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Outer Automorphism Group of the Ergodic Equivalence Relation Generated by Translations of Dense Subgroup of Compact Group on its Homogeneous Space

By

Sergey L. GEFTER*

Abstract

We study the outer automorphism group $\mathrm{Out}R_\Gamma$ of the ergodic equivalence relation R_Γ generated by the action of a lattice Γ in a semisimple Lie group on the homogeneos space of a compact group K. It is shown that $\mathrm{Out}R_\Gamma$ is locally compact. If K is a connected simple Lie group, we prove the compactness of $\mathrm{Out}R_\Gamma$ using the D. Witte's rigidity theorem. Moreover, an example of an equivalence relation without outer automorphisms is constructed.

Introduction

An important problem in the theory of full factors [Sak], [Con 1] and in the orbit theory for groups with the T-property [Kaz] is that of studying the group of outer automorphisms of the corresponding object as a topological group. Unlike the amenable case, the outer automorphism group is a Polish space in the natural topology (see Section 1) and its topological properties are algebraic invariants of the factor and orbital invariants of the dynamical system. The first results in this sphere were obtained by A. Connes [Con 2] in 1980 who showed that the outer automorphism group of the II_1-factor of an ICC-group with the T-property is discrete and at most countable. Various examples of factors and ergodic equivalence relations with locally compact groups of outer automorphisms were constructed in [Cho 1], [EW], [GGN], [Gol], [GG 1] and [GG 2]. In [Gol] and [GG 2] the topological properties of the outer automorphism group were used to construct orbitally nonequivalent

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*Kharkov State University, Department of Mechanics and Mathematics, Freedom sq. 4, Kharkou, 310077, Ukraine.

ergodic actions of arithmetic groups with the T-property. The group of outer automorphisms was first calculated in the explicit form in [GG 1] for the equivalence relation given by translations of $SO(n, \mathbb{Q})$ on $SO(n, \mathbb{R})$, $n \geq 5$ (see Section 4 [GG 2]).

The present paper deals with studying the group of outer automorphisms of the ergodic equivalence relation generated by the translations of a lattice in a semisimple Lie group on the homogeneous space of compact group. Section 1 presents a preliminary information on the groups of outer automorphisms of equivalence relations. Section 2 contains the proof of the general theorem about the local compactness of outer automorphism group of the equivalence relation generated by the action of a group with the T-property on the homogeneous space of an arbitrary compact group (Theorem 2.3). As it is shown in the example of the action of $SL(n, \mathbb{Z})$ on $SL(n, \mathbb{Z}_p)$, the group of outer automorphisms can be noncompact (Remark 2.8).

The most important case where the lattice acts on a homogeneous space of a connected conpact simple Lie group is considered in Section 3. In this case the D. Witte's results [Wit] play an important role. We use them to obtain the explicit description of all automorphisms of the equivalence relation (Theorem 3.3). It is proved that the outer automorphisms group is compact, Moreover, we find some conditions which ensure this group to be finite (Corollary 3.4) and trivial (Corollary 3.5). Section 4 involves one of the principal results of the present paper, namely the construction of equivalence relations of type II_1 and II_∞ without outer automorphisms (Theorem 4.2 and Corollary 4.3). It shoud be noted that the problem of the existence of a type II factor and type III equivalence relation without outer automorphisms remains unsolved.

The main results are reported in [Gef].

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§1. Preliminaries

Let Γ be a countable group, (X, ν) a free ergodic Γ-space with finite invariant measure. Denote the equivalence relation generated by Γ as R_Γ. Let $\mathrm{Aut}R_\Gamma$ be the group of its automophisms, Int R_Γ the inner automorphism subgroup [FM 1], $M=M(R_\Gamma)$ the von Neumann algebra constructed by R_Γ [FM 2], and $A=L^\infty(X, \nu)$. We endow $\mathrm{Aut}M$ with the topology of pointwise convergence in norm on M_* [Haa]. Then

$$\mathrm{Aut}(M, A)=\{\theta \in \mathrm{Aut}M: \theta(A)=A\}$$

and

$$Z(M, A) = \{\theta \in \mathrm{Aut}M : \theta(a) = a,\ a \in A\}$$

are closed (hence Polish) subgroups of Aut$M$. Since AutR_Γ can be identified with Aut$(M, A)/Z(M, A)$ [FM 2], it is a Polish group itself.

Remark 1.1. Consider the natural σ-finite measure m using the set R_Γ[FM 1] and associate the unitary operator U_θ in the space $L^2(R_\Gamma, m)$ with the automorphism θ of AutR_Γ, by

$$(U_\theta \xi)(x, y) = \xi(\theta^{-1}(x), \theta^{-1}(y)).$$

It can be shown that this topology on AutR_Γ is induced by the weak operator topology when AutR_Γ is embedded into the unitary group of $L^2(R_\Gamma, m)$. Moreover, this topology coincides with the Polish topology on AutR_Γ which was considered by T. Hamachi and M. Osikawa [HO] (see [Dan, section 3]).

If the group Γ is amenable then R_Γ is an approximately finite equivalence relation [CFW] and the group IntR_Γ of its inner automorphisms is dense in AutR_Γ [HO]. The T-property is diametrically opposite to the amenability [Kaz, Zim 4]. If Γ contains an ergodic subgroup Γ_0 with the T-property and Γ is an ICC-group with respect to Γ_0, i.e. the set

$$\{\gamma g \gamma^{-1} : \gamma \in \Gamma_0\}$$

is infinite for all $g \neq e$, then IntR_Γ is a closed subgroup of AutR_Γ [GG 1, GG 2]. Thus, the group of outer automorphisms

$$\mathrm{Out}R_\Gamma = \mathrm{Aut}R_\Gamma / \mathrm{Int}R_\Gamma$$

of R_Γ is a Polish group. It shoud be noted that this result also follows from the more general statement about the closeness of IntR_Γ in AutR_Γ (see [JS, p.113]), since it can be shown that Γ with the above properties is not inner amenable [Cho 2, Pr. 6(b)].

For an equivalence relation with the closed subgroup of inner auto-morphisms, the problem of studying the outer automorphism group as a topological group seems natural and important. In the general case the Polish group can be extremely "large".

Example 1.2. Let (S, μ) be a Lebesgue apace with the probability measure and Γ an ICC-group with the T-property. Consider the Bernoulli action of Γ on the space

$$(X, \nu) = \prod_{\gamma \in \Gamma} (S, \mu).$$

Since any μ-preserving automorphism of the space S generates an outer automorphism of the equivalence relation R_Γ the group OutR_Γ is not locally compact.

In the following section we prove that if Γ acts by translations on a homogeneous space of a compact group, then the outer automorphism group of R_Γ is locally compact.

§2. The Action of a Countable Dense Subgroup of a Compact Group on its Homogeneous Space

Let K be a compact group with the Haar measure μ, Γ its countable dense subgroup, and L a closed subgroup of K. Consider the action of Γ via left translations on the homogeneous space $X = K/L$ with the invariant meaure ν:

$$\gamma(kL) = \gamma kL, \quad k \in K, \quad \gamma \in \Gamma.$$

Since Γ is dense in K, we have that (X, ν) is an ergodic Γ-space.

Lemma 2.1. *Assume that K is a compact connected simple Lie group with the trivial center. Then the action of Γ on (X, ν) is free.*

Proof. Consider the following sets for $\gamma \in \Gamma$

$$S\gamma = \{x \in X : \gamma x = x\} \text{ and}$$
$$T\gamma = \{k \in K : \gamma kL = kL\}.$$

Assume that $\nu(S\gamma) > 0$. Then $\mu(T\gamma) > 0$. Since K and L are real algebraic groups, the set T_γ is a real algebraic subvariety in K. Consequently, $T_\gamma = K$. Hence

$$\gamma \in \widetilde{L} = \bigcap_{k \in K} kLk^{-1}.$$

One can easily see that $\widetilde{L}$ is a normal subgroup in K. Consequently, $\widetilde{L} \subset Z(K) = \{e\}$, i.e. $\gamma = e$. $\square$

Let $N_K(L)$ be the normalizer of L in K. If $t \in N_k(L)$, then the right translation r_t is well defined on (X, ν), as follows:

$$r_t(x) = xt^{-1}, \quad x \in X.$$

It is obvious that $r_t = \mathrm{id}$ if and only if $t \in L$. Moreover, the automorphism r_t lies in the centralizer of the Γ-action.

Lemma 2.2. *The centralizer of the Γ-action coincides with the group of right translations, i.e. isomorphic to $N_K(L)/L$.*

Proof. Assume the Borel map $\beta: X \to X$ commutes with left translations of Γ. Denote by p the projection from K onto $X = K/L$ and consider the map

$$\phi: K \to X, \quad \phi(k) = k^{-1}\beta(p(k)).$$

According to the conditions, $\beta(\gamma x) = \gamma\beta(x)$ for all $\gamma \in \Gamma$ at a.a. $x \in X$. Consequently, $\phi(\gamma k) = \phi(k)$ for all $\gamma \in \Gamma$ and a.a. $k \in K$. Since the Γ-action is ergodic on (K, μ) there are $t \in K$ and $K_0 \subset K$ so that $\mu(K_0) = 1$ and $\phi(k) = p(t^{-1})$ for all $k \in K_0$. Therefore

$$\beta(p(k)) = k\phi(k) = p(kt^{-1}), \quad k \in K_0. \tag{2.1}$$

We show that $t \in N_k(L)$. Let $h \in L$. Since $\mu(K_0 h^{-1} \cap K_0) = 1$, there is an element $k \in K_0$ with $kh \in K_0$. Then

$$\beta(p(k)) = \beta(p(kh)) = khp(t^{-1}), \text{ i.e.}$$
$$p(t^{-1}) = hp(t^{-1}) \text{ and } tht^{-1} \in L.$$

Thus, $tLt^{-1}. \subset L$. Using the compactness of K one can demonstrate that $t^{-1}Lt \subset L$, i.e. $t \in N_K(L)$. Now, in view of (2.1), $\beta(x) = r_t(x)$ for a.a. $x \in X$. □

Consider the ergodic equivalence relation R_Γ generated by left translations of Γ on (X, ν) and denote by ε the projection from $\mathrm{Aut}R_\Gamma$ outo $\mathrm{Out}R_\Gamma$. The following theorem about the topological structure of $\mathrm{Out}R_\Gamma$ is a generalization of the results of Section 3 in [GG 2], where the case $L = \{e\}$ is considered.

Theorem 2.3. *Assume that Γ acts freely on (X, ν) and contains a dense of subgroup Γ_0 in K with the T-property, and Γ is an ICC-group with respect to Γ_0, i.e. the set $\{\gamma g \gamma^{-1} : \gamma \in \Gamma_0\}$ is infinite for all $g \neq e$. Then $\varepsilon(\{r_t : t \in N_K(L)\})$ is an open subgroup in $\mathrm{Out}R_\Gamma$, which is topologically isomorphic to $N_k(L)/L$. Hence, $\mathrm{Out}R_\Gamma$ is a locally compact group.*

To prove Theorem 2.3, we shall provide several considerations.

Given $\gamma \in \Gamma$ and $t \in N_k(L)$, we denote by $\widetilde{l}_\gamma$ and $\widetilde{r}_t$ the automorphisms of $A = L^\infty(X, \nu)$ generated by γ-left and t-right translations:

$$\widetilde{l}_\gamma(a)(x) = a(\gamma^{-1}x), \quad \widetilde{r}_t(a)(x) = a(xt), \ a \in A, \ x \in X.$$

Consider the crossed product $M = W^*(A, \widetilde{l}_\gamma, \Gamma)$. The algebra M is generated by operators $\pi(a)$ and λ_γ $(a \in A, \gamma \in \Gamma)$ of the space $L^2(X, \nu) \otimes \ell^2(\Gamma)$ given by

$$(\pi(a)\xi)(x, g) = a(gx)\xi(x, g),$$
$$(\lambda_\gamma \xi)(x, g) = \xi(x, \gamma^{-1}g).$$

Then

$$\pi(\widetilde{l}_\gamma(a)) = \lambda_\gamma \pi(a) \lambda_\gamma^*, \quad a \in A, \quad \gamma \in \Gamma.$$

Since (X, ν) is a free Γ-space, the algebra $M = W^*(A, l, \Gamma)$ is isomorphic to the algebra $M(R_\Gamma)$ constructed by R_Γ. Set up

$$C_{\Gamma 0} = \{\theta \in \mathrm{Aut} M : \theta(\lambda_\gamma) = \lambda_\gamma, \ \gamma \in \Gamma\}.$$

Lemma 2.4. *If $\theta \in C_{\Gamma_0}$, then the automorphism θ maps the algebra A into itself, i.e.*

$$C_{\Gamma_0} \subset \mathrm{Aut}(M, A).$$

Proof. Let $\widehat{K}$ be the dual space of the group K. Denote by $\widehat{K}_L$ the set of irreducible representations $\sigma \in \widehat{K}$ which admit a nontrivial L-invariant vector in the space of representations H_σ. For each $\sigma \in \widehat{K}$ we select an orthonormal basis $\{\xi_i^\sigma : i = 1, \cdots, \dim \sigma\}$ of H_σ in such a way that the first q_σ its elements form a basis of the space of L-invariant vectors. Determine the matrix elements of the representation σ as

$$\sigma_{ij}(k) = (\sigma(k)\xi_j^\sigma \mid \xi_j^\sigma), \quad i, j = 1, \cdots, \dim \sigma. \tag{2.2}$$

Then $\sigma_{ij}(kh) = \sigma_{ij}(k)$ if $h \in L$, $k \in K$ and $j \leq q_\sigma$. Consequently, for $j \leq q_\sigma$ the matrix elements σ_{ij} can be considered as functions on the homogeneous space $X = K/L$. The system

$$\{\sigma_{ij} : \sigma \in \widehat{K}_L, \ i = 1, \cdots, \dim \sigma, \ j = 1, \cdots, q_\sigma\}$$

is an orthogonal basis in $L^2(X, \nu)$ with $\sigma_{ij} \in A = L^\infty(X, \nu)$.

For $\gamma \in \Gamma$ we have

$$\sigma_{ij}(\gamma^{-1}k) = \sum_{p=1}^{\dim\sigma} \sigma_{ip}(\gamma^{-1})\sigma_{pj}(k), \text{ i.e. } \widetilde{l}_\gamma(\sigma_{ij}) = \sum_{p=1}^{\dim\sigma} \overline{\sigma_{pi}(\gamma)}\sigma_{pj}. \tag{2.3}$$

Let $\theta \in C_{\Gamma_0}$ and

$$\theta(\pi(\sigma_{ij})) = \sum_{g\in\Gamma} \sum_{\rho\in\widehat{K}_L} \sum_{m=1}^{\dim\rho} \sum_{n=1}^{q\sigma} c(g; m, n, \rho)\pi(\rho_{mn})\lambda_g. \tag{2.4}$$

Since the algebra A is generated by the matrix elements σ_{ij}, it suffices to prove that

$$\theta(\pi(\sigma_{ij})) \in \pi(A), \sigma \in \widehat{K}_L,$$
$$i = 1, \cdots, \dim\sigma, \ \ j = 1, \cdots, q_\sigma.$$

By (2.3), the equality

$$\lambda_\gamma\theta(\pi(\sigma_{ij}))\lambda_\gamma^* = \theta(\pi(\widehat{l}_\gamma(\sigma_{ij}))), \gamma \in \Gamma_0,$$

the ICC-property, and convergence of series (2.4), we can obtain $c(g; m, n, \rho) = 0$ for $g \neq e$ and hence, $\theta(\pi(\sigma_{ij})) \in \pi(A)$. □

For $t \in N_K(L)$ denote by R_t the automorphism of the factor M given by

$$R_t(\pi(a)) = \pi(\widetilde{r}_t(a)), \ \ R_t(\lambda_\gamma) = \lambda_\gamma, \ \ a \in A, \ \ \lambda\gamma \in \Gamma.$$

Besides, given a cocycle $c \in Z^1(\Gamma, U(A))$ we define an automorphism θ_c of M by setting

$$\theta_c(\pi(a)) = \pi(a), \ \theta_c(\lambda_\gamma) = \pi(c_\gamma)\lambda_\gamma$$

(see [FM 2]).

Lemma 2.5. *Let $\theta \in C_{\Gamma_0}$. Then there exist $t \in N_K(L)$ and $c \in Z^1(\Gamma, U(A))$ such that $\theta = \theta_c R_t$. The cocycle c here is defined uniquely while t is defined up to an element of L.*

Proof. In view of Lemma 2.4, θ determines the automorphism $\widetilde{\beta}$ of the algebra $A : \pi \circ \widetilde{\beta} = \theta \circ \pi$. By the assumption, $\theta(\lambda_\gamma) = \lambda_\gamma$ and so $\widetilde{\beta}\widetilde{\lambda}_\gamma = \widetilde{\lambda}_\gamma\widetilde{\beta}$, $\gamma \in \Gamma$. According to 2.2, $\widetilde{\beta} = \widetilde{r}_t$ for a some $t \in N_K(L)$. Set up $\theta_1 = \theta R_t^{-1}$.

Then $\theta_1(\pi(a)) = \pi(a)$, $a \in A$. Therefore there is $c \in Z^1(\Gamma, U(A))$, such that $\theta_1 = \theta_c$ (see [FM 2]). Hence, $\theta = \theta_c R_t$. □

Lemma 2.6. *Let Γ be an* ICC-*group, (X, μ) a free ergodic Γ-space with finite invariant measure and $\theta: X \to X$ an automorphism such that $\theta(\gamma x) = \gamma\theta(x)$. If $\theta \neq$ id, then θ is an outer automorphism of the equivalence relation R_Γ.*

Proof. Suppose that $\theta \in \mathrm{Int}R_\Gamma$. Set up

$$E_\alpha = \{x \in X : \theta(x) = \alpha x\}.$$

Then $\gamma E_\alpha = E_{\gamma\alpha\gamma^{-1}}$ and, hence,

$$\mu(E_{\gamma\alpha\gamma^{-1}}) = \mu(E_\alpha),\ \gamma \in \Gamma.$$

Since by our assumption $\theta \in \mathrm{Int}R_\Gamma$, there is $\gamma_0 \in \Gamma$ with $\mu(E_{\gamma_0}) > 0$. However, the set $\{\gamma\gamma_0\gamma^{-1} : \gamma \in \Gamma\}$ is infinite if $\gamma_0 \neq e$. Therefore $\gamma_0 = e$. Consequently, $\gamma E_e = E_e$ for all $\gamma \in \Gamma$, i.e., $\mu(E_e) = \mu(X)$ and $\theta =$ id. □

Proof of Theorem 2.3.

Denote by $C^M_{\Gamma_0}$ the subgroup of Aut$M$ which is algebraically generated by C_{Γ_0} and Int$M$. According to 2.3 [GG 2], $C^M_{\Gamma_0}$ is an open subgroup of Aut$M$. Therefore $C^M_{\Gamma_0} \cap \mathrm{Aut}(M, A)$ is an open subgroup of Aut(M, A). Furthermore, in view of Lemma 2.4, $C_{\gamma_0} \subset \mathrm{Aut}(M, A)$ and, hence, the subgroup $C^M_{\Gamma_0} \cap$ Aut (M, A) is generated by C_{Γ_0} and $\mathrm{Int}(M, A) = \mathrm{Aut}(M, A) \cap \mathrm{Int}M$. Using Lemma 2.5 we deduce that the subgroup generated algebraically by Int(M, A), $Z(M, A) \approx Z^1(\Gamma, U(A))$, and $\{R_t : t \in N_K(L)\}$ is open in Aut(M, A). Consequently, in view of Lemma 2.1 [GG 2], the subgroup generated by IntR_Γ and $\{r_t : t \in N_k(L)\}$ is open in AutR_Γ. Now it should be noted that

$$\{r_t : t \in N_K(L)\} \cap \mathrm{Int}R_\Gamma = \{\mathrm{id}\}$$

(see Lemma 2.6). Thus, $\varepsilon(\{r_t : t \in N_k(L)\})$ is an open subgroup of OutR_Γ isomorphic algebraically to $N_K(L)/L$. Since OutR_Γ is a Polish group and ε is continuous, the groups $\varepsilon(\{r_t : t \in N_k(L)\})$ and $N_k(L)/L$ are topologicaly isomorphic. □

Remark 2.7. Note the two significant cases where the group Γ is an ICC-group with respect to the subgroup Γ_0 [GG 2, 1.6 and 3.7]:

(a) Γ is isomorphic to the subgroup in the connected semisimple Lie group with trivial center and without compact factors, Γ_0 being a lattice in this

Lie group;

(b) K is a connected group with trivial center.

Remark 2.8. The group OutR_Γ can be noncompact. For example, let

$$K=SL(n,\mathbf{Z}_p),\ \Gamma=SL(n,\mathbf{Z}),\ n=2j+1\geq 3,\ L=\{e\}$$

($\mathbf{Z}_p$ is the ring of p-adic integers). It can be shown that OutR_Γ contains an open subgroup which is topologically isomorphic to the group $H=SL(n,\mathbf{Q}_p)$. Let $\Lambda=SL(n,\mathbf{Z}[\frac{1}{p}])$. Then Λ is a dense subgroup in H and $K\cap\Lambda=\Gamma$. Consider the ergodic equivalence relation R_Λ of the type II_∞ generated by left translations of Λ on the space $(H,\tilde{\mu})$ where $\tilde{\mu}$ is the Haar mesure on H. Since K is an open subgroup in H, $R_\Lambda\approx R_\Gamma\times\mathrm{I}_\infty$ where I_∞ is the transitive equivalence relation on H/K. Besides, the map $\phi: K\times H/K\to H$ which is responsible for isomorphism of $R_\Gamma\times\mathrm{I}_\infty$ and R_Λ can be chosen so that

$$\phi(r_t\times\mathrm{id})\phi^{-1}=r_t\quad\text{for}\quad t\in K.$$

For $h\in H$, denote the right translation on the space $(H,\tilde{\mu})$ by $r_h: r_h(s)=sh^{-1}$, $s\in H$. We claim that if $h\neq e$, r_h is an outer automorphism of R_Λ. In fact, assume that $r_h\in\mathrm{Int}R_\Lambda$. Then there exists $\alpha\in\Lambda$ for which the set $E_\alpha=\{s\in H: sh^{-1}=\alpha s\}$ has the positive measure. Consider the closed subgroup F_α generated by the set $\{s_2^{-1}s_1: s_1, s_2\in E_\alpha\}$. If $s_1, s_2\in E_\alpha$, then $s_1h^{-1}=\alpha s_1$, $hs_2^{-1}=s_2^{-1}\alpha^{-1}$, and hence $hs_2^{-1}s_1h^{-1}=s_2^{-1}s_1$, i. e. h lies in the subgroup F_α centralizer. Since F_α is a closed subgoup of positive measure, F_a is open in H. However, in this case F_α contains an open compact subgroup of the form $K(m)=\overline{\Gamma(m)}$, where

$$\Gamma(m)=\{\gamma\in\Gamma:\equiv 1\ (\mathrm{mod}\ p^m)\}.$$

Here $\Gamma(m)$ is the lattice in $SL(n,\mathbf{R})$. Consequently, groups $\Gamma(m)$ and $F_\alpha\supset\Gamma(m)$ are Zariski dense in $H=SL(n,\mathbf{Q}_p)$ [Wan]. Hence, we obtain that $h\in Z(H)$, and it means that $h=\alpha^{-1}\in\Lambda$, i.e. $h\in Z(\Lambda)=\{e\}$. Thus, H is embedded into Out$R_\Lambda\approx$ Out$(R_\Gamma\times\mathrm{I}_\infty)\approx$ OutR_Γ (see [GG 2, Corollary B.3]).

It should be noted that if K is a connected group and $L=\{e\}$, OutR_Γ is a compact group [GG 2, Theorem 4.9]. Using the D. Witte's results [Wit] the following section will give a proof of the compactness of OutR_Γ for the action of Γ on a homogeneous space of a connected Lie group.

§3. Automorphisms of the Equivalence Relation Generated by Lattice Translations in the Lie Group on a Homogeneous Space of a Compact Lie Group

This Section is concerned with automorphisms of the equivalence relation constructed by the Γ-action on the space $X=K/L$ in the most important case where Γ is a lattice in a simple Lie group and K a compact simple Lie group. We obtain the explicit description of automorphisms (Theorem 3.3) using the D. Witte's argument [Wit, 9.4].

We need the following general statement which can be deduced from R. Zimmer's rigidity theorem [Zim 2, 4.3 and 4.5].

Proposition 3.1. *Let G be a connected noncompact simple Lie group with trivial center and $\mathbb{R}$-rank $(G) \geq 2$, Γ a lattice in G and (X, ν) a free ergodic Γ-space with the finite invariant measure. Let $\theta \in \mathrm{Aut}R_\Gamma$ and $\alpha: X \times \Gamma \to \Gamma$ be the cocycle corresponding to θ, i.e. $\theta(\gamma x)=\alpha(x, \gamma)\theta(x)$ for all $\gamma \in \Gamma$ and a. a. $x \in X$. Then there exist Borel function $\phi: X \to G$ and $\sigma_1 \in \mathrm{Aut}G$ such that $\alpha(x, \gamma)=\phi(\gamma x)^{-1}\sigma_1(\gamma)\phi(x)$.*

Proof. Using the action of Γ, let us construct that of G (the induced action [Zim 1]) and an automorphism $\widetilde{\theta} \in \mathrm{Aut}R_G$ corresponding to $\theta \in \mathrm{Aut}R_\Gamma$. Let $\pi: G \to \Gamma\backslash G$ be the natural projection, $\pi(g)=\Gamma g$, and $\omega: \Gamma\backslash G \to G$ its Borel section, i.e., $\pi \circ \omega=\mathrm{id}$. Determine Borel maps $s: G \to G$ and $f: G \to \Gamma$ as follows

$$s(g)=\omega(\pi(g)),\ f(g)=gs(g)^{-1},\ \ g \in G.$$

Set up $S=\omega(\Gamma\backslash G)=s(G)$. Then S is a Borel set, and each element g of G is uniquely represented in the form

$$g=f(g)s(g),\ \ f(g) \in \Gamma,\ \ s(g) \in S. \tag{3.1}$$

Now consider the diagonal action of Γ on the space $X \times G$, i.e.

$$\gamma(x, h)=(\gamma x, \gamma h),\ \ \gamma \in \Gamma,\ \ x \in X,\ \ h \in G.$$

The orbit partition of $X \times G$ of this action is measurable, and the orbit space $Y=(X \times G)/\Gamma$ can be identified with the space $X \times (\Gamma\backslash G)$ via the Borel isomorphism

$$\Phi: X \times (\Gamma\backslash G) \to (X \times G)/\Gamma$$

given by

$$\Phi((x, \pi(h))) = [x, s(h)], \tag{3.2}$$

where $[x, h]$ is the image of $(x, h) \in X \times G$ in the space $Y = (X \times G)/\Gamma$. It should be noted that

$$[\gamma x, \gamma h] = [x, h] = [f(h)^{-1}x, s(h)], \ x \in X, \ h \in G, \ \gamma \in \Gamma. \tag{3.3}$$

Using the isomorphism Φ, we introduce the measure $\rho = \Phi_*(\nu \times \lambda)$ on Y, where λ is the finite G-invariant measure on $\Gamma \backslash G$. Now determine the induced action of group G on (Y, ρ) [Zim1]:

$$g[x, h] = [x, hg^{-1}], \ x \in X, \ h \in G. \tag{3.4}$$

It can be easily verified that Φ transforms this action to the following action on the space $X \times (\Gamma \backslash G)$ ([Zim 1, 2.2]):

$$g(x, \pi(h)) = (c(g, \pi(h))x, \pi(h)g^{-1}),$$

where $c(g, \pi(h)) = s(hg^{-1})gs(h)^{-1} \in \Gamma$, $h \in G$, $x \in X$. Thus, the action of G on (Y, ρ) is free, properly ergodic, and preserves the finite measure ρ [Zim 1,2.7]. Denote by R_G the equivalence relation generated by the action of G on (Y, ρ). Let I be the transitive equivalence relation generated by translations of G on $\Gamma \backslash G$. It can be readily seen that Φ (see (3.2)) carries out isomorphism of $R_\Gamma \times I$ and R_G. Using the automorphism $\theta \in \mathrm{Aut} R_\Gamma$, one can construct an automorphism $\widetilde{\theta} \in R_G$: $\widetilde{\theta} = \Phi(\theta \times \mathrm{id})\Phi^{-1}$. The direct calculation shows that

$$\widetilde{\theta}([x, h]) = [\theta(f(h)^{-1}x), s(h)] \tag{3.5}$$

(see (3.1)-(3.3)).

Consider the cocycle $\widetilde{\alpha}: Y \times G \to G$ associated to $\widetilde{\theta}$:

$$\widetilde{\theta}(gy) = \widetilde{\alpha}(y, g)\widetilde{\theta}(y), \ g \in G, \ y \in Y.$$

In view of the Zimmer's rigidity theorem, $\widetilde{\alpha}$ is cohomologous to the restriction of an automorphism of G [Zim 2]. Thus, there exist a Borel function $\psi: (X \times G)/\Gamma \to G$ and $\widetilde{\sigma}_1 \in \mathrm{Aut} G$ are such that

$$\widetilde{\alpha}(y, g) = \psi^{-1}(gy)\widetilde{\sigma}_1(g(\psi(y)))$$

for any $g \in G$ at a.a. $y \in Y$. Set up

$$\widetilde{\theta}_1(y) = \psi(y)\,\widetilde{\theta}(y),\ y \in Y.$$

According to [Zim 2, Prop. 2.4] and [Zim 3, Lemma 3.5], $\widetilde{\theta}_1$ is an automorphism of (Y, ρ) and

$$\widetilde{\theta}_1(gy) = \widetilde{\sigma}_1(g)\,\widetilde{\theta}_1(y) \tag{3.6}$$

for each $g \in G$ at a.a. $y \in Y$. Herewith, $\widetilde{\theta}_1 \in \mathrm{Aut}R_G$ and $\widetilde{\theta}\,\widetilde{\theta}_1^{-1} \in \mathrm{Int}R_G$. For $g \in G$ and $y = [x, h] \in Y$ we have (see (3.4) and (3.5)):

$$\begin{aligned}\widetilde{\theta}_1(gy) &= \psi(gy)\,\widetilde{\theta}_1(gy) = \psi([x, hg^{-1}])\,\widetilde{\theta}([x, hg^{-1}])\\ &= \psi([x, hg^{-1}])\,[\theta(f(hg^{-1})^{-1}x),\ s(hg^{-1})]\\ &= [\theta(f(hg^{-1})^{-1}x),\ s(hg^{-1})\,\psi([x, hg^{-1}])^{-1}].\end{aligned}$$

On the other hand,

$$\begin{aligned}\widetilde{\sigma}_1(g)\,\widetilde{\theta}_1(g) &= \widetilde{\sigma}_1(g)\,\psi([x, h])\,[\theta(f(h)^{-1}x),\ s(h)]\\ &= [\theta(f(h)^{-1}x),\ s(h)\,\psi([x, h])^{-1}\,\widetilde{\sigma}_1(g)^{-1}].\end{aligned}$$

Now we obtain from (3.6)

$$\begin{aligned}&[\theta(f(hg^{-1})^{-1}x,\ s(hg^{-1}\psi([x, hg^{-1}])^{-1})]\\ &\qquad = [\theta(f(h)^{-1}x),\ s(h)\,\psi([x, h])^{-1}\sigma_1(g)^{-1}]\end{aligned} \tag{3.7}$$

for all $g \in G$ and a.a. $[x, h] \in Y$.

Since $\theta(\gamma x) = \alpha(x, \gamma)\,\theta(x)$, we rewrite (3.7) as follows

$$\begin{aligned}&[\alpha(x, f(hg^{-1})^{-1})\,\theta(x),\ s(hg^{-1})\,\psi([x, hg^{-1}])^{-1}]\\ &\quad = [\alpha(x, f(h)^{-1})\,\theta(x),\ s(h)\,\psi([x, h])^{-1}\,\widetilde{\sigma}_1(g)^{-1}],\end{aligned}$$

or

$$\begin{aligned}&[\theta(x),\ \alpha(x, f(hg^{-1})^{-1})^{-1}\ s(hg^{-1})\,\psi([x, hg^{-1}])^{-1}]\\ &\quad = [\theta(x),\ \alpha(x, f(h)^{-1})^{-1}\ s(h)\,\psi([x, h])^{-1}\,\widetilde{\sigma}_1(g)^{-1}].\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\alpha(x, f(hg^{-1})^{-1})^{-1}s(hg^{-1})\,\psi([x, hg^{-1}])^{-1}\\ &\quad = \alpha(x, f(h)^{-1})^{-1}s(h)\,\psi([x, h])^{-1}\,\widetilde{\sigma}_1(g)^{-1},\end{aligned}$$

or

$$\alpha(x, f(hg^{-1})^{-1})^{-1}s(hg^{-1})\psi([x, hg^{-1}])^{-1}\tilde{\sigma}_1(hg^{-1})^{-1}$$
$$= \alpha(x, f(h)^{-1})^{-1}s(h)\psi([x, h])^{-1}\tilde{\sigma}_1(h)^{-1} \tag{3.8}$$

for all $g \in G$ and a.a. $[x, h] \in Y$. Then for a.a. $x \in X$ (3.8) is satisfied at a.a. $g, h \in G$. Consequently, for a.a. $x \in X$ we have

$$\alpha(x, f(h)^{-1})^{-1}s(h)\psi([x, h])^{-1}\tilde{\sigma}_1(h)^{-1} = c(x) \text{ at a. a. } h \in G. \tag{3.9}$$

Since the set $S \subset G$ has positive Haar measure (see (3.1)), it follows that for a. a. $x \in X$

$$\alpha(x, f(\gamma^{-1}s(h))^{-1})^{-1}s(\gamma^{-1}s(h))\psi([x, \gamma^{-1}s(h)])^{-1}\tilde{\sigma}_1(\gamma^{-1}s(h))^{-1}$$
$$= \alpha(x, f(s(h))^{-1})^{-1}s(s(h))\psi([x, s(h)])^{-1}\tilde{\sigma}_1(s(h))^{-1}$$

for all $\gamma \in \Gamma$ and a. a. $s(h) \in S$ (we have substituted $h_1 = \gamma^{-1} s(h)$ and $h_2 = s(h)$ into (3.9)). Now it should be noted that

$$f(\gamma^{-1}s(h)) = \gamma^{-1},\ s(\gamma^{-1}s(h)) = s(h),\ \psi([x, \gamma^{-1}s(h)]) = \psi([\gamma x, s(h)]).$$

Hence,

$$\alpha(x, \gamma)^{-1}s(h)\psi([\gamma x, s(h)])^{-1}\tilde{\sigma}_1(s(h))^{-1}\tilde{\sigma}_1(\gamma)$$
$$= s(h)\psi([x, s(h)])^{-1}\tilde{\sigma}_1(s(h)),$$

or

$$\alpha(x, \gamma) = s(h)\psi([\gamma x, s(h)])^{-1}\tilde{\sigma}_1(s(h)^{-1}\gamma s(h))\psi([x, s(h)]s(h)^{-1} \tag{3.10}$$

for all $\gamma \in \Gamma$ and a. a. $x \in X$ and $h \in G$.

Take an element $s(h) \in S$ such that (3.10) is satisfied for all $\gamma \in \Gamma$ and a. a. $x \in X$ and put

$$\phi(x) = \psi([x, s(h)])s(h)^{-1},\ \sigma_1(g) = \tilde{\sigma}_1(s(h)^{-1}gs(h)).$$

Then $\alpha(x,\gamma) = \phi(\gamma x)^{-1}\sigma_1(\gamma)\phi(x)$ for all $\gamma \in \Gamma$ and a. a. $x \in X$. □

While studying the group OutR_Γ, we need an interesting analogue of Lemma 2.6.

Lemma 3.2. *Let Γ be an irreducible lattice in a connected semisimple Lie group G with trivial center, without compact factors, and $\mathbb{R}$-rank$(G) \geq 2$, (X, μ) a free ergodic Γ-space with finite invariant measure and $\theta : X \to X$ an automorphism such that $\theta(\gamma x) = \sigma(\gamma)\theta(x)$ for some $\sigma \in \mathrm{Aut}\Gamma$. If θ is an inner automorphism of the equivalence relation R_Γ, there is an element $\gamma_0 \in \Gamma$ such that $\theta(x) = \gamma_0 x$ and $\sigma(\gamma) = \gamma_0\gamma\gamma_0^{-1}$.*

Proof. We for $\alpha \in \Gamma$ set up $E_\alpha = \{x \in X : \theta(x) = \alpha x\}$. Then $\gamma E_\alpha = E_{\sigma(\gamma)\alpha\gamma^{-1}}$ and, thus,

$$\mu(E_{\sigma(\gamma)\alpha\gamma^{-1}}) = \mu(E_\alpha), \ \ \gamma \in \Gamma.$$

Since $\theta \in \mathrm{Int}R_\Gamma$, there is $\gamma_0 \in \Gamma$ with $\mu(E_{\gamma_0}) > 0$. Consequently, the set $\{\sigma(\gamma)\gamma_0\gamma^{-1} : \gamma \in \Gamma\}$ is finite. Let

$$\Gamma_0 = \{\gamma \in \Gamma : \sigma(\gamma) = \gamma_0\gamma\gamma_0^{-1}\} = \{\gamma \in \Gamma : \sigma(\gamma)\gamma_0\gamma^{-1} = \gamma_0\}.$$

Then Γ_0 is a finite index subgroup in Γ. Therefore Γ_0 and $\sigma(\Gamma_0)$ are irreducible lattices in Γ. According to the Mostow-Margulis rigility theorem [Zim 4, 5.1.1], σ is uniquely extended up to an automorphism of G, i. e. $\sigma(g) = \gamma_0 g\gamma_0^{-1}$, $g \in G$. In particular, $\sigma(\gamma) = \gamma_0\gamma\gamma_0^{-1}$, $\gamma \in \Gamma$. Consequently, $\gamma E_{\gamma_0} = E_{\gamma_0}$ for all $\gamma \in \Gamma$, i. e. $\mu(E_{\gamma_0}) = \mu(X)$ and $\theta(x) = \gamma_0 x$. □

Now turn again to the equivalence relation R_Γ generated by left translations of the group Γ on the homogeneous space $X = K/L$. Let

$$\mathrm{Aut}(K, \Gamma) = \{\sigma \in \mathrm{Aut}K : \sigma(\Gamma) = \Gamma\},$$
$$Q = \{(\sigma, t) : \sigma \in \mathrm{Aut}(K, \Gamma),\ t\sigma(L)t^{-1} = L\}\cdot$$

For $(\sigma, t) \in Q$ the automorphism $r_{(\sigma,t)}$ of the space (X, ν):

$$r_{(\sigma,t)}(kL) = \sigma(k)t^{-1}L$$

is well defined. Since $\sigma(\Gamma) = \Gamma$, $r_{(\sigma,t)} \in \mathrm{Aut}R_\Gamma$.

The next theorem gives the explicit description of the automorphisms of R_Γ, which is similar to [GG 2, Theorem 4.4]. We use the argument of D. Witte [Wit, 9.4] in its proof.

Theorem 3.3. *Let K be a connected compact simple Lie group with trivial center, L its closed subgroup, and Γ the lattice in a connected simple Lie group G with trivial center, and $\mathbb{R}$-rank$(G) \geq 2$. Suppose that Γ is densely embedded into*

K, and consider the equivalence relation R_Γ generated by left translations of Γ on the homogeneous space $X=K/L$. Let $\theta \in$ AutR_Γ. *Then there exist $t \in K$, $\sigma \in$* Aut (K, Γ) and *$w \in$* IntR_Γ *such that $(\sigma, t) \in Q$* and *$\theta = wr_{(\sigma,t)}$.*

Proof. According to Lemma 2.1, Γ acts on (X, ν) freely. Consider the G-action induced by that of Γ. Let $\Delta : \Gamma \rightarrow K \times G$ be the diagonal embedding. Then the space of the induced action $(X \times G)/\Gamma$ can be identified with the double coset space $\Delta(\Gamma)\backslash(K \times G)/L$ (we also denote the subgroup $L \times \{e\} \subset K \times G$ by L). The identification is given by the map

$$\begin{aligned} &\mathrm{id}: (X \times G)/\Gamma \rightarrow \Delta(\Gamma)\backslash(K \times G)/L, \\ &\mathrm{id}([x, h]) = \Delta(\Gamma)(k, h)L, \ x = kL. \end{aligned} \tag{3.11}$$

Now the induced action coincides with that of G by right translations on $\Delta(\Gamma)\backslash(K \times G)/L$ (see (3.4)), i. e. $gy = \Delta(\Gamma)(k, hy^{-1})L$, where $y = \Delta(\Gamma)(k, h)L$. Let $\theta \in \mathrm{Aut}R_\Gamma$, $\alpha : X \times \Gamma \rightarrow X$ be the cocycle corresponding to θ and $\widetilde{\theta}_1 \in \mathrm{Aut}R_G$ the automorphism constructed while proving Proposition 3.1 (see (3.5) and (3.4)):

$$\widetilde{\theta}_1([x, h]) = \psi([x, h])\,\widetilde{\theta}([x, h]) = [\theta(f(h)^{-1}x), s(h)\psi([x, h])^{-1}], \quad [x, h] \in Y. \tag{3.12}$$

Then,

$$\widetilde{\theta}_1(gy) = \widetilde{\sigma}_1(g)\,\widetilde{\theta}_1(y) \tag{3.13}$$

for all $g \in G$ and a. a. $y \in Y$. Since (Y, ρ) is an ergodic G-space, $\widetilde{\theta}_1$ preserves the measure. Similarly to the proof of Corollary 9.4 [Wit], consider the following G-equivalent projections:

$$\begin{aligned} &p: \Delta(\Gamma)\backslash(K \times G)/L \rightarrow \Gamma\backslash G, \\ &p(\Delta(\Gamma)(k, h)L) = \Gamma h \ \text{ and} \\ &\pi_L: \Delta(\Gamma)\backslash(K \times G) \rightarrow \Delta(\Gamma)\backslash(K \times G)/L, \\ &\pi_L: (\Delta(\Gamma)(k, h)) = \Delta(\Gamma)(k, h)L. \end{aligned}$$

Taking into consideration identification (3.11), define the map $F: \Delta(\Gamma)\backslash(K \times G) \rightarrow \Gamma\backslash G$, $F = p \circ \widetilde{\theta}_1 \circ \pi_L$. In view of (3.13) F is an affine map for $\{e\} \times G$ [Wit, Def. 2.2], i. e.

$$F(\Delta(\Gamma)(k, h)g) = F(\Delta(\Gamma)(k, h))\,\widetilde{\sigma}_1(g)$$

at each $g \in G$ for a. a. $(k, h) \in K \times G$. Moreover, F preserves the measure.

Since $\Delta(\Gamma)$ is a lattice in $K\times G$ and $\Delta(\Gamma)$ is densely projected in K, in view of the reasoning given in the proof of Corollary 9.1 [Wit], F is an affine map for the entire group $K\times G$, i.e.

$$F(\Delta(\Gamma)(kt, hg)) = F(\Delta(\Gamma)(k, h))\,\delta(t, g),$$

where δ is an homomorphism from $K\times G$ to G with $\delta(e, g) = \tilde{\sigma}_1(g)$, $g \in G$. Hence, $\delta(t, g) = \tilde{\sigma}_1(g)$ and

$$F(\Delta(\Gamma)(kt, hg)) = F(\Delta(\Gamma)(k, h)\,\tilde{\sigma}_1(g) \tag{3.14}$$

at all $(t, g) \in K\times G$ for a. a. $(k, h) \in K\times G$. Consider the map $B: K\times G \to \Gamma\backslash G$, $F(\Delta(\Gamma)(k, h))\,\tilde{\sigma}_1(h)^{-1}$. Using (3.14) we obtain

$$\begin{aligned} B(kt, hg) &= F(\Delta(\Gamma)(kt, hg))\,\tilde{\sigma}_1(hg)^{-1} \\ &= F(\Delta(\Gamma)(k,h))\,\tilde{\sigma}_1(g)\,\tilde{\sigma}_1(h)^{-1} = B(k, h) \end{aligned}$$

for all $(t, g) \in K\times G$ and a. a. $(k, h) \in K\times G$. Consequently, there exists an element of $h_0 \in G$ such that $B(k, h) = \Gamma h_0$ for a. a. $(k, h) \in K\times G$. Thus, $F(\Delta(\Gamma)(k, h)) = \Gamma h_0 \tilde{\sigma}_1(h)$ for a. a. $(k, h) \in K\times G$. Furthermore, using (3.11), (3.12) and definitions of the projections p and π_L we obtain

$$\begin{aligned} F(\Delta(\Gamma)(k, h)) &= (p\cdot\tilde{\theta}_1)(\Delta(\Gamma)(k, h)L) \\ &= p([\theta(f(h)^{-1}x), s(h)\psi([x, h])^{-1}]) \\ &= \Gamma s(h)\psi([x, h])^{-1}, \end{aligned}$$

where $x = kL \in X$. Consequently,

$$\Gamma s(h)\psi([x, h])^{-1} = \Gamma h_0\,\tilde{\sigma}_1(h)$$

for a. a. $h \in G$ and $x \in X$. Hence, we obtain for a. a. $h = s(h) \in S$

$$h_0\tilde{\sigma}_1(s(h))\psi([x, s(h)]s(h)^{-1} \in \Gamma.$$

Set up $\psi_1([x, h]) = h_0\tilde{\sigma}_1(s(h))\psi([x, s(h)])s(h)^{-1}$. Then

$$\psi([x, s(h)]) = \tilde{\sigma}_1(s(h))^{-1}h_0^{-1}\psi_1([x, s(h)])s(h) \tag{3.15}$$

and $\psi_1([x, h]) \in \Gamma$ for a. a. $x \in X$ and $h \in G$. Now substitute (3.15) into (3.10) for the cocycle α (see the Proof of Proposition 3.1):

$$\alpha(x,\gamma)=s(h)\,\psi([\gamma x,\ s(h)])^{-1}\,\widetilde{\sigma}_1(s(h)^{-1}\gamma s(h))\,\psi([x,s(h)])\,s(h)^{-1}$$
$$=\psi_1([\gamma x,\ s(h)])^{-1}h_0\,\widetilde{\sigma}_1(\gamma)\,h_0^{-1}\psi_1([x,\ s(h)])$$

for all $\gamma\in$ and a. a. $x\in X$ and $h\in G$.

Fix the element $s(h)\in S$ such that (3.15) is satisfied for a. a. $x\in X$ and set up

$$\eta(x)=\psi_1([x,\ s(h)],\ \sigma_0(\gamma))=h_0\,\widetilde{\sigma}_1(\gamma)\,h_0^{-1}.$$

Then for all $\gamma\in\Gamma$ and a. a. $x\in X$ we obtain

$$\alpha(x,\ \gamma)=\eta(\gamma x)\,\sigma_0(\gamma)\,\eta(x),\ \eta(x)\in\Gamma \text{ and } \sigma_0\in \mathrm{Aut}\Gamma$$

since $\sigma_0(\gamma)=\eta(\gamma x)\,\alpha(x,\ \gamma)\,\eta(x)^{-1}\in\Gamma$.

Now let $\theta_1(x)=\eta(x)\,\theta(x)$. Then $\theta_1\in \mathrm{Aut}R_\Gamma$ and

$$\theta_1(\gamma x)=\sigma_0(\gamma)\,\theta_1(x). \tag{3.16}$$

Let us show that $\sigma_0\in \mathrm{Aut}\Gamma$ can be extended up to an automorphism $\sigma\in$ Aut $(K,\ \Gamma)$. Consider the representation U of K in the space $L^2(X,\ \nu)$:

$$(U_k\xi)(x)=\xi(k^{-1}x),\ \ k\in K,\ \ \ x\in X=K/L.$$

Define also the unitary operator V:

$$(V\xi)(x)=\xi(\theta_1(x)),\ \ x\in X.$$

Then from (3.16) we obtain that

$$U\gamma V=VU_{\sigma_0(\gamma)}\ \text{ or }\ U_{\sigma_0(\gamma)}=V^*U_\gamma V,\ \ \gamma\in\Gamma.$$

Set up $U_{\sigma(k)}=V^*U_kV$, $k\in K$. Since U is a strongly continuous representation and Γ is a dence subgroup in K, $U_{\sigma(k)}\in\{U_t:t\in K\}$. Thus, σ is an automorphism of K and $\sigma(\gamma)=\sigma_0(\gamma)$ for $\gamma\in\Gamma$, i.e. $\sigma\in$ Aut $(K,\ \Gamma)$. Furthermore, consider the Borel map

$$\beta:K\to X,\ \beta(k)=\sigma(k)^{-1}\theta_1(kL).$$

In view of (3.16)

$$\beta(\gamma k) = \sigma(\gamma k)^{-1}\theta_1(\gamma kL) = \beta(k)$$

for all $\gamma \in \Gamma$ and a. a. $k \in K$. Consequently, there exists an element $t \in K$ such that $\beta(k) = t^{-1}L$, i.e. $\theta_1(kL) = \sigma(k)t^{-1}L$ for a. a. $k \in K$. We show that $t\sigma(L)t^{-1} = L$. Let $l \in L$. Then there is $k \in K$ with $\theta_1(kL) = \sigma(k)t^{-1}L$ and $\theta_1((kl)L) = \sigma(kl)t^{-1}L$. Therefore $\sigma(k)t^{-1}L = \sigma(kl)t^{-1}L$, i.e. $t\sigma(l)t^{-1} \in L$. Consequently, $t\sigma(L)t^{-1} \subset L$. In view of the compactness of L we obtain $t\sigma(L)t^{-1} = L$. Thus, $\theta_1 = r_{(\sigma,t)}$, where $(\sigma, t) \in Q$. Now for $w = \theta\theta_1^{-1}$ we obtain $w \in \mathrm{Int}R_\Gamma$ and $\theta = wr_{(\sigma t)}$. □

Corollary 3.4. *Let all the conditions of Theorem* 3.3 *are satisfied. Then the group* $\mathrm{Out}R_\Gamma$ *is compact. Moreover, if L is a finite index in $N_K(L)$,* $\mathrm{Out}R_\Gamma$ *is finite.*

Proof. Set up $H_1 = N_K(L)/L$, $H_2 = N_K(\Gamma)/\Gamma$. Using the explicit form of the automorphism from $\mathrm{Aut}R_\Gamma$ obtained in Theorem 3.3 and Lemma 2.6, it can be readily verified that H_1 and H_2 are embedded into $\mathrm{Out}R_\Gamma$ as normal subgroups. Besides, in view of Lemma 3.2 $H_1 \cap H_2 = \{e\}$ in $\mathrm{Out}R_\Gamma$. Let us show that the group $((\mathrm{Out}R_\Gamma)/H_1)/H_2$ is finite. If (σ_1, t_1), $(\sigma_2, t_2) \in Q$, the automorphisms $r_{(\sigma_1,t_1)}$ and $r_{(\sigma_2,t_2)}$ differ by the right translation r_{t_0}, where $t_0 = t_1t_2^{-1} \in N_k(L)$. Further, $\mathrm{Out}K$ is a finite, therefore $N_k(\Gamma)$ is a finite index subgroup in $\mathrm{Aut}(K, \Gamma)$ (remind that $Z(K) = \{e\}$). Now the finiteness of $((\mathrm{Out}R_\Gamma)/H_1)/H_2$ follows from the fact that each element of $\mathrm{Out}R_\Gamma$ is the image of $r_{(\sigma,t)}$ where $(\sigma, t) \in Q$ (Theorem 3.3). To complete the proof, it suffices to show that the H_2 is finite. Since $Z(K) = Z(\Gamma) = \{e\}$, we have that H_2 is embedded into $\mathrm{Out}\Gamma$. However, it follows from the finiteness of $\mathrm{Out}G$ ([Bou, Ch. III, Section 9, Prop. 30 (ii)] and [Mos, p. 254]), $N_G(\Gamma)/\Gamma$ ([Rag, 5.17]) and the Mostow-Margulis rigidity theorem ([Zim 4, 5.1.1]) that $\mathrm{Out}\Gamma$ is finite. □

Corollary 3.5. *Assume that* $\mathrm{Aut}K = \mathrm{Int}K$, $N_k(L) = L$ *and* $N_k(\Gamma) = \Gamma$. *Then* $\mathrm{Aut}R_\Gamma = \mathrm{Int}R_\Gamma$, i. e. $\mathrm{Out}R_\Gamma$ *is trivial.*

Proof. Let $\theta \in \mathrm{Aut}R_\Gamma$. According to Theorem 3.3, $\theta = wr_{(\sigma,t)}$, where $w \in \mathrm{Int}R_\Gamma$ and $r_{(\sigma,t)} = \sigma(k)t^{-1}L$ with $\sigma \in \mathrm{Aut}(K, \Gamma)$ and $t\sigma(L)t^{-1} = L$. Since $\mathrm{Aut}K = \mathrm{Int}K$, we have $\sigma(k) = k_0kk_0^{-1}$, $k_0 \in K$. Hence, $k_0 \in N_k(\Gamma) = \Gamma$, $tk_0 \in NK(L) = L$, and $r_{(\sigma,t)}(kL) = k_0kk_0^{-1}t^{-1}L = k_0k(tk_0)^{-1}L = k_0kL$, i. e. $r_{(\sigma,t)} \in \mathrm{Int}R_\Gamma$. Consequently, $\theta = wr_{(\sigma,t)} \in \mathrm{Int}R_\Gamma$. □

§4. An Example of Ergodic Equivalence Relation without Outer Automorphisms

Let us proceed to the construction of an example of an equivalence relation without outer automorphisms.

Consider the quadratic form

$$f(x) = x_1^2 + x_2^2 + \cdots + x_{n-2}^2 - \sqrt{2}x_{n-1}^2 - \sqrt{2}x_n^2$$

where $n = 2l + 1 \geq 5$. Let G be a connected component of the identity in $SO(f, \mathbf{R})$ and $\Gamma_0 = SO(f, \mathbf{Z}[\sqrt{2}]) \cap G$. Then G is a connected simple Lie group with trivial center, $\mathbf{R}$-rank$(G) = 2$, and Γ_0 is a lattice in G (see [Zim 4, 5.2.12 and 6.1.5] and [Sul]). Denote the nontrivial automorphism of the field $\mathbf{Q}(\sqrt{2})$ by $\tau: \tau(a + b\sqrt{2}) = a - b\sqrt{2}$.

Consider now the quadratic form

$$f^\tau(x) = x_1^2 + x_2^2 + \cdots + x_{n-2}^2 + \sqrt{2}x_{n-1}^2 + \sqrt{2}x_n^2$$

and let $K = SO(f^\tau, \mathbf{R})$. Then K is a connected compact group which is isomorphic to $SO_n(\mathbf{R})$ and the group $SO(f, \mathbf{Z}[\sqrt{2}])$ is embedded into K via τ as a dense subgroup [Sul]. Since Γ_0 is a finite index subgroup in $SO(f, \mathbf{Z}[\sqrt{2}])$ it is also densely embedded into K. We will identity Γ_0 with its image in K.

Lemma 4.1.

(1) $N_k(\Gamma_0) \subset SO(f^\tau, \mathbf{Q}(\sqrt{2}))$;

(2) $N_G(\Gamma_0) \subset SO(f, \mathbf{Q}(\sqrt{2}))$.

Proof. We prove (1). Set up $H = SO(f^\tau, \mathbf{C})$. Then H is a simple algebraic group defined over the field $\mathbf{Q}(\sqrt{2})$. Let $k \in N_k(\Gamma_0)$. Since Γ_0 is dense in K, and K is Zariski dense in H, Γ_0 is also Zariski dense in H. Therefore the inner automorphism $h \to khk^{-1}$ of H is defined over $\mathbf{Q}(\sqrt{2})$ [Zim 4, 3.1.10]. Consequently, $\mathrm{Ad}k \in (\mathrm{Ad}(H))_{\mathbf{Q}(\sqrt{2})}$, where $\mathrm{Ad}(H)$ is the adjoint group. Since the center of H is trivial, $\mathrm{Ad}: H \to \mathrm{Ad}(H)$ is an isomorphism defined over $\mathbf{Q}(\sqrt{2})$. Consequently,

$$k \in H_{\mathbf{Q}(\sqrt{2})} = SO(f^\tau, \mathbf{Q}(\sqrt{2})).$$

Similarly proposition (2) is proved. □

Set up $\Gamma_1 = N_G(\Gamma_0)$. Then Γ_1 is a discrete subgroup and, hence, a lattice in G [Rag, 5.17]. Besides, $\Gamma_1 \subset SO(f, \mathbf{Q}(\sqrt{2}))$. In view of Lemma 4.1 we can identify Γ_1 with $N_K(\Gamma_0)$ via τ. For Γ_1 there are also embeddings

$$N_k(\Gamma_1) \subset SO(f^\tau, \mathbf{Q}(\sqrt{2})) \text{ and}$$
$$N_G(\Gamma_1) \subset SO(f, \mathbf{Q}(\sqrt{2})).$$

Continuing this construction, we obtain the sequence of lattices $\{\Gamma_m\}$ in G with $\Gamma_{m+1} = N_G(\Gamma_m)$. Moreover, all the groups $\{\Gamma_m\}$ are embedded into K via τ as dense subgroups, and we have $\Gamma_{m+1} = N_K(\Gamma_m)$. According to [Rag, 9.8], there is m_0 with $\Gamma_m = \Gamma_{m_0}$ for all $m \geq m_0$. Put $\Gamma = \Gamma_{m_0}$. Then $N_K(\Gamma) = \Gamma$. Further more, identifying K with $SO_n(\mathbf{R})$, we denote by L the subgroup in $SO_n(\mathbf{R})$ consisting of matrices of the form

$$\left[\begin{array}{ccc|c} & & & 0 \\ & * & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & \pm 1 \end{array}\right].$$

Consider the Γ-action on the homogeneous space $X = K/L$.

Theorem 4.2. *All the automorphisms of the equivalence relation R_Γ, are inner.*

Proof. According to [Die, ch. IV], Aut$K$ = Int$K$. Let us show that $N_k(L) = L$. Indeed, consider the projection $p =$

$$= \left[\begin{array}{ccc|c} & & & 0 \\ & 0 & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right].$$

Then

$$L = \{k \in K : kpk^{-1} = p\}.$$

If $k \in N_K(L)$, then for $l \in L$ we have $klk^{-1}p = pklk^{-1}$, i. e. $l(k^{-1}pk) = (k^{-1}pk)\,l$. Hence, $k^{-1}pk = p$, i. e. $k \in L$. Thus, all the conditions of Statement 3.5 are satisfied. □

Corollary 4.3. *Let* I_∞ *be the transitive equivalence relation on* $(\mathbf{Z}, \delta)$ *where* $\delta(\{n\}) = 1$. *Then all the automorphisms of the* II_∞*-equivalence relation* $\widetilde{R}_\Gamma = R_\Gamma$

$\times$ I_∞ *are inner.*

Proof. According to [GG 2, Corollary B.3] all the automorphisms of Aut $\widetilde{R}_\Gamma$ preserve the measure. Therefore, the embedding of AutR_Γ into Aut $\widetilde{R}_\Gamma$, $\theta \rightarrow \theta \times$ id generate an isomorphism of OutR_Γ and Aut $\widetilde{R}_\Gamma$. □

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