorem 2

Throughout this § we assume (A2).

Let $\psi(s)$ be a positive smooth function of $s\geq 0$ satisfying

(3.1) $\psi(s)$ is bounded, monotone increasing in s, and $\psi(s)\geq s\psi'(s)$.

We multiply by $\psi(r)(w_r+\frac{N-1}{2r}w)$ on both sides of equation (1.1). It then follows that

(3.2) $$\tilde{X}_t+\nabla\cdot\tilde{Y}+\tilde{Z}=0,$$

where

$$\widetilde{X}=\phi w_t\left(w_r+\frac{N-1}{2r}w\right),$$
$$\widetilde{Y}=-\frac{1}{2}\phi\left\{\frac{x}{r}\left(w_t^2-|\nabla w|^2+\frac{N-1}{2r}w^2\right)+2\nabla w\left(w_r+\frac{N-1}{2r}w\right)\right\};$$
$$\widetilde{Z}=\phi bw_t\left(w_r+\frac{N-1}{2r}w\right)$$
$$+(r^{-1}\phi-\phi')\left\{|\nabla w|^2-w_r^2+\frac{(N-1)(N-3)}{4r^2}w^2\right\}$$
$$+\frac{1}{2}\phi'\left\{w_t^2+\left|\nabla w+\frac{N-1}{2r}\frac{x}{r}w\right|^2+\frac{(N-1)(N-3)}{4r^2}w^2\right\}.$$

Lemma 3.1. *Let $w(t)$ be the solution to (1.1) with finite energy. Then*

(3.3)
$$\frac{1}{2}\int_0^t\int_\Omega\phi'\left\{w_t^2+\left|\nabla w+\frac{N-1}{2r}\frac{x}{r}w\right|^2+\frac{(N-1)(N-3)}{4r^2}w^2\right\}dxd\tau$$
$$+\int_0^t\int_\Omega\phi bw_t\left(w_r+\frac{N-1}{2r}w\right)dxd\tau\le 2\sup_{t>0}\int_\Omega|\widetilde{X}(x,t)|dx.$$

Proof. Integrate by parts the both sides of (3.2) over $\Omega\times(0,t)$. Then since $N\ge 3$ and $r^{-1}\phi-\phi'\ge 0$, we have

$$\int_\Omega\widetilde{X}dx\Big|_{\tau=0}^{\tau=t}+\int_0^t\int_{\partial\Omega}\nu\cdot\widetilde{Y}dSd\tau+\int_0^t\int_\Omega\phi bw_t\left(w_r+\frac{N-1}{2r}w\right)dxd\tau$$
$$+\frac{1}{2}\int_0^t\int_\Omega\phi'\left\{w_t^2+\left|\nabla w+\frac{N-1}{2r}\frac{x}{r}w\right|^2+\frac{(N-1)(N-3)}{4r^2}w^2\right\}dxd\tau\le 0,$$

where ν is the outer unit normal to the boundary $\partial\Omega$. By means of the boundary condition $w|_{\partial\Omega}=0$,

$$\int_0^t\int_{\partial\Omega}\nu\cdot\widetilde{Y}dSd\tau=\frac{1}{2}\int_0^t\int_{\partial\Omega}\phi\left\{\left(\nu\cdot\frac{x}{r}\right)|\nabla w|^2-2(\nu\cdot\nabla w)\left(\frac{x}{r}\cdot\nabla w\right)\right\}dSd\tau$$
$$=-\frac{1}{2}\int_0^t\int_{\partial\Omega}\phi\left(\nu\cdot\frac{x}{r}\right)|\nu\cdot\nabla w|^2dSd\tau.$$

Here we have $(\nu\cdot x/r)\le 0$ since the origin $\mathbf{R}^N\setminus\Omega$ is starshaped with respect to the origin. Thus, (3.3) holds. □

Lemma 3.2. *There exists a $C_\phi>0$ such that*

(3.4)
$$\int_\Omega|\widetilde{X}(x,t)|dx\le C_\phi\|w(0)\|_E^2 \text{ for any } t\ge 0.$$

Proof. By the Schwarz inequality we have

$$\int_\Omega |\widetilde{X}(x, t)|dx \le \sup_{s>0} \psi(s) \int_\Omega \left(w_t^2 + \left|w_r + \frac{N-1}{2r} w\right|^2\right) dx.$$

Thus, (3.4) follows if we use the well known inequality

$$\frac{(N-2)^2}{4} \int_\Omega \frac{1}{r^2} w^2\, dx \le \int_\Omega w_r^2\, dx$$

and (1.4).

Lemma 3.3. *Let* $w_0(t)$ *be the solution to* (1.7) *with finite energy. Then*

(3.5) $$\int_0^\infty \int_\Omega \{(e_n + r)\cdots\log^{[n-1]}(e_n + r)[\log^{[n]}(e_n + r)]^\gamma\}^{-1} w_{0t}^2\, dx dt \le C\|f\|_E^2,$$

where $\gamma > 1$ *and* $C = C(n, \gamma)$ *is a positive constant independent of* $w(t)$.

Proof. We put

$$\psi(r) = 1 - \alpha\{\log^{[n]}(e_n + r)\}^{-\gamma+1}$$

where $0 < \alpha \le \gamma^{-1} < 1$. Then

(3.6) $$\psi'(r) = \alpha(\gamma - 1)\{(e_n + r)\cdots\log^{[n-1]}(e_n + r)[\log^{[n]}(e_n + r)]^\gamma\}^{-1},$$

and it follows that

$$r^{-1}\psi(r) \ge (1-\alpha)(e_n + r)^{-1} \ge \alpha(\gamma - 1)(e_n + r)^{-1} \ge \psi'(r).$$

Thus, (3.1) is satisfied for this $\psi(r)$.

We apply Lemmas 3.1 and 3.2 with this ψ to the free solution $w_0(t)$. Then noting $b(x, t) \equiv 0$, we have

$$\int_0^t \int_\Omega \psi' w_{0t}^2\, dx d\tau \le 4C_\psi \|w_0(0)\|_E^2.$$

Since $w_0(0) = f$, this and (3.6) show (3.5). □

Our proof of Theorem 2 is based on Lemma 3.3 and the following usual energy equation.

Lemma 3.4. *We have*

$$(3.7)\qquad 2(w(t),\, w_0(t))_E+\int_0^t\int_\Omega b(x,\, t)\, w_t\, w_{0t}dxd\tau=2(w(0),\, w_0(0))_E$$

for any $t>0$, *where*

$$(3.8)\qquad 2(w(t),\, w_0(t))_E=\int_\Omega\{w_tw_{0t}+\nabla w\cdot\nabla w_0\}\, dx.$$

Proof. Differentiate (3.8) and use equations (1.1) and (1.7). Then integrations by parts give

$$2\partial_t(w(t),\, w_0(t))_E=-\int_\Omega b(x,\, t)\, w_t\, w_{0t}\, dx.$$

Thus, integrating both sides over $(0,\, t)$, we obtain (3.7). □

Proof of Theorem 2 (a). For the solution $w_0(t)$ of (1.7), $w_0(t+\sigma)$ also satisfies (1.7) with $\{f_1,\, f_2\}$ replaced by $\{w_0(\sigma),\, w_{0t}(\sigma)\}$. So, it follows from (1.10) and (3.7) that

$$(3.9)\quad 2(w_\sigma(t),\, w_0(t+\sigma))_E+\int_0^t\int_\Omega b(x,\, \tau)\, w_{\sigma t}(\tau)\, w_{0t}(\tau+\sigma)\, dxd\tau=2\|w_0(\sigma)\|_E^2.$$

Contrary to the conclusion, assume that $\|w_\sigma(t)\|_E\to 0$ as $t\to\infty$. Then since $\|w_0(t)\|_E$ is independent of t, letting $t\to\infty$ in (3.9), we obtain

$$(3.10)\qquad \int_0^\infty\int_\Omega b(x,\, t)\, w_\sigma(t)\, w_0(t+\sigma)\, dxdt=2\|w_0(\sigma)\|_E^2.$$

Thus, by the Schwarz inequality and (1.4),

$$\int_0^\infty\int_\Omega b(x,\, t)\, w_0(t+\sigma)^2 dxdt\geq 4\|w_0(\sigma)\|_E^2.$$

Since $\|w_0(\sigma)\|_E=\|f\|_E$, this contradicts to (1.9) under our reqirement (A2) on $b(x,\, t)$.

Theorem 2 (a) is thus proved. □

Proof of Theorem 2 (b). Let $U_0(t)$, $t\in\mathbf{R}$, be the unitary operator in the energy space E which represents the solution $w_0(t)$ to (1.7):

$$\{w_0(t),\ w_{0t}(t)\} = U_0(t)f.$$

Then it follows from (3.7) that

$$(U_0(-t)w(t) - U_0(-s)w(s),\ f)_E = -\int_s^t \int_\Omega b(x,\ t)\, w_t\, w_{0t}\, dx d\tau$$

for any $0 \leq s < t$, where $w(t)$ stands for the pair $\{w(t),\ w_t(t)\}$. By the Schwarz inequality and (3.5) we have

$$(3.11)\qquad |(U_0(-t)w(t) - U_0(-s)w(s),\ f)_E| \leq C\left\{\int_s^t \int_\Omega b(x,\ t)\, w_t^2\, dx d\tau\right\}^{1/2} \|f\|_E.$$

$f = \{f_1,\ f_2\}$ being any pair in E, we see from (3.11) that

$$\|U_0(-t)w(t) - U_0(-s)w(s)\|_E \to 0 \quad \text{as} \quad s,\ t \to \infty,$$

and $U_0(-t)w(t)$ converges in E as $t \to \infty$. Put

$$f^+ \equiv \{f_1^+,\ f_2^+\} = s\text{-}\lim_{t\to\infty} U_0(-t)w(t).$$

Then $f^+ \in E$ and we have

$$\|w(t) - U_0(t)f^+\|_E = \|U_0(-t)w(t) - f^+\|_E \to 0 \quad \text{as} \quad t \to \infty.$$

Theorem 2 (b) is thus proved. □

§4. Energy Decay for Quasilinear Wave Equations

In this § we remark that our proof of the energy decay can be applied to some quasilinear wave equations.

Consider the Cauchy problem

$$(4.1)\qquad \begin{cases} w_{tt} - \nabla \cdot \{\sigma(|\nabla w|^2)\nabla w\} + b(x,\ t)w_t = 0, & (x,\ t) \in \mathbf{R}^N(0,\ \infty) \\ w(x,\ 0) = w_1(x),\ w_t(x,\ 0) = w_2(x) & x \in \mathbf{R}^N, \end{cases}$$

where $\sigma(s) = 1/\sqrt{1+s}$ and $b(x,\ t) \geq 0$. For the sake of simplicity, we assume

$$(4.2)\qquad \{w_1(x),\ w_2(x)\} \in C_0^\infty(\mathbf{R}^N) \times C_0^\infty(\mathbf{R}^N).$$

The energy of solutions at time t is defined by

$$\|w(t)\|_E^2 = \frac{1}{2}\int_{\mathbf{R}^N}\{w_t(t)^2 + \sigma_1(|\nabla w(t)|^2)\}\,dx,$$

and a weighted energy of solutions at time t is defined by

$$\|w(t)\|_{E_\varphi}^2 = \frac{1}{2}\int_{\mathbf{R}^N}\varphi(r+t)\{w_t^2 + \sigma_1(|\nabla w|^2)\}\,dx,$$

where

$$\sigma_1(\eta) = \int_0^\eta \sigma(s)\,ds$$

and $\varphi(s)$ is a function satisfying (2.1) $\sim$ (2.3).

Lemma 4.1. *For any $t>0$ and $0<\epsilon<1$, the solution $w(t)$ of (4.1) admits the inequality*

$$\begin{aligned}(4.3)\qquad &(1-\epsilon)\|w(t)\|_{E_\varphi}^2 + \frac{1}{2}\int_{\mathbf{R}^N}(-2\varphi'' + \varphi' b - \epsilon^{-1}\varphi^{-1}\varphi'^2)w^2 dx \\ &+ \int_0^t\int_{\mathbf{R}^N}\left\{(\varphi b - 2\varphi')w_t^2 - \frac{1}{2}(\varphi' b)_t w^2\right\}dx d\tau \\ &\le (1+\epsilon)\|w(0)\|_{E_\varphi}^2 + \frac{1}{2}\int_{\mathbf{R}^N}(-2\varphi'' + \varphi' b + \epsilon^{-1}\varphi^{-1}\varphi'^2)w_1^2 dx.\end{aligned}$$

Proof. Multiply by $\{\varphi(r+t)w\}_t$ on both sides of (4.1). Then as in § 2, it follows that

$$(4.4)\qquad X_t + \nabla\cdot Y + Z = 0,$$

where

$$\begin{aligned} X &= \frac{1}{2}\varphi\{w_t^2 + \sigma_1(|\nabla w|^2)\} + \varphi' w_t w + \frac{1}{2}(\varphi' b - \varphi'')w^2 \\ Y &= -(\varphi w_t + \varphi' w)\sigma(|\nabla w|^2)\nabla w, \\ Z &= (\varphi b - 2\varphi')w_t^2 + \frac{1}{2}\varphi'|w_t + \sigma(|\nabla w|^2)w_r + \varphi'^{-1}\varphi'' w|^2 \\ &\quad + \frac{1}{2}\varphi'\{-\sigma_1(|\nabla w|^2) + 2\sigma(|\nabla w|^2)|\nabla w|^2 - \sigma(|\nabla w|^2)^2 w_r^2\} \\ &\quad + \frac{1}{2}\{\varphi''' - \varphi'^{-1}\varphi''^2 - (\varphi' b)_t\}w^2 - \varphi'' w_t w.\end{aligned}$$

Since we have

$$-\sigma_1(s)+2\sigma(s)s-\sigma(s)^2s=\left(1-\frac{1}{\sqrt{1+s}}\right)^2\geq 0;$$
$$-\varphi''w_tw=-\frac{1}{2}\partial_t[\varphi''w^2]+\frac{1}{2}\varphi'''w^2,$$

it follows that

$$Z\geq(\varphi b-2\varphi')w_t^2-\frac{1}{2}(\varphi'b)_tw^2-\frac{1}{2}\partial_t[\varphi''w^2]. \tag{4.5}$$

Integrate by parts (4.4) over $\mathbb{R}^N\times(0,t)$. Then since $w(t)$ has a finite propagation speed, noting (4.2) and (4.5), we can follow the proof of Lemma 2.1 to conclude the assertion.

As in §2, we can easily prove the following theorem with this lemma.

Theorem 3. *Assume* (A1) *with* $\Omega=\mathbb{R}^N$, *let* $\{w_1, w_2\}$ *satisfy* (4.2) *and let* $w(t)$ *be the corresponding solution to* (4.1). *If* $w(t)$ *is global, then its energy decays as t goes to infinity. More precisely, there exists a* $K=K(w_0, w_1, n)>0$ *such that*

$$\|w(t)\|_E^2\leq K\{\log^{[n]}(e_n+t)\}^{-\mu}, \tag{4.6}$$

where $\mu=\min\{1, b_0/2\}$.

Remark. A similar result on the energy decay can be obtained for equations with nonlinear dissipation $b(x,t)|w_t|^{\rho-1}w_t$ under suitable restrictions on $b(x,t)$ and $\rho>1$ as given in [4], where is studied decay and nondecay properties for semilinear equations.

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