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On the K-Theory of Cuntz-Krieger Algebras

By

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Abstract

We extend the uniqueness and simplicity results of Cuntz and Krieger to the countably infinite case, under a row-finite condition on the matrix A. Then we present a new approach to calculating the K-theory of the Cuntz-Krieger algebras, using the gauge action of $\mathbf{T}$, which also works when A is a countably infinite 0-1 matrix. This calculation uses a dual Pimsner-Voiculescu six-term exact sequence for algebras carrying an action of $\mathbf{T}$. Finally, we use these new results to calculate the K-theory of the Doplicher-Roberts algebras.

§1. Introduction

In [4], [5], [3], Cuntz and Krieger studied the C^*-algebras generated by a family of n non-zero partial isometries S_i, satisfying the Cuntz-Krieger relations

$$S_i^* \ S_k = \delta_{i,k} \sum_{j=1}^{n} A(i,j) \ S_j S_j^*, \ 1 \leq i,k \leq n, \tag{1}$$

where A is an $n \times n$, 0-1 matrix with no zero row or column. Let $\sum_A$ denote the set of finite sequences $\mu = (\mu_1, \cdots, \mu_k)$ with $1 \leq \mu_i \leq n$ and $A(\mu_i, \mu_{i+1}) = 1$ for $i = 1, \cdots, k-1$. The length k of the sequnce μ is denoted by $|\mu| = k$. We may also think of $\mu \in \sum_A$ as a finite path in the infinite graph with vertices at each level labelled $1, \cdots, n$, in which for $k \geq 1$ there is an edge joining vertex i on the k^{th} level to vertex j on the $k+1^{\text{st}}$ level if and only if $A(i, j) \neq 0$. For instance, the k^{th} and $k+1^{\text{st}}$ levels of such a graph may look like the diagram given below:

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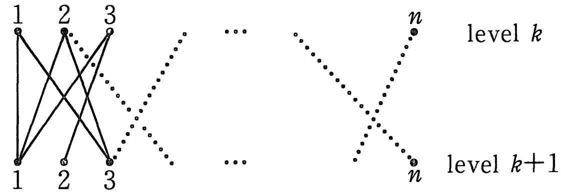

Definition 1.1. *Let Σ_0 denote the set of i_0. $1 \leq i_0 \leq n$ such that there are at least two different elements $\mu, \nu \in \Sigma_A$ such that $\mu_1 = \nu_1 = \mu_{|\mu|} = \nu_{|\nu|} = i_0$, and μ_p, $\nu_q \neq i_0$ for $1 < p < |\mu|$, $1 < q < |\nu|$. The matrix A satisfies condition* (I) *provided, for each i, with $1 \leq i \leq n$, there is some $m \geq 1$ and $i_0 \in \Sigma_0$ with $A^m(i, i_0) \neq 0$.*

In particular, if A is irreducible (i.e. for all i,j, there is a strictly positive integer $m = m(i, j)$ such that $A^m(i, j) \neq 0$) and is not a permutation matrix, then A satisfies condition (I). It was shown in [5] that when A satisfies (I), the C^*-algebra generated by the S_i, $i = 1, \cdots, n$ is independent of the choice of the partial isometries S_i, and simple whenever A is irreducible. It may therefore be denoted by $\mathcal{O}_A$. More precisely, in [5, 2.13 and 2.14] it was shown that:

Theorem 1.2. (i) *Suppose that A is a finite* {0,1} *matrix satisfying* (I) *and S_i, T_i, $i = 1, \cdots, n$, are two families of non-zero partial isometries satisfying the same Cuntz-Krieger relations* (1). *Then the map $S_i \mapsto T_i$ extends to an isomorphism of $C^*(S_1, \cdots, S_n)$ onto $C^*(T_1, \cdots, T_n)$.*

(ii) *If the matrix A is irreducible and not a permutation matrix, then the C^*-algebra $\mathcal{O}_A = C^*(S_1, \cdots, S_n)$ is simple.*

In the next section we give conditions (J) on a countably infinite 0-1 matrix A, under which the following theorem holds:

Theorem 1. (i) *Suppose that A is a countably infinite* {0,1} *matrix satisfying* (J) *and S_i, T_i, $i \in \mathbf{N}$ are two families of non-zero partial isometries satisfying the same infinite Cuntz-Krieger relations. Then the map $S_i \mapsto T_i$ extends to an isomorphism of $C^*(S_i)$ onto $C^*(T_i)$.*

(ii) *If the matrix A row-finite and irreducible then the C^*-algebre $\mathcal{O}_A = C^*(S_i)$ is simple.*

While condition (J) is analogous to condition (I), in order to get the simplicity result, we must assume irreducibility and a finiteness condition to ensure that certain topological obstacles do not occur. In the third section we review the proof of the following theorem of Kishimoto and Takai, [9, Theorem 2], since we shall need explicit details of the isomorphism later.

Theorem 2. *Let B be unital C^*-algebra and β a strongly continuous action of a compact group G which has large spectral subspaces, then the fixed point algebra B^β is stably isomorphic to $B\times_\beta G$.*

In the fourth section, we calculate the K-theory of the $\mathcal{O}_A$ defined in the first section, generalising the results for finite matrices (see [5], [3]). In particular, we prove the following:

Theorem 3. *If A is a countably infinite 0-1 matrix which satisfies condition (J) and is row-finite, then there is an exact sequence*

$$0 \longrightarrow K_1(\mathcal{O}_A) \longrightarrow \tilde{\mathbf{Z}}^\infty \xrightarrow{1-A^t} \tilde{\mathbf{Z}}^\infty \xrightarrow{i_*} K_0(\mathcal{O}_A) \longrightarrow 0,$$

so that $K_1(\mathcal{O}_A) \cong \operatorname{Ker}\{(1-A^t) : \widetilde{\mathbf{Z}}^\infty \to \widetilde{\mathbf{Z}}^\infty\}$ and $K_0(\mathcal{O}_A) \cong \widetilde{\mathbf{Z}}^\infty/\operatorname{Im}(1-A^t)\ \widetilde{\mathbf{Z}}^\infty$. The map i_* carries each canonical generator ξ_j of $\widetilde{\mathbf{Z}}^\infty$ to the projection $[P_j] = [S_j\ S_j^*] = [S_j^*\ S_j]$ in $K_0(\mathcal{O}_A)$ for each $j \in \mathbf{N}$. Thus we see that $K_0(\mathcal{O}_A)$ is generated by the projections $[P_i]$, for each $i \in \mathbf{N}$, subject to $[P_i] = \sum_{j=1}^\infty A(i, j)\ [P_j]$ induced from the Cuntz-Krieger relations.

While this result is not suprising, and could conceivably be deduced from the known results finite A, we feel our proof is of some interest even in the finite case. We bypass the natural realisation of $\mathcal{O}_A \otimes \mathcal{K}$ as a crossed product by $\mathbf{Z}$, using instead the gauge action of $\mathbf{T}$ on $\mathcal{O}_A$, and the six-term exact sequence in K-theory dual to the Pimsner-Voiculescu of [11, §3] (which curiously was described in [1, §10.6] as of limited use). Thus our argument give an alternative approach to the calculation of $K_*(\mathcal{O}_A)$ which may be slightly more accessible.

Our original motivation for this work was to calculate the K-theory of the algebras $\mathcal{O}_\rho$ appearing in the Doplicher-Roberts duality theory for compact groups. In [10] , it was shown that $\mathcal{O}_\rho$ was isomorphic to a corner in a C^*-algebra generated by an infinite Cuntz-Krieger family; now we know by Theorem 1 that this Cuntz-Krieger algebra $\mathcal{O}_{A_\rho}$ is simple, we have $K_*(\mathcal{O}_\rho) \cong K_*(\mathcal{O}_{A_\rho})$, and we can use Theorem 3 to compute $K_*(\mathcal{O}_\rho)$. In fact, we can do better: we can identify $\widetilde{\mathbf{Z}}^\infty$ with the representation ring $\mathcal{R}(G)$, and $K_*(\mathcal{O}_\rho)$ with the kernel and cokernel of the map $[\pi] \mapsto [\pi \otimes \rho]$ on $\mathcal{R}(G)$. In the fifth section we shall briefly discuss this approach.

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We originally learned about Doplicher-Roberts duality and the computation of $K_*(\mathcal{O}_\rho)$ in seminars, discussions and joint work with Colin Sutherland (cf. [10], [12]). We thank him for his help and interest, and particularly for suggesting to us that we should interpret our results in terms of the representation

ring.

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§2. Infinite Cuntz-Krieger Algebras

2.1. Uniqueness and Simplicity

The following definitions and results are extensions of those given in [5, p.253], where it is claimed that their results for finite matrices carry over to the infinite case. Upon closer inspection, this does not seem as straightforward as originally thought. Given a countably infinite 0-1 matrix, A, with no zero row or column, we consider the C^*-algebra generated by non-zero partial isometries S_i, $i \in \mathbb{N}$, satisfying the Cuntz-Krieger relations,

$$S_i^* S_k = \delta_{i,k} \sum_{j=1}^{\infty} A(i, j) S_j S_j^*, \quad \text{for all } i, k \in \mathbb{N} \tag{2}$$

where the infinite sum above converges in the strong operator topology. If we have $\mu = (\mu_1, \cdots, \mu_k) \in \Sigma_A$, we write $S_\mu = S_{\mu_1} \cdots, S_{\mu_k}$, and then each S_μ is a partial isometry with range projection denoted by $P_\mu = S_\mu S_\mu^*$. In particular, $P_i = S_i S_i^*$ denotes the range projection of each partial isometry S_i, for all $i \in \mathbb{N}$.

Definition 2.1.1. *Let Σ_∞ denote the set of $i_\infty \in \mathbb{N}$ such that there are at least two distinct paths $\mu, \nu \in \Sigma_A$, such that $\mu_1 = \nu_1 = \mu_{|\mu|} = \nu_{|\nu|} = i_\infty$, and μ_p, $\nu_q \neq i_\infty$ for $1 < p < |\mu|$, $1 < q < |\nu|$. The matrix A satisfies condition* (J) *provided there is a finite subeset $\Sigma_1 \subseteq \Sigma_\infty$ such that for each $i \in \mathbb{N}$ there is some $m \geq 1$ and $i_1 \in \Sigma_1$ with $A^m(i, i_1) \neq 0$.*

Note 2.1.2. As in [5, p.254] we note that if the countably infinite 0-1 matrix A is irreducible, then it satisfies condition (J), because irreducibility implies that there is at least one vertex with at least two edges eminating from it, from which we may construct the required paths μ, ν. Here we can dispense with the requirement for A not to be a permutation matrix as the above construction relies on the infinite nature of the graph as well as the irreducibility of the matrix A.

Also, we note that, for countably infinite 0-1 matrices, condition (J) is stronger than the full countably infinite version of condition (I) where we do not demand the existence of the finite subset Σ_1.

Example 2.1.3. Consider the matrices

$$A_1=\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} A_2=\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} A_3=\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then A_1 satisfies condition (J), with $\Sigma_\infty=\Sigma_1=\{1, 2\}$. On the other hand A_2 satisfies the full countably infinite version of condition (I), but does not satisfy condition (J), because $\Sigma_\infty=\mathbf{N}$, and each $i_\infty\in\Sigma_\infty$ is connected to precisely two indices. The irreducible matrix A_3 satisfies condition (J), even though $\Sigma_\infty=\mathbf{N}$; we can take any finite subset for Σ_1.

We shall henceforth assume that our infinite 0-1 matrix A satisfies condition (J).

Lemma 2.1.4. *For each $n\in\mathbf{N}$, there is a positive integer m_n and a partition I_n^k, $1\leq k\leq m_n$ of $\mathbf{N}$ such that:*

(i) *$I_n^k=\{k\}$ if $1\leq k\leq n$.*

(ii) *For $1\leq i\leq n$, there are subsets K_i of $\{1,\cdots,m_n\}$ such that for all $j\in\mathbf{N}$*

$$A(i, j)=1 \text{ if and only if } j\in\bigcup_{k\in K_i} I_n^k.$$

Proof. Add to $I_n^k=\{k\}$, $1\leq k\leq n$, the partition of $\{n+1, n+2,\cdots,\}$ generated by the sets

$$J_i=\{j>n : A(i, j)=1\}$$

for $i=1,\cdots,n$ and the set $\mathbf{N}\setminus(\cup_{i=1}^n J_i)$, giving m_n sets in all. Roughly speaking, I_n^k, for $k>n$ represent the vertices which can be reached in one step from each of some, possibly the subsets of $\{1,\cdots,n\}$. The sets K_i, $i=1,\cdots,n$ consist of those superscripts k of the subsets I_n^k whose vertices are reached from vertex i in exactly one step. We may see that each K_i is non-empty from the definition of the J_i. Finally, the number m_n is finite for each n, since the number of disjoint subsets of $\mathbf{N}$ which the J_i, $i=1,\cdots,n$ can generate is finite. $\square$

Definition 2.1.5. *For each $n \in \mathbf{N}$ define an $m_n \times m_n$ 0-1 matrix B_n by*

$$B_n(k, l) = \begin{cases} 1 \text{ if } A(i, j) = 1 \text{ for some } i \in I_n^k, \ j \in I_n^k, \\ 0 \text{ otherwise.} \end{cases}$$

We think of the new vertices $\{1, \cdots, m_n\}$ as those obtained by identifying all vertices comprising I_n^k, and joining new vertex k to new vertex l if any vertex in I_n^k is joined to any vertex from I_n^l in the original graph. Note that, by construction, no row or column of B_n is zero.

Lemma 2.1.6.

(i) *If n is large enough, then the matrix B_n satisfies condition* (I).

(ii) *If $1 \leq k \leq n$, then $K_k = \{l : B_n(k, l) = 1\}$.*

Proof. From 2.1.1, let $\Sigma_1 \subset \mathbf{N}$ be given for the countably infinite 0-1 matrix A. Choose n sufficiently large so that $n \geq i_1$ for each $i_1 \in \Sigma_1$ and each of the designated paths μ, $\nu \in \Sigma_A$ for i_1 only visit the first n vertices. This means that for this n, we have $i_1 \in \Sigma_0$ for B_n since we are not identifying any edges used in the paths μ, ν. Hence the set Σ_0 for the matrix B_n contains Σ_1. We know from condition (J), that all vertices in the original graph connect to a vertex in Σ_1. Thus, since in constructing B_n we are effectively adding new paths to the original graph, each of the new vertices must connect to a vertex in $\Sigma_1 \subseteq \Sigma_0$, which is sufficient for condition (I) (see 1.1).

For (ii), if $k \leq n$, then by definition, $B_n(k, l) = 1$ if and only if there is some $j \in I_n^l$ such that $A(k, j) = 1$. Hence this is so if and only if

$$I_n^l \subset \bigcup_{m \in K_k} I_n^m,$$

that is, if and only if $l \in K_k$. $\square$

Example 2.1.7. For the matrices A_1, A_2 and A_3 given in 2.1.3, we see how the above proof gives rise to the need for condition (J) over the full infinite version of condition (I). For A_1 if we choose $n=2$, then we have

$$B_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

which satisfies condition (I). However, for A_2, whenever $n\geq 2$, say 2, then we have

$$B_2=\begin{pmatrix}0&1&0\\1&1&0\\0&0&1\end{pmatrix}$$

which does not satisfy condition (I), as the third vertex only connects to itself. For matrix A_3, whenever $n\geq 2$, say 2, we obtain

$$B_2=\begin{pmatrix}0&1&0&0\\1&1&1&0\\0&1&1&1\\0&0&1&1\end{pmatrix}$$

which does satisfy condition (I).

Theorem 2.1.8. *Suppose that A is a countably infinite 0-1 matrix satisfying condition (J) and that $\{S_i\}$, $\{T_i\}$ are two families of non-zero partial isometries satisfying the same Cuntz-Krieger relations (2). Then there is an isomorphism ϕ of $C^*(S_i)$ onto $C^*(T_i)$ such that $\phi(S_i)=T_i$ for all $i\in\mathbf{N}$.*

Proof. Define $S_{k,n}=S_k$, for $1\leq k\leq n$, and for $n<k\leq m_n$ take $S_{k,n}$ to be a partial isometry with range projection

$$S_{k,n}S_{k,n}^*=\sum_{j\in I_n^k}^{s.o.} S_j S_j^*, \tag{3}$$

and initial projection

$$S_{k,n}^* S_{k,n}=\sum_{l=1}^{m_n} B_n(k,l)\Big(\sum_{j\in I_n^l}^{s.o.} S_j S_j^*\Big).$$

Since the inside sum is over a finite set, the right hand side converges in the strong operator topology, as do all the sums which follow. With the above definitions, we claim that the partial isometries $S_{k,n}$, $1\leq k\leq m_n$ satisfy

$$S_{k,n}^* S_{k,n}=\sum_{l=1}^{m_n} B_n(k,l)S_{l,n}S_{l,n}^*.$$

To see this, for $1 \leq k \leq n$, we have that

$$\begin{aligned} S_{k,n}^{*} S_{k,n} = S_{k}^{*} S_{k} &= \sum_{j \in J_k} A(k, j) S_j S_j^{*} \\ &= \sum_{j \in \cup_{l \in K_k} I_n^l} S_j S_j^{*} \text{ by 2.1.4ii)} \\ &= \sum_{l \in K_k} \Big(\sum_{j \in I_n^l} S_j S_j^{*} \Big) \\ &= \sum_{l=1}^{m_n} B_n(k, l) \Big(\sum_{j \in I_n^l} S_j S_j^{*} \Big) \quad \text{by 2.1.6ii)} \\ &= \sum_{l=1}^{m_n} B_n(k, l) \; S_{l,n} S_{l,n}^{*} \quad \text{by (3).} \end{aligned}$$

If $k > n$ then the condition holds by definition of the $S_{k,n}$.

If we do the same construction for the T_i's; then, since by 2.1.6 i) B_n satisfies condition (I) we may apply the Cuntz-Krieger theorem 1.2 i) to give an isomorphism of $C^*(S_{1,n}, \cdots, S_{m_n,n})$ onto $C^*(T_{1,n}, \cdots, T_{m_n,n})$ carrying $S_{k,n}$ to $T_{k,n}$ for $1 \leq k \leq m_n$. This restricts to an isomorphism ϕ_n of $C^*(S_1, \cdots, S_n)$ onto $C^*(T_1, \cdots, T_n)$, and hence we have a countable family of isomorphisms ϕ_n defining an isomorphism

$$\phi : \bigcup_n C^*(S_1, \cdots, S_n) \to \bigcup_n C^*(T_1, \cdots, T_n).$$

Each ϕ_n is isometric, hence so is ϕ, and ϕ extends to the closure and has the desired properties. □

Corollary 2.1.9. *If A is a countably infinite matrix which satisfies condition* (J), *then there is an action α (called the gauge action) of $\mathbb{T}$ on $\mathcal{O}_A$ such that*

$$\alpha_z(S_i) = z S_i \quad \textit{for all} \quad i \in \mathbb{N}.$$

Proof. To see this, we note that the partial isometries $T_i = zS_i$, $i \in \mathbb{N}$ also satisfy condition (J) and the infinite Cuntz-Krieger relations (2). Thus Theorem 2.1.8 gives an isomorphism α_z of $\mathcal{O}_A$. The map $z \mapsto \alpha_z(a)$ is continuous when a lies in the $*$-subalgebra generated by the S_i and hence for all $a \in \mathcal{O}_A$.

Definition 2.1.10. *The countably infinite* 0-1 *matrix* A *is said to be row-finite if, for each* $i \in \mathbf{N}$ *the number of* $j \in \mathbf{N}$ *with* $A(i, j) \neq 0$ *is finite.*

Note 2.1.11. The row-finite condition defined above implies that the sums occurring in the infinite Cuntz-Krieger relations (2) are finite, and hence trivially converge. Also, from [15, Lemma 1.5] the row-finite condition on A is a necessary and sufficient condition for the one-sided infinite path space X_A^+ to be given a locally compact topology. Irreducibility then implies that this topology has no isolated points, cf. [5, p.254].

Corollary 2.1.12. *Suppose the countably infinite* 0-1 *matrix* A *is row-finite and irreducible, then the* C^**-algebra* $\mathcal{O}_A = C^*(S_i)$ *is simple.*

Proof. Suppose $I \lhd \mathcal{O}_A$ is a proper closed 2-sided $*$-ideal. We claim that no S_i can belong to I. For if $S_i \in I$ for some $i \in \mathbf{N}$, then $S_i^* S_i \in I$, in which case $S_j = S_i^* S_i S_j$ belongs to I whenever $A(i, j) = 1$. By induction we would then have $S_\mu \in I$ whenever $\mu \in \sum_A$ satisfied $\mu_1 = i$ or $A(i, \mu_1) = 1$. Since A is irreducible by hypothesis, we may thus show that $S_k \in I$ for all $k \in \mathbf{N}$ since $S_k = S_\mu^* S_\mu S_k \in I$ for any path $\mu \in \sum_A$ with $A(\mu_{|\mu|}, k) = 1$ and $\mu_1 = i$ say. Since we know that $\mathcal{O}_A = C^*(S_i)$, this would imply that $I = \mathcal{O}_A$, which is a contradiction.

Since $S_i \notin I$, for all $i \in \mathbf{N}$ and the sums in (2) are finite, the images $q(S_i) = S_i + I$ in the quotient algebra $\mathcal{O}_A / I$ are non-zero partial isometries satisfying the infinite Cuntz-Krieger relations (2). Thus there is an isomorphism ϕ of $\mathcal{O}_A / I$ onto $\mathcal{O}_A$, such that $\phi(S_i + I) = S_i$ for all $i \in \mathbf{N}$. But then the composition $\phi \circ q$ must be the identity, so $I = \{0\}$, as required. □

Remark 2.1.13. (i) This last argument will not work if A is not row-finite, since the quotient map need not respect the infinite Cuntz-Krieger relations (2), because they involve strong operator convergence.
(ii) For the algebra $\mathcal{O}_\infty$, the countably infinite 0-1 matrix A consists entirely of 1's, and it may be considered as the direct limit of the $\mathcal{O}_{B_n}$. Thus we can immediately deduce the simplicity of $\mathcal{O}_\infty$ and compute its K-theory (using the continuity of K-theory). In general, our proof of Theorem 2.1.8 does not show how to compute $K_0(\mathcal{O}_A)$, because the algebras $\mathcal{O}_{B_n}$ lie partly outside $\mathcal{O}_A$.

2.2. The AF-core

Throughout this section A will be row-finite, for such A, as in [5, Lemma 2.2], every word in S_i and S_i^* is a linear combination of terms of the form $S_\mu P_j S_\nu^*$ for some $j \in \mathbf{N}$ and $\mu, \nu \in \sum_A$. Following [5, p.253], for each $i \in \mathbf{N}$ and $k \geq 0$, we let $\mathcal{F}_k(i)$ be the C^*-algebra generated by all elements of the form $E^i_{\mu,\nu} = S_\mu P_i S_\nu^*$ where $|\mu| = |\nu| = k$. Since A is row finite, $\mathcal{F}_k(i)$ is a finite dimensional full matrix algebra, since there can only be finitely many paths in $\sum_A$ of length k ending at vertex i.

Definition 2.2.1. *For $j\in\mathbf{N}$, $k\geq 0$, let $V_k^j=\{i\in\mathbf{N} : A^k(h, i)\neq 0$ for some $1\leq h\leq j\}$, and for $i\in V_k^j$ let $\mathcal{F}_k^j(i)$ be the C^*-algebra generated by all elements of the form $E_{\mu,\nu}^i$ where $|\mu|=|\nu|=k$, and $1\leq\mu_1, \nu_1\leq j$. We denote by $\mathcal{F}_A$ the closure of the infinite union $\cup_{k,j}\cup_{i\in V_k^j}\mathcal{F}_k^j(i)$.*

Lemma 2.2.2. *For each $j\in\mathbf{N}$, $k\geq 0$ and $i\in V_k^j$ the C^*-algebra $\mathcal{F}_k^j(i)$ is a full matrix algebra with matrix units $E_{\mu,\nu}^i$, where $|\mu|=|\nu|$, $1\leq\mu_1, \nu_1\leq j$ and $\mu, \nu\in\sum_A$. The C^*-algebra $\mathcal{F}_A$ is an AF-algebra.*

Proof. As in [5, Proposition 2.3] the elements $E_{\mu,\nu}^i$, with $|\mu|=|\nu|=k$ satisfy

$$E_{\mu,\nu}^i E_{\kappa,\sigma}^j=\delta_{i,j}\delta_{\nu,\kappa}E_{\mu,\sigma}^i \tag{4}$$

for μ, ν, κ, $\sigma\in\sum_A$. This implies in particular that, for fixed i, the $E_{\mu,\nu}^i$ form a system of matrix units; since A is row-finite there are only finitely many paths $\mu\in\sum_A$ with $1\leq\mu_1\leq j$ and $\mu_k=i$, hence $\mathcal{F}_k^j(i)$ is a full matrix algebra.

From the definition we know that

$$\mathcal{F}_A=\overline{\operatorname{span}}\{S_\mu P_i S_\nu^* : |\mu|=|\nu|=k, A(\mu_k, i)=A(\nu_k, i)=1\}.$$

We order these elements first by fixing the level k at which we operate, then by fixing the terminating vertex at level k and finally by restricting the starting points of our paths $\mu,\nu\in\sum_A$ to a range $1\leq\mu_1, \nu_1\leq j$. As j increases, we just add more matrix units $E_{\mu,\nu}^i$ to the collection spanning $\mathcal{F}_k^j(i)$. Thus the increasing union of matrix algebras, $\mathcal{F}_k(i)=\cup_j\mathcal{F}_k^j(i)$ is either itself a matrix algebra, or a copy of the compacts.

The algebras $\mathcal{F}_k(i)$, for $i\in\mathbf{N}$ are mutually orthogonal by (4), so $\mathcal{F}_k=\cup_i\cdot\mathcal{F}_k(i)$ is actually a C^*-algebraic direct sum $\bigoplus_i\mathcal{F}_k(i)$ of C^*-algebras isomorphic to $\mathcal{K}(\mathcal{H})$ for some, possibly finite dimensional Hilbert space. In particular, each $\mathcal{F}_k$ is an AF algebra, and hence so is $\mathcal{F}_A$, which completes the proof. $\square$

Lemma 2.2.3. *With the above notation, we have that $\mathcal{F}_A=\mathcal{O}_A^\alpha$.*

Proof. For $a\in\mathcal{O}_A$ the operator $P_\alpha(a)=\int_{\mathbf{T}}\alpha_z(a)\,dz$ is a conditional expectation of $\mathcal{O}_A$ onto $\mathcal{O}_A^\alpha$ of norm 1. By definition of α, each $S_\mu P_i S_\nu^*$ with $|\mu|=|\nu|$ lies in $\mathcal{O}_A^\alpha$, hence $\mathcal{F}_k^j(i)\subset\mathcal{O}_A^\alpha$ for all $j\in\mathbf{N}$, $k\geq 0$, $i\in V_k^j$. Thus by 2.2.2

we have that $\mathscr{F}_A \subseteq \mathcal{O}_A^\alpha$.

Any $a \in \mathcal{O}_A$ may be approximated by finite linear combinations of $\{S_\mu S_\nu^* : \mu, \nu \in \sum_A\}$. From the definition of P_α we see that

$$P_\alpha(S_\mu S_\nu^*) = \int_{\mathbf{T}} z^{|\mu|-|\nu|} S_\mu S_\nu^* \, dz$$

which is non-zero if and only if $|\mu| = |\nu|$. So, if $a = P_\alpha(a) \in \mathcal{O}_A^\alpha$, continuity of P_α implies that a can be approximated by linear combinations of $\{ S_\mu S_\nu^* : \mu, \nu \in \sum_A, |\mu| = |\nu|\}$. But each linear combination belongs to $\mathscr{F}_A$, which gives us that $\mathcal{O}_A^\alpha \supseteq \mathscr{F}_A$, and completes the proof. □

§3. A Result of Kishimoto and Takai

3.1. General Theory

Convention 3.1.1. Throughout this section, B will be a unital C^*-algebra with identity 1, the identity map on B will be denoted by i, G a compact abelian group with normalised Haar measure, and discrete dual group $\widehat{G}$. $\mathscr{H}$ will denote an infinite dimensional separable Hilbert space, and $\mathscr{K}$ the C^*-algebra of compact operators on $\mathscr{H}$, generated by matrix units e_{ij}, $i, j \in \mathbf{N}$. The compact operators on the Hilbert space $L^2(G)$ will be denoted $\mathscr{K}(L^2(G))$.
We shall use the following definition which is to be found in [11, §2]:

Definition 3.1.2. *Let β be a (strongly continuous) action of a compact abelian group G on a C^*-algebra B, and B^β its fixed point algebra. For a character $\chi \in \widehat{G}$, we let $B^\beta(\chi)$ denote the spectral subspace $\{b \in B : \beta_t(b) = \chi(t) b$ for all $t \in G\}$. We say that β has large spectral subspaces if $\overline{B^\beta(\chi)^* B^\beta(\chi)} = B^\beta$ for each $\chi \in \widehat{G}$.*

Definition 3.1.3. *Let λ, ρ denote the left, right regular representations of a compact group G on $L^2(G)$, i.e.*

$$(\lambda_s \xi)(t) = \xi(s^{-1} t) \text{ and } (\rho_s \xi)(t) = \xi(ts)$$

for $s, t \in G$ and all $\xi \in L^2(G)$. Also, let M denote the representation of $C_o(G)$ as multiplication operators on $L^2(G)$ given, for $f \in C_o(G)$, by

$$(M_f \xi)(s) = (f\xi)(s) \text{ for } \xi \in L^2(G).$$

Let τ denote the action of G on $C_0(G)$ by left translation, that is $\tau_s(f)(t) = f(st)$

for $f \in C_0(G)$, $s, t \in G$.

Here we shall consider the group C^**-algebra* $C^*(G)$ *to be the closed linear span of* $\{\lambda(\gamma) : \gamma \in \widehat{G}\}$ *in* $\mathcal{L}(L^2(G))$, *where* $\lambda(\gamma)$ *is the operator given by* $\int_G \gamma(s)\lambda_s\, ds$ (*this differs from the definition used in* [9, *p.* 387], *we believe that this version is more common*). *Finally, let* $Ad\ \rho$ *denote the adjoint action of* $G = \widehat{\widehat{G}}$ *on* $\mathcal{K}(L^2(G))$, *given, for* $T \in \mathcal{K}(L^2(G))$ *and* $s \in G$ *by* $Ad\ \rho_s(T) = \rho_s T \rho_s^*$.

Lemma 3.1.4. *The algebra* $\mathcal{K}(L^2(G))$ *is the closed span of* $\{M_\chi \lambda(\gamma) : \chi, \gamma \in \widehat{G}\}$; *note that*

$$\lambda(\gamma) M_\chi = M_\chi \lambda(\overline{\chi}\gamma). \tag{5}$$

The spectral subspaces of the action $Ad\ \rho$ *on* $\mathcal{K}(L^2(G))$ *are given by*

$$\mathcal{K}(L^2(G))^{Ad\rho}(\chi) = M_\chi \lambda(C^*(G)), \tag{6}$$

for all $\chi \in \widehat{G}$.

Proof. We note, from [14, Example 4] that the triple $(\mathcal{K}(L^2(G)), M, \lambda)$ is a crossed product for $(C_0(G), G, \tau)$. Thus from [14, p.322] we know that

$$\overline{span}\ \{M_f \lambda(z) : f \in C_0(G),\ z \in C_c(G)\} = \mathcal{K}(L^2(G))$$

and by Stone-Weierstrass we also know that the closed span of $\widehat{G}$ is dense in $C_0(G)$ and $C_c(G)$. Hence the operators $M_\chi \lambda(\gamma)$ span a dense subspace of $\mathcal{K}(L^2(G))$. Equation (5) is an easy calculation. For the last part, note that ρ_s commutes with λ_t for $s, t \in G$, and hence with $\lambda(C^*(G))$, and

$$Ad\ \rho_s\ (M_\chi) = \chi(s) M_\chi,$$

so $M_\chi \lambda(C^*(G))$ is certainly contained in the spectral subspace. On the other hand the projection P_χ onto $\mathcal{K}(L^2(G))^{Ad\rho}(\chi)$ is given by

$$P_\chi(T) = \int_G Ad\rho_t(T)\overline{\chi(t)}\, dt,$$

and hence

$$P_\chi(M_\kappa \lambda(\gamma)) = \begin{cases} M_\kappa \lambda(\gamma) & \text{if } \kappa = \chi \\ 0 & \text{otherwise.} \end{cases}$$

Since the $M_\kappa \lambda(\gamma)$ span a dense subspace of $\mathcal{K}(L^2(G))$, and $M_\chi \lambda(C^*(G))$ is closed, it follows that $\mathcal{K}(L^2(G)^{Ad\rho}(\chi)\subseteq M_\chi \lambda(C^*(G))$, which completes the proof. □

3.2. The Main Theorem

Lemma 3.2.1. *Let $\beta: G\rightarrow AutB$ be a (strongly continuous) action of a compact abelian group G. Then there is an isomorphism Φ of the crossed product $B\times_\beta G$ onto $(B\otimes\mathcal{K}(L^2(G)))^{\beta\otimes Ad\rho}$, such that*

$$\Phi(i_B(b)i_G(\gamma))=b\otimes M_{\bar{\chi}}\lambda(\gamma), \tag{7}$$

where γ, $\chi\in\widehat{G}$, $b\in B^\beta(\chi)$.

Proof. Define a map $\beta^{-1}: B\rightarrow C_b(G, B)\subset M(B\otimes C_0(G))$ by the formula $\beta^{-1}(b)(t)=\beta_{t^{-1}}(b)$. Now let

$$j_B=(i\otimes M)\circ\beta^{-1}: B\rightarrow M(B\otimes\mathcal{K}(L^2(G)))$$

and let

$$j_G=1\otimes\lambda: G\rightarrow UM(B\otimes\mathcal{K}(L^2(G))).$$

A straightforward calculation, as at the top of [14, p.326], shows that (j_B, j_G) is covariant on (B, G, β). Then, [14, Proposition 2 (2)] gives us a non-degenerate homomorphism $\Phi=j_B\times j_G$ of $B\times_\beta G$ into $M(B\otimes\mathcal{K}(L^2(G)))$, such that $\Phi\circ i_B=j_B$ and $\Phi\circ i_G=j_G$. For $b\in B^\beta(\chi)$, $\chi,\in\widehat{G}$, we have $\beta^{-1}(b)=b\otimes\bar{\chi}$, and an easy calculation gives (7). It follows that Φ takes values in $B\otimes\mathcal{K}(L^2(G))$ and another calculation using (6) shows that the image is fixed under $\beta\otimes Ad\rho$.

Since $\mathcal{K}(L^2(G))=\overline{\text{span}}\,\{M_\chi\lambda(\gamma):\chi,\gamma\in\widehat{G}\}$, $(B\otimes\mathcal{K}(L^2(G)))^{\beta\otimes Ad\rho}$ is spanned by elements of the form

$$\int_G\beta_s\otimes Ad\rho_s(c\otimes M_{\bar{\chi}}\lambda(\gamma))\ ds=\int_G\overline{\chi(s)}\beta_s(c)\,ds\otimes M_{\bar{\chi}}\lambda(\gamma)\,ds,$$

where $c\in B$ and γ, $\chi\in\widehat{G}$. But $b=\int_G\overline{\chi(s)}\beta_s(c)\,ds$ lies in $B^\beta(\chi)$, so this shows that Φ has dense range and hence maps $B\times_\beta G$ onto $(B\otimes\mathcal{K}(L^2(G)))^{\beta\otimes Ad\rho}$.

Finally, we note that since Φ is the regular representation of $B\times_\beta G$ induced from $i: B\rightarrow B$, and G is amenable, we have that Φ is a faithful representation of $B\times_\beta G$ (see [13, 7.7.8]). Thus Φ is injective, and this completes our proof. □

Let ι denote the trivial action of G on $\mathcal{K}$. Now we may use the above to prove the following result of Kishimoto and Takai ([9, Theorem 2]).

Theorem 3.2.2. *Let $\beta : G \rightarrow AutB$ be a (strongly continuous) action of a compact abelian group G with large spectral subspaces, then the C^*-algebra $B^\beta \otimes \mathcal{K} \otimes \mathcal{K}(L^2(G))$ is isomorphic to $(B \otimes \mathcal{K}) \times_{\beta \otimes \iota} G$.*

Proof. We write $\overline{\beta} = \beta \otimes \iota$ for the product action of G on $B \otimes \mathcal{K}$. It is easy to see that $B^\beta \otimes \mathcal{K} \otimes \mathcal{K}(L^2(G)) = (B \otimes \mathcal{K})^{\overline{\beta}} \otimes \mathcal{K}(L^2(G))$. [9, Lemma 4.7] gives the existence of unitaries $v_\tau \in M(B \otimes \mathcal{K})^{\overline{\beta}}(\tau)$, $\tau \in \widehat{G}$ such that

$$(B \otimes \mathcal{K})^{\overline{\beta}}(\tau) = (B \otimes \mathcal{K})^{\overline{\beta}}\ v_\tau. \tag{8}$$

(In the next section, we shall find v_1 explicitly for $B = \mathcal{O}_A$, and we can then take $v_n = v_1^n$, thus we do not actually use Kishimoto and Takai's lemma). Given the unitaries v_τ, we may now describe the map which implements the isomorphism of $(B \otimes \mathcal{K})^{\overline{\beta}} \otimes \mathcal{K}(L^2(G))$ with $(B \otimes \mathcal{K}) \times_{\overline{\beta}} G$ in two stages:

Firstly, for each χ and all $\gamma \in \widehat{G}$, $b \in B^\beta$, $i, j \in \mathbf{N}$ we have an isomorphism

$$m_\chi : (B \otimes \mathcal{K})^{\overline{\beta}} \otimes M_\chi \lambda(C^*(G)) \rightarrow (B \otimes \mathcal{K})^{\overline{\beta}}\ (\overline{\chi}) \otimes \mathcal{K}(L^2(G))^{Ad\rho}(\chi)$$

which is given by

$$m_\chi(b \otimes e_{ij} \otimes M_\chi\ \lambda(\gamma)) = (v^*_{\chi\gamma}(b \otimes e_{ij})\, v_\gamma) \otimes M_\chi\ \lambda(\gamma). \tag{9}$$

For each $\tau \in \widehat{G}$, $\lambda(\tau)$ is the rank one projection in $\mathcal{K}(L^2(G))$ onto the subspace spanned by τ, hence the series $\sum_{\tau \in \widehat{G}} v_\tau \otimes \lambda(\tau)$ converges strictly in $M(B \otimes \mathcal{K} \otimes \mathcal{K}(L^2(G)))$ to a unitary V. Conjugating by V gives an isomorphism

$$m : (B \otimes \mathcal{K})^{\overline{\beta}} \otimes \mathcal{K}(L^2(G)) \rightarrow \overline{span}\ \{(B \otimes \mathcal{K})^{\overline{\beta}}(\overline{\chi}) \otimes \mathcal{K}(L^2(G))^{Ad\rho}(\chi) : \chi \in \widehat{G}\} \tag{10}$$

which restricts to m_χ on $(B \otimes \mathcal{K})^{\overline{\beta}} \otimes M_\chi\ \lambda(C^*(G))$. By [9, Proposition 3.1], the closed span on the right hand side of (10) is precisely $(B \otimes \mathcal{K} \otimes \mathcal{K}(L^2(G)))^{\overline{\beta} \otimes Ad\rho}$.

Secondly, we note that if the action β on B has large spectral subspaces, then so does the action $\overline{\beta}$ on $B \otimes \mathcal{K}$, thus we may apply the previous lemma to give us an isomorphism of $(B \otimes \mathcal{K} \otimes \mathcal{K}(L^2(G)))^{\overline{\beta} \otimes Ad\rho}$. with $(B \otimes \mathcal{K}) \times_{\overline{\beta}} G$. Under this isomorphism we see that

$$(v_{\chi\gamma}^*(b\otimes e_{ij})v_\gamma)\otimes M_\chi\lambda(\gamma)\mapsto i_{B\otimes\mathscr{K}}(v_{\chi\gamma}^*(b\otimes e_{ij})v_\gamma))i_G(\gamma), \tag{11}$$

where by definition, $i_G(\gamma)$ is the extension of $i_G: G\rightarrow UM(B\otimes\mathscr{K})$ to $\widehat{G}\subset C(\widehat{G})\subset L^1(G)$, which completes the proof. □

3.3. Construction of v_1 for $\mathscr{O}_A$

We now specialise to the case of the guage action on the Cuntz Krieger algebra $\mathscr{O}_A$ of a row finite matrix A satisfying (J). Here we construct a unitary operator $v_1\in M(\mathscr{O}_A\otimes\mathscr{K})^{\bar{\alpha}}(1)$ which implements the 1-grading of the algebra $\mathscr{O}_A\otimes\mathscr{K}$; i.e. $(\mathscr{O}_A\otimes\mathscr{K})^{\bar{\alpha}}(1)=(\mathscr{O}_A\otimes\mathscr{K})^{\bar{\alpha}}v_1$ where we again write $\bar{\alpha}$ for the product action $\alpha\otimes\iota$ of $\mathbf{T}$ on $\mathscr{O}_A\otimes\mathscr{K}$.

Before we start, we give some background details, which may be found in [2] that will be used frequently throughout the construction. For a C^*-algebra B, the strict topology of $M(B)$ is generated by the seminorms $\lambda_b(x)=\|bx\|$ and $\rho_b(x)=\|xb\|$ for each $b\in B$ and $x\in M(B)$. Since $M(B)$ is complete in the strict topology, a routine $\frac{\epsilon}{3}$ argument gives the following lemma:

Lemma 3.3.1. *Let $\mathscr{B}$ be a dense subset of B, and $\{w_n\}$ a norm-bounded sequence in $M(B)$ such that $\{w_n b\}$ and $\{bw_n\}$ are Cauchy sequences for all $b\in\mathscr{B}$. Then $\{w_n\}$ converges strictly in $M(B)$.*

Now we carry out the contruction of v_1. First notice that for each $i\in\mathbf{N}$ the operators $S_i\otimes e_{1i}$ are partial isometries with mutually orthogonal initial spaces $\mathscr{H}\otimes e_i$ and mutually orthogonal range spaces $(S_i\mathscr{H})\otimes e_1$. Hence the infinite sum $\sum_{i=1}^{\infty}(S_i\otimes e_{1i})$, converges strongly to an isometry u of $\mathscr{H}\otimes\mathscr{H}=\text{span}\{\mathscr{H}\otimes e_i\}$ onto $\mathscr{H}\otimes e_1$. We claim that this series in fact converges strictly in $M(\mathscr{O}_A\otimes\mathscr{K})^{\bar{\alpha}}(1)$.

To apply 3.3.1, we first note that the partial sums $u_n=\sum_{i=1}^{n}(S_i\otimes e_{1i})$ are all partial isometries, and hence $\|u_n\|=1$ for all n. For the dense subalgebra required in the Lemma, we take $\mathscr{B}=\text{span}\{S_\mu P_r S_\nu^*\otimes e_{jk}\}$, where $r, j, k\in\mathbf{N}$, and $\mu,\nu\in\sum_A$ (which is dense in $\mathscr{O}_A\otimes\mathscr{K}$ since A is row finite). For a given generator $c=S_\mu P_r S_\nu^*\otimes e_{jk}$, provided $n\geq j$ we have that

$$\begin{aligned}
u_n c &= \left(\sum_{i=1}^{n}S_i\otimes e_{1i}\right)(S_\mu P_r S_\nu^*\otimes e_{jk})\\
&= \begin{cases}(S_j\otimes e_{1j})(S_\mu P_r S_\nu^*\otimes e_{jk}) & \text{if } |\mu|\geq 1 \text{ and } A(\mu_1, j)\neq 0,\\ (S_j\otimes e_{1j})(P_r S_\nu^*\otimes e_{jk}) & \text{if } |\mu|=0 \text{ and } A(r, j)\neq 0,\\ 0 & \text{otherwise.}\end{cases}\\
&= \begin{cases}(S_j\otimes e_{1j})c & \text{or}\\ 0.\end{cases}
\end{aligned}$$

It follows that for any b in the dense subalgebra $\mathscr{B}=\text{span}\,\{S_\mu P_r S_\nu \otimes e_{jk}\}$, the sequence $\{u_n b\}$ is eventually constant and hence trivially convergent. Similarly

$$cu_n = (S_\mu P_r S_\nu^* \otimes e_{jk})\Big(\sum_{i=1}^{n} S_i \otimes e_{1i}\Big)$$

$$= \begin{cases} 0 & \text{if } k \neq 1, \\ (S_\mu P_r S_{\nu_{|\nu|}}^* \cdots\cdots S_{\nu_2}^* \otimes e_{j\nu_{|\nu|}}) & \text{if } |\nu| \geq 0,\ |\nu_1| = i \text{ and } k=1, \\ (S_\mu S_r \otimes e_{jr}) & \text{if } |\nu| = 0 \text{ and } k=1,\ i=r, \end{cases}$$

which is constant if $n \geq \nu_1$ (or $n \geq r$ if $|\nu|=0$), and so by Lemma 3.3.1 the sequence $\{u_n\}$ converges strictly to $u \in M(\mathcal{O}_A \otimes \mathcal{K})^{\bar{\alpha}}(1)$.

Because the summands are mutually orthogonal projections, the expansions

$$u^* u = \sum_{i,j=1}^{\infty} (S_j^* S_i \otimes e_{j1} e_{1i}) = \sum_{i=1}^{\infty} (S_i^* S_i \otimes e_{ii})$$

$$uu^* = \sum_{i,j=1}^{\infty} (S_i S_j^* \otimes e_{1i} e_{j1}) = \sum_{i=1}^{\infty} (S_i S_i^* \otimes e_{11}),$$

certainly converge strongly, and applications of Lemma 3.3.1 like that in the previous paragraph show that these too converge strictly in $M(\mathcal{O}_A^\alpha \otimes \mathcal{K})$.

Following the construction in [9, §4], we tensor with another copy of $\mathcal{K}$, which allows us the freedom to find isometries v, $w \in M(\mathcal{O}_A^\alpha \otimes \mathcal{K} \otimes \mathcal{K})$ such that $x = v^* (u \otimes 1_{\mathscr{H}}) w$ is unitary. The infinite sums which appear below all consist of partial isometries with mutually orthogonal initial and range spaces and hence all their partial sums have norm 1. When we pre- or post-multiply by a generator $S_\mu P_r S_\nu^* \otimes e_{jk} \otimes e_{pq}$ in the dense subalgebra span $\{S_\mu P_r S_\nu^* \otimes e_{jk} \otimes e_{pq}\}$ of $\mathcal{O}_A^\alpha \otimes \mathcal{K} \otimes \mathcal{K}$ these sums are eventually constant, so Lemma 3.3.1 implies that they converge strictly in $M(\mathcal{O}_A^\alpha \otimes \mathcal{K} \otimes \mathcal{K})$.

Choose an isometry $v: \mathscr{H} \otimes \mathscr{H} \to e_{11}(\mathscr{H}) \otimes \mathscr{H}$, such that

$$v\,(e_{11} \otimes e_{11})\,v^* = e_{11} \otimes e_{11}. \tag{12}$$

$$w = \sum_{i=1}^{\infty} (S_i S_i^* \otimes V_i) \in M(\mathcal{O}_A^\alpha \otimes \mathcal{K} \otimes \mathcal{K}),$$

where we note that $S_i S_i^* \otimes V_i$ are non-zero on orthogonal subspaces of $\mathscr{H} \otimes \mathscr{H} \otimes \mathscr{H}$ and so $\|\sum_i S_i S_i^* \otimes V_i\| = 1$ for all finite partial sums. Hence the product

$$w^* w = \sum_{i,j=1}^{\infty} (S_j S_j^* S_i S_i^* \otimes V_j^* V_i)$$

converges strictly, which gives us that

$$w^* w = \sum_{i=1}^{\infty} (S_i S_i^* \otimes V_i^* V_i) = \sum_{i=1}^{\infty} (S_i S_i^* \otimes 1_{\mathscr{H}} \otimes 1_{\mathscr{H}}) = 1_{\mathscr{H}} \otimes 1_{\mathscr{H}} \otimes 1_{\mathscr{H}}$$

since the V_i are isometries. We also have that

$$w w^* = \sum_{i,j=1}^{\infty} (S_i S_i^* S_j S_j^* \otimes V_i V_j^*) = \sum_{i=1}^{\infty} (S_i S_i^* \otimes V_i V_i^*),$$

this becomes,

$$\sum_{i=1}^{\infty} \left(S_i S_i^* \otimes \left(\sum_{j=1}^{\infty} A(j, i) e_{jj} \right) \otimes 1_{\mathscr{H}} \right) = \left(\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} A(j, i) S_i S_i^* \right) \otimes e_{jj} \right) \otimes 1_{\mathscr{H}} \right)$$

finally, applying the Cuntz-Krieger relation (2) gives us that

$$w w^* = \sum_{j=1}^{\infty} (S_j^* S_j \otimes e_{jj} \otimes 1_{\mathscr{H}}).$$

Thus we have shown that w is an isometry of $\mathscr{H} \otimes \mathscr{H} \otimes \mathscr{H}$, onto the initial space of $u \otimes 1_{\mathscr{H}}$, and thus

$$v_1 = (1_{\mathscr{H}} \otimes v^*) (u \otimes 1_{\mathscr{H}}) w \tag{13}$$

is a unitary in $M(\mathcal{O}_A \otimes \mathcal{K} \otimes \mathcal{K})^{\overline{\overline{\alpha}}}$ (1), where $\overline{\overline{\alpha}} = \alpha \otimes \iota \otimes \iota$. As in [9, 4.7], we could identify $(\mathcal{O}_A \otimes \mathcal{K} \otimes \mathcal{K}, \mathbb{T}, \overline{\overline{\alpha}})$ with $(\mathcal{O}_A \otimes \mathcal{K}, \mathbb{T}, \overline{\alpha})$, to get the required unitary v_1, we shall replace $\mathcal{K}$ by $\mathcal{K} \otimes \mathcal{K}$ and use (13) at the crucial steps.

§4. Computing the K-Theory

4.1. The Dual Pimsner-Voiculescu Sequence

Consider the gauge action α of $\mathbb{T}$ on $\mathcal{O}_A$ given in 2.1.9. From [1, §10.6] (see also [11,§3]), there is a dual Pimsner-Voiculescu exact sequence for this action;

$$\begin{array}{ccccc} K_0(\mathcal{O}_A \times_\alpha \mathbb{T}) & \xrightarrow{1-\widehat{\alpha}_*} & K_0(\mathcal{O}_A \times_\alpha \mathbb{T}) & \xrightarrow{p} & K_0(\mathcal{O}_A) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(\mathcal{O}_A) & \xleftarrow{p} & K_1(\mathcal{O}_A \times_\alpha \mathbb{T}) & \xleftarrow{1-\widehat{\alpha}_*} & K_1(\mathcal{O}_A \times_\alpha \mathbb{T}) \end{array}$$

Figure 1.

where $\widehat{\alpha}$ is the homomorphism induced by the generator of the dual action of $\mathbb{Z}$ on $\mathcal{O}_A \times_\alpha \mathbb{T}$. We want to replace $\mathcal{O}_A \times_\alpha \mathbb{T}$ by the fixed point algebra $\mathcal{O}_A^\alpha$, so we need:

Lemma 4.1.1. *The gauge action α on $\mathcal{O}_A$ has large spectral subspaces.*

Proof. It suffices to check that $\overline{\mathcal{O}_A^\alpha(r)^* \mathcal{O}_A^\alpha(r)} = \mathcal{O}_A^\alpha$ for each $r \in \mathbb{Z}$. We claim that $\mathcal{O}_A^\alpha(r) \supseteq \operatorname{span}\{S_\mu S_\nu^* : |\mu| - |\nu| = r\}$; this follows easily since $\alpha_z(S_\mu S_\nu^*) = z^{|\mu|-|\nu|} S_\mu S_\nu^*$. Hence $\overline{\mathcal{O}_A^\alpha(r)^* \mathcal{O}_A^\alpha(r)}$ contains all norm limits of elements of the form $(S_\mu S_\nu^*)^* (S_\kappa S_\sigma^*)$ where $|\mu| - |\nu| = |\kappa| - |\sigma| = r$. Choosing $\mu = \kappa$, and ν, σ freely in the above (note this implies that $|\nu| = |\sigma|$), we may thus construct any norm limit of $S_\mu S_\nu^*$, $|\mu| = |\nu|$, whose span is dense in $\mathcal{O}_A^\alpha$. Thus we have that $\overline{\mathcal{O}_A^\alpha(r)^* \mathcal{O}_A^\alpha(r)} \supseteq \mathcal{O}_A^\alpha$. Since $\overline{\mathcal{O}_A(r)^* \mathcal{O}_A^\alpha(r)} \subseteq \mathcal{O}_A^\alpha$ by definition and the continuity of α, this completes the proof. $\square$

Hence we may apply the results from the previous section to give us an isomorphism $\vartheta : \mathcal{O}_A^\alpha \otimes \mathcal{K} \otimes \mathcal{K}(L^2(\mathbb{T})) \to (\mathcal{O}_A \otimes \mathcal{K}) \times_{\overline{\alpha}} \mathbb{T} = (\mathcal{O}_A \times_\alpha \mathbb{T}) \otimes \mathcal{K}$. It is well known that for any rank one projection $e \in \mathcal{K}$, and any C^*-algebra B, the map $t : p \mapsto p \otimes e$ induces an isomorphism $t_* : K_0(B) \to K_0(B \otimes \mathcal{K})$, independent of the choice of e. Since, from 2.2.3, we know that $\mathcal{O}_A^\alpha$ is an AF algebra, we thus have that

$$K_*(\mathcal{O}_A \times_\alpha \mathbb{T}) \cong K_*(\mathcal{O}_A^\alpha) = \begin{cases} K_0(\mathcal{O}_A^\alpha) & \text{if } * = 0, \\ 0 & \text{if } * = 1. \end{cases}$$

Applying this to the exact sequence in Figure 1, we obtain:

$$\begin{tikzcd} 0 \arrow[r] & K_1(\mathcal{O}_A) \arrow[r, "\partial"] \arrow[dr] & K_0(\mathcal{O}_A \times_\alpha \mathbf{T}) \arrow[r, "1-\widehat{\alpha}_*"] & K_0(\mathcal{O}_A \times_\alpha \mathbf{T}) \arrow[r, "p"] & K_0(\mathcal{O}_A) \arrow[r] & 0 \\ & & K_0(\mathcal{O}_A^\alpha) \arrow[u, "\eta_*"'] \arrow[r, "\phi"] & K_0(\mathcal{O}_A^\alpha) \arrow[u, "\eta_*"] \arrow[ur, "i"'] & & \end{tikzcd}$$

Figure 2.

where η_* is the composition of the isomorphisms, ϑ_*, t_* mentioned above, i is the homomorphism induced from commutativity of the right-hand triangle, and ϕ is induced to make the central square commute. From the lower exact sequence we may deduce that

$$K_0(\mathcal{O}_A) = K_0(\mathcal{O}_A^\alpha)/\mathrm{Im}\ \phi \quad \text{and} \quad K_1(\mathcal{O}_A) = \mathrm{Ker}\ \phi.$$

In order to make any calcualtions, we need to know what $K_0(\mathcal{O}_A^\alpha)$, ϕ and i are.

Proposition 4.1.2. *With notation as above, we have*

$$K_0(\mathcal{O}_A^\alpha) = \varinjlim (\widetilde{\mathbf{Z}}^\infty, A^t),$$

where $\widetilde{\mathbf{Z}}^\infty = \coprod_{i=1}^\infty \mathbf{Z}$, *the additive group of all infinite sequences with integer coefficients which are eventually zero.*

Proof. Recall from 2.2.2 and 2.2.3 that $\mathcal{O}_A^\alpha = \mathcal{F}_A$, is the direct limit $\overline{\cup_k \mathcal{F}_k}$ of a sequence of algebras, each of which is the countable direct sum $\mathcal{F}_k = \bigoplus i \mathcal{F}_k(i)$ of algebras isomorphic to $\mathcal{K}(\mathcal{H})$. Since $K_0(\mathcal{K})$ is generated by any minimal projection, to get a set of generators for $K_0(\mathcal{F}_k)$ it suffices to write down a minimal projection in each $\mathcal{F}_k(i)$. For this, choose any path $\mu(i)$ of length k with $A(\mu_k(i), i) = 1$, and take $[S_{\mu(i)} P_i S_{\mu(i)}^*]$. Thus, the map θ given by

$$\{n_i\}_{i\in\mathbf{N}} \mapsto \sum_{i\in\mathbf{N}} n_i \left[S_{\mu(i)} P_i S_{\mu(i)}^* \right],$$

is an isomorphism of the infinite direct sum $\widetilde{\mathbf{Z}}^\infty$ onto $K_0(\mathcal{F}_k)$. If we can compute the inclusions $\varphi_{k*} : K_0(\mathcal{F}_k) \hookrightarrow K_0(\mathcal{F}_{k+1})$, we can use continuity to get $K_0(\mathcal{F}_A)$.

The embedding $\varphi_k : \mathcal{F}_k \hookrightarrow \mathcal{F}_{k+1}$ sends $S_\mu P_i S_\mu^*$ to

$$\sum_{\{j:A(i,j)=1\}} S_\mu S_i S_j S_j^* S_i^* S_\mu^*.$$

If we write $\tilde{\mu}(i)$ for the path $(\mu_1, \cdots, \mu_k, i) \in \Sigma_A$, we have

$$[S_{\mu(i)} P_i S_{\mu(i)}^*]_{K_0(\mathcal{F}_{k+1})} = \Big[\sum_{\{l:A(i,l)=1\}} S_{\tilde{\mu}(i)} P_l S_{\tilde{\mu}(i)}^*\Big]_{K_0(\mathcal{F}_{k+1})}$$

$$= \sum_{l=1}^{\infty} A(i,l)\,[S_{\tilde{\mu}(i)} P_l S_{\tilde{\mu}(i)}^*]_{K_0(\mathcal{F}_{k+1})}.$$

Thus

$$\sum_{i=1}^{\infty} n_i [S_{\mu(i)} P_i S_{\mu(i)}^*]_{K_0(\mathcal{F}_k)} \mapsto \sum_{l=1}^{\infty}\Big(\sum_{i=1}^{\infty} A(i,l)\,n_i\Big)[S_{\tilde{\mu}(i)} P_l S_{\tilde{\mu}(i)}^*]_{K_0(\mathcal{F}_{k+1})}$$

and so we have a commuting square:

$$\begin{tikzcd} K_0(\mathcal{F}_k) \arrow[r, "\varphi_{k*}"] & K_0(\mathcal{F}_{k+1}) \\ \tilde{\mathbb{Z}}^\infty \arrow[u, "\vartheta"] \arrow[r, "A^t"] & \tilde{\mathbb{Z}}^\infty \arrow[u, "\vartheta"'] \end{tikzcd}$$

which gives the result. $\square$

4.2. The Computation

In order to proceed, we must calculate the effect of the map ϕ on $K_0(\mathcal{O}_A^\alpha)$ induced from $1-\hat{\alpha}_*$. To do this, we examine the central commuting square in Figure 2 above. In particular, we must calculate the effect of the isomorphisms comprising η_* on the generators of $K_0(\mathcal{O}_A^\alpha)$ as well as the dual action $1-\hat{\alpha}_*$. Expanding all the components comprising η_* we have:

$$\begin{tikzcd}
K_0(\mathcal{O}_A \times_\alpha \mathbf{T}) \arrow[r, "1-\widehat{\alpha}_*"] \arrow[d, "m_*"'] \arrow[dr, phantom, "(1)"] & K_0(\mathcal{O}_A \times_\alpha \mathbf{T}) \arrow[d, "m_*"] \\
K_0((\mathcal{O}_A \times_\alpha \mathbf{T}) \otimes \mathcal{K}) \arrow[r, "(1-\widehat{\alpha}_*)\otimes\iota_*"] \arrow[dr, phantom, "(2)"] & K_0((\mathcal{O}_A \times_\alpha \mathbf{T}) \otimes \mathcal{K}) \\
K_0((\mathcal{O}_A \otimes \mathcal{K}) \times_{\bar{\alpha}} \mathbf{T}) \arrow[u, "l_*"] \arrow[r, "1-\widehat{\bar{\alpha}}_*"] & K_0((\mathcal{O}_A \otimes \mathcal{K}) \times_{\bar{\alpha}} \mathbf{T}) \arrow[u, "l_*"'] \\
K_0(\mathcal{O}_A^\alpha \otimes \mathcal{K} \otimes \mathcal{K}(L^2(\mathbf{T}))) \arrow[u, "\vartheta_*"] \arrow[r, phantom, "(3)"] & K_0(\mathcal{O}_A^\alpha \otimes \mathcal{K} \otimes \mathcal{K}(L^2(\mathbf{T}))) \arrow[u, "\vartheta_*"'] \\
K_0(\mathcal{O}_A^\alpha) \arrow[u, "t_*"] \arrow[r, "\phi"] & K_0(\mathcal{O}_A^\alpha) \arrow[u, "t_*"']
\end{tikzcd}$$

Figure 3.

The maps m_*, t_* are induced by the homomorphism $a \mapsto a \otimes e$ where e is a rank one projection. The map l_* is induced from the isomorphism of $(\mathcal{O}_A \otimes \mathcal{K}) \times_{\bar{\alpha}} \mathbf{T}$ and $(\mathcal{O}_A \times_\alpha \mathbf{T}) \otimes \mathcal{K}$, and the map ϑ_* is described by equation (11). The square (1) in Figure 3 above commutes by the naturality of the maps, and square (2) commutes by the functoriality of K_0. Finally, the map ϕ is by definition the homomorphism which makes the square (3) commute.

Lemma 4.2.1. *The map $\phi: K_0(\mathcal{O}_A^\alpha) \to K_0(\mathcal{O}_A^\alpha)$ induces the following commutative diagram*:

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{\mathbf{Z}}^\infty \arrow[r, "A^t"] \arrow[d, "1-A^t"'] & \widetilde{\mathbf{Z}}^\infty \arrow[r, "A^t"] \arrow[d, "1-A^t"'] & \widetilde{\mathbf{Z}}^\infty \arrow[r] \arrow[d, "1-A^t"'] & \cdots K_0(\mathcal{O}_A^\alpha) \arrow[d, "\phi"'] \\
\cdots \arrow[r] & \widetilde{\mathbf{Z}}^\infty \arrow[r, "A^t"] & \widetilde{\mathbf{Z}}^\infty \arrow[r, "A^t"] & \widetilde{\mathbf{Z}}^\infty \arrow[r] & \cdots K_0(\mathcal{O}_A^\alpha).
\end{tikzcd}$$

Proof. Since $K_0(\mathcal{O}_A^\alpha)=\varinjlim K_0(\mathcal{F}_k)=\varinjlim\ (\widetilde{\mathbb{Z}}^\infty, A^t)$, it is enough to compute ϕ on the image of $K_0(\mathcal{F}_k)\ \cong\ \widetilde{\mathbb{Z}}^\infty$. Under this isomorphism the generators are $[S_{\mu(i)}\ P_i\ S_{\mu(i)}^*]$, where $i\in\mathbb{N}$ and $\mu(i)\in\sum_A$ is any path of length k ending at i. We write $\mu=\mu(i)$ and express $\phi([S_\mu\ P_i\ S_\mu^*])$ as a combination of classes of the same form. The isomorphism t_* comes from tensoring by any rank-one projection $e\in\mathcal{K}\otimes\mathcal{K}(L^2(\mathbb{T}))$, and we can in particular choose $e=e_{11}\otimes\lambda(s^q)$, where s^q denotes the function $s\mapsto s^q$ on $\mathbb{T}$. Thus we have that

$$[t_*(S_\mu\ P_i\ S_\mu^*)]=[S_\mu\ P_i\ S_\mu^*\otimes e_{11}\otimes\lambda(s^q)].$$

Next, we must examine the effect of the Kishimoto-Takai isomorphism ϑ_* on our element. From equations (9) and (11), we have

$$\vartheta(S_\mu\ P_i\ S_\mu^*\otimes e_{11})\otimes\lambda(s^q)=i_{\mathcal{O}_A\otimes\mathcal{K}}(v_q^*\ (S_\mu\ P_i\ S_\mu^*\otimes e_{11})\,v_q)\,i_{\mathbb{T}}(s^{-q}), \tag{14}$$

where

$$(\mathcal{O}_A\otimes\mathcal{K})\times_{\overline{\alpha}}\mathbb{T}=\overline{span}\ \{i_{\mathcal{O}_A\otimes\mathcal{K}}(x)\,i_{\mathbb{T}}(z(s))\}\ \text{for}\ x\in\mathcal{O}_A\otimes\mathcal{K},\ z(s)\in C(\mathbb{T}),$$

as in 3.2.2, and $v_q=v_1^q$ is a unitary operator in $M(\mathcal{O}_A\otimes\mathcal{K})^{\overline{\alpha}}(q)$. Taking $q=0$, in which case v_0 is the identity operator, we have that

$$\begin{aligned}\vartheta_*\circ t_*([S_\mu\ P_i\ S_\mu^*])\ &=\ \vartheta_*([S_\mu^* P_i\ S_\mu^*\otimes e_{11}\otimes\lambda(s^0)])\\ &=\ [i_{\mathcal{O}_A\otimes\mathcal{K}}(S_\mu\ P_i\ S_\mu^*\otimes e_{11})\,i_{\mathbb{T}}(s^0)],\end{aligned} \tag{15}$$

in $K_0((\mathcal{O}_A\otimes\mathcal{K})\times_{\overline{\alpha}}\mathbb{T})$.

Next we calculate the effect of the dual action $\widehat{\overline{\alpha}}_*$ on the class of our projection, and return to $1-\widehat{\overline{\alpha}}_*$ later. Since $\widehat{\overline{\alpha}}$ is given on $C_C(\mathbb{T})$ by $\widehat{\overline{\alpha}}(f(s))=sf(s)$, we have $\widehat{\overline{\alpha}}(i_{\mathbb{T}}(s^0))=i_{\mathbb{T}}\ (s^1)$, and so

$$\widehat{\overline{\alpha}}_*(i_{\mathcal{O}_A\otimes\mathcal{K}}(S_\mu\ P_i\ S_\mu^*\otimes e_{11})\,i_{\mathbb{T}}(s^0))=i_{\mathcal{O}_A\otimes\mathcal{K}}(S_\mu\ P_i\ S_\mu^*\otimes e_{11})\,i_{\mathbb{T}}(s^1).$$

In order to reverse the isomorphism ϑ_* on the right hand side of Figure 3, we see from (14) that we must now conjugate $S_\mu\ P_i\ S_\mu^*\otimes e_{11}$ by $v_{-1}=v_1^*$. Thus we have that

$$\vartheta_*^{-1}\circ\widehat{\overline{\alpha}}_*\circ t_*([S_\mu\ P_i\ S_\mu^*])=[v_1^*\ (S_\mu\ P_i\ S_\mu^*\otimes e_{11})\,v_1)\otimes\lambda(s^{-1})].$$

In order to apply the formula for v_1 from 3.3, we expand $\mathcal{K}$ to $\mathcal{K}\otimes\mathcal{K}$ and put $v_1=(1\otimes v^*)\,(u\otimes 1)\ w$. For $\mu=(\mu_1,\cdots,\mu_k)$, we have that

$$v_1^*(S_\mu P_i S_\mu^* \otimes e_{11} \otimes e_{11}) v_1 = w^*(u^* \otimes 1)(S_\mu P_i S_\mu^* \otimes e_{11} \otimes e_{11})(u \otimes 1) w \quad \text{by (12)}$$
$$= w^*(S_{\mu_2} \cdots S_{\mu_k} P_i S_{\mu_k}^* \cdots S_{\mu_2}^* \otimes e_{\mu_1\mu_1} \otimes e_{11}) w$$
$$= (S_{\mu_2} S_{\mu_2}^* S_{\mu_2})\ S_{\mu_3} \cdots S_{\mu_k} P_i S_{\mu_k}^* \cdots S_{\mu_3}^* (S_{\mu_2}^* S_{\mu_2} S_{\mu_2}^*) \otimes V_{\mu_2}^* (e_{\mu_1\mu_1} \otimes e_{11}) V_{\mu_2}.$$

Since, $S_\mu \neq 0$ we have $A(\mu_2, \mu_1) = 1$, and this becomes

$$(S_{\mu_2} \cdots S_{\mu_k} P_i S_{\mu_k}^* \cdots S_{\mu_2}^*) \otimes e,$$

where e is some other rank 1 projection on $\mathcal{H} \otimes \mathcal{H}$. But this has the same class in $K_0(\mathcal{O}_A^\alpha \otimes \mathcal{K} \otimes \mathcal{K})$ as

$$(S_{\mu_2} \cdots S_{\mu_k} P_i S_{\mu_k}^* \cdots S_{\mu_2}^*) \otimes e_{11} \otimes e_{11}.$$

Thus if we go back from $(\mathcal{K} \otimes \mathcal{K}, e_{11} \otimes e_{11})$ to $(\mathcal{K}, e_{11})$, we obtain

$$[v_1^*(S_\mu P_i S_\mu^* \otimes e_{11})\ v_1] = [S_{\mu_2} \cdots S_{\mu_k} P_i S_{\mu_k}^* \cdots S_{\mu_2}^* \otimes e_{11}].$$

Hence,

$$\vartheta_*^{-1} \circ \widehat{\widehat{\alpha}}_* \circ t_*([S_\mu P_i S_\mu^*]) = [(v_1^*(S_\mu P_i S_\mu^* \otimes e_{11}) v_1) \otimes \lambda(s^{-1})]$$

which, in $K_0(\mathcal{O}_A^\alpha \otimes \mathcal{K} \otimes \mathcal{K}(L^2(\mathbf{T})))$ is

$$[(S_{\mu_2} \cdots S_{\mu_k} P_i S_{\mu_k}^* \cdots S_{\mu_2}^*) \otimes e_{11} \otimes \lambda(s^{-1})].$$

We note that the length of the path μ has been decreased by 1; to rewrite this in terms of projections in the original $\mathcal{F}_k$ we may use (cf. [3, p.32]) the infinite Cuntz-Krieger relation (2), to write it as

$$\left[\left(\sum_{j=1}^{\infty} A(i, j) S_{\mu_2} \cdots S_{\mu_k} S_i P_j S_i^* S_{\mu_k}^* \cdots S_{\mu_2}^*\right) \otimes e_{11} \otimes \lambda(s^{-1})\right].$$

Provided $|\nu(j)| = k$, the class of the projection $S_{\nu(j)} P_j S_{\nu(j)}^*$ in $K_0(\mathcal{F}_k)$ is determined completely by j; thus $\vartheta_*^{-1} \circ (1 - \widehat{\alpha}_*) \circ \vartheta_* \circ t_*([S_{\mu(i)} P_i S_{\mu(i)}^*])$ is given by

$$[S_{\mu(i)} P_i S_{\mu(i)}^* \otimes e_{11} \otimes \lambda(s^0)] - \sum_{j=1}^{\infty} A(i, j)\,[S_{\nu(j)} P_j S_{\nu(j)}^* \otimes e_{11} \otimes \lambda(s^{-1})].$$

Finally, since the map t_* is independent of the choice of projection, applying t_*^{-1}, gives

$$\phi([S_{\mu(i)} P_i S_{\mu(i)}^*]) = [S_{\mu(i)} P_i S_{\mu(i)}^*] - \sum_{j=1}^{\infty} A(i,j)\,[S_{\nu(j)} P_j S_{\nu(j)}^*].$$

Thus ϕ maps the image of $K_0(\mathcal{F}_k)$ in $K_0(\mathcal{O}_A^\alpha)$ into itself, and is given on $K_0(\mathcal{F}_k) \cong \widetilde{\mathbb{Z}}^\infty$ by $1-A^t$. So we have a commuting diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \widetilde{\mathbb{Z}}^\infty & \xrightarrow{A^t} & \widetilde{\mathbb{Z}}^\infty & \xrightarrow{A^t} & \widetilde{\mathbb{Z}}^\infty & \longrightarrow \cdots & K_0(\mathcal{O}_A^\alpha) \\
1-A^t & \Big\downarrow & 1-A^t & \Big\downarrow & 1-A^t & \Big\downarrow & \phi & \Big\downarrow \\
\cdots \longrightarrow & \widetilde{\mathbb{Z}}^\infty & \xrightarrow{A^t} & \widetilde{\mathbb{Z}}^\infty & \xrightarrow{A^t} & \widetilde{\mathbb{Z}}^\infty & \longrightarrow \cdots & K_0(\mathcal{O}_A^\alpha)
\end{array}$$

as required. □

Lemma 4.2.2. *The map i in Figure 2 is the homomorphism induced by the inclusion $i_A : \mathcal{O}_A^\alpha \hookrightarrow \mathcal{O}_A$.*

Proof. As in the previous result, we need only to check this on the generators of $K_0(\mathcal{O}_A^\alpha)$, namely $S_\mu P_i S_\mu^* \in K_0(\mathcal{F}_k)$, for $j \in \mathbb{N}$, $|\mu| = k \geq 0$. We have already calculated in (15) that

$$\vartheta_* \circ t_*([S_\mu P_i S_\mu^*]) = [i_{\mathcal{O}_A \otimes \mathcal{K}}(S_\mu P_i S_\mu^* \otimes e_{11})\; i_{\mathbb{T}}(s^0)]$$

in $K_0((\mathcal{O}_A \otimes \mathcal{K}) \times_{\overline{\alpha}} \mathbb{T})$. Since $\overline{\alpha} = \alpha \otimes \iota$, l_* maps this into

$$[(i_{\mathcal{O}_A}(S_\mu P_i S_\mu^*)\; i_{\mathbb{T}}(s^0)) \otimes e_{11}].$$

Finally m_* strips off the rank one projection e_{11} hence $\eta_*([S_\mu P_i S_\mu^*])$ is the class of $i_{\mathcal{O}_A}(S_\mu P_i S_\mu^*)\; i_{\mathbb{T}}(s^0)$ in $K_0(\mathcal{O}_A \times_\alpha \mathbb{T})$.

Next, the homomorphism p shown in Figure 2 is induced from the embedding of $\mathcal{O}_A \times_\alpha \mathbb{T}$ in $(\mathcal{O}_A \times_\alpha \mathbb{T}) \times_{\hat{\alpha}} \mathbb{Z}$, which gives the Takai isomorphism $(\mathcal{O}_A \times_\alpha \mathbb{T}) \times_{\hat{\alpha}} \mathbb{Z} \cong \mathcal{O}_A \otimes \mathcal{K}$ and the identification of $K_0(\mathcal{O}_A \otimes \mathcal{K})$ with $K_0(\mathcal{O}_A)$ (see [14, Theorem 6]). From [14, p.326], we see that the embedding is given by

$$j_{\mathcal{O}_A} \times j_{\mathbb{T}}(i_{\mathcal{O}_A}(S_\mu P_i S_\mu^*) i_{\mathbb{T}}(s^0)) = j_{\mathcal{O}_A}(S_\mu P_i S_\mu^*) j_{\mathbb{T}}(s^0),$$

and so

$$p\circ\eta_*([S_\mu P_i S_\mu^*])=[(S_\mu P_i S_\mu^*\otimes 1)(1\otimes\lambda(s^0))]=[S_\mu P_i S_\mu^*\otimes f],$$

where $f=\lambda(s^0)$ is a rank one projection in $\mathcal{K}(L^2(\mathbb{T}))$. The identification of $K_0(\mathcal{O}_A\otimes\mathcal{K})$ with $K_0(\mathcal{O}_A)$ removes this projection and so $p\circ\eta_*([S_\mu P_i S_\mu^*])$ is the class of $S_\mu P_i S_\mu^*$, viewed as a projection in $\mathcal{O}_A$ rather than $\mathcal{O}_A^\alpha$. This completes the proof of the lemma. □

Corollary 4.2.3. *$K_0(\mathcal{O}_A)$ is generated by the equivalence classes* $\{[P_i]: i\in\mathbf{N}\}$.

Proof. From the previous lemma, and the exactness of the sequence given in Figure 2 we may deduce that the images of $[S_\mu P_i S_\mu^*]$ generate $K_0(\mathcal{O}_A)$. However, within $\mathcal{O}_A$ (though not in $\mathcal{O}_A^\alpha$) this projection is Murray-von Neumann equivalent to $[P_i]$, for each $i\in\mathbf{N}$. Thus, the map i_{A*} is many to one, and sends the class of each $S_\mu P_i S_\mu^*$ which generate $K_0(\mathcal{O}_A^\alpha)$ to $[P_i]$ in $K_0(\mathcal{O}_A)$, which is the required result. □

As in [3], we may represent the inductive limit $\varinjlim(\tilde{\mathbf{Z}}^\infty, A^t)$ as the set of equivalence classes of sequences $[\{x^j\}_{j\in\mathbf{N}}]$, where $x^j\in\tilde{\mathbf{Z}}^\infty$ and $x^{j+1}=A^t x^j$ for j greater than some j_0 where two sequences are identified if they differ only at a finite number of points. With this understanding, we have that $\phi([\{x^j\}])=[\{x^j-A^t x^j\}]$, and again following [3, pp.32-33], we have:

Theorem 4.2.4. *Let A be a countably infinite 0-1 matrix which is row finite and satisfies condition* (J), *then the map $\omega:\tilde{\mathbf{Z}}^\infty\to\varinjlim(\tilde{\mathbf{Z}}^\infty, A^t)$ given by $\omega(x)=[\{(A^t)^j x\}]$ induces an isomorphism of $\tilde{\mathbf{Z}}^\infty/(1-A^t)\tilde{\mathbf{Z}}^\infty$ onto $\varinjlim(\tilde{\mathbf{Z}}^\infty, A^t)/\operatorname{Im}\phi\cong K_0(\mathcal{O}_A)$, and $\operatorname{Ker}\{(1-A^t):\tilde{\mathbf{Z}}^\infty\to\tilde{\mathbf{Z}}^\infty\}$ onto $\operatorname{Ker}\phi\cong K_1(\mathcal{O}_A)$.*

Proof. For the first part, note that $\omega((1-A^t)y)\in\operatorname{Im}\phi$ for all $y\in\tilde{\mathbf{Z}}^\infty$, so the induced homomorphism is well defined. If $[\{x^j\}]\in\varinjlim(\tilde{\mathbf{Z}}^\infty, A^t)/\operatorname{Im}\phi$, then we see that

$$\begin{aligned}[\{x^j\}]+\operatorname{Im}\phi &= [\{A^t x^j\}]+[\{x^j-A^t x^j\}]+\operatorname{Im}\phi\\ &= [\{A^t x^j\}]+\operatorname{Im}\phi. \end{aligned}\tag{16}$$

For sufficiently large j_0, we have that $\{x^j\}=\{x^1, x^2, \cdots, x^{j_0}, A^t x^{j_0}, (A^t)^2 x^{j_0}, \cdots,\}$. From j_0 applications of (16) we see that $[\{x^j\}]$ is equivalent, modulo $\operatorname{Im}\phi$ to

$$\begin{aligned}[\{(A^t)^{j_0} x^j\}] &= [\{(A^t)^{j_0} x^1, (A^t)^{j_0} x^2, \cdots, (A^t)^{j_0} x^{j_0}, (A^t)^{j_0} A^t x^{j_0}, \cdots\}] \\ &= [\{x^{j_0}, (A^t) x^{j_0}, \cdots, (A^t)^{j_0} x^{j_0}, (A^t)^{j_0+1} x^{j_0}, \cdots\}] \\ &= \omega(x^{j_0}).\end{aligned}$$

Thus we have shown that the homomorphism ω is surjective.

Now suppose that $\omega(x) \in \operatorname{Im} \phi$, that is, there exists $[\{x^j\}] \in \varinjlim (\tilde{\mathbb{Z}}^\infty, A^t)$ such that $[\{(A^t)^j x\}] = [\{x^j - A^t x^j\}]$. Then, for large k, we have $(A^t)^k x = x^k - A^t x^k$, and so

$$\begin{aligned}x &= x - (A^t)^k x + (A^t)^k x \\ &= (1 - A^t)(1 + A^t + \cdots + A^t)^{k-1}) x + (1 - A^t) x^k\end{aligned}$$

which belongs to $(1 - A^t)\tilde{\mathbb{Z}}^\infty$, which implies that the map induced from ω is injective, and completes the proof of the first part.

For the second part we note that the induced map is well-defined, since if $x \in \operatorname{Ker} \{(1 - A^t) : \tilde{\mathbb{Z}}^\infty \to \tilde{\mathbb{Z}}^\infty\}$ then

$$\begin{aligned}\phi(\omega(x)) &= \phi([\{(A^t)^j x\}]) \\ &= [\{(A^t)^j x - (A^t)^{j+1} x\}] \\ &= [\{(A^t)^j (1 - A^t) x\}] \\ &= [\{0\}].\end{aligned}$$

If $\phi([\{x^j\}]) = 0$, then $x^j = A^t x^j = x^{j+1}$ for $j \geq j_0$. Hence $\{x^j\}$ is equivalent to the constant sequence $\{x^{j_0}, x^{j_0}, \cdots\}$, which is in $\operatorname{Ker}\{(1 - A^t) : \tilde{\mathbb{Z}}^\infty \to \tilde{\mathbb{Z}}^\infty\}$. Thus the induced map of $(1 - A^t)\tilde{\mathbb{Z}}^\infty$ to $\operatorname{Ker} \phi$ is surjective. Finally we note that the induced map is faithful, since, if $\omega(x) = [\{0\}]$ then $(A^t)^k x = 0$ for large k, which implies $x = 0$ because $A^t x = x$. □

In the case of K_0, our calculations actually say more:

Corollary 4.2.5. *$K_0(\mathcal{O}_A)$ is generated, as an abelian group by the family $[S_i S_i^*]$, subject only to the relations*

$$[S_i S_i^*] = \sum_{j=1}^{\infty} A(i, j) [S_j S_j^*]$$

induced by the Cuntz-Krieger relation (2).

Proof. What we have actually proved above was that the diagram

$$\begin{array}{ccccc} \tilde{\mathbf{Z}}^\infty & \xrightarrow{\omega} & \varinjlim(\tilde{\mathbf{Z}}^\infty, A^t) & \cong & K_0(\mathcal{O}_A^\alpha) \\ {\scriptstyle 1-A^t}\downarrow & & \downarrow & & \downarrow{\scriptstyle \phi} \\ \tilde{\mathbf{Z}}^\infty & \xrightarrow{\omega} & \varinjlim(\tilde{\mathbf{Z}}^\infty, A^t) & \cong & K_0(\mathcal{O}_A^\alpha) \end{array}$$

induces an isomorphism of $\tilde{\mathbf{Z}}^\infty/(1-A^t)\tilde{\mathbf{Z}}^\infty$ onto Coker $\phi \cong K_0(\mathcal{O}_A)$. Since the composition $\tilde{\mathbf{Z}}^\infty \to K_0(\mathcal{O}_A^\alpha) \to K_0(\mathcal{O}_A)$ is given by

$$\{n_i\}_{i\in\mathrm{N}} \mapsto \sum_{i=1}^\infty n_i[S_{\mu(i)}P_i S_{\mu(i)}^*] \in K_0(\mathcal{F}_k) \mapsto \sum_{i=1}^\infty n_i[P_i] \in K_0(\mathcal{O}_A),$$

this implies the result. □

§5. Applications

5.1 Doplicher-Roberts Algebras

The principal motivation for making the calculations given in this paper was to calculate the K-theory of the Doplicher-Roberts Algebras, $\mathcal{O}_\rho$ which are central to their nonabelian duality theory, [6], [7], [8]. These C^*-algebras are built from spaces of intertwiners between tensor powers of a given faithful representation $\rho: G \to SU(\mathcal{H})$, where G is a compact group and $1 < dim(\mathcal{H}) < \infty$. We refer to [10] for further details of their construction. Decomposing the tensor powers of ρ into irreducible components yields a countable 0-1 matrix A_ρ, which may be shown to be irreducible and row finite. From 2.1.12 $\mathcal{O}_{A_\rho}$ is simple, and so the map $\phi: \mathcal{O}_\rho \to \mathcal{O}_{A_\rho}$ given in [10, Theorem 2.1] is an isomorphism onto full corner of $\mathcal{O}_{A_\rho}$, and hence $\mathcal{O}_\rho$ is Morita equivalent to $\mathcal{O}_{A_\rho}$. Thus we have the following result:

Corollary 5.1.1. *Let $\rho: G \to SU(\mathcal{H})$ be a faithful representation of a compact group, with $1 < dim(\mathcal{H}) < \infty$, then*

$$K_*(\mathcal{O}_\rho) \cong K_*(\mathcal{O}_{A_\rho}) \cong \begin{cases} \text{Coker } (1-A_\rho^t) & * = 0 \\ \text{Ker}(1-A_\rho^t) & * = 1, \end{cases}$$

where $1-A_\rho^t$ *is considered as a linear operator on* $\tilde{\mathbb{Z}}^\infty$.

We may identify $\tilde{\mathbb{Z}}^\infty$ with the representation ring, $\mathcal{R}(G)$ of G as follows: given a list of representatives of $\hat{G}$, $\{\pi_i\}_{i\in\mathbb{N}}$ and the canonical basis $\{\varepsilon_i\}_{i\in\mathbb{N}}$ of $\tilde{\mathbb{Z}}^\infty$, define a map $\Phi : \tilde{\mathbb{Z}}^\infty \to \mathcal{R}(G)$ by $\xi_i \mapsto [\pi_i]$. It may be shown that Φ extends to an isomorphism of additive abelian groups, and that the map A_ρ^t on $\tilde{\mathbb{Z}}^\infty$ induces the map β_ρ on $\mathcal{R}(G)$, where

$$\beta_\rho : [\pi_i] \mapsto [\pi_i \otimes \rho].$$

With this identification, we may restate the result 5.1.1 as:

Theorem 5.1.2. *Let* $\rho: G \to SU(\mathcal{H})$ *be a faithful representation of a compact group, with* $1 < dim(\mathcal{H}) < \infty$, *then*

$$K_*(\mathcal{O}_\rho) \cong \begin{cases} \text{Coker } \{(1-\beta_\rho) : \mathcal{R}(G) \to \mathcal{R}(G)\} & *=0 \\ \text{Ker } \{(1-\beta_\rho) : \mathcal{R}(G) \to \mathcal{R}(G)\} & *=1, \end{cases}$$

where β_ρ *is the linear operator on* $\mathcal{R}(G)$ *given by* $[\pi_i] \mapsto [\pi_i \otimes \rho]$, *for* $i \in \mathbb{N}$.

5.2. Examples

Just for completeness, we calculate the K-groups of the infinite Cuntz-Krieger algebras $\mathcal{O}_{A_1}$, $\mathcal{O}_{A_3}$ we considered in Section 2. Firstly we consider the linear operators $1-A_1^t$ and $1-A_3^t$ acting on $\tilde{\mathbb{Z}}^\infty$ where

$$1-A_1^t = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots \\ -1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad 1-A_3^t = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots \\ -1 & 0 & -1 & 0 & \cdots \\ 0 & -1 & 0 & -1 & \cdots \\ 0 & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to show that $\text{Ker}\,(1-A_1^t) = \text{Ker}\,(1-A_3^t) = \{0\}$ as operators on $\tilde{\mathbb{Z}}^\infty$, even though A_3^t has fixed points in the full infinite product of copies of $\mathbb{Z}$.

For $1-A_1^t$, we see that $\underline{y} = (y_1, y_2, \cdots) \in \text{Im}\,(1-A_1^t)$ provided $y_1+y_2=y_3$, in which case $\tilde{\mathbb{Z}}^\infty/\text{Im}\,(1-A_1^t) \cong \mathbb{Z}$. For $1-A_3^t$ we see that $\underline{y} = (y_1, y_2, \cdots) \in \text{Im}\,(1-A_3^t)$ provided $\sum_{i=1}^\infty (-1)^i (y_{2i}+y_{2i-1}) = 0$, in which case $\tilde{\mathbb{Z}}^\infty/\text{Im}\,(1-A_3^t) \cong \mathbb{Z}$ as well. Thus we have shown that

$$\begin{aligned} K_0(\mathcal{O}_{A_1}) &= \mathbb{Z} \qquad & K_1(\mathcal{O}_{A_1}) &= 0 \\ K_0(\mathcal{O}_{A_3}) &= \mathbb{Z} \qquad & K_1(\mathcal{O}_{A_3}) &= 0. \end{aligned}$$

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